Functional Analysis
Notes from lectures given in SMI Perugia 2021

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Abstract. One of the aims of Functional Analysis is to develop general tools on linear and/or topological spaces to tackle practical applications in Analysis. This class will be devoted to a study of some selected topics in the field. In the last part, some applications to the variational formulation and resolution of partial differential equations will be given.

These notes are strongly influenced by the courses taught by the author at Université de Lorraine (France) and the books cited in the references.

Prerequisite: Topology (especially of metric spaces), Topology of linear normed spaces including projection on convex in Hilbert spaces, Differential Calculus, Integration. Some reminders will be given during the lectures.

Contents

Prologue: What is the purpose of this class?

Functional Analysis is a field in which we develop general tools on linear and/or topological spaces to tackle practical applications in Analysis (see the introduction of Rudin [6]). In these notes, we will present several classical tools and theorems with numerous applications. Our leitmotiv throughout these chapters will be to solve ordinary differential equations (ODEs for short) like

$$\begin{cases} -u'' = f(u) & \text{in an interval } I \text{ (possibly } \mathbb{R}) \\ + \text{ boundary conditions} \end{cases} \tag{E}$$

Of course, there is the famous Cauchy-Lipschitz Theorem (see Theorem 16.5) that yields existence and uniqueness of solutions to such equations with initial boundary conditions (namely $u(0)$ and $u'(0)$). However, for some applications, it is natural to investigate solutions on an interval like $[0,1]$ where $u(0)$ and $u(1)$ are prescribed. And even sometimes on $\mathbb{R}$ where the limits at $\pm\infty$ are prescribed.

It will be convenient to rewrite (E) as a fixed point problem like $u = T(u)$, or as a variational problem like $F'(u) = 0$. Here, the functions T and F are defined on spaces of functions. It is at this point that Functional Analysis is coming into play: to develop abstract setting to find, for instance, fixed points or critical points. For instance, to find a critical point $F'(u) = 0$, a good strategy is to minimize F, which is convenient when we have some compactness properties... In finite dimensions, we often use the Bolzano-Weierstrass Theorem that allows to make every sequence converge (up to extraction) as soon as it is bounded: this is not true anymore on spaces of functions that are infinite-dimensional. This is why we will study several compactness issues, including the introduction of a weak topology for which a kind of Bolzano-Weierstrass theorem holds.

These notes are organized as follows:

In **Part 1**, we study spaces of continuous functions with a stress on Picard's/Banach's fixed point theorem and Ascoli's Theorem. This will allow us to solve some differential equations.

In **Part 2**, we will focus on two classical theorems, namely Baire's Lemma and the Hahn-Banach theorems.

Part 3 will be about the use of continuous linear maps. It is a mix between linear algebra and topology, the connection being made by norms. We will see the method of continuity that allows to preserves bijectivity of maps, thus allowing to solve "continuously" some problems. We will also introduce the weak topology that will let us be able to get a Bolzano-Weierstrass type property, which will allow us to get minimization results .

Part 4 will be devoted to Sobolev spaces , essentially in dimension 1. Here, the differential equations will be written in variational forms, and the Hilbert structure will be used.

Part 5 will be about the Fourier transform on $\mathbb{R}$. This transform will change the nature of ODEs by turning them (at least partially) into a polynomial form that will be more convenient to handle when one works on a topological group like $\mathbb{R}$.

The appendix is a collection of definitions, properties and theorem. They must be considered as prerequisite.

Exercises are directly included in the text. Solutions are at the end of these notes.

These notes are strongly inspired by the classical references by Brezis [1], Dixmier [2] and Rudin [6, 7] and the classes taught by the author in the Université de Lorraine in the past years.

Part 1

Continuous fonctions

CHAPTER 1

Picard's fixed point theorem

1. Completeness (quick review)

1.1. Definition.

DEFINITION 1.1. *Let (X, d) be a metric space. We say that a sequence $(x_n)_n \in X^{\mathbb{N}}$ is a Cauchy sequence if*

$$\forall \epsilon > 0\, \exists n_\epsilon \in \mathbb{N} \text{ such that for all } p, q \in \mathbb{N},\ \{p, q \geq n_\epsilon \Rightarrow d(x_p, x_q) < \epsilon\}.$$

We say that a metric space (X, d) is complete if every Cauchy sequence in X converges in X. A Banach space is a linear normed space complete for the associated metric.

1.2. Examples:

- $\mathbb{R}$, $\mathbb{R}^n$ $(n \geq 1)$ and any finite-dimensional linear space is complete for the unique normed space topology.
- Let X be a topological space and (Y, d_Y) be a metric spaces. We assume that X is compact and that (Y, d_Y) is complete. Then $C^0(X, Y)$ is complete when endowed with the metric
$$d(f, g) := \sup_{x \in X} d_Y(f(x), g(y)) \text{ for } f, g \in C^0(X, Y).$$
We will often write $C^0(X) := C^0(X, \mathbb{R})$.
- Let X be a topological compact space and $(F, \|\cdot\|_F)$ be a Banach spaces. Then $C^0(X, F)$ is a Banach space when endowed with the norm
$$\|f\|_\infty := \sup_{x \in X} \|f(x)\|_F \text{ for } f \in C^0(X, F).$$
- Let $(F, \|\cdot\|_F)$ be a Banach space. Then $C^k([0,1], F)$ is a Banach space when endowed with the norm
$$\|f\|_{C^k([0,1],F)} := \sum_{i=0}^{k} \|f^{(i)}\|_\infty \text{ for } f \in C^k([0,1], F).$$
- The spaces $(L^p(\Omega), \|\cdot\|_p)$ for $1 \leq p \leq \infty$.

1.3. Property.

Proposition 1.1. *Let (X, d) be a complete metric space. We take $F \subset X$. Then F is complete (with the induced metric) if and only if F is closed in X.*

1.4. Decreasing sequences of closed subspaces.

Proposition 1.2. *Let (X, d) be a complete metric space. Let $(F_n)_{n\in\mathbb{N}}$ be a sequence such that*

- $F_n \subset X$ *is closed for all* $n \in \mathbb{N}$
- $F_n \neq \emptyset$ *for all* $n \in \mathbb{N}$
- $F_{n+1} \subset F_n$ *for all* $n \in \mathbb{N}$
- *diam*$(F_n) \to 0$ *as* $n \to +\infty$.

Then there exists $x \in X$ such that $\bigcap_{n\in\mathbb{N}} F_n = \{x\}$.

PROOF. For any $n \in \mathbb{N}$, we fix $x_n \in F_n$. For $p, q \in \mathbb{N}$ such that $p \geq q$, we have that $F_p \subset F_q$, so $x_p, x_q \in F_q$ and then $d(x_p, x_q) \leq \text{diam}(F_q)$. We fix $\epsilon > 0$, so that there exists $q_0 \in \mathbb{N}$ such that $\text{diam}(F_q) < \epsilon$ for all $q \geq q_0$. Therefore $d(x_p, x_q) < \epsilon$ for all $p \geq q \geq q_0$. Then (x_n) is a Cauchy sequence in X that is complete, so it converges to some $x \in X$. We fix $n \in \mathbb{N}$ and $p \geq n$: we have that $x_p \in F_p \subset F_n$. So $x_p \in F_n$ for all $p \geq n$: since (x_p) converges to x and F_n is closed, we have that $x \in F_n$ for all $n \in \mathbb{N}$. Therefore $x \in \bigcap_{n\in\mathbb{N}} F_n$. We now take $y \in \bigcap_{n\in\mathbb{N}} F_n$, so that $y \in F_n$ for all $n \in \mathbb{N}$, and then $x, y \in F_n$ and then $d(x, y) \leq \text{diam}(F_n)$ for all $n \in \mathbb{N}$. Passing to the limit yields $d(x, y) \leq 0$, and then $x = y$. This proves that $\bigcap_{n\in\mathbb{N}} F_n = \{x\}$. $\square$

1.5. Why do we need complete spaces?? In this notes, the objective is to develop general tools to solve some problems in Analysis. "Solving" often means here "finding a solution to". With completeness, we get the existence of "something", and this will hopefully be a solution that we look for. But in general, you will not know anything more than the existence.

Keep the following motto in mind:

"When you have a problem to solve a problem in Analysis, always start with general theorems on completeness."

For instance, think of the well-known statement: "every nondecreasing sequence bounded from above in $\mathbb{R}$ converges in $\mathbb{R}$". You have a limit, but you do not know its value. To prove this result, you can prove that your sequence is a Cauchy sequence, so it converges.

2. Statement and proof of the Picard/Banach's fixed-point theorem

2.1. Definitions. Let $T : X \to X$ be a function from a set X into itself. A fixed point for T is a point $x \in X$ such that $T(x) = x$. Let (X, d) be a metric space. A contracting map on X is a map $T : X \to X$ such that there exists $k \in [0, 1)$ such that $d(T(x), T(y)) \leq kd(x, y)$ for all $x, y \in X$. In other words, T is a $k-$Lipschitz map with $k < 1$.

