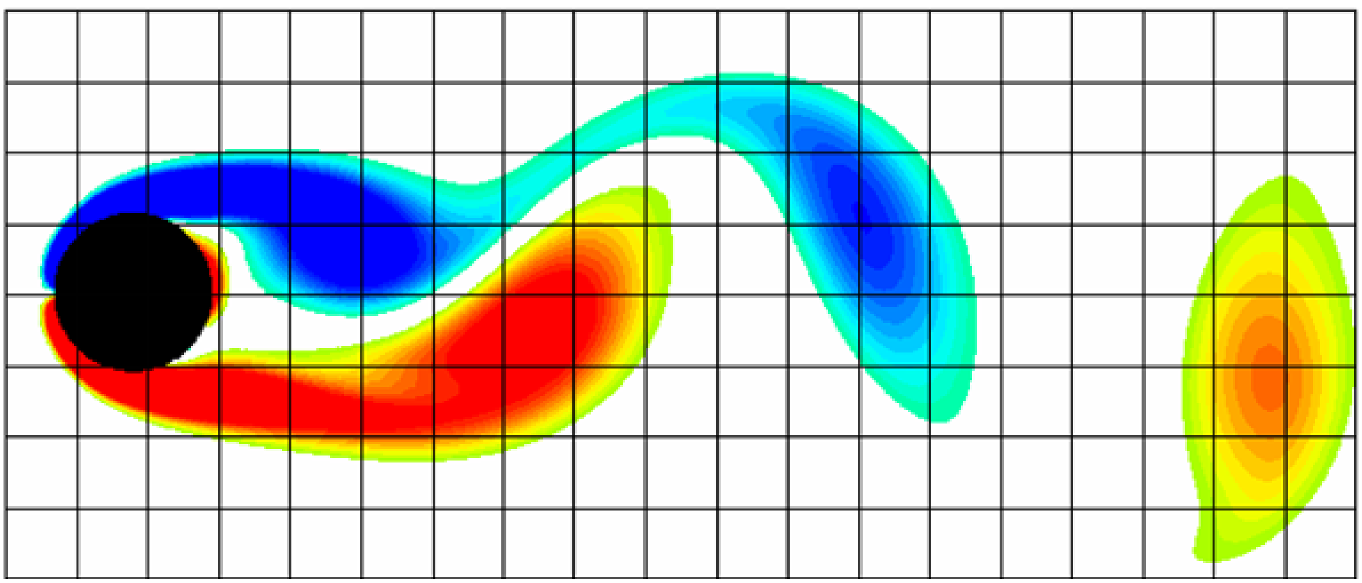
COMPUTATIONAL FLUID DYNAMICS

Solution to the Exercises - Master Energy & Mechanics



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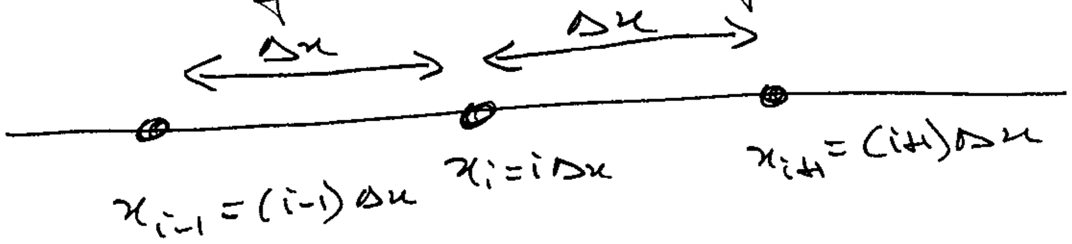
Exercise: FD formulas on uniform mesh



Notation: $u_i = u(x_i)$

1) The functions $u(x) = 1$, x and x^2 have the exact derivatives $\frac{d}{dx}u(x) = 0$, 1 and $2x$. Is it the same for the FD central derivatives δ_x^0 (and also for the second-order derivative δ_{xx}^0).

ⓐ First order central Derivative $\delta_x^0 u_i = \dfrac{u_{i+1} - u_{i-1}}{2\Delta x}$

- $u(x) = 1$

$$\delta_x^0(1) = \frac{(1)_{i+1} - (1)_{i-1}}{2\Delta x} = \frac{1-1}{2\Delta x} = 0 \Rightarrow \text{exact.}$$

- $u(x) = x$ (in this case $u_i = x_i = i\Delta x$)

$$\delta_x^0(x) = \frac{x_{i+1} - x_{i-1}}{2\Delta x} = \frac{(x_i + \Delta x) - (x_i - \Delta x)}{2\Delta x}$$

$$= 1 \Rightarrow \text{exact.}$$

- $u(x) = x^2$

$$\delta_x^0(x^2) = \frac{(x_i + \Delta x)^2 - (x_i - \Delta x)^2}{2\Delta x} = \frac{4\Delta x\, x_i}{2\Delta x}$$

$$= 2x_i \Rightarrow \text{exact.}$$

Conclusion: Why is the LTE $\tau_i = 0$ for these polynomials? Remember that: (2)

$$\delta^0_x u_i = \frac{d}{dx} u_i + \underbrace{\frac{\Delta x^2}{2}\left(\frac{\partial^3 u}{\partial x^3}\right)_i}_{\tau_i}$$

Hence, for polynomials $u(x) = x^k$ with $k \leq 2$, we have $\tau_i = 0$. When $k \geq 3$, the 3rd derivative is non-zero and thus central formula would give numerical errors ($\tau_i \neq 0$).

b) Second-order central derivative: $\delta^0_{xx} u_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}$

$$\delta^0_{xx}(1) = 0, \quad \delta^0_{xx}(x) = 0$$

$$\delta^0_{xx}(x^2) = \frac{(x_i + \Delta x)^2 - 2x_i + (x_i - \Delta x)^2}{\Delta x^2} = 2,$$

the exact derivatives are found.

Conclusion: For δ^0_{xx}, the LTE is $\tau_i = \frac{\Delta x^2}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_i$

Hence, $\tau_i = 0$ for polynomials of order $k \leq 3$.

2) Let us determine the truncation error τ_i:

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} = u''_i + \tau_i$$

by the use of Taylor Tables.

Reminder: $u_{i+1} = u(x_i + \Delta x) = u_i + \Delta x\, u'_i + \frac{\Delta x^2}{2!} + \frac{\Delta x^3}{3!} u_i^{(3)} \cancel{+ \frac{\Delta x^4}{4!}}$
$$+ \frac{\Delta x^4}{4!} u_i^{(4)} + O(\Delta x^5)$$

⚠ Taylor expansion should be done up to $O(\Delta x^4)$ because the exercise tells us that $\tau_i = O(\Delta x^2)$.

After a similar expansion for u_{i-1}, we obtain the table;

	$1\times u_{i+1}$	$-2u_i$	$1\times u_{i-1}$	$u_{i+1} - 2u_i + u_{i-1}$ (sum of the rows)
$O(1)$	$1\times u_i$	$-2\times u_i$	$1\times u_{i-1}$	0
$O(\Delta x)$	$1\times \Delta x\, u'_i$		$1\times -\Delta x\, u'_i$	0
$O(\Delta x^2)$	$1\times \frac{\Delta x^2}{2} u''_i$		$1\times \frac{\Delta x^2}{2} u_i^{(2)}$	$\frac{\Delta x^2}{2} u''_i$
$O(\Delta x^3)$	$1\times \frac{\Delta x^3}{6} u'''_i$		$1\times - \frac{\Delta x^3}{6} u_i^{(3)}$	0
$O(\Delta x^4)$	$1\times \frac{\Delta x^4}{24} u_i^{(4)}$		$1\times \frac{\Delta x^4}{24} u_i^{(4)}$	$\frac{\Delta x^4}{12} u_i^{(4)}$

From this table, we get:

$$u_{i+1} - 2u_i + u_{i-1} = \Delta x^2 u''_i + \frac{\Delta x^4}{12} u_i^{(4)} + O(\Delta x^6)$$

$$\Rightarrow \delta^0_{xx} u_i = u''_i + \underbrace{\frac{\Delta x^2}{12} u_i^{(4)}}_{\tau_i}$$

3)

$$\delta_x^-\left(\delta_x^+ u_i\right) = \delta_x^-\left[\frac{u_{i+1}-u_i}{\Delta x}\right] = \frac{1}{\Delta x}\left[\delta_x^-(u_{i+1}) - \delta_x^-(u_i)\right]$$

$$= \frac{1}{\Delta x}\left[\frac{u_{i+1}-u_i}{\Delta x} - \frac{u_i - u_{i-1}}{\Delta x}\right]$$

$$= \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}$$

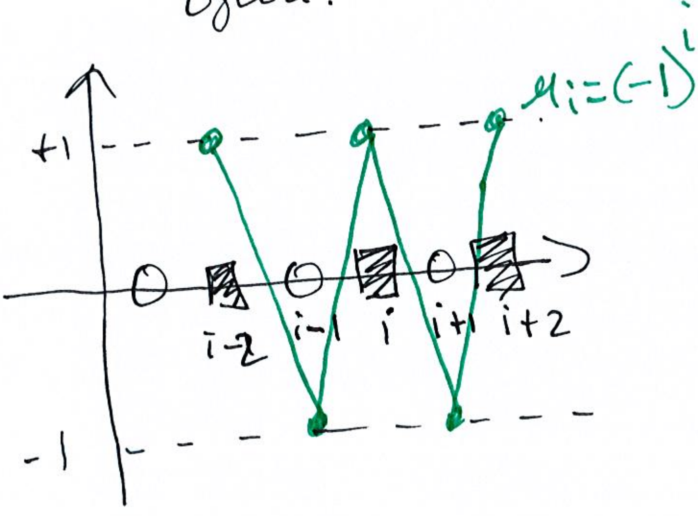
The formula $\delta_x^-\delta_x^+$ recovers, thus formula δ^0_{xx} of the lectures.

$$\delta_x^0\left(\delta_x^0 u_i\right) = \delta_x^0\left(\frac{u_{i+1}-u_{i-1}}{2\Delta x}\right) = \frac{1}{2\Delta x}\left[\delta_x^0 u_{i+1} - \delta_x^0 u_{i-1}\right]$$

$$= \frac{1}{2\Delta x}\left[\frac{u_{i+2}-u_i}{2\Delta x} - \frac{u_i - u_{i-2}}{2\Delta x}\right]$$

$$= \frac{u_{i+1} - 2u_i + u_{i-2}}{(2\Delta x)^2}$$

This formula correspond to the central formula δ^0_{xx} on the a mesh of twice the size $(2\Delta x)$. Thus, the formula decouples the odd and even indices on the grid.



For the oscillatory function $u_i = (-1)^i$:

$$\delta_x^0\left(\delta_x^0 u_i\right) = \frac{(-1)^{i+2} - 2(-1)^i + (-1)^{i-2}}{(2\Delta x)^2}$$

$$= 0.$$

The result is not physically realistic.

For the central formula of the Lecture:

$$\delta^0_{xx} u_i = \frac{(-1)^{i+1} - 2(-1)^{i} + (-1)^{i-1}}{\Delta x^2} = (-1)^i[-1-2-1]$$

$$= 4(-1)^{i+1} \neq 0.$$ The result is realistic.