2.2. Statement.

THEOREM 1.3 (Picard's/Banach's fixed point theorem). *Let (X, d) be a complete space and let $T : X \to X$ be a contracting map with ratio $k \in [0, 1)$. Then T has a unique fixed point $x \in X$. In addition, given a point $z \in X$, we define the sequence $x_0 := z$ and $x_{n+1} = T(x_n)$ for all $n \in \mathbb{N}$. Then $(x_n)_n$ converges to x in (X, d) and we have the control*

$$d(x_n, x) \leq \frac{k^n}{1-k} d(x_1, x_0) \text{ for all } n \in \mathbb{N}. \tag{1}$$

2.3. Optimality of the hypotheses. The theorem does not hold if:

- T does not maps X onto itself. Example: $T : [0,1] \to [1, \frac{3}{2}]$ with $T(x) = 1 + \frac{x}{2}$ for all $x \in [0,1]$
- X is not complete. Example: $T : (0,1) \to (0,1)$ with $T(x) = \frac{x}{2}$ for all $x \in (0,1)$
- T is not contracting. Example: $T : [0,+\infty) \to [0,+\infty)$ with $T(x) = 2x+1$ for all $x \in [0,+\infty)$.

2.4. Proof of Theorem 1.3. We let $T : X \to X$, k and $(x_n)_n$ as in the statement of the theorem.

Uniqueness. Assume that $x, y \in X$ are fixed points. Then $d(x,y) = d(T(x), T(y)) \leq kd(x,y)$, and then $(1-k)d(x,y) \leq 0$. Since $k < 1$, we get that $d(x,y) \leq 0$ and then $x = y$. This proves uniqueness.

Existence. *Step 1:* For $n \in \mathbb{N}$, $n \geq 1$, we have that $d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \leq kd(x_n, x_{n-1})$. With an immediate recurrence, we then get that

$$d(x_{n+1}, x_n) \leq k^n d(x_1, x_0) \text{ for all } n \in \mathbb{N}.$$

We now fix $p \geq 1$. Via the triangle inequality, we have that

$$\begin{aligned} d(x_{n+p}, x_n) &\leq \sum_{l=0}^{p-1} d(x_{n+l+1}, x_{n+l}) \leq \sum_{l=0}^{p-1} k^{n+l} d(x_1, x_0) \\ &= k^n d(x_1, x_0) \sum_{l=0}^{p-1} k^l = k^n d(x_1, x_0) \frac{1-k^p}{1-k} \text{ since } k \neq 1 \\ &\leq \frac{k^n}{1-k} d(x_1, x_0) \text{ since } 0 \leq k < 1. \end{aligned} \tag{2}$$

Step 2: let us show that (x_n) is a Cauchy sequence. We fix $\epsilon > 0$. Since $|k| < 1$, then $k^n \to 0$ as $n \to +\infty$, and then there exists $n_0 \in \mathbb{N}$ such that $\frac{k^n}{1-k} d(x_1, x_0) < \epsilon$ for all $n \geq n_0$. Therefore $d(x_{n+p}, x_n) < \epsilon$ for all $n \geq n_0$ and $p \in \mathbb{N}$. So (x_n) is a Cauchy sequence. Since X is complete, then there exists $x \in X$ such that $x_n \to x$ when $n \to +\infty$.

Step 3: let us prove that x is a fixed point. We have that $x_{n+1} = T(x_n)$ for all $n \in \mathbb{N}$. Since T is Lipschitz, it is continuous. Passing to the limit yields $x = T(x)$ and x is a fixed point. Passing to the limit $p \to +\infty$ in (2), we get (1). This ends the proof of Theorem 1.3.

Exercise 1: Let (X,d) be a metric space. We fix a sequence $(x_n)_n$ such that $x_n \in X$ for all $n \in \mathbb{N}$. We assume that (x_n) converges to some $x \in X$. We assume that there exists $l \in X$ and $\alpha \in \mathbb{R}$ such that $d(x_n, l) \leq \alpha$ for all $n \in \mathbb{N}$. Prove that $d(x,l) \leq \alpha$.

2.5. A few direct applications.

- There exists $x \in \mathbb{R}$ such that $x = \sqrt{1+x}$.

  PROOF. Of course, we do not expect to square the identity to solve a quartic... If there is a solution, then $x \geq 0$. Define $T : [0,+\infty) \to [0,+\infty)$ such that $T(x) = \sqrt{1+x}$ for all $x \geq 0$. This is well-defined, differentiable and $0 \leq T'(x) = \frac{1}{2\sqrt{1+x}} \leq \frac{1}{2}$ for all $x \in \mathbb{R}$. It then follows from the Mean Value Inequality (see Theorem 16.12) that $|T(x) - T(y)| \leq \frac{1}{2}|x-y|$ for all $x, y \in [0, +\infty)$. It follows from Picard's theorem that there exists a unique solution to $x = T(x) = \sqrt{1+x}$. Remark: solving the quartic, we have that $x = \frac{1+\sqrt{5}}{2}$. $\square$

- Fix $k \in (0,1)$ and $h \in C^1(\mathbb{R})$ be such that $|h'(x)| \leq k$ for all $x \in \mathbb{R}$. Define

$$\left\{ \begin{array}{cccc} \phi : & \mathbb{R}^2 & \to & \mathbb{R}^2 \\ & (x,y) & \mapsto & (x-h(y), y-h(x)) \end{array} \right. .$$

  Then ϕ is a C^1-diffeomorphism.

Exercise 2: Prove this statement. Here are some indications:

- Fix $(X,Y) \in \mathbb{R}^2$. Show that proving the existence of $(x,y) \in \mathbb{R}^2$ such that $\phi(x,y) = (X,Y)$ is equivalent to find a fixed point for a suitable map $T : \mathbb{R}^2 \to \mathbb{R}^2$.
- Using a suitable norm on $\mathbb{R}^2$, and the Mean Value Inequality, prove that T is contracting.
- Use the Global Inverse Functions Theorem (see Theorem 16.4) to conclude.

2.6. More sophisticated applications: a (selective) review. The Picard's theorem is used in countless contexts:

- **Proof of the Inverse Functions Theorem** (see Theorem 16.3). Given y around $f(x_0)$ in a suitable space, finding x close to x_0 such that $y = f(x)$ rewrites $x = T(x)$ with $T(x) = x - f(x) + y$. Assuming that $df_{x_0} = Id$ (this is always possible), T is contracting around x_0: this yield local injectivity and surjectivity.
- **Proof of the Cauchy-Lipschitz Theorem for ODEs**. In order to solve $x'(t) = f(x(t))$ with $x(0) = x_0$, we rewrite $x(t) = x_0 + \int_0^t f(x(s))\,ds$ for all t. Then the function x is a solution to $x = T(x)$ with $T(x)(t) := x_0 + \int_0^t f(x(s))\,ds$ for all t. The difficulty is then to find a suitable space X to get $T : X \to X$ that is contracting. See Exercise 3 below for a proof in a specific case.
- **In numerics**, reducing a problem to a fixed point problem is often a first step. Indeed, the speed of convergence is explicit, which is a good thing. Unfortunately, this is not enough in general, and specialists have to develop much better methods to get a better convergence.
- **In economics**, people often reduce problems to fixed point with contracting maps. Since this is a big constraint, they are often asking for more sophisticated methods: this is a job ad!

Exercise 3: The Cauchy-Lipschitz theorem in a specific case. Let $f \in C^0(\mathbb{R}, \mathbb{R})$ be such that there exists $K > 0$ such that

$$|f(x) - f(y)| \leq K|x-y| \text{ for all } x, y \in \mathbb{R}.$$

We fix $x_0 \in \mathbb{R}$ and we investigate the existence of $\delta > 0$ and $x \in C^1([-\delta, \delta], \mathbb{R})$ such that

$$\left\{ \begin{array}{ll} x'(t) = f(x(t)) & \text{for all } t \in [-\delta, \delta] \\ x(0) = x_0 & \end{array} \right. \tag{3}$$

We let $\delta > 0$ be a positive number that will be fixed later.

- Let $x \in C^0([-\delta,\delta],\mathbb{R})$ be a continous function. Show that we have the equivalence:

$$\{x \text{ is } C^1 \text{ and solves (3)}\} \Leftrightarrow \left\{x(t) = x_0 + \int_0^t f(x(s))\,ds \text{ for all } t \in [-\delta,\delta]\right\}.$$

Define

$$\begin{array}{rccc} T: & C^0([-\delta,\delta],\mathbb{R}) & \to & C^0([-\delta,\delta],\mathbb{R}) \\ & x & \mapsto & \left(t \mapsto x_0 + \int_0^t f(x(s))\,ds\right) \end{array}$$

- Prove that T is well-defined and that for $\delta > 0$ well chosen, it is a contracting map when $C^0([-\delta,\delta],\mathbb{R})$ is endowed with the $\|\cdot\|_\infty$ norm.
- Prove existence and uniqueness of solutions to (3).

2.7. Continuity with respect to parameters.

Proposition 1.4. *Let (X,d) be a complete metric space and $(E,\mathcal{T})$ be a topological space. Let $F: E\times X \to X$ be a continuous function (for the product topology, see Definiton 0.5). We assume that there exists $k \in [0,1)$ such that for all $\lambda \in E$, $T(\lambda,\cdot): X \to X$ is $k-$Lipschitz continuous. For any $\lambda \in E$, it then follows from Picard's Theorem that there exists a unique $x_\lambda \in X$ that is a fixed point of $T(\lambda,\cdot)$. Then the map $\lambda \to x_\lambda$ from $E \to X$ is continuous.*

Exercise 4: Prove Proposition 1.4. Before tackling the general case, start with the following situations:

- $(E,\mathcal{T})$ is a metric space (E,d_E) and there exists $L > 0$ such that T satisfies
$$d(T(\lambda,x),T(\lambda',x')) \leq Ld_E(\lambda,\lambda') + kd(x,x')$$
for all $\lambda,\lambda' \in E$ and $x,x' \in X$.
- $(E,\mathcal{T})$ is a metric space.
- General case.

Exercise 5: We take the notations of Exercise 3. Prove that the map $(x_0,t) \mapsto x(t)$ is continuous on $\mathbb{R}\times[-\delta,\delta]$, up to taking $\delta > 0$ smaller.

3. Application: attractive and repulsive fixed points

Let I be an intervalle of $\mathbb{R}$ and fix $f \in C^1(I,I)$. We assume that there exists $\alpha \in I$ that is a fixed point of f. We define

$$A := \{(x_n)_n /\, x_n \in I \text{ for all } n \in \mathbb{N} \text{ and } x_{n+1} = f(x_n) \text{ for all } n \in \mathbb{N}\}.$$

We say that

- The fixed point α is said to be *attractive* if there exists $\delta > 0$ such that for all sequence $(x_n)_n \in A$ such that there exists $n_0 \in \mathbb{N}$ such that $|x_{n_0} - \alpha| < \delta$, then (x_n) converges to α.
- The fixed point α is said to be *repulsive* if any sequence $(x_n)_n \in A$ that converges to α is indeed stationary at α (that is there exists $n_1 \in \mathbb{N}$ such that $x_n = \alpha$ for all $n \geq n_1$.

1. We assume that f is C^1 and that $|f'(\alpha)| < 1$. Show that α is attractive.
2. We assume that f is C^1 and that $|f'(\alpha)| > 1$. Show that α est repulsive.
3. Show that if f is C^1 and that $|f'(\alpha)| = 1$, then anything can happen.
4. Generalize to $f: U \to U$ with U an open set of $\mathbb{R}^n$, $n \geq 1$.

4. Newton's method

Let $[a,b]$ be a compact nonempty intervalle of $\mathbb{R}$. Let $f:[a,b]\to\mathbb{R}$ be a C^2-function. We assume that

(1) There exists $\alpha\in]a,b[$ such that $f(\alpha)=0$,
(2) $f'(x)>0$ for all $x\in[a,b]$.

We define

$$F:\begin{array}{ccc}[a,b] & \to & \mathbb{R}\\ x & \mapsto & x-\frac{f(x)}{f'(x)}\end{array}$$

1. Compute F'.

2. Show that there exists $\epsilon>0$ such that for all $z\in[\alpha-\epsilon,\alpha+\epsilon]\subset[a,b]$, then the sequence $(x_n)_n$ defined by $\{x_0=z\,;\,x_{n+1}=F(x_n)\text{ for all }n\}$ is well-defined and converges to α.

3. We assume that f is in C^3. Show that, up to taking $\epsilon>0$ small enough, there exists $K>0$ such that for all $z\in[\alpha-\epsilon,\alpha+\epsilon]$, the sequence (x_n) defined in **2.** satisfies

$$|x_{n+1}-\alpha|\le K|x_n-\alpha|^2\text{ for all }n\in\mathbb{N}.$$

Deduce that $K|x_n-\alpha|\le(K|z-\alpha|)^{2^n}$ for all $n\in\mathbb{N}$. Is this convergence of interest? Generalize to $f\in C^2$.

4. Generalize to $f:U\to\mathbb{R}^n$ where U is an open subset of $\mathbb{R}^n$.

5. Solution of the exercises of Chapter 1

Solution of Exercise 1: Via the triangle inequality, for any $n\in\mathbb{N}$, we have that $d(x,l)\le d(x,x_n)+d(x_n,l)\le d(x,x_n)+\alpha$. Fix $\epsilon>0$. Then there exists $n_0\in\mathbb{N}$ such that $d(x,x_n)<\epsilon$ for all $n\ge n_0$. So for $n\ge n_0$, we have that $d(x,l)\le\epsilon+\alpha$. This is valid for all $\epsilon>0$. Letting $\epsilon\to0$ yields the result. $\square$

Solution of Exercise 2: For $(X,Y)\in\mathbb{R}^2$ and $(x,y)\in\mathbb{R}^2$, we have that

$$\phi(x,y)=(X,Y)\Leftrightarrow(x,y)=(x,y)-\phi(x,y)+(X,Y):=T(x,y)$$

where

$$\left\{\begin{array}{cccc}T: & \mathbb{R}^2 & \to & \mathbb{R}^2\\ & (x,y) & \mapsto & (X+h(y),Y+h(x))\end{array}\right..$$

We endow $\mathbb{R}^2$ with the norm $\|(x,y)\|_\infty:=\max\{|x|,|y|\}$ for $(x,y)\in\mathbb{R}^2$. The mean value inequality (see Theorem 16.12) yields $|h(x)-h(y)|\le k|x-y|$ for all $x,y\in\mathbb{R}$. Therefore, given $(x,y),(x',y')\in\mathbb{R}^2$, we have that

$$\begin{aligned}\|T(x,y)-T(x',y')\|_\infty &= \max\{|h(y')-h(y)|,|h(x')-h(x)|\}\\ &\le k\max\{|y'-y|,|x'-x|\}\le k\|(x,y)-(x',y')\|_\infty\end{aligned}$$

Since $0\le k<1$, $T:\mathbb{R}^2\to\mathbb{R}^2$ is contracting for $\|\cdot\|_\infty$. Since $(\mathbb{R}^2,\|\cdot\|_\infty)$, we get that T has a unique fixed point. Therefore for any $(X,Y)\in\mathbb{R}^2$, there exist a unique point $(x,y)\in\mathbb{R}^2$ such that $\phi(x,y)=(X,Y)$. Then ϕ is a bijection.

It is clear that $\phi\in C^1(\mathbb{R}^2,\mathbb{R}^2)$ and the Jacobian is

$$\mathrm{Jac}(\phi)_{(x,y)}=\det(\partial_x\phi,\partial_y\phi)=\begin{vmatrix}1 & -h'(y)\\ -h'(x) & 1\end{vmatrix}=1-h'(x)h'(y)\ge1-k^2>0.$$

So $d\phi_{(x,y)}$ is invertible for all $(x,y)\in\mathbb{R}^2$. Since ϕ is $1-1$ and $\phi(\mathbb{R}^2)=\mathbb{R}^2$, it follows from Theorem 16.4 that ϕ is a C^1-diffeomorphism. $\square$

Solution of Exercise 3:

- ($\Rightarrow$) We just write that for any $t \in [-\delta, \delta]$, we have that $x(t) = x(0) + \int_0^t x'(s)\,ds = x_0 + \int_0^t f(x(s))\,ds$.
  ($\Leftarrow$) Let $x \in C^0([-\delta, \delta], \mathbb{R})$ be such $x(t) = x_0 + \int_0^t f(x(s))\,ds$ for all $t \in [-\delta, \delta]$. Since $f \circ x$ is continuous, we get that $t \mapsto \int_0^t f(x(s))\,ds$ is C^1. Therefore $x \in C^1$ and $x'(t) = 0 + f(x(t))$ for all t and $x(0) = x_0 + \int_0^0 \ldots = x_0$.
- Since $f \circ x$ is continuous, then $t \mapsto \int_0^t f(x(s))\,ds$ is C^1, so in $C^0([-\delta, \delta], \mathbb{R})$. Then T is well defined. For $x, y \in C^0([-\delta, \delta], \mathbb{R})$ and $t \in [-\delta, \delta]$, we have that

$$\begin{aligned} |(T(x) - T(y))[t]| &= \left| x_0 + \int_0^t f(x(s))\,ds - \left(x_0 + \int_0^t f(y(s))\,ds \right) \right| \\ &= \left| \int_0^t (f(x(s)) - f(y(s))\,ds \right| \leq \left| \int_0^t |f(x(s)) - f(y(s)|\,ds \right| \\ &\leq K \left| \int_0^t |x(s) - y(s)|\,ds \right| \leq K \left| \int_0^t \|x - y\|_\infty\,ds \right| \\ &\leq K|t|\|x - y\|_\infty \leq K\delta\|x - y\|_\infty \end{aligned}$$

  Taking the supremum with respect to $t \in [-\delta, \delta]$ yields

$$\|T(x) - T(y)\|_\infty \leq K\delta\|x - y\|_\infty \text{ for all } x, y \in C^0([-\delta, \delta], \mathbb{R})$$

  Then, with $0 < \delta < \frac{1}{K}$, the map T is contracting.
- Since $(C^0([-\delta, \delta], \mathbb{R}), \|\cdot\|_\infty)$ is complete (even Banach), T has a unique fixed point $x \in C^0([\delta, \delta], \mathbb{R})$, which is by the first question the unique solution to (3).