4) Let's find the coefficients a, b, c of the formula:

$$\Delta^+_x u_i = a\, u_{i+2} + b\, u_{i+1} + c\, u_i . \qquad (1)$$

Such that:

$$\Delta^+_x u_i = \left(\frac{\partial u}{\partial x}\right)_i + \tau_i \qquad (2)$$

with a LTE τ_i as small as possible.

The Taylor table is:

	$a \times u_{i+2}$	$b \times u_{i+1}$	$c \times u_i$
	$\parallel$	$\parallel$	$\parallel$
$O(1)$	u_i	u_i	u_i
$O(\Delta x)$	$(2\Delta x)\, u'_i$	$\Delta x\, u'_i$	
$O(\Delta x^2)$	$\frac{(2\Delta x)^2}{2!} u''_i$	$\frac{\Delta x^2}{2!} u''_i$	
$O(\Delta x^3)$	$\frac{(2\Delta x)^3}{3!} u'''_i$	$\frac{\Delta x^3}{3!} u^{(3)}_i$	

From this table we get:

$$\Delta^+_x u_i = (a+b+c) u_i + (2a\Delta x + b\Delta x) u'_i + \left(\frac{4a}{3}\Delta x^3 + \frac{b}{6}\Delta x^3\right) u^{(3)}_i$$

$$+ \left(2a\Delta x^2 + b\frac{\Delta x^2}{2}\right) u''_i$$

From this table we get:

$$\Delta_x^+ u_i = \overbrace{(a+b+c)}^{=0} u_i + \overbrace{(2a+b)\Delta x}^{=1} u'_i + \overbrace{\left(2a+\frac{b}{2}\right)\Delta x^2}^{=0} u''_i + \underbrace{\left(\frac{4a}{3}+\frac{b}{6}\right)}_{\text{the LTE}} \Delta x^3 u^{(3)}_i + O(\Delta x^4) \qquad (3).$$

To obtain the 3 coefficients a, b, c, we need to write 3 ~~alge~~ algebraic equations from the first terms of Eq. (2). The 4th term will give the LTE.

By comparing (1) and (2), we get:

$$\begin{cases} a+b+c = 0 \\ 2a\cancel{\Delta x} + b\cancel{\Delta x} = 1/\Delta x \\ 2a\cancel{\Delta x^2} + \frac{b}{2}\cancel{\Delta x^2} = 0. \end{cases} \qquad (4)$$

and the truncation error is given by:

$$\tau_i = \left(\frac{4a}{3}+\frac{b}{6}\right)\Delta x^3 \left(\frac{\partial u}{\partial x}\right)_i + O(\Delta x^4). \qquad (5)$$

The solution of system (4) is:

$$a = -\frac{1}{2\Delta x}, \quad b = \frac{2}{\Delta x}, \quad c = -\frac{3}{2\Delta x}$$

which gives the FD formula:

$$\Delta_x^+ u_i = \frac{-u_{i+2}+4u_{i+1}-3u_i}{2\Delta u}$$

whose LTE is given by (5):

$$\tau_i = -\frac{\Delta u^2}{3}\left(\frac{\partial^3 u}{\partial u^3}\right)_i = O(\Delta u^2).$$

5) For a method of order p, we have:

$$\tau_i = O(\Delta x^p) = c\,\Delta x^p \; ; \; c \text{ is a constant.}$$

Thus: $\tau_i(\Delta x) = c(\Delta x)^p$, $\tau_i(\Delta x/2) = c\left(\frac{\Delta x}{2}\right)^p$

$\Rightarrow$ the reduction factor is $R = \dfrac{\tau_i(\Delta x)}{\tau_i(\Delta x/2)} = 2^p$

6) <u>Application of q5).</u>

$$\tau = c\,\Delta x^p \Rightarrow \underbrace{\log \tau}_{y} = p\,\underbrace{\log \Delta x}_{x} + \underbrace{\log c}_{b}$$

$\Rightarrow$ similar to $y = px + b$

$\Rightarrow$ In the log-log plot of Fig A.1, $\tau(\Delta x)$ is a straight line of slope p.

The table in Fig. A.1 is completed with the formula $R = \dfrac{\tau(\Delta x)}{\tau(\frac{\Delta x}{2})}$

	R for scheme A	R for scheme B
$\Delta x = 0{,}25$	—	—
$\Delta x/2$	$4.55/1.88 = 2.82$	$\frac{16{,}2}{8{,}85} = 1.83$
$\Delta x/4$	3.64	1.91
$\Delta x/8$	3.68	1.95
$\Delta x/16$	3.84	1.97

A method of order $p = 1 \Rightarrow R = 2$

order $p = 2 \Rightarrow R = 4$

For scheme A: $R \simeq 3.84 \Rightarrow$ Scheme A has order $p = 2$

—— B: $R \simeq 1.97 \Rightarrow$ Scheme B —— $p = 1$.

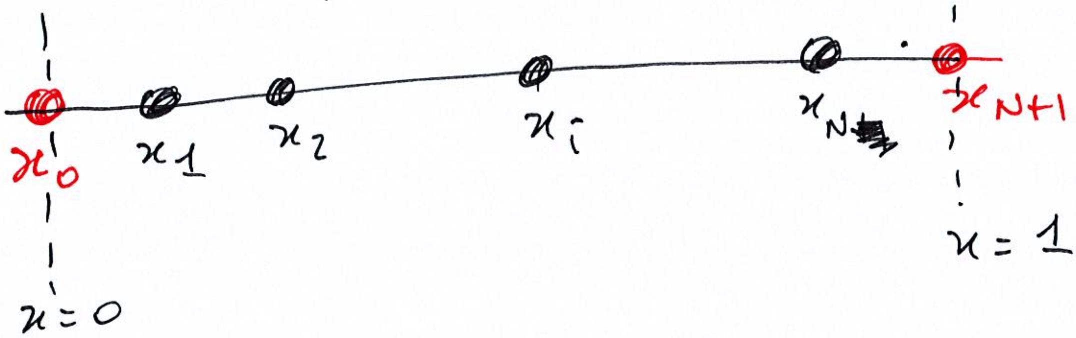
Exercise A.3

Laplace Eq:

$$\begin{cases} -u''(x) = f(x) \quad \text{in } \Omega =]0,1[& (1) \\ u(0) = u_L \quad , \quad u(1) = u_R & (2). \end{cases}$$

Remark: if $u_L = u_R = 0$, the Dirichlet BC would be homogeneous.

Mesh: $\Omega_i = \{ x_i = i\Delta x, \; i = 0, \dots N+1 \}$



1) The PDE (1) is discretized on the inner nodes:

$$\forall \; i = 1, \dots, N : \quad -\delta^0_{xx} u_i = f_i \qquad (3)$$

and the BCs are set on the boundary nodes x_0 and x_{N+1}:

$$u_0 = u_L \quad , \quad u_{N+1} = u_R .$$

At points x_1 and x_N, the scheme (3) needs to incorporate these BCs:

$$i = 1 \qquad -\frac{u_2 - 2u_1 + \overbrace{u_0}^{= u_L}}{\Delta x^2} = f_1$$

$$\Rightarrow \frac{1}{\Delta x^2} \left[-u_2 + 2u_1 \right] = f_1 + \frac{u_L}{\Delta x^2} . \qquad (4)$$

Similarly, we obtain on node $i = N$:

$$i = N \qquad \frac{1}{\Delta x^2} \left[2u_N - u_{N-1} \right] = f_N + \frac{u_R}{\Delta x^2} . \qquad (5)$$

② At points $i = 2, \dots, N-1$, the scheme is simply:

$$\frac{1}{\Delta x^2}\left[\underline{u_{i+1} - 2u_i + u_{i-1}} \right] = f_i \qquad (6).$$

$$\Leftrightarrow \frac{1}{\Delta x^2}\left[\underbrace{K_{i,i+1}}_{1} u_{i+1} + \underbrace{K_{i,i}}_{-2} u_i + \underbrace{K_{i,i-1}}_{1} u_{i-1} \right] = f_i .$$

The scheme can be written in tensor formulation:

$$(7) \qquad \frac{1}{\Delta x^2} \sum_{j=i-1}^{i+1} \underline{\underline{K}}_{i,j} u_j = \underline{F}_i \quad , \quad \forall i = 1, \dots, N.$$

with $\underline{\underline{K}} = \text{tridiag}(-1, 2, -1)$ a matrix of size N.

and the vector
$$\underline{F} = \begin{bmatrix} f_1 + u_L/\Delta x^2 \\ f_2 \\ \vdots \\ f_i \\ \vdots \\ f_{N-1} \\ f_N + u_L/\Delta x^2 \end{bmatrix}$$

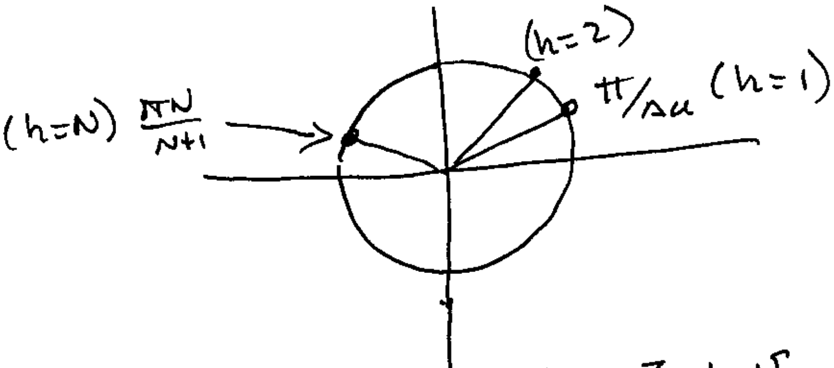
Eq (7) can also written as the linear system in matrix formulation:

$$\frac{1}{\Delta x^2} \underline{\underline{K}}\, \underline{U} = \underline{F} \quad , \text{with } \underline{U} = \begin{bmatrix} u_1 \\ \vdots \\ u_i \\ \vdots \\ u_N \end{bmatrix}$$

Each line of the system correspond to an inner node in the mesh.

2) For $a = c = -1$ and $b = 2$, according to the hint in the exercise:

For $k = 1, \dots, N$: $\lambda_k = 2 - 2\cos(k\pi\Delta x)$ $\quad \Delta x = \frac{1}{N+1}$.



We have $\cos(k\pi\Delta x) \in]-1, 1[$, hence:

- $\lambda_k > 0$, $k = 1 \dots N$: All eigenvalues of $\underline{\underline{K}}$ are strictly positive.
- $\underline{\underline{K}}$ is symetric: $\underline{\underline{K}} = \underline{\underline{K}}^T$

$\Rightarrow$ The matrix $\underline{\underline{K}}$ is symetric positive-definite. It can be efficiently solved by iterative methods such as Jacobi, Gauss-Seidel, or, better methods such as Conjugate Gradient.

3) $\chi_A = \frac{\lambda_{max}}{\lambda_{min}}$: the ratio of the maximal and minimal eigenvalues of $\underline{\underline{K}}$.

~~If χ_A close to~~ When χ_A is large, the iterative methods need more and more iterations to reach convergence, hence the ~~efficiency~~ performance of the method is reduced.

For $\underline{\underline{K}}$, according to 2) $\lambda_{max} = \lambda_N = 2 + 2\cos(\pi\Delta x)$

$\lambda_{min} = \lambda_1 = 1 - \cos(\pi\Delta x)$

$\Rightarrow \chi(A) = \frac{1 + \cos\pi\Delta x}{1 - \cos\pi\Delta x} \Rightarrow \chi(\underline{A}) \simeq \frac{4}{(\pi\Delta x)^2} \to +\infty$ when $\Delta x \to 0$.