□

Solution of Exercise 4:

- In the first case, for $\lambda, \lambda' \in E$, we have that

$$d(x_\lambda, x'_\lambda) = d(T(\lambda, x_\lambda), T(\lambda', x_{\lambda'})) \leq L d_E(\lambda, \lambda') + k d(x_\lambda, x_{\lambda'}).$$

  And then $d(x_\lambda, x_\lambda) \leq \frac{L}{1-k} d_E(\lambda, \lambda')$. Therefore $\lambda \mapsto x_\lambda$ is Lipschitz continuous, and then continuous.
- Assume that $(E, \mathcal{T}) = (E, d_E)$ is a metric space. For $\lambda, \lambda_0 \in E$, we have that

$$\begin{aligned} d(x_\lambda, x_{\lambda_0}) &= d(T(\lambda, x_\lambda), T(\lambda_0, x_{\lambda_0})) \leq d(T(\lambda, x_\lambda), T(\lambda, x_{\lambda_0})) + d(T(\lambda, x_{\lambda_0}), T(\lambda_0, x_{\lambda_0})) \\ &\leq k d(x_\lambda, x_{\lambda_0}) + d(T(\lambda, x_{\lambda_0}), T(\lambda_0, x_{\lambda_0})) \end{aligned}$$

  and therefore

$$d(x_\lambda, x_{\lambda_0}) \leq \frac{d(T(\lambda, x_{\lambda_0}), T(\lambda_0, x_{\lambda_0}))}{1 - k}.$$

  For $\lambda_0 \in E$ that is fixed, the map $\lambda \to T(\lambda, x_{\lambda_0})$ is continuous (since T is continuous). Fix $\epsilon > 0$. Then there exists $r > 0$ such that for $\lambda \in E$, then $d_E(\lambda.\lambda_0) < r \Rightarrow d(T(\lambda, x_{\lambda_0}), T(\lambda_0, x_{\lambda_0})) < (1 - k)\epsilon$, and then $d(x_\lambda, x_{\lambda_0}) < \epsilon$. So $\lambda \mapsto x_\lambda$ is continuous.

- For a general topological space $(E, \mathcal{T})$, the proof is similar except for a tiny difference in the final argument. For $\lambda_0 \in E$ that is fixed, the map $\lambda \to T(\lambda, x_{\lambda_0})$ is continuous (since T is continuous). Fix $\epsilon > 0$. Then there exists an open set $x_{\lambda_0} \in U \subset E$ such that for $\lambda \in U$, we have that $d(T(\lambda, x_{\lambda_0}), T(\lambda_0, x_{\lambda_0})) < (1-k)\epsilon$, and then $d(x_\lambda, x_{\lambda_0}) < \epsilon$. So $\lambda \mapsto x_\lambda$ is continuous.

□

Solution of Exercise 5: Define $T(x_0, x) := \left\{t \mapsto x_0 + \int_0^t f \circ x(s)\, ds\right\}$ for $x \in C^0([-\delta, \delta], \mathbb{R})$. Arguing as in Exercise 3, we get that

$$\|T(x_0, x) - T(y_0, y)\|_\infty \leq |x_0 - y_0| + K\delta \|x - y\|_\infty \text{ for all } x, y \in C^0([-\delta, \delta], \mathbb{R}).$$

So $T : \mathbb{R} \cap C^0([-\delta, \delta], \mathbb{R}) \times C^0([-\delta, \delta], \mathbb{R})$ is continuous and for $0 < \delta < \frac{1}{K}$, $T(x_0, \cdot)$ is $K\delta-$Lipschitz. So we are in the first case of Exercise 4, and $x_0 \mapsto \{x \in C^0([-\delta, \delta], \mathbb{R})\text{solution to (3)}\}$ is continuous $\mathbb{R} \to C^0([-\delta, \delta], \mathbb{R})$. Then $(x_0, t) \mapsto x(t)$ is also continuous. □

Solution of "Application: attractive and repulsive fixed points":

1. Fix $k \in \mathbb{R}$ such that $|f'(\alpha)| < k < 1$. Since f' is continuous, there exists $\delta > 0$ such that for all $x \in [\alpha - \delta, \alpha + \delta] \cap I$, we have that $|f'(x)| < k$. It then follows from the mean value theorem that $|f(x) - f(y)| \leq k|x - y|$ for all $x \in [\alpha - \delta, \alpha + \delta] \cap I$.

We claim that $f([\alpha - \delta, \alpha + \delta] \cap I) \subset [\alpha - \delta, \alpha + \delta] \cap I$, that is $[\alpha - \delta, \alpha + \delta] \cap I$ is stable by f.
We prove the claim. Take $x \in [\alpha - \delta, \alpha + \delta] \cap I$. Then $|f(x) - \alpha| = |f(x) - f(\alpha)| \leq k|x - \alpha| \leq k\delta \leq \delta$. Since $f(I) \subset I$, we then have that $f(x) \in [\alpha - \delta, \alpha + \delta] \cap I$ and the claim is proved.

Up to taking $\delta > 0$ smaller, $[\alpha - \delta, \alpha + \delta] \cap I$ is a closed intervalle. Since f is contracting on $[\alpha - \delta, \alpha + \delta] \cap I$ which is stable, it follows from Picard's/Banach's theorem that for all $x \in A$, if there exists n_0 such that $x_{n_0} \in [\alpha - \delta, \alpha + \delta]$, then $x_n \to \alpha$ as $n \to +\infty$. So α is attractive.

2. We take $k \in \mathbb{R}$ such that $|f'(\alpha)| > k > 1$. Since f' is continuous, there exists $\delta > 0$ such that for all $x \in [\alpha - \delta, \alpha + \delta] \cap I$, $|f'(x)| \geq k$. So wih the mean value theorem, we have that $|f(x) - f(y)| \geq k|x - y|$ for all $x, y \in [\alpha - \delta, \alpha + \delta] \cap I$. Take $(x_n)_n \in A$ such that $x_n \to \alpha$ as $n \to +\infty$. Then there exists n_0 such that $x_n \in [\alpha - \delta, \alpha + \delta]$ for all $n \geq n_0$. Then $|x_{n+1} - \alpha| = |f(x_n) - f(\alpha)| \geq k|x_n - \alpha|$ for all $n \geq n_0$. So by induction, $|x_n - \alpha| \geq k^{n-p}|x_p - \alpha|$ for all $n \geq p \geq n_0$. Since $x_n \to \alpha$ and $k > 1$, letting $n \to +\infty$ yields $x_p = \alpha$ for all $p \geq n_0$. So (x_n) is stationary, and then α is repulsive.

3. Here are a few examples. In all these examples, $f \in C^1(\mathbb{R})$, $f(0) = 0$ and $f'(0) = 1$

- $f(x) = x$ for all $x \in \mathbb{R}$. Then for all $x_0 \in \mathbb{R}$, $x_n = x_0$ for all $n \in \mathbb{N}$. Neither attractive nor repulsive.
- $f(x) = x - x^2$ for $x > 0$ and $f(x) = x + x^2$ for $x < 0$. Then 0 is attractive. Indeed, as one checks, $f([0, 1]) \subset [0, 1]$. So take $(x_n)_n \in A$ be such that $0 \leq x_0 \leq 1$. Then $0 \leq x_n \leq 1$ for all $n \geq 1$ and $x_{n+1} = x_n - x_n^2 \leq x_n$, so (x_n) is non-increasing and nonnegative, so it converges to a limit that is a fixed point of f, that is 0. This is similar if $-1 \leq x_0 \leq 0$. So for $-1 \leq x_0 \leq 1$, then $(x_n)_n \to 0$ and then 0 is attractive.

- $f(x) = x + x^2$ for $x > 0$ and $f(x) = x - x^2$ for $x < 0$. Then 0 is repulsive. Take $(x_n) \in A$ be such that $x_0 > 0$. Then $x_{n+1} = f(x_n) = x_n + x_n^2 \geq x_n$, and then (x_n) is increasing. If it converges, let l be its limit: we have that $l \geq x_0$ and $f(l) = l$, which is $l = 0$. This is a contradiction so (x_n) does not converge and it goes to $+\infty$. Similarly, if $x_0 < 0$, then $(x_n) \to -\infty$. This proves that 0 is repulsive.

4. On $E = \mathbb{R}^n$, take a norm $\|\cdot\|$. Case 1 corresponds to $\|df_\alpha\|_{E\to E} < 1$. For Case 2, one must assume that $\|df_\alpha(x)\|_E > \|x\|_E$ for all $x \in E \setminus \{0\}$. $\square$

Solution of "Newton's method":

1. Since $f \in C^2$, then F is C^1 and we get that

$$F'(x) = 1 - \frac{f'(x) \times f'(x) - f(x)f''(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2} \text{ for all } x \in [a,b].$$

2. Since $f(\alpha) = 0$, we get that $F(\alpha) = \alpha$ and $F'(\alpha) = 0$: it then follows from the results about attractive/repulsive points that α is an attractive fixed-point for F, which yields the existence of ϵ (remember that $a < \alpha < b$).