Exercise A-4)

θ- scheme for ODE $\dot{y} = F(y,t)$

$$(1) \quad \frac{y^{n+1} - y^{n}}{\Delta t} = \theta F(t_{n+1}, y^{n+1}) + (1-\theta) F(t_n, y^n) \qquad \theta \in [0,1].$$

will be applied in FDM for the heat Eqt:

$$(2) \quad \frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} = f(t,x).$$

1) MOL has 2 steps:

Step 1: Spatial discretization of Eq. (2):

$\forall x_i$ on the mesh: $$\frac{d u_i(t)}{dt} - \nu\, \delta^0_{xx} u_i(t) = f(x_i, t)$$

2) Step 2: Use θ-scheme $^{(1)}$ for the system of ODEs:

$$\underbrace{\frac{d}{dt} u_i(t)}_{\frac{d}{dt} y(t)} = \underbrace{\nu\, \delta^0_{xx} u_i(t) + f(x_i, t)}_{F(t,y)}$$

Application of the θ-scheme gives:

$$(3) \quad \boxed{\frac{u_i^{n+1} - u_i^{n}}{\Delta t} - \nu\theta\, \delta^0_{xx} u_i^{n+1} - \nu(1-\theta)\, \delta^0_{xx} u_i^{n} = \theta f_i^{n+1} + (1-\theta) f_i^{n}}$$

θ- scheme for the Heat Equation.

- the scheme has 2-levels: u_i^n and u_i^{n+1}
- The iterative form of the scheme is:

$$u_i^{n+1} - \nu\theta\Delta t\, \delta^0_{xx} u_i^{n+1} = \nu(1-\theta)\Delta t\, \delta^0_{xx} u_i^{n} + \Delta t\,\theta f_i^{n+1} + \Delta t(1-\theta) f_i^{n} \qquad (4)$$

- For $\theta=0$, no equations to solve: the scheme is <u>explicit</u>

- For $\theta \neq 1$: the scheme is <u>implicit</u>.

- ~~For~~ the scheme (3) for $\theta=0$ is:

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} - \nu\,\delta^0_{xx} u_i^n = f_i^n$$

$\Rightarrow$ FE scheme for the heat eq^n.

- the scheme (3) for $\theta=1$ is:

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} - \nu\,\delta^0_{xx} u_i^{n+1} = f_i^{n+1}$$

$\Rightarrow$ BE scheme for the heat Eq^n.

Remark Q3) to 5) are preliminary questions that will allow you to obtain the LTE in Q6) of θ-scheme (3). All LTE should be given up to $\mathcal{O}(\Delta t^3, \Delta x^3)$.

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3) $$\phi^{n+1} = \phi(t_n+\Delta t) = \phi^n + \Delta t \frac{\partial}{\partial t}\phi^n + \frac{\Delta t^2}{2}\frac{\partial^2}{\partial t^2}\phi^n + \mathcal{O}(\Delta t^3)$$

Hence we obtain:

$$\theta\phi^{n+1} + (1-\theta)\phi^n = \phi^n + \theta\,\Delta t\frac{\partial}{\partial t}\phi^n + \theta\frac{\Delta t^2}{2}\frac{\partial^2}{\partial t^2}\phi^n + \mathcal{O}(\Delta t^3) \quad (5)$$

4) According to the lectures:

$$\delta^0_{xx} u_i = \left(\frac{\partial^2 u}{\partial x^2}\right)_i + \frac{\Delta x^2}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_i + \mathcal{O}(\Delta x^4) \quad (6)$$

$$\frac{u^{n+1}-u^n}{\Delta t} = \left(\frac{\partial u}{\partial t}\right)^n + \frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)^n + \frac{\Delta t^2}{6}\left(\frac{\partial^3 u}{\partial t^3}\right)^n + \mathcal{O}(\Delta t^3) \quad (7)$$

5) Perform the LTE of the θ-scheme:

$$(8)\quad \underbrace{\frac{u_i^{n+1}-u_i^{n}}{\Delta t}}_{\text{given by (7)}} - \nu \underbrace{\left[\theta\, \delta^0_{xx} u_i^{n+1} + (1-\theta)\, \delta^0_{xx} u_i^{n}\right]}_{\text{given by (5) and (6), details are given below.}} - \left[\theta\, \underbrace{f_i^{n+1} + (1-\theta) f_i^{n}}_{\text{given by (5)}}\right]$$

$\left(\frac{\partial u}{\partial t}\right)_i^n$	$-\nu$	$\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n$	$-$	f_i^n
$+\frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)_i^n$		$+\theta \Delta t \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n$		$\theta \Delta t \frac{\partial}{\partial t} f_i^n$
$+\frac{\Delta t^2}{6}\left(\frac{\partial^3 u}{\partial t^3}\right)_i^n$		$+\theta \frac{\Delta t^2}{2}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n$		$\frac{\theta}{2}\Delta t^2 \frac{\partial^2}{\partial t^2} f_i^n$
		$+\frac{\Delta x^2}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_i^n$		
$+O(\Delta t^3)$		$+O(\Delta t^3)+O(\Delta t\,\Delta x^2)$		$+O(\Delta t^3)$

The first line of the Taylor Table ~~gives~~ recovers the heat equation (2). The following lines gives the LTE, ordered in increasing powers of Δt and Δx.

Remark: The LTE of the diffusion term is performed as:

$$\theta\, \delta^0_{xx} u_i^{n+1} + (1-\theta)\, \delta^0_{xx} u_i^n = \delta^0_{xx} u_i^n + \theta \Delta t \frac{\partial}{\partial t}\delta^0_{xx} u_i^n + \cancel{\frac{\theta \Delta t^2}{2}} + \theta\frac{\Delta t^2}{2}\frac{\partial^2}{\partial t^2}\delta^0_{xx} u_i^n, \text{ according to (5).}$$

Then, according to (6):

$$= \left(\frac{\partial^2 u}{\partial x^2}\right)_i^n + \frac{\Delta x^2}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_i^n + \theta \Delta t \frac{\partial^3}{\partial t \partial x^2} u_i^n + \theta \frac{\Delta t^2}{2}\frac{\partial^4}{\partial t^2 \partial x^2} u_i^n + O(\Delta t\,\Delta x^2) + O(\Delta t^2 \Delta x^2) + O(\Delta t^3).$$

According to the Taylor table, we obtain:

$$\theta\text{-scheme } (8) = \left\{ \frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} - f \right\}_i^n + \tau_i^n$$

where the LTE is given by:

$$\tau_i^n = \frac{\Delta t}{2} \frac{\partial}{\partial t} \left\{ \frac{\partial u}{\partial t} - 2\theta \left[\nu \frac{\partial^2 u}{\partial x^2} + f \right] \right\}_i^n$$
$$+ \frac{\Delta t^2}{2} \frac{\partial^2}{\partial t^2} \left\{ \frac{1}{3} \frac{\partial u}{\partial t} - \nu \theta \frac{\partial^2 u}{\partial x^2} - f \right\}_i^n$$
$$- \nu \frac{\Delta x^2}{12} \left\{ \frac{\partial^4 u}{\partial x^4} \right\}_i^n + O(\Delta t^3, \Delta t \Delta x^2).$$

5) For all $\theta \in [0,1]$, we have $\tau_i^n \to 0$ when $\Delta t, \Delta x \to 0$, so the scheme is consistent.

- For $\theta = \frac{1}{2}$, we have a special case: the term in $O(\Delta t)$ of the LTE ~~...~~ cancels:

$$\frac{\Delta t}{2} \frac{\partial}{\partial t} \underbrace{\left\{ \frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} - f \right\}_i^n}_{=0 \text{ by using Eq. (2)}} = 0$$

and thus the LTE is:

$$\tau_i^n = O(\Delta t^2, \Delta x^2).$$

The θ-scheme with $\theta = \frac{1}{2}$ is called the Crank-Nicolson scheme

- For $\theta \neq \frac{1}{2}$, we have $\tau_i^n = O(\Delta t, \Delta x^2)$, so the θ-scheme is first-order in time, 2nd order in space.

Q7) For $\theta = \frac{1}{2}$ the θ-scheme corresponds to the CN scheme:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{\nu}{2}\left[\delta_{xx}^0 u_i^{n+1} + \delta_{xx}^0 u_i^n\right] = \frac{1}{2} f_i^{n+1} + \frac{1}{2} f_i^n$$

the iterative form of the scheme is:

$$u_i^{n+1} - \frac{\nu \Delta t}{2} \delta_{xx}^0 u_i^{n+1} = u_i^n + \frac{\nu \Delta t}{2} \delta_{xx}^0 u_i^n + \frac{1}{2} f_i^{n+1} + \frac{1}{2} f_i^n \quad (1)$$

By using $r = \frac{\nu \Delta t}{\Delta x^2}$ and $\delta_{xx}^0 u_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}$,

Eq (1) becomes:

$$\underbrace{-\frac{\nu \Delta t}{2\Delta x^2}}_{B^1_{i,i+1} = -\frac{r}{2}} u_{i+1}^{n+1} + \underbrace{\left(1 + \frac{\nu \Delta t}{\Delta x^2}\right)}_{B^1_{i,i} = 1+r} u_i^{n+1} \underbrace{- \frac{\nu \Delta t}{2\Delta x^2}}_{B^1_{i,i-1} = -\frac{r}{2}} u_{i-1}^{n+1}$$

$$= \underbrace{\frac{\nu \Delta t}{2\Delta x^2}}_{B^2_{i,i+1} = \frac{r}{2}} u_{i+1}^n + \underbrace{\left(1 - \frac{\nu \Delta t}{\Delta x^2}\right)}_{B^2_{i,i} = 1-r} u_i^n + \underbrace{\frac{\nu \Delta t}{2\Delta x^2}}_{B^2_{i,i-1} = \frac{r}{2}} u_{i-1}^n$$

$$+ \frac{1}{2} f_i^{n+1} + \frac{1}{2} f_i^n$$

which can be written in matrix form:

$$\sum_{j=i-1}^{i+1} B^1_{i,j} u_j^{n+1} = \sum_{j=i-1}^{i+1} B^2_{i,j} u_j^n + \frac{1}{2} f_i^{n+1} + \frac{1}{2} f_i^n$$

with $\underline{\underline{B}}^1 = \text{tridiag}\left(-\frac{r}{2}, 1+r, -r/2\right)$

$\underline{\underline{B}}^2 = \text{tridiag}\left(\frac{r}{2}, 1-r, r/2\right)$

Q8) For $f_i = 0$ the iterative form of the CN scheme is:

$$-\frac{r}{2}u_{i+1}^{n+1} + (1+r)u_i^{n+1} - \frac{r}{2}u_{i-1}^{n+1} = \frac{r}{2}u_{i+1}^{n} + (1-r)u_i^{n} + \frac{r}{2}u_{i-1}^{n}$$

Assuming $u_i^n = \hat{u}^n e^{ihi\Delta x}$, we obtain:

$$\hat{u}^{n+1} e^{ihi\Delta x}\left[(1+r) - \frac{r}{2}\left(e^{ih\Delta x} + e^{-ih\Delta x}\right)\right]$$
$$= \hat{u}^{n} e^{ihi\Delta x}\left[(1-r) + \frac{r}{2}\left(e^{ih\Delta x} + e^{-ih\Delta x}\right)\right]$$

which gives the amplification factor:

$$g(h\Delta x) = \frac{\hat{u}^{n+1}}{\hat{u}^{n}} = \frac{1 - r(1-\cos h\Delta x)}{1 + r(1-\cos h\Delta x)}$$

Using the auxiliary variable $\xi = \frac{1-\cos h\Delta x}{2}$, the VN stability condition is:

$$|g(\xi)| = \left|\frac{1-2r\xi}{1+2r\xi}\right| \leq 1 \quad \forall \xi \in [0,1].$$

this inequality reads:

$$-1-2r\xi \leq 1-2r\xi \leq 1+2r\xi$$
$$-2 \leq 0 \leq 4r\xi$$

which is always verified since $\xi > 0$ and $r = \nu\frac{\Delta t}{\Delta x^2} > 0$.