3. Since f is C^3, then $F \in C^2([a,b])$ and define $M := \max_{x\in[a,b]} |F''(x)| < +\infty$. Given $n \in \mathbb{N}$, Taylor's formula yields the existence of $c_n = c(x_n, \alpha) \in (\alpha, x_n)$ such that

$$x_{n+1} = F(x_n) = F(\alpha) + F'(\alpha)\cdot(x_n - \alpha) + \frac{F''(c_n)}{2}(x_n - \alpha)^2$$

Since $F(\alpha) = \alpha$ and $F'(\alpha) = 0$, we then get that

$$|x_{n+1} - \alpha| \leq K|x_n - \alpha|^2 \text{ for all } n \in \mathbb{N} \text{ with } K := M/2.$$

Defining $t_n := K|x_n - \alpha|$, we then get that $0 \leq t_{n+1} \leq t_n^2$ for all $n \in \mathbb{N}$. An immediate induction yields $t_n \leq t_0^{2^n}$ for all $n \in \mathbb{N}$, and then

$$|x_n - \alpha| \leq K^{-1}(K|z - \alpha|)^{2^n} \text{ for all } n \in \mathbb{N}.$$

Therefore, if $|z - \alpha| < 1/K$, we get a very fast convergence of $x_n \to \alpha$, much faster than the rate for $k-$contracting maps (k^n). BUT for this, we need a control of K, which amount to get a good control on $f, f', f'', f^{(3)}$.

If $f \in C^2$, then $F \in C^1$ and for $n \in \mathbb{N}$, there exists $d_n \in (x_n, \alpha)$ such that

$$x_{n+1} - \alpha = F(x_n) - F(\alpha) = F'(d_n)\cdot(x_n - \alpha) = \frac{f(d_n)f''(d_n)}{(f'(d_n))^2}\cdot(x_n - \alpha)$$

In addition,

$$|f(d_n)| = |f(d_n) - f(\alpha)| \leq (\max_{[0,1]}|f'|)\cdot|d_n - \alpha| \leq (\max_{[0,1]}|f'|)\cdot|x_n - \alpha| \text{ since } d_n \in (x_n, \alpha)$$

and then, putting all these inequalities together, we get that

$$|x_{n+1} - \alpha| \leq \frac{\max_{[0,1]}|f'|\cdot\max_{[0,1]}|f''|}{\min_{[0,1]}|f'|}(x_n - \alpha)^2$$

and then we have an estimate similar to the case $f \in C^3$, and we can proceed with the same strategy.

4. In the general case $f : U \to \mathbb{R}^n$ where U is an open subset of $\mathbb{R}^n$, we assume that some $\alpha \in U$ is such that $f(\alpha) = 0$, that $f \in C^1(U, \mathbb{R}^n)$ and that $df_x \in Gl(\mathbb{R}^n)$ for all $x \in U$. We then define

$$F(x) = x - (df_x)^{-1}(f(x)) \text{ for all } x \in U.$$

Then all the previous analysis works by using several variables calculus. □

CHAPTER 2

Stone-Weierstrass and density

Main issue:

Do we have "good" functions dense in C^0? For instance smooth (C^∞) functions or polynomials?

Apart for the conceptual interest of this issue, let us mention two reasons to get interested in this question:

- Some properties that are true on smooth functions can sometimes be extended to continuous functions via density arguments.
- In these notes, we will often use "separable" metric spaces, that is metric spaces that have a countable dense subset .

1. The Stone-Weierstrass Theorem

1.1. Statement.

THEOREM 2.1. *[Stone-Weierstrass Theorem] Let $(X, \mathcal{T})$ be a compact nonempty topological space. We let $\mathcal{H} \subset C^0(X, \mathbb{R})$ be such that*

- *$\mathbb{R} \subset \mathcal{H}$ (in the sense that $\mathcal{H}$ contains the constant functions)*
- *For all $u, v \in \mathcal{H}$, $u + v \in \mathcal{H}$ and $uv \in \mathcal{H}$*
- *For all $x, y \in X$ such that $x \neq y$, then there exists $u \in \mathcal{H}$ such that $u(x) \neq u(y)$.*

Then $\mathcal{H}$ is dense in $C^0(X, \mathbb{R})$ for the norm $\|u\|_\infty := \sup_{x \in X} |u(x)|$.

A "compact" statement is that "Any separing subalgebra of $C^0(X, \mathbb{R})$ containing the constants is dense."

1.2. Proof.

1.2.1. We start with a Proposition:

Proposition 2.2. *Let $(X, \mathcal{T})$ be a compact topological space and $\mathcal{H} \subset C^0(X, \mathbb{R})$ be such that*

- *$\mathcal{H}$ is stable via sup and inf (that is for any $u, v \in \mathcal{H}$, we have that $\sup\{u, v\}, \inf\{u, v\} \in \mathcal{H}$)*
- *For all $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$ (with $\alpha = \beta$ is $x = y$), there exists $u \in \mathcal{H}$ such that $u(x) = \alpha$ and $u(y) = \beta$.*

Then $\mathcal{H}$ is dense in $C^0(X, \mathbb{R})$ for the norm $\|u\|_\infty := \sup_{x \in X} |u(x)|$.

PROOF. We fix $f \in C^0(X, \mathbb{R})$ and $\epsilon > 0$.

Step 1: We claim that for all $x_0 \in X$, there exists $u \in \mathcal{H}$ such that $u(x_0) = f(x_0)$ and $u > f - \epsilon$.

Proof of Step 1: For any $y \in X$, there exists $u_y \in \mathcal{H}$ such that $u_y(y) = f(y)$ and $u_y(x_0) = f(x_0)$. We define

$$U_y := \{x \in X \text{ such that } u_y(x) > f(x) - \epsilon\}.$$

As one checks, $y \in U_y$ and $U_y := u_y^{-1}((f(x) - \epsilon, +\infty))$ is open for all $y \in X$. Therefore $X = \bigcup_{y \in X} U_y$. Since X is compact, we extract a finite covering

$$X = U_{y_1} \cup ... \cup U_{y_N} \text{ for some } y_1, ..., y_N \in X.$$

Define $u := \sup\{u_{y_1}, ..., u_{y_N}\}$. Then $u \in \mathcal{H}$ (this deserves a little proof) and $u(x_0) = \sup\{u_{y_1}(x_0), ..., u_{y_N}(x_0)\} = f(x_0)$. Let $x \in X$ be arbitrary. Then there exists $i \in \{1, ..., N\}$ such that $x \in U_{y_i}$, and therefore $u(x) \leq u_{y_i}(x) > f(x) - \epsilon$. This yields Step 1.

Step 2: We claim that there exists $v \in \mathcal{H}$ such that $f + \epsilon > v > f - \epsilon$.

Proof of Step 2: It follows from Step 1 that for any $x \in X$, there exists $v^x \in \mathcal{H}$ such that $v^x(x) = f(x)$ and $v^x > f - \epsilon$. We define

$$V^x := \{z \in X \text{ such that } v^x(z) < f(x) + \epsilon\}.$$

As above, $(V^x)_{x \in X}$ is an open covering of the compact X, so there exists $x_1, ..., x_M \in X$ such that $X = V^{x_1} \cup ... \cup V^{x_M}$. We define $v := \inf\{v^{x_1}, ..., v^{x_M}\}$. Arguing as in Step 1, we get that $v < f + \epsilon$. Moreover, since $v^{x_i} > f - \epsilon$ for all $i = 1, ..., M$, we have that $f + \epsilon > v > f - \epsilon$. This yields Step 2.

Step 3: We fix $R > 0$ and we define the function $p : [0, R] \to \mathbb{R}$ as $p(t) = \sqrt{t}$ for all $0 \leq t \leq R$. We claim thet p is the uniform limit of a sequence of polynomials.

Proof of Step 3: We define a sequence of functions as follows:

$$\begin{cases} p_0(t) = 0 & \text{for all } t \in \mathbb{R} \\ p_{n+1}(t) := p_n(t) + \frac{1}{2}(t - p_n(t)^2) & \text{for all } t \in \mathbb{R} \text{ and } n \in \mathbb{N} \end{cases}$$

Show that

- For all $n \in \mathbb{N}$, p_n is a polynomial.
- For all $n \in \mathbb{N}$, $0 \leq p_n(t) \leq \sqrt{t}$ for all $t \in [0, 1]$. Use an induction argument.

  Solution: For $n \in \mathbb{N}$ and $t \geq 0$, we have that

$$p_{n+1}(t) - \sqrt{t} = (p_n(t) - \sqrt{t}) \left(1 - \frac{1}{2}(p_n(t) + \sqrt{t})\right). \tag{4}$$

  We set the induction hypothesis $HR_n := \{0 \leq p_n(t) \leq \sqrt{t} \text{ for all } t \in [0, 1]\}$. Clearly HR_0 holds. Fix $n \in \mathbb{N}$ and assume that HR_n holds. We take $t \geq 0$. We have that $p_n(t) \geq 0$ and $t - p_n(t)^2 \geq 0$, the definition then yields $p_{n+1}(t) \geq 0$. The identity (4) then yields $p_{n+1}(t) - \sqrt{t} \leq 0$. Then HR_{n+1} holds. Therefore HR_n holds for all $n \in \mathbb{N}$ and the claim is proved.
- For all $t \geq 0$, $(p_n(t))_n$ converges to $\sqrt{t}$ as $n \to +\infty$.