Conclusion: the CN stable is stable for all $\Delta t \geq 0$.

Q10) Summary:

- $\theta = 0$ is the FE1 scheme.
  - $\tau_i = \mathcal{O}(\Delta t, \Delta x^2)$
  - Explicit scheme.
  - Stable if $\Delta t \leq \dfrac{\Delta x^2}{2\nu}$ (see lectures).

- $\theta = \frac{1}{2}$ is the CN scheme
  - $\tau_i = \mathcal{O}(\Delta t^2, \Delta x^2)$
  - Implicit scheme
  - Stable for all $\Delta t \geq 0$.

- $\theta = 1$ is the BE1 scheme
  - $\tau_i = \mathcal{O}(\Delta t, \Delta x^2)$
  - Implicit scheme
  - Stable for all $\Delta t \geq 0$.

Remark:

- Those 3 schemes have 2 levels: u_i^n and u_i^{n+1}
- All other values of θ have no interest.

Exo II   Schéma de Richardson:

$$\frac{u_i^{n+1}-u_i^{n-1}}{2\Delta t} - \nu\,\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} = 0$$

1) 3 niveaux de temps: ~~n~~, n−1, n and n+1

$$\cancel{(\varepsilon_h)_i^n = L u_i^n - L_h u_i^n}$$

$$\varphi_i^n = +\frac{\Delta t^2}{6}\left(\frac{\partial^3 u}{\partial t^3}\right)_i^n - \nu\,\frac{\Delta x^2}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_i^n + O(\Delta t^3, \Delta x^4).$$

car $$\frac{u_i^{n+1}-u_i^{n-1}}{2\Delta t} = \left(\frac{\partial u}{\partial t}\right)_i^n + \frac{\Delta t^2}{6}\left(\frac{\partial^3 u}{\partial t^3}\right)_i^n + O(\Delta t^3).$$

le schéma est en $O(\Delta t^2, \Delta x^2)$.

2) $u_i^{n+1} = u_i^{n-1} + 2r\left(u_i^n - 2u_i^n + u_{i-1}^n\right)$

en posant $\zeta = \frac{1-\cos k\Delta x}{2}$, $u_i^n = \hat{u}^n e^{Ikx_i}$

on obtient
$$\hat{u}^{n+1} = \hat{u}^{n-1} - 8r\zeta\,\hat{u}^n$$

On utilise la variable auxiliaire $\hat{v}^{n+1} = \hat{u}^n$, d'où

$$\begin{cases} \hat{u}^{n+1} = \hat{v}^n - 8r\zeta\,\hat{u}^n \\ \hat{v}^{n+1} = \hat{u}^n \end{cases}$$

qui s'écrit matriciellement

$$\begin{pmatrix} \hat{u}^{n+1} \\ \hat{v}^{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}}_{G} \begin{pmatrix} \hat{u}^n \\ \hat{v}^n \end{pmatrix} \quad \text{avec } a = -8r\zeta \le 0.$$

les v.p. sont $\lambda = \dfrac{a \pm \sqrt{4+a^2}}{2}$??? ok [Maple le 3 Nov 04]

le rayon spectral est $\rho(G) = -\frac{a}{2} + \sqrt{1+\frac{a^2}{4}} \ge 1$ car $a \le 0$.

Donc la cond. de V.N $\rho(G) \le 1$ ne peut être vérifiée.

Exercise: Steady convection-diffusion PDE:

$$(1)\begin{cases} a\dfrac{du}{dx} - \nu\dfrac{d^2u}{dx^2} = 0 \quad , \Omega = (0,1) \\ u(0) = 0 , u(1) = 1 . \quad (2) \end{cases}$$

1) We use the dimensionless variables:

$$u' = \frac{u}{a} \quad , \quad x' = \frac{x}{L} \quad \Rightarrow dx = L\,dx'$$

thus $\dfrac{du}{dx} = \dfrac{a}{L}\dfrac{du'}{dx'}$, $\dfrac{d^2u}{dx^2} = \dfrac{d}{dx}\left(\dfrac{a}{L}\dfrac{du'}{dx'}\right) = \dfrac{a}{L^2}\dfrac{d^2u'}{dx'^2}$

Eq. (1) then reads:

$$\frac{du'}{dx'} - \frac{1}{P_e}\frac{d^2u}{dx^2} = 0 \quad (2)$$

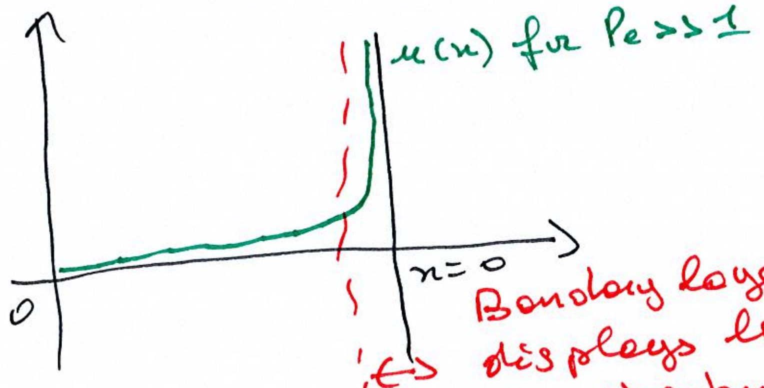
with the Péclet number $P_e = \dfrac{aL}{\nu}$ (3).

Péclet number is a dimensionless number, just like the Reynolds number for N.S eq^t:

$$\underbrace{\frac{\partial u}{\partial t}}_{\text{unsteadiness}} + \underbrace{\vec{u}\cdot\nabla\vec{u}}_{\text{convection}} - \underbrace{\frac{1}{Re}\nabla^2\vec{u}}_{\text{diffusion.}} = 0 \quad , Re = \frac{UL}{\nu} .$$

Pe and Re measures the ratio of convection to diffusion. The larger they are, convection effects dominate diffusion effects.

2) As shown in the Figure, when Pe is large ($Pe >> 1 \Rightarrow aL >> \nu$) the solution exhibits a boundary layer near boundary $x=1$, just like fluid velocity near a solid boundary when Re is large (see Lecture on turbulent flows and boundary layers).



Boundary layer where $u(x)$ displays large variations. The thickness $\varepsilon \to 0$ when $Pe \nearrow +\infty$.

3)

Eq (1) is discretized at the inner nodes $\{i = 1, \dots, N\}$ with central formulas:

$$a\, \delta^0_x u_i - \nu\, \delta^0_{xx} u_i = 0,$$

that yields:

$$\left(\frac{a}{2\Delta x} - \frac{\nu}{\Delta x^2}\right) u_{i+1} + \frac{2\nu}{\Delta x^2} u_i - \left(\frac{a}{2\Delta x} + \frac{\nu}{\Delta x^2}\right) u_{i-1} = 0.$$

After using the cell Peclet number $Pe_h = \frac{a\Delta x}{\nu}$, the linear system reads:

$$(4) \quad \frac{\nu}{2\Delta x^2}\left[(Pe_h - 2) u_{i+1} + 4u_i - (Pe_h + 2) u_{i-1}\right] = 0.$$

Eq. (4) needs to be modified at nodes $i=1$ and $i=N$ to put the BCs (2):

$$i=1 : \quad \underbrace{4}_{A_{1,1}} u_1 + \underbrace{(Pe_h - 2)}_{A_{1,2}} u_2 = \underbrace{(Pe_h+2) \underbrace{u_0}_{=0}}_{F_1} .$$

$$i=N : \quad \underbrace{-(Pe_h+2)}_{A_{N,N-1}} u_{N-1} + \underbrace{4}_{A_{N,N}} u_N = \underbrace{-(Pe_h-2) \underbrace{u_{N+1}}_{=0}}_{F_N} .$$

And at $i = 2, \dots, N-1$ the linear system reads:

$$\underbrace{-(Pe_h+2)}_{A_{i,i-1}} u_{i-1} + \underbrace{4}_{A_{i,i}} u_i + \underbrace{(Pe_h-2)}_{A_{i,i+1}} u_{i+1} = \underbrace{0}_{F_i} .$$

~~Finally~~, In summary the linear system reads:

For $i = 1, \dots, N$ $$\sum_{j=i-1}^{i+1} \underline{\underline{A}}_{i,j} u_j = \underline{F}_i ,$$

where each line of the system corresponds to one grid point.

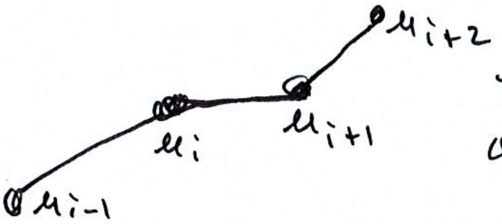
For example, the matrix $\underline{\underline{A}}$ reads:

$$\underline{\underline{A}} = \begin{bmatrix} 4 & (Pe_h-2) & & (0) \\ -(Pe_h+2) & 4 & \ddots & \\ & \ddots & \ddots & Pe_h-2 \\ (0) & & -(Pe_h+2) & 4 \end{bmatrix} = \text{tridiag}\left[-(Pe_h+2), 4, Pe_h-2\right] .$$

The LTE of the scheme is in $O(\Delta x^2)$:

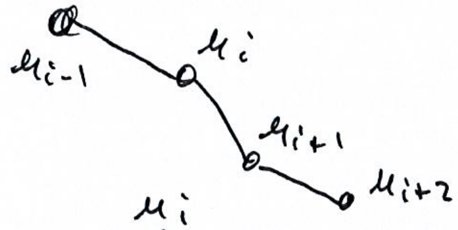
$$\tau_i = \Delta x^2 \left[\frac{a}{6} \left(\frac{\partial^3 u}{\partial x^3}\right)_i - \frac{\nu}{12} \left(\frac{\partial^4 u}{\partial x^4}\right)_i \right]$$

4) The purpose of this question is to detect spurious (unphysical) solutions in FD schemes.
To motivate this analysis, let us investigate 3 cases:



a) The solution grows monotonously: the ratio between 2 ~~consecutive~~ consecutive gradients is:

$$R = \frac{\nabla u_i}{\nabla u_{i-1}} = \frac{u_{i+1} - u_i}{u_i - u_{i-1}} > 0.$$



b) the solution decays monotonously:

$$R = \frac{\nabla u_i}{\nabla u_{i-1}} > 0$$

c) the solution has node-to-node oscillations: $R = \dfrac{\nabla u}{\nabla u_{i-1}} < 0$.