  Solution: The definition and $0 \leq p_n(t) \leq \sqrt{t}$ yields $0 \leq p_n(t) \leq p_{n+1}(t) \leq \sqrt{t}$ for all $n \in \mathbb{N}$ and $t \geq 0$. So the sequence $(p_n(t))_t$ is nondecreasing and bounded from above, and then it converges to some limit, say $p_\infty(t) \geq 0$. Passing to the limit in the definition yields $p_\infty(t) := p_\infty(t) + \frac{1}{2}(t - p_\infty(t)^2)$, and then $p_\infty(t) = \sqrt{t}$. This ends the proof of the claim.

- Conclude with Dini's Theorem (see Theorem 16.11).

  Solution: It is straightforward.
- Generalize to $[0, R]$ by writing that $\sqrt{t} = \sqrt{R} \times \sqrt{\frac{t}{R}}$.

Step 4: Proof of the Theorem of Stone-Weierstrass. We define $H := \overline{\mathcal{H}}$ where $\mathcal{H}$ is as in the statement of the Theorem.

Exercise 6: We are proving that H satisfies the hypothesis of Proposition 2.2.

- Prove that for all $u \in H$, then $|u| \in H$.
- Using that $\inf\{u, v\} = \frac{1}{2}(u+v-|u-v|)$ and $\sup\{u, v\} = \frac{1}{2}(u+v+|u-v|)$, prove that H is stable via inf and sup.
- Fix $x, y \in X$ such that $x \neq y$ and $\alpha, \beta \in \mathbb{R}$ (with $\alpha = \beta$ if $x = y$). Let $u \in \mathcal{H}$ be such that $u(x) \neq u(y)$. With a suitable choice of $a, b \in \mathbb{R}$, prove that $v := au + b \in \mathcal{H}$ and that $v(x) = \alpha$, $v(y) = \beta$. What about the case $x = y$ and $\alpha = \beta$?
- Conclude to prove the Theorem.

□

2. Applications of the Stone-Weierstrass Theorem

2.1. Complex version.

THEOREM 2.3. *[Stone-Weierstrass Theorem] Let $(X, \mathcal{T})$ be a compact nonempty topological space. We let $\mathcal{H} \subset C^0(X, \mathbb{C})$ be such that*

- *$\mathbb{C} \subset \mathcal{H}$ (in the sense that $\mathcal{H}$ contains the constant functions)*
- *For all $u, v \in \mathcal{H}$, $u + v \in \mathcal{H}$, $uv \in \mathcal{H}$ and $\bar{u} \in \mathcal{H}$*
- *For all $x, y \in X$ such that $x \neq y$, then there exists $u \in \mathcal{H}$ such that $u(x) \neq u(y)$.*

Then $\mathcal{H}$ is dense in $C^0(X, \mathbb{C})$ for the norm $\|u\|_\infty := \sup_{x\in X} |u(x)|$.

A "compact" statement is that "Any separing subalgebra of $C^0(X, \mathbb{C})$ stable by conjugasion and containing the constants is dense."

PROOF. The idea is to use the real version by taking the real part. The main remark is that the real and imaginary parts satisfy $Re(f) = \frac{1}{2}(f + \bar{f})$ and $Im(f) = \frac{1}{2i}(f - \bar{f})$ for all $f \in C^0(X, \mathbb{C})$. Define

$$\mathcal{H}_r := \{Re(f)/\ f \in \mathcal{H}\} \text{ and } \mathcal{H}_i := \{Im(f)/\ f \in \mathcal{H}\}.$$

The steps of the proof are in the following exercise:

Proof of the complex density theorem

- Prove that $\mathcal{H}_r = \mathcal{H} \cap C^0(X, \mathbb{R})$
- Prove that $\mathcal{H}_r$ is dense in $C^0(X, \mathbb{R})$.
- Similarly, prove that $\mathcal{H}_i$ is dense in $C^0(X, \mathbb{R})$.
- Conclude to prove that $\mathcal{H}$ is dense in $C^0(X, \mathbb{C})$.

□

2.2. Density of polynomials.

Proposition 2.4. *The polynomials are dense in* $C^0([0,1],\mathbb{R})$*. In particular,* $C^\infty([0,1],\mathbb{C})$ *is dense in* $C^0([0,1],\mathbb{R})$*.*

This is a direct application by taking $\mathcal{H} = \mathbb{R}[X] \cap C^0([0,1],\mathbb{R})$ (with a slight abuse of notation).

2.3. Density of trigonometric polynomials.

Proposition 2.5. *Define* $\mathcal{T} := \{t \mapsto P(e^{it}) / \ P \in \mathbb{C}[X]\}$ *the space of trigonometric polynomials. Then* $\mathcal{T}$ *is dense in* $C^0_{2\pi}(\mathbb{R},\mathbb{C})$ *(the set of* $2\pi-$*periodic continuous functions* $\mathbb{R} \to \mathbb{C}$*) for* $\|\cdot\|_\infty$*.*

Proof. The proof uses the notion of topology induced by a quotient, which is not in the scope of this lecture. However, here are the main steps:

- Define the quotient $\mathbb{T} := \mathbb{R}_{/2\pi\mathbb{Z}}$ and the canonical projection $p : \mathbb{R} \to \mathbb{T}$ such that $p(x) = \bar{x}$ (the class) for all $x \in \mathbb{R}$.
- We say that a subset $\Omega \subset \mathbb{T}$ is open if $p^{-1}(\Omega)$ is open in $\mathbb{R}$. This defined a topology on $\mathbb{T}$ for which $\mathbb{T}$ is a compact topological space.
- Define $i : C^0(\mathbb{T},\mathbb{C}) \to C^0_{2\pi}(\mathbb{R},\mathbb{C})$ as $i(f) := f \circ p$ for all $f \in C^0(\mathbb{T},\mathbb{C})$. Then i is an isometry for the associated $\|\cdot\|_\infty$ norms.
- Via Stone-Weierstarss's theorem, $p^{-1}(\mathcal{T})$ is dense in $C^0(\mathbb{T},\mathbb{C})$.
- Via the isometry p, $\mathcal{T}$ is dense in $C^0_{2\pi}(\mathbb{R},\mathbb{C})$.

$\square$

3. Solution of the exercises of Chapter 2

Solution of Exercise 6:

- For simplicity, we first assume that $u \in \mathcal{H}$. Then $|u| = \sqrt{u^2}$. Since u^2 is continuous on a compact set, it is bounded and there exists $R > 0$ such that $u^2(x) \in [0,R]$ for all $x \in X$. It then follows from Step 3 that $(p_n(u^2))$ converges to $\sqrt{u^2} = |u|$ uniformly on X as $n \to +\infty$. Since $X \to p_n(X^2)$ is a polynomial and $u \in \mathcal{H}$, we get that $p_n(u^2) \in \mathcal{H}$ for all $n \in \mathbb{N}$, and then $|u| \in \overline{\mathcal{H}} = H$.

  We now tackle the general case $u \in H$. We let $(u_n)_n$ such that $u_n \in \mathcal{H}$ for all $n \in \mathbb{N}$ and $u_n \to u$ uniformly as $n \to +\infty$. Since $||u_n| - |u|| \leq |u_n - u|$, we have that $|u_n| \to |u|$ uniformly. It follows from the first case that $|u_n| \in H$, so $|u| \in \overline{H} = H$. This proves the claim.
- H contains the constants and is stable by product and sum, so $au+bv \in H$ for all $a, b \in \mathbb{R}$ and $u, v \in H$. Since $|u-v| \in H$ for all $u, v \in H$, we then get the result.
- It is the resolution of the system $au(x) + b = \alpha$ and $au(y) + b = \beta$, which yields $a = \frac{\alpha-\beta}{u(x)-u(y)}$ and $b = \frac{\beta u(x) - \alpha u(y)}{u(x)-u(y)}$. When $x = y$ and $\alpha = \beta$, just take $v = \alpha$ (the constant function).
- The space H satisfies the hypothesis of Proposition 2.2. Therefore $C^0(X,\mathbb{R}) = \overline{H} = H = \overline{\mathcal{H}}$ (H is closed).

$\square$

Solution of Exercise "Complex density Theorem" :

- ($\subset$) take $\varphi \in H_r$. So there exists $f \in \mathcal{H}$ such that $\varphi = (f + \bar{f})/2$. Since $\bar{f} \in \mathcal{H}$, then $f + \bar{f} \in \mathcal{H}$. Since $1/2 \in \mathcal{H}$, then $\varphi \in H$. Since it is real valued, we have that $\varphi \in \mathcal{H} \cap C^0(X, \mathbb{R})$.

  ($\supset$) take $\varphi \in \mathcal{H} \cap C^0(X, \mathbb{R})$. So $\varphi = Re(\varphi)$ and then it is in $\mathcal{H}_r$.
- By writing $\mathcal{H} = H \cap C^0(X, \mathbb{R})$, we get that it is stable via product and sum (intersection of algebras). Take a constant $\lambda \in \mathbb{R}$, so $\lambda \in \mathcal{H}$ and $\lambda = Re(\lambda) \in H_r$. So it contains the constants. Let us deal with separation. Fix $x \neq y$ in X. Then there exists $u \in \mathcal{H}$ such that $u(x) \neq u(y)$. If $Re(u)(x) \neq Re(u)(y)$, then $Re(u) \in H_r$ separates x and y. If not, then $Im(u)(x) \neq Im(u)(y)$. Since $iu \in \mathcal{H}$, then $Re(iu) = -Im(u) \in H_r$, so $Im(u) \in H_r$ separates x and y. Then we apply Theorem 2.1 to get that $\mathcal{H}_r$ is dense.
- we can follow the same proof, but there is an easier way. Indeed, we have that $\mathcal{H}_r = \mathcal{H}_i$ by using the proof above. More precisely, take $f \in \mathcal{H}_i$, then $f = Im(u)$ for some $u \in \mathcal{H}$. But $iu \in \mathcal{H}$ and then $Im(u) = Re(-iu) \in H_r$, so $f \in \mathcal{H}_r$. This yields $\mathcal{H}_i \subset \mathcal{H}_r$. The converse is similar.
- Fix $f \in C^0(X, \mathbb{C})$ and $\epsilon > 0$. Let $f_1, f_2 \in C^0(X, \mathbb{R})$ be such that $f = f_1 + if_2$. Since $\mathcal{H}_r = \mathcal{H}_i$ is dense in $C^0(X, \mathbb{R})$, there exists $g_1, g_2 \in \mathcal{H}_r = \mathcal{H}_i$ such that $\|f_1 - g_1\|_\infty < \epsilon/\sqrt{2}$ and $\|f_2 - g_2\|_\infty < \epsilon/\sqrt{2}$. Define $g = g_1 + ig_2 \in \mathcal{H}$. For $x \in X$, we have that

  $$|(f - g)(x)|^2 = ((f_1 - g_1)(x))^2 + ((f_2 - g - 2)(x))^2 < 2\epsilon^2/2 = \epsilon^2,$$

  and then $\|f - g\|_\infty < \epsilon$, with $g \in \mathcal{H}$. So $\mathcal{H}$ is dense in $C^0(X, \mathbb{C})$.

$\square$

CHAPTER 3

Relatively compact subsets and Ascoli's compactness theorem

1. Prologue

In previous analysis classes you have attended, compactness has been permanently used. For instance to minimize a function (and then to find critical points), or to get uniform continuity (Heine's theorem): essentially, we use that we can always extract a converging subsequence from a sequence valued in a compact set.

Keep in mind that this is what we want from the beginning of this lecture: approximate a problem, find a solution to the approximation, and manage to make it converge. If you are in a compact set, this will work. So it is essential to know the compact subsets of spaces of functions like $C^0(X)$.

Definition 1.1. *Let $(X, \mathcal{T})$ be a topological space. A subspace $A \subset X$ is relatively compact if $\overline{A}$ is compact in X.*

In finite dimension, there is a simple criterion

Theorem 3.1. *Let $(E, \|\cdot\|_E)$ be a finite-dimensional linear normed space endowed with the induced topology. Then a subset of E is compact if and only is it is closed and bounded. In particular, a set is relatively compact iff it is bounded.*

As a consequence, combining the Bolzano-Weierstrass theorem (bounded sequence on $\mathbb{R}$ converge up to a subsequence) and the equivalence of the norms in finite dimension, we have that

Proposition 3.2. *Let $(E, \|\cdot\|_E)$ be a finite-dimensional linear normed space endowed with the induced topology. Then any sequence bounded in E converges up to a subsequence.*

The situation is much more intricate in infinite dimension where the above statement does not hold due to Riesz's theorem (The unit closed ball $\bar{B}_E(0,1)$ is compact iff E is finite-dimensional). Here we will be interested in spaces of continuous functions, for instance $C^0([0,1], \mathbb{R})$.

Here is a classical example. For $n \geq 1$, we define $f_n \in C^0([0,1], \mathbb{R})$ by

- $f_n(0) = 0$, $f_n\left(\frac{1}{n}\right) = 1$ and f_n is affine on $\left[0, \frac{1}{n}\right]$,
- $f_n\left(\frac{2}{n}\right) = 0$ and f_n is affine on $\left[\frac{1}{n}, \frac{2}{n}\right]$,
- $f_n(t) = 0$ for $t \in \left[\frac{2}{n}, 1\right]$

We have that $f_n \in \overline{B}_{\|\cdot\|_\infty}(0,1)$. However, no subsequence of $(f_n)_n$ converges in $C^0([0,1], \mathbb{R})$.

Exercise 7: Prove that no subsequence of $(f_n)_n$ converges in $C^0([0,1], \mathbb{R})$.

2. Equicontinuous functions

2.1. Definition.

DEFINITION 2.1. *Let (X, d_X) and (Y, d_Y) be two metric spaces. We say that a family $\mathcal{F} \subset C^0(X, Y)$ is equicontinuous if*

$$\forall x_0 \in X,\, \epsilon > 0,\, \exists \alpha_{x_0,\epsilon} > 0 \text{ s.t. for all } x \in X\, \{d_X(x, x_0) < \alpha_{x_0,\epsilon} \Rightarrow d_Y(f(x), f(x_0) < \epsilon \text{ for all } f \in \mathcal{F}\}$$

Note that $\alpha_{x_0,\epsilon}$ depends on both ϵ and x_0.

2.2. Examples. In $C^0(\mathbb{R}, \mathbb{R})$, the set of $1-$Lipschitz functions is equicontinuous. Let (X, d) be a metric space and fix $\alpha \in (0, 1]$. We define the Hölder space

$$C^{0,\alpha}(X) := \{u \in C^0(X, Y) \text{ s.t. } \exists K > 0 \text{ s.t. } d_Y(u(x), u(y)) \leq K d_X(x, y)^\alpha \text{ for all } x, y \in X\}$$

that we endow with the norm

$$\|u\|_{C^{0,\alpha}} := \|u\|_\infty + \sup_{x,y \in X,\, x \neq y} \frac{d_Y(u(x), u(y))}{d_X(x, y)^\alpha}$$

Then the set $\{u \in C^{0,\alpha}(X, Y)/\, \|u\|_{C^{0,\alpha}} \leq 1\}$ is equicontinuous. Note that $C^{0,1}(X, Y)$ is the set of Lipschitz continuous functions from $X \to Y$.

2.3. Uniform equicontinuity.

DEFINITION 2.2. *Let (X, d_X) and (Y, d_Y) be two metric spaces. We say that a family $\mathcal{F} \subset C^0(X, Y)$ is uniformly equicontinuous if*

$$\forall \epsilon > 0,\, \exists \alpha_\epsilon > 0 \text{ s.t. for all } x, y \in X, \{d_X(x, y) < \alpha_\epsilon \Rightarrow d_Y(f(x), f(y) < \epsilon \text{ for all } f \in \mathcal{F}\}$$

Here, $\alpha_\epsilon > 0$ is independant of the choice of x or y. As for continuous functions, we have Heine's theorem:

Proposition 3.3. *Let (X, d_X) and (Y, d_Y) be two metric spaces. We assume that (X, d_X) is compact. Then any equicontinuous family in $C^0(X, Y)$ is uniformly equicontinuous.*

Exercise 8: Prove Proposition 3.3. This goes as the classical Heine's theorem.

3. Ascoli's Theorem

3.1. Statement.

THEOREM 3.4 (Ascoli's Theorem). *Let (X, d_X) and (Y, d_Y) be two topological spaces and let $\mathcal{F} \subset C^0(X, Y)$ be a family of continuous functions. We assume that (X, d_X) is compact. Then the two following assertions are equivalent:*

- *$\mathcal{F}$ is relatively compact in $C^0(X, Y)$*
- *$\mathcal{F}$ is uniformly equicontinuous in $C^0(X, Y)$ and for all $x \in X$, $\{f(x)/\, f \in \mathcal{F}\}$ is relatively compact in Y.*

A more convenient context is when (Y, d_Y) is a finite-dimensional

THEOREM 3.5 (Ascoli's Theorem, version 2). *Let (X, d_X) be a compact topological space and let $\mathcal{F} \subset C^0(X, \mathbb{R}^n)$, $n \geq 1$, be a family of continuous functions. Then the two following assertions are equivalent:*

- *$\mathcal{F}$ is relatively compact in $C^0(X, \mathbb{R}^n)$*
- *$\mathcal{F}$ is bounded and uniformly equicontinuous in $C^0(X, \mathbb{R}^n)$.*

Here, note the difference with finite dimension: being bounded is not enough to ensure relative compactness.

3.2. Proof. We do not prove the Theorem in its full generality here. We prove that

$$\mathcal{F} := \{f \in C^0([0,1]) \text{ such that } |f(x)| \leq 1 \text{ and } |f(x)-f(y)| \leq |x-y| \text{ for all } x,y \in [0,1]\}$$

is compact in $(C^0([0,1]), \|\cdot\|_\infty)$. Since we work on metric spaces, we are using the sequential characterization of compactness. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence such that $f_n \in \mathcal{F}$ for all $n \in \mathbb{N}$. We are going to prove that a subsequence converges in $\mathcal{F}$.

Step 1: separability. We claim that there exists $(x_k)_{k\in\mathbb{N}}$ such that $x_k \in [0,1]$ for all k and $\{x_k/k \in \mathbb{N}\}$ is dense in $[0,1]$. In other words, $[0,1]$ is separable.