To detect this behaviour, let us write the solution u_i as the normal mode $u_i = q^i$, where q depends on the coefficients of the FD scheme. After observing that

$$R = \frac{u_{i+1} - u_i}{u_i - u_{i-1}} = \frac{q^{i+1} - q^i}{q^i - q^{i-1}} = q$$

we can state that:

- if $q < 0$: the FD solution oscillates.
- if $q \geq 0$: no oscillations.

For simple FD schemes like in this exercises, it is possible to solve analytically the linear system and detect the cases when $q \geq 0$ or $q < 0$.

Let us consider the central scheme (4):

$$(Pe_h-2)\, u_{i+1} + 4\, u_i - (Pe_h+2)\, u_{i-1} = 0.$$

When $u_i = q^i$, this scheme reads:

$$\underbrace{(Pe_h-2)}_{a}\, q^2 + \underbrace{4}_{b}\, q - \underbrace{(Pe_h+2)}_{c} = 0. \quad (5)$$

This quadratic equation have 2 solutions:

$$q_{+/-} = -\frac{b \pm \sqrt{\Delta}}{2a}, \quad \Delta = b^2 - 4ac = (2\cancel{4}\, Pe)^2$$

So the particular solutions to ~~the central scheme~~ (5) are: $\cancel{u_i = \alpha q_+ + \beta q_-}$

~~with~~ $$q_+ = \frac{-b + \sqrt{\Delta}}{2a} = 1$$

$$q_- = \frac{2 + Pe_h}{2 - Pe_h}.$$

The general solution of the central scheme is thus:

$$u_i = \alpha q_+^i + \beta q_-^i = \alpha + \beta \left(\frac{2+Pe_h}{2-Pe_h}\right)^i. \quad (5)$$

The ~~scheme is~~ solution has oscillates when:

$$q_- = \frac{2+Pe_h}{2-Pe_h} < 0 \Rightarrow \boxed{Pe_h \geq 2}$$

Remark: Coefficients α and β are given by the BC (2):

$$i = 0,\ u_0 \Rightarrow \alpha = -\beta$$

$$i = N+1,\ u_{N+1} = 1 \Rightarrow \beta = \frac{1}{1 - \left(\frac{2+Pe_h}{2-Pe_h}\right)^{N+1}}$$

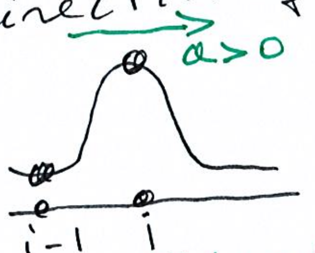
5) For $a = 1$, $\nu = 10^{-3}$ and $L = 1$, the Péclet number $Pe = aL/\nu = 10^{+3}$ is large $\Rightarrow$ Solution has boundary layer near $x = 1$.

The numerical solution with central scheme does not oscillate if $Pe_h < 2$, i.e. the ~~Number of points~~ grid size Δx should be:

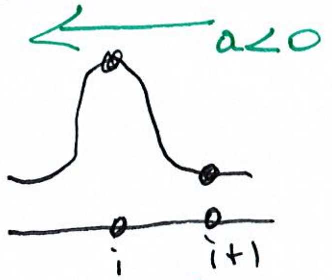
$$\boxed{\Delta x < 2\nu/a}$$

<u>Application</u>: Since $\Delta x = \frac{1}{N+1}$, the number of points of the mesh should be $N > \frac{1}{\Delta x} - 1 \simeq 599$ to have non-oscillatory solutions.

6) In the 1970s, CFD practitioners observed that node-to-node oscillations are caused by the central approximation of convection ($a\,\delta_x^0 u_i$), but ~~also~~ that they oscillations can be suppressed (or at least reduced) by using backward (δ_x^-) or forward (δ_x^+) FD formulas according to the direction of the convection velocity:



use $\delta_x^- u_i = \dfrac{u_i - u_{i-1}}{\Delta x}$



use $\delta_x^+ u_i = \dfrac{u_{i+1} - u_i}{\Delta x}$

This concept is called <u>upwinding</u>.

For $a>0$, the upwind scheme for Eq. (1) is: 7

$$a\,\delta_x^- u_i - \nu\,\delta_{xx}^0 u_i = 0 . \qquad (6)$$

<u>Remark</u>: Diffusion term is always discretized with central FD.

The LTE of scheme (6) is:

$$\tau_i = -a\frac{\Delta x}{2}\left(\frac{\partial u}{\partial x}\right)_i - \nu\frac{\Delta x^2}{2}\left(\frac{\partial^3 u}{\partial x^3}\right)_i = O(\Delta x),$$

so the upwind discretization reduces the accuracy compared to the central scheme of (3).

By assuming $u_i = \varphi^i$ in (6) we obtain:

$$u_{i+1} - (Pe_h+2)\,u_i + (1+Pe_h)\,u_{i-1} = 0$$

$$\Rightarrow \varphi^2 - (Pe_h+2)\,\varphi + (1+Pe_h) = 0$$

$$\Delta = b^2 - 4ac = Pe^2, \quad \varphi^+ = Pe+1, \quad \varphi^- = 1$$

$$u_i = \alpha\,(1+Pe_h)^i + \beta$$

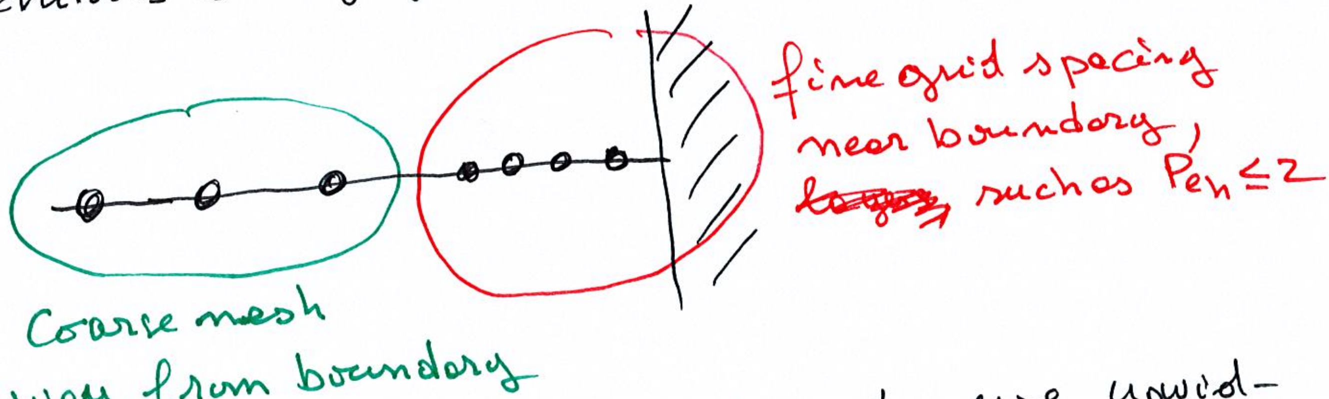
Since $1+Pe_h>0$, the solution never oscillates.

Conclusions for the exercise.

- Central methods for convection-dominated flows are plagued by spurious oscillations unless the grid size verifies the condition: $Pe_h = \frac{a \Delta x}{\nu} < 2$.

- ~~this~~ when the Péclet (or Reynolds) number $Pe = \frac{aL}{\nu}$ is large (e.g. Turbulent flows), this condition leads to very fine meshes. For this reason, central methods are rarely used by CFD codes for engineering applications.

- One remedy is to use non-uniform meshes with fine resolution in flow regions where the solution exhibits strong gradients (e.g. boundary layers)



- Another widely used remedy is to use upwind-biased discretization for the convection term. Upwind scheme has low accuracy (the LTE is $O(\Delta x)$), so CFD codes uses upwind-biased methods of order at least $O(\Delta x^2)$, see the following lectures.

1

Exercise: Analysis of the Errors of the upwind scheme for the transport eq^t:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \qquad a > 0. \quad (1)$$

With Taylor expansions, we can show that:

$$\text{Upwind-scheme} = \underbrace{[\text{Eq. }(1)]_i + T_i}_{\text{modified Equation of upwind scheme}}$$

The purpose of the exercise is to clearly write the modified equation as a PDE (Q1 and 2), then solve the PDE by Fourier method (see lecture 1), in order to investigate the dissipation and dispersion errors of the upwind scheme.

1) By using Taylor expansion, we have:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\partial u}{\partial t}\Big|_i^n + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}\Big|_i^n + \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3}\Big|_i^n + O(\Delta t^3) \quad (2)$$

$$\frac{u_i^n - u_{i-1}^n}{\Delta x} = \frac{\partial u}{\partial x}\Big|_i^n - \frac{\Delta x}{2}\frac{\partial^2 u}{\partial x^2}\Big|_i^n + \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3}\Big|_i^n + O(\Delta x^3) \quad (3)$$

After inserting these expansions into the upwind scheme:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\left(\frac{u_i^n - u_{i-1}^n}{\Delta x}\right) = 0 \quad (4)$$

$$(3)\quad \underbrace{\frac{\partial u_i^n}{\partial t} + a\frac{\partial u_i^n}{\partial x}}_{\text{PDE (1)}} + \underbrace{\underbrace{\frac{\Delta t}{2}\frac{\partial^2 u_i^n}{\partial t^2} - \frac{a\Delta x}{2}\frac{\partial^2 u}{\partial x^2}\Big|_i^n}_{\text{Leading terms of the LTE}} + \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3}\Big|_i^n + a\frac{\Delta x^2}{6}\frac{\partial^3 u_i^n}{\partial x^3} + O(\Delta t^3, \Delta x^3)}_{\text{LTE}}$$

In particular, ~~we no~~ the leading terms of the LTE shows that the scheme is first-order accurate:

$$\tau_i^n = \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}\Big|_i^n - \frac{a\Delta x}{2}\frac{\partial^2 u}{\partial x^2}\Big|_i^n + O(\Delta t^2, \Delta x^2)$$

2) We are going to express the time derivatives $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^3 u}{\partial t^3}$ in the LTE (3) by using the transport equation (1): $\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x}$.

As in the lecture on L-W scheme, we get:

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}, \qquad \frac{\partial^3 u}{\partial t^3} = -a^3\frac{\partial^3 u}{\partial x^3}.$$

Using these expressions, Eq. (3) reads:

$$(4)\quad \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \eta\frac{\partial^2 u}{\partial x^2} + \Delta\frac{\partial^3 u}{\partial x^3} + O(\Delta t^3, \Delta x^3)$$

with:

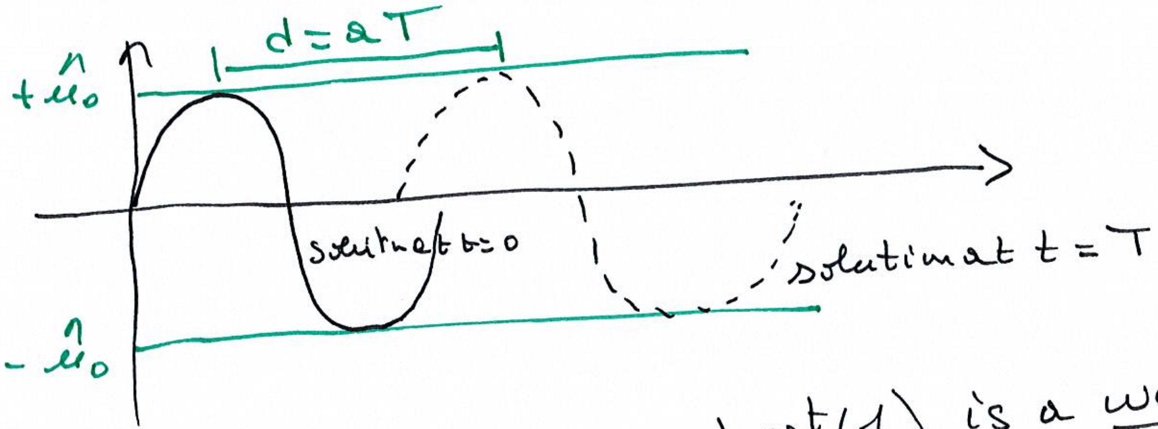
$$(5)\quad \begin{aligned} \eta &= a^2\frac{\Delta t}{2} - a\frac{\Delta x}{2} = a\Delta x(C-1) \\ \Delta &= -a^3\frac{\Delta t^2}{6} + a\frac{\Delta x^2}{6} = a\frac{\Delta x^2}{6}(1-C^2) \end{aligned}$$

<u>Remark</u>:
- For $a>0$, $C = \frac{a\Delta t}{\Delta x} > 0$.
- We have $\eta \le 0$ and $\Delta \geq 0$ when the CFL condition $0 \le C \le 1$ is verified.

Reminder of lecture: Inserting the Fourier mode $u(t,x) = \hat{u}(t)\, e^{ikx}$ in PDE (1) gives the ODE:

$$\frac{d\hat{u}}{dt} + a\,ik\,\hat{u} = 0 \Rightarrow \hat{u}(t) = \hat{u}_0\, e^{-ikat}$$

$$\Rightarrow u(t,x) = \hat{u}_0\, e^{ik(x-at)} \qquad (6)$$



The solution to the transport eq^t (1) is a wave travelling at speed a with no deformation.

3) Inserting the Fourier mode $u(t,x) = \hat{u}(t)\, e^{ikx}$ in the equivalent Eq. (4):

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \eta\frac{\partial^2 u}{\partial x^2} + \Delta\frac{\partial^3 u}{\partial x^3} = 0 \qquad (7)$$

yields the ODE:

$$\frac{d\hat{u}}{dt} + \left[ik(a - \Delta k^2) - \eta k^2\right]\hat{u}(t) = 0. \qquad (8)$$

$$\Rightarrow \hat{u}(t) = \hat{u}_0\, e^{\eta k^2 t}\, e^{ik[-(a-\Delta k^2)t]}$$

The Fourier mode is finally:

$$u(t,x) = \underbrace{\hat{u}_0\, e^{\eta k^2 t}}_{u_0'(t)}\, e^{ik[x-(\underbrace{a-\Delta k^2}_{a'})t]} \qquad (9)$$

4) The amplitude of the solution (a) is:

$$|u(t,x)| = |\hat{u}_0|\, e^{\eta k^2 t}$$

with viscosity η given by (5): $\eta = a\,\Delta x\, c(c-1)$.

When the CFL condition $0 < c < 1$ is verified:

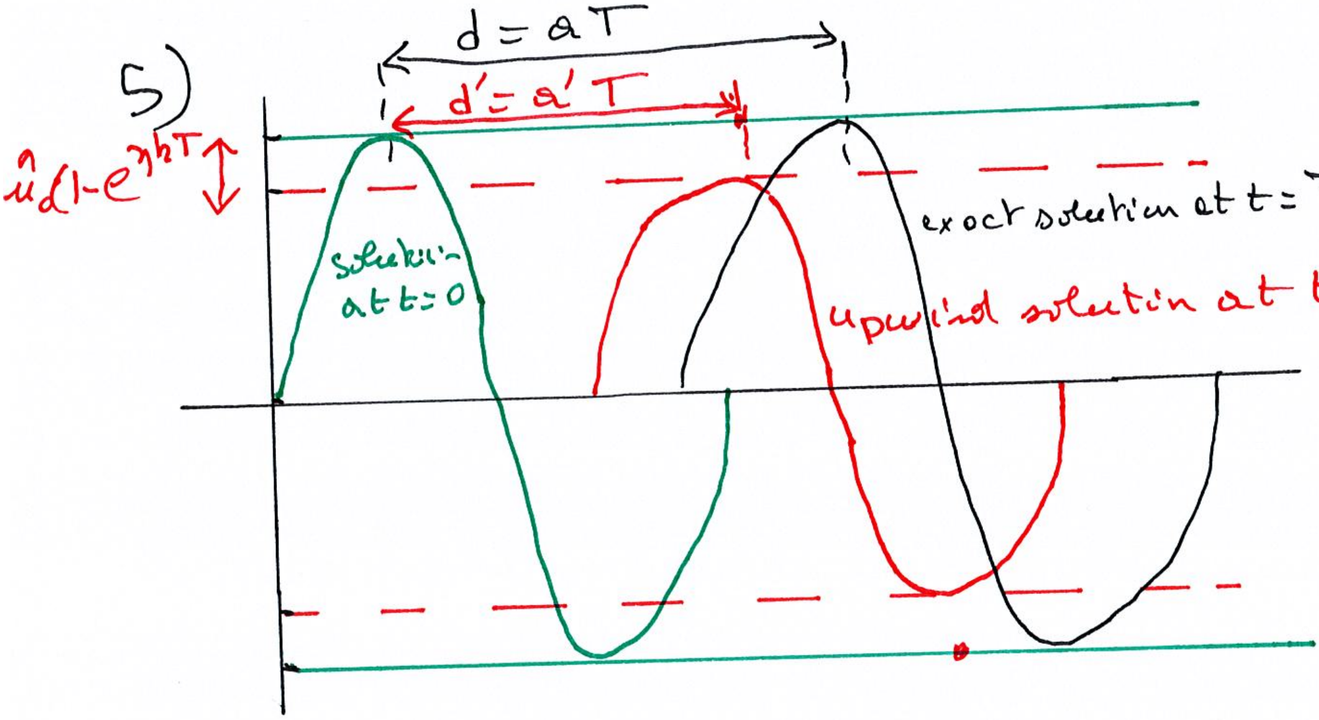
$$\lim_{t \to +\infty} |u(t,x)| = 0 \quad \text{since } \eta < 0.$$

Thus this term represents the numerical dissipation of the Upwind scheme.

When $c > 1$, we have $\eta > 0 \Rightarrow \lim_{t\to+\infty} |u(t,x)| = +\infty$

which shows the scheme gives unstable solution.

5)



Fourier solution of the upwind scheme:

$$u(t,x) = \hat{u}_0 e^{\eta k^2 t} e^{ik(x-a't)}$$

- Term $e^{\eta k^2 t}$ is the numerical dissipation term which damps the solution during the simulation
  artificial viscosity: $\eta = a\frac{\Delta x}{2}(c-1) = O(\Delta x)$
  $\Rightarrow$ First-order dissipation.

- Term $e^{ik(x-a't)}$ shows that the upwind solution has speed a' different than the exact speed $\Rightarrow$ numerical dispersion.

$$a' = a - \Delta k^2 \quad \text{with } \Delta = a\frac{\Delta x^2}{6}(1-c^2) = O(\Delta x^2)$$

- The dispersion is second-order: dissipation dominates in the upwind solution.
- Under CFL condition $c \leq 1$, $\Delta > 0 \Rightarrow a' < a$: the numerical speed a' is lower than exact speed a.

Application: The equivalent equation of the LW scheme is:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \underbrace{a\frac{\Delta x^2}{6}(1-c^2)}_{\Delta}\frac{\partial^3 u}{\partial x^3} + \underbrace{ac\frac{\Delta x^3}{8}(1-c^2)}_{\eta}\frac{\partial^4 u}{\partial x^4}$$

odd-order derivatives $\Rightarrow$ complex numbers $\Rightarrow$ dispersion terms.
even-order — $\Rightarrow$ real numbers $\Rightarrow$ dissipation terms.

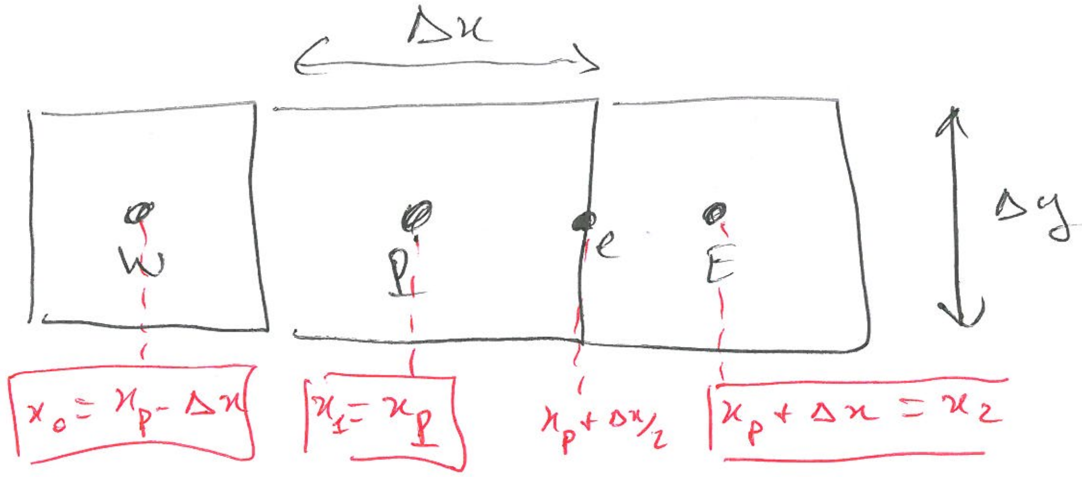
- $\Delta = O(\Delta x^2)$ and $\eta = O(\Delta x^3) \Rightarrow$ dissipation is lower than dispersion.
- Under CFL condition $|c| < 1$, $\eta < 0$: stable scheme, and $\Delta > 0 \Rightarrow a' < a$.

12 Mars 2014

1

Exo 5

k-scheme



$$(F_x^x)_e \simeq u_e \, \Delta y \, \Phi_e .$$

Q1) Quick scheme.

1-c)

$$\phi_e = \frac{\phi_{i+1} + \phi_i}{2} - \frac{\Delta x^2}{2} \, \delta_{xx}^{c} \phi_i$$

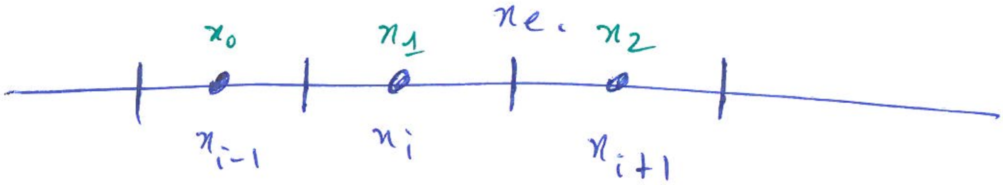
$$= \frac{\phi_{i+1} + \phi_i}{2} - \frac{\Delta x^2}{2} \left(\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x^2} \right)$$

$$= \phi_{i+1} \underbrace{\left(\frac{1}{2} - \frac{1}{8} \right)}_{3/8} + \phi_i \underbrace{\left(\frac{1}{2} + \frac{1}{4} \right)}_{3/4} - \frac{1}{8} \phi_{i-1}$$

CQFD.

Interpretation: We had a diffusive term to CDS interpolation to smooth the oscillation of central scheme by adding numerical dissipation.

n-Schéma b) | Quick-interpolation |



$$x_0 = x_{i-1} = x_i - \Delta x.$$

$$x_1 = x_i$$

$$x_2 = x_{i+1} = x_i + \Delta x.$$

$$x_e = x_i + \frac{\Delta x}{2}.$$

$$\Pi_2[\phi(x)] = \sum_{j=0}^{2} f(x_j) L_j(x)$$

$$= L_0(x_e)\Phi_{i-1} + L_1(x_e)\Phi_i + L_2(x_e)\Phi_{i+1}$$

avec $L_0(x) = \dfrac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}.$

$$L_0(x_e) = \frac{(x_i + \Delta x/2 - x_i)(x_i + \Delta x/2 - x_i - \Delta x)}{(x_i - \Delta x - x_i)(x_i - \Delta x - x_i - \Delta x)}$$

$$= \frac{\frac{1}{2}\Delta x \times -\frac{1}{2}\Delta x}{-\Delta x \times -2\Delta x} = -\frac{1}{8} \text{ /ok.}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$L_1(x_e) = \frac{(x_i + \Delta x/2 - x_i + \Delta x)(x_i + \Delta x/2 - x_i - \Delta x)}{(x_i - x_i + \Delta x)(x_i - x_i - \Delta x)}$$

$$= \frac{\frac{3}{2}\Delta x \times -\frac{1}{2}\Delta x}{\Delta x \times -\Delta x} = +\frac{3}{4}.$$

h-schme 7)

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

$$L_2(x_e) = \frac{(x_i + \Delta x/2 - x_i + \Delta x)(x_i + \Delta x/2 - x_i)}{(x_i + \Delta x - x_i + \Delta x)(x_i + \Delta x - x_i)}$$

$$= \frac{\frac{3}{2} \times \frac{1}{2}}{2 \times 1} = \frac{3}{8} \, / m .$$

$$\Rightarrow \phi_e = \frac{3}{8} \phi_{i+1} + \frac{3}{4} \phi_i - \frac{1}{8} \phi_{i-1} .$$

φ 1d

$$\phi_e = \frac{3}{8}\phi_{i+1} + \frac{3}{4}\phi_i - \frac{1}{8}\phi_{i-1}$$

DL en $x = x_e$ avec $x_{i+1} = x_e + \Delta x/2$

$x_i = x_e - \Delta x/2$

$x_{i+1} = x_e - \frac{3\Delta x}{2}$

$$\frac{3}{8}\times\phi_{i+1} = \phi_e + \frac{\Delta x}{2}\phi'_e + \frac{\Delta x^2}{8}\phi''_e + \frac{\Delta x^3}{48}\phi'''_e + O(\Delta x^4)$$

$$+\frac{3}{4}\times\phi_i = \phi_e - \frac{\Delta x}{2}\phi'_e + \frac{\Delta x^2}{8}\phi''_e - \frac{\Delta x^3}{48}\phi'''_e + O(\Delta x^4)$$

$$-\frac{1}{8}\times\phi_{i-1} = \phi_e - \frac{3\Delta x}{2}\phi'_e + \frac{9\Delta x^2}{8}\phi''_e - \frac{9}{4}\Delta x^3\phi'''_e + O(\Delta x^4)$$

$27/48$

$$\phi_{\text{recherché}} = \underbrace{\left(\frac{3}{8}+\frac{3}{4}-\frac{1}{8}\right)}_{1}\phi_e + \Delta x\underbrace{\left(\frac{3}{16}-\frac{3}{8}+\frac{3}{16}\right)}_{0}\phi'_e$$

$$+ \Delta x^2\underbrace{\left(\frac{3}{64}+\frac{3}{32}-\frac{9}{64}\right)}_{=0}\phi''_e$$

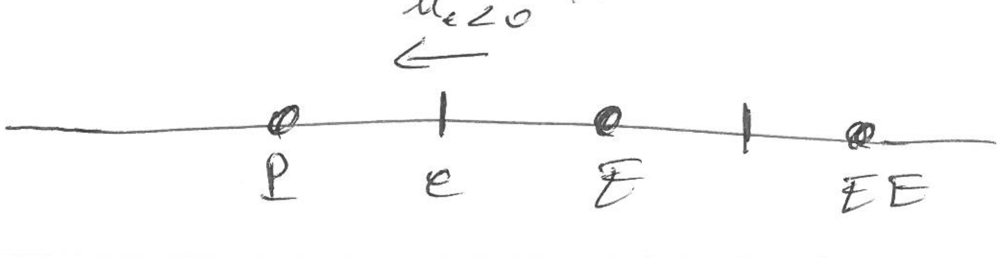
$$+ \Delta x^3\left(\frac{3}{8}\times\frac{1}{48} - \frac{3}{4}\times\frac{1}{48} + \cancel{\frac{9}{4\times 8}}\right)\phi'''_e$$

$\frac{35}{128}$ double ! Faux

$\frac{1}{16}\Delta x^3$

A rechercher

e)



2)

$u_e = u_w = u > 0$. 5

$$\phi_e = \frac{\phi_{i+1} + \phi_i}{2} + \frac{(k-1)}{4}\left[\phi_{i+1} - \overset{2\phi_i}{\cancel{\phi_i - (\phi_i}} + \phi_{i-1})\right]$$

$$\phi_w = \frac{\phi_i + \phi_{i-1}}{2} + \frac{(k-1)}{4}\left[\phi_i - \overset{2\phi_{i-1}}{\cancel{\phi_{i-1} - (\phi_{i-1}}} + \phi_{i-2})\right]$$

$$u D_x^{(u)} \phi_i = \frac{u}{\Delta x}\left[\phi_e - \phi_w\right]$$

$$= \phi_{i+1}\left(\frac{1+k}{4}\right) + \phi_i\left(\underbrace{\frac{1}{2} - 2\frac{(k-1)}{4} - \frac{1}{2} - \frac{(k-1)}{4}}_{\frac{3}{4}(1-k)}\right)$$

$$+ \phi_{i-1}\left(\underbrace{-\frac{(1-k)}{4} - \frac{1}{2} + 2\frac{(k-1)}{4}}_{\frac{1}{4}(3k-5).}\right)$$

$$+ \phi_{i-2}\left(+\frac{1}{4}(1-k)\right).$$

$k = 1$:

$$u D_x^{(u)} \phi = \frac{1}{\Delta x}\left[\frac{1}{2}\phi_{i+1} - \phi_{i-1}\right]$$

$k = -1$:

$$D_x^{(u)} \phi = \frac{a}{4\Delta x}\left[6\phi_i - 8\phi_{i-1} + 2\phi_{i-2}\right]$$

1-Schem 4

$$\left.\frac{du}{dx}\right]_n \frac{F^c_{i+\frac{1}{2}} - F^c_{i-\frac{1}{2}}}{\Delta u} = \frac{a}{4\Delta u}\left[(1+h)\phi_{i+1} + 3(1-h)\phi_i + (3h-5)\phi_{i-1} + (1-h)\phi_{i-2}\right]$$

$$(1+h)\times \phi_{i+1} = \phi_i + \Delta u\, \phi_i' + \frac{\Delta u^2}{2}\phi_i'' + \frac{\Delta u^3}{6}\phi_i''' + \frac{\Delta u^4}{24}\phi_i^{(4)}$$

$$3(1-h)\times \phi_i = \phi_i$$

$$(3h-5)\times \phi_{i-1} = \phi_i - \Delta u\, \phi_i' + \frac{\Delta u^2}{2}\phi_i'' - \frac{\Delta u^3}{6}\phi_i''' + \frac{\Delta u^4}{24}\phi_i^{(4)}$$

$$(1-h)\times \phi_{i-2} = \phi_i - 2\Delta u\, \phi_i' + 2\Delta u^2 \phi_i'' - \frac{4\Delta u^3}{3}\phi_i''' + \frac{2\Delta u^4}{3}\phi_i^{(4)}$$

$$\phi_i\,\big(\underbrace{1+h+3(1-h)+(3h-5)+(1-h)}_{\text{ok } 0.}\big)\times\frac{1}{4}$$

$$\phi_{i\,\Delta x}'\,\big(\underbrace{(1+h)-(3h-5)-2(1-h)}_{4}\big)\times\frac{1}{4}$$

$$+\,\phi_i''\,\Big(\underbrace{\frac{(1+h)}{2}+\frac{(3h-5)}{2}+2(1-h)}_{\to\ (h-\frac{1}{3})}\Big)\frac{\Delta x}{4}$$

$$+\,\phi_i'''\Big[\underbrace{\frac{(1+h)}{6}-\frac{1}{6}(3h-5)-\frac{4}{3}(1-h)}_{0}\Big]\frac{\Delta u^2}{4}$$

h-schema-3 |

$$+ \phi_i''' \left[\frac{(1+h)}{24} + \frac{(3h-5)}{24} + \frac{2}{3}(1-h) \right] \frac{\Delta x^3}{4}$$

$$\frac{1}{24} - \frac{5}{24} + \frac{2}{3} = \frac{1-5+16}{24} = \frac{1}{2}.$$

$$\frac{h}{24} + \frac{3h}{24} - \frac{2}{3}h = \frac{h}{24}(1+3-16) = h \cdot -\frac{1}{2} \times$$

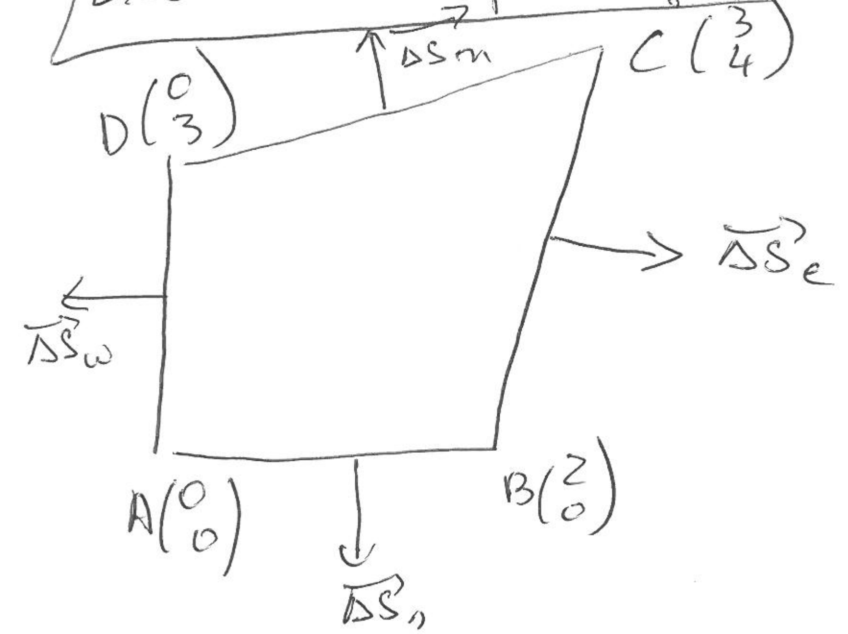
$$\phi_i''' \Delta x^3 \frac{(-h+1)}{8}$$

$$\left(\frac{\partial \phi}{\partial x}\right)\Big|_i^h = \phi_i' + \left(h - \frac{1}{3}\right)\frac{\Delta x^2}{4}\phi_i''' + \frac{(1-h)}{8}\Delta x^3 \phi_i^{(4)}$$

Tous instable

pr EP1 ~~avec~~ pr $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x_i}$?

12 Mars 2014 | Exo cell – Quadrangle (1)



1) $\vec{\Delta S}_e = \begin{cases} \Delta S^e_x = y_C - y_B = 4 \\ \Delta S^e_y = x_B - x_C = -1 \end{cases}$

$\vec{\Delta S}_n = \begin{pmatrix}-1\\3\end{pmatrix}$ $\vec{\Delta S}_w = \begin{pmatrix}-3\\0\end{pmatrix}$ $\vec{\Delta S}_s = \begin{pmatrix}0\\-2\end{pmatrix}$

Comme $\int_\Omega d\vec{n}\, dS = 0$ pour toute surface Ω fermée,

on a bien

$\Delta S^e_x + \Delta S^w_x + \Delta S^n_x + \Delta S^s_x = 0$

idem dans la direction y.

2)

face	u	v	ΔS_x	ΔS_y	$\dot m$
e	11	0	4	−1	
n	6	3	−1	3	3
w	5	2	−3	0	−15
s	5	2	0	−2	−4

2) a face south:

$$\dot{m}_s = \int_{\Gamma_s} \vec{v} \cdot \overrightarrow{\Delta S} \overset{?}{=} u_s \, \Delta S_x^s + v_s \, \Delta S_x^s$$
$$= -4$$

$$\dot{m}_n = 3, \qquad \dot{m}_w = -15.$$

3) the Continuity eqn reads:

$$\sum_{faces} \dot{m}_f = 0$$

so: $\dot{m}_e = -(\dot{m}_n + \dot{m}_w + \dot{m}_s)$

with $\dot{m}_e = u_e \, \Delta S_x^e + \underbrace{v_e \, \Delta S_y^e}_{=0}.$

$$\Rightarrow u_e = \frac{-(\dot{m}_n + \dot{m}_w + \dot{m}_s)}{\Delta S_x^e} = \frac{-(-16)}{4}$$
$$= 4.$$

4) Volume of the cell is:

$$V = \frac{1}{2}\Big(\underbrace{\vec{A} \cdot \overrightarrow{\Delta S}_s}_{\binom{0}{0} \times \binom{0}{-2}} + \underbrace{\vec{B} \cdot \overrightarrow{\Delta S}_e}_{\overline{\binom{2}{0} \times \binom{4}{-1}} \; 8} + \overbrace{C \cdot \overrightarrow{\Delta S}_n}^{\overline{\binom{3}{4} \times \binom{-1}{3}} \; 9} + \underbrace{\vec{D} \cdot \overrightarrow{\Delta S}_w}_{\binom{-3}{0} \times \binom{0}{3}} \Big)$$

$$= \frac{17}{2}$$

6)

$\forall$ all a curve $\mathcal{C}$ connecting A and B,

$$\psi_A - \psi_B = \int_A^B \vec{v}\cdot\vec{n}\,dS$$

the mass flux is ~~...~~ identical

$$\psi_B = \psi_A - \dot{m}_s = +4$$

$$\psi_B - \psi_C = \dot{m}_e \Rightarrow \psi_C = -\dot{m}_e + \psi_B$$
$$= -12$$

$$\psi_C - \psi_D = \dot{m}_n \Rightarrow \psi_D = \psi_C - \dot{m}_n$$
$$= -12 - 3 = -15$$

$$\psi_D - \psi_A = \dot{m}_w \Rightarrow \psi_A = \psi_D - \dot{m}_w$$
$$= -15 + 15.$$

ok

5) $$\nabla\vec{u}\Big|_V = \frac{1}{V}\sum_f \vec{u_f}\cdot\overrightarrow{\Delta S_f}$$

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)\Big|_V = \frac{1}{V}\Big(u_e \Delta S^x_e + u_n \Delta S^x_n + u_s \Delta S^x_s + u_w \Delta S^x_w\Big)$$

$$\left(\frac{\partial u}{\partial y}\right)_V = \frac{1}{V}\Big(u_e \Delta S^y_e + u_n \Delta S^y_n + \cdots\Big)$$

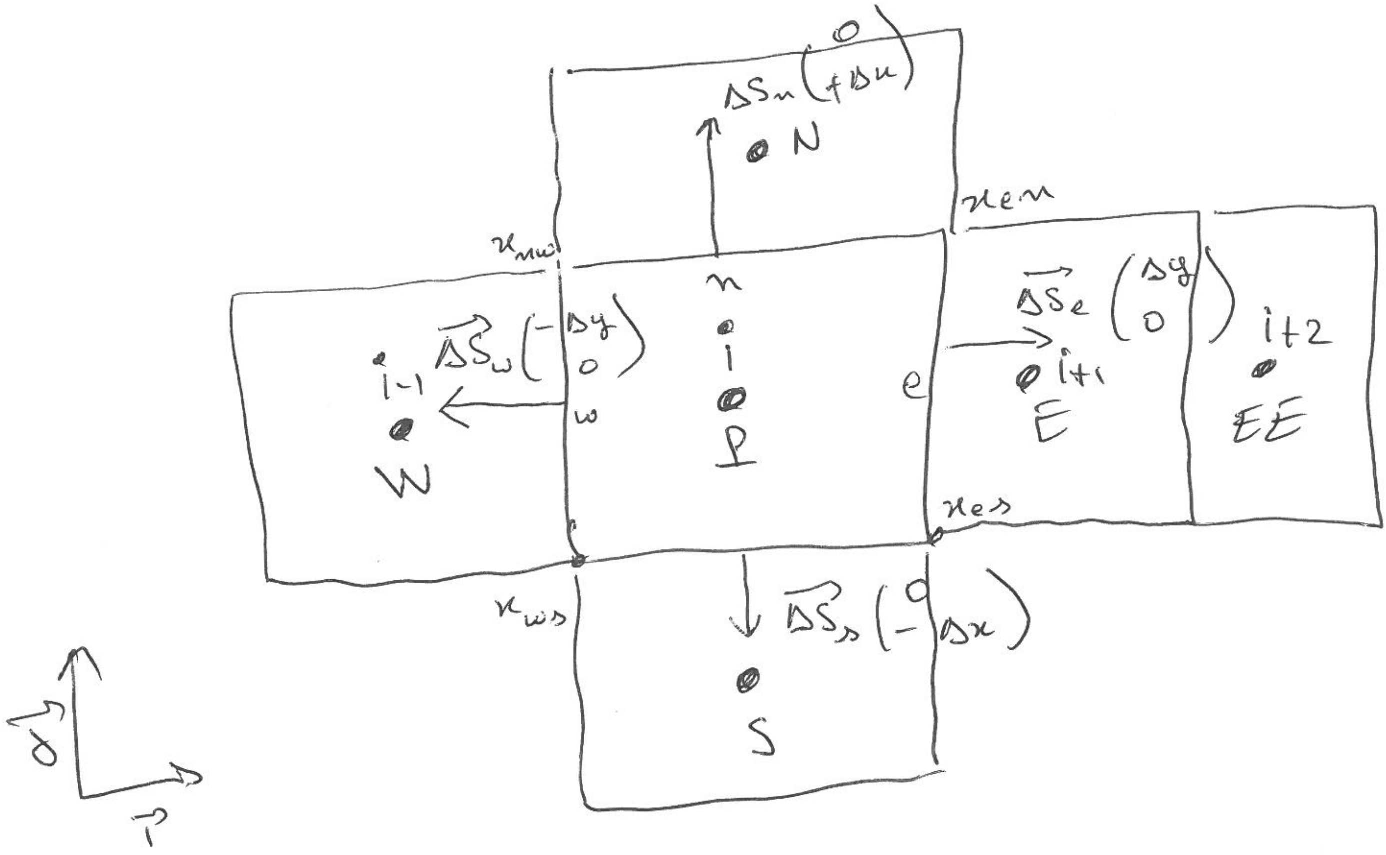
<u>A.N</u>

$$\left(\frac{\partial u}{\partial x}\right)_V = \frac{-5}{V} \quad , \quad \left(\frac{\partial u}{\partial y}\right)_V = \frac{4}{V}$$

$$\left(\frac{\partial v}{\partial x}\right)_V = \frac{-9}{V} \quad , \quad \left(\frac{\partial v}{\partial y}\right) = \frac{5}{V}.$$

Exo A.6

Gradient approximation & Spurious oscillations



1) cf Dessin. $\sum_f \vec{\Delta S}_f = 0$.

2) $V_P = \frac{1}{2} \sum_{faces} \vec{x}_f \cdot \vec{\Delta S}_f$

$= \frac{1}{2} \left(\vec{x}_{es} \cdot \vec{\Delta S}_e + \vec{x}_{en} \cdot \vec{\Delta S}_n + \vec{x}_{nw} \cdot \vec{\Delta S}_w + x_{ws} \cdot \vec{\Delta S}_s \right)$

avec $\vec{x}_{ws} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\vec{x}_{es} \begin{pmatrix} \Delta x \\ 0 \end{pmatrix}$ $\vec{x}_{ne} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$ $\vec{x}_{nw} \begin{pmatrix} 0 \\ \Delta y \end{pmatrix}$

$V_P = \frac{1}{2} \left(\Delta x \Delta y + \Delta y \Delta x \right) = \Delta x \Delta y$. ok!

3)

$$\overline{\nabla \Phi}_P = \frac{1}{V_P} \sum_f \Phi_f \overrightarrow{\Delta S}_f$$

$$= \frac{1}{\Delta x \Delta y} \left[(\Phi_e - \Phi_w) \Delta y \vec{j} + (\Phi_n - \Phi_s) \Delta x \vec{i} \right]$$

puis $\Phi_e = \frac{1}{2} \Phi_E + \Phi_P$, etc....

$$\overline{\nabla \Phi}_P = \begin{pmatrix} \delta^0_x \Phi_P \\ \delta^0_y \Phi_P \end{pmatrix} \quad \text{avec } \delta^0_x \phi_P = \frac{\phi_E - \phi_W}{2\Delta x}$$

4)

$$\left(\frac{\partial \phi}{\partial x}\right)_e = \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)_P + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)_E$$

$$= \frac{1}{2}\left[\frac{\phi_E - \phi_W}{2\Delta x} + \frac{\phi_{EE} - \phi_P}{2\Delta x}\right]$$

$$= \frac{1}{2}\left[\frac{\phi_{i+2} + \phi_{i+1} - (\phi_i + \phi_{i-1})}{2\Delta x}\right]$$

si $\phi_i = c + (-1)^i$

$$\left(\frac{\partial \phi}{\partial x}\right)_e = \frac{1}{4}(-1)^i\left[(-1)^2 + (-1)^1 - ((-1)^0 + (-1)^{-1})\right] = 0.$$

minimum correct approach: $N = \overrightarrow{\Delta S_e}$, $\vec{T} = \vec{0}$

$$\left(\frac{\partial \phi}{\partial x}\right)_e = \frac{\phi_E - \phi_P}{\Delta x} \quad \text{formule habituelle !}$$