Proof: The set $D = \mathbb{Q} \cap [0,1] = \left\{\frac{p}{q}\,/p,q \in \mathbb{N} \text{ and } 0 < p \leq q\right\}$ is dense in $[0,1]$. It is countable since $\mathbb{Q}$ is countable. This yields the existence of $(x_k)_k$ and yields the claim. □

Step 2: Cantor's diagonal argument. We claim that there exists a subsequence $(f_{j(n)})_n$ such that for all $k \in \mathbb{N}$, then $(f_{j(n)}(x_k))_n$ converges when $n \to +\infty$.

Proof: Note that we have that $|f_n(x)| \leq 1$ for all $n \in \mathbb{N}$ and all $x \in [0,1]$. We claim that there exists $j_0, j_1, ... : \mathbb{N} \to \mathbb{N} \nearrow$ such that for any $k \in \mathbb{N}$,

$$(f_{j_0\circ\cdots j_k(n)}(x_k))_n \text{ converges as } n \to +\infty. \tag{5}$$

We prove this claim by induction:

- We have that $f_n(x_0) \in \mathbb{R}$ and $|f_n(x_0)| \leq 1$ for all $n \in \mathbb{N}$. It then follows from Bolzano-Weierstrass's theorem that there exists a subsequence $(f_{j_0(n)}(x_0))_n$ that is convergent in $\mathbb{R}$.
- Similarly, we have that $(f_{j_0(n)}(x_1))_n$ is real-valued and bounded by 1. Then there exists a subsequence $(f_{j_0\circ j_1(n)}(x_1))_n$ that is convergent in $\mathbb{R}$.
- We fix $K \in \mathbb{N}$ and we assume that we have constructed $j_0, ..., j_K$ such that for all $k \in \{0,...,K\}$, $(f_{j_0\circ\cdots\circ j_k(n)}(x_k))_n$ converges in $\mathbb{R}$. Again, since $(f_{j_0\circ\cdots\circ j_K(n)}(x_{K+1}))_n$ is reals-valued and bounded, there exists $n \mapsto j_{K+1}(n)$ such that $(f_{j_0\circ\cdots\circ j_{K+1}(n)}(x_{K+1}))_n$ converges in $\mathbb{R}$.
- By induction, this yields (5)[1]

We define $j(n) := j_0 \circ j_1 \circ \cdots \circ j_n(n)$ for all $n \in \mathbb{N}$. As one checks, $j : \mathbb{N} \to \mathbb{N}$ is increasing and for any $k \in \mathbb{N}$, $(f_{j(n)}(x_k))_n$ is a subsequence of $(f_{j_0\circ\cdots\circ j_k(n)}(x_k))_n$ for $n \geq k$. So it converges and this ends Step 2. □

Exercise 9: Prove this latest assertions.

Step 3: Pointwise convergence. We choose $(f_{j(n)})$ given in Step 2. We claim that for any $x \in [0,1]$, then $(f_{j(n)}(x))_n$ converges as $n \to +\infty$.

Proof: We fix $x \in [0,1]$. We are going to prove that $(f_{j(n)}(x))_n$ is a Cauchy sequence in $\mathbb{R}$, so it will converge. We fix $\epsilon > 0$ and we take $p,q \in \mathbb{N}$. We let $k \in \mathbb{N}$ to be fixed later. With the triangle inequality and using that f_n is $1-$Lipschitz for all

[1] Indeed, there is still an argument to give here. We have proved that for any $k \in \mathbb{N}$, there are $j_0, ..., j_k$ that satisfy our hypothesis. However, we want a family (j_k) defined for all k. The key is to take a maximal family and prove via a countable Zorn's Lemma that it is defined for all index.

$n \in \mathbb{N}$, we have that

$$\begin{aligned} |f_{j(p)}(x) - f_{j(q)}(x)| &\leq |f_{j(p)}(x) - f_{j(p)}(x_k)| + |f_{j(p)}(x_k) - f_{j(q)}(x_k)| \\ &\quad + |f_{j(p)}(x_k) - f_{j(q)}(x)| \\ &\leq 2|x - x_k| + |f_{j(p)}(x_k) - f_{j(q)}(x_k)| \end{aligned} \tag{6}$$

With the density of Step 1, we choose $k = k_x \in \mathbb{N}$ such that $|x - x_k| < \frac{\epsilon}{3}$. Since $(f_{j(n)}(x_k))_n$ converges (see Step 2), it is a Cauchy sequence, so there exists $n_0 = n_0(k, \epsilon) = n_0(x, \epsilon)$ such that for $p, q \geq n_0$, we have that $|f_{j(p)}(x_k) - f_{j(q)}(x_k)| < \frac{\epsilon}{3}$. Therefore, we have that

$$|f_{j(p)}(x) - f_{j(q)}(x)| < 2 \times \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \text{ for } p, q \geq n_0.$$

Therefore $(f_{j(n)}(x))_n$ is a Cauchy sequence in $\mathbb{R}$ that is complete, so it converges as $n \to +\infty$. This ends Step 3. □

However, since $n_0 = n_0(x, \epsilon)$ depends on x, this is not enough to get uniform convergence. So we need to improve a little this proof.

Step 4: Uniform convergence (improvement of Step 3, indeed). The sequence $(f_{j(n)})_n$ converges uniformly in $C^0([0,1])$ as $n \to +\infty$.

Proof: We are going to prove that $(f_{j(n)})_n$ is a Cauchy sequence in $C^0([0,1])$. We fix $\epsilon > 0$. Since $\{x_k / k \in \mathbb{N}\}$ is dense in $[0,1]$, we have that

$$[0,1] = \bigcup_{k \in \mathbb{N}} B\left(x_k, \frac{\epsilon}{3}\right) = \bigcup_{k \in \mathbb{N}} \left]x_k - \frac{\epsilon}{3}, x_k + \frac{\epsilon}{3}\right[\cap [0,1].$$

Since $[0,1]$ is compact, we then get that there exists $k_1, ..., k_N \in \mathbb{N}$ such that

$$[0,1] = \bigcup_{i=1,...,N} \left]x_{k_i} - \frac{\epsilon}{3}, x_{k_i} + \frac{\epsilon}{3}\right[\cap [0,1].$$

We fix $x \in [0,1]$. Then there exists $i \in \{1, ..., N\}$ such that $|x - x_{k_i}| < \frac{\epsilon}{3}$. We take $p, q \in \mathbb{N}$. It follows from (6) that

$$|f_{j(p)}(x) - f_{j(q)}(x)| \leq 2|x - x_{k_i}| + |f_{j(p)}(x_{k_i}) - f_{j(q)}(x_{k_i})|$$

Since $(f_{j(n)}(x_{k_i}))_n$ converges (see Step 2), it is a Cauchy sequence, so there exists $n_0(k_i, \epsilon) \in \mathbb{N}$ such that for $p, q \geq n_0(k_i, \epsilon)$, we have that $|f_{j(p)}(x_{k_i}) - f_{j(q)}(x_{k_i})| < \frac{\epsilon}{3}$. Therefore, we have that

$$|f_{j(p)}(x) - f_{j(q)}(x)| < 2 \times \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \text{ for } p, q \geq n_0(k_i, \epsilon).$$

We then take $n_0(\epsilon) := \max\{n_0(k_1, \epsilon), ..., n_0(k_N, \epsilon)\}$, so that

$$|f_{j(p)}(x) - f_{j(q)}(x)| < 2 \times \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \text{ for } p, q \geq n_0(\epsilon),$$

which is valid for all $x \in [0,1]$, so $\|f_{j(p)} - f_{j(q)}\|_\infty < \epsilon$ for $p, q \geq n_0(\epsilon)$. So it is a Cauchy sequence in the complete space $C^0([0,1])$. Therefore it converges to some function $f \in C^0([0,1])$. This proves Step 4 . □

As one checks, $f \in \mathcal{F}$, so this yield the compactness of $\mathcal{F}$. This ends the proof of Theorem 3.4 in our particular case.

Remark: Step 3 can be skipped. Its sole objective is to outline the uniformity that is needed in the proof.

Exercise 10: Let $(E, \|\cdot\|)$ be a finite-dimensional normed space. We fix $\alpha \in (0,1)$ and $K, M > 0$. By following the proof above, prove that

$$\mathcal{F} := \{f \in C^0([0,1], E) \text{ such that } \|f(x)\| \leq M \text{ and } \|f(x)-f(y)\| \leq K|x-y|^\alpha \text{ for all } x, y \in [0,1]\}$$

is compact in $(C^0([0,1], E), \|\cdot\|_\infty)$.

Exercise 11: Prove Theorem 3.4 in the general case by following the proof above.

4. Applications

4.1. An exercise. We consider the solutions $u \in C^1([0,1])$ of the ODE

$$\left\{ \begin{array}{ll} u'(t) = t + \frac{1}{1+u(t)^2} & \text{in } [0,1] \end{array} \right\} \qquad (E)$$

We take for granted that such solutions exist (this is a consequence of the Cauchy-Lipschitz Theorem 16.5).

- We fix $M > 0$. We take $u \in C^1([0,1])$ that is a solution to (E) such that $|u(0)| \leq M$. Show that there exists $C > 0$ independent of u such that $|u'(t)| \leq C$ for all $t \in [0,1]$.

  We set

  $A_M := \{u \in C^1([0,1]) / \, u \text{ is a solution to } (E) \text{ and } |u(0)| \leq M.\}$
- Show that A_M is relatively compact in $C^0([0,1])$.
- We take a sequence $(u_n)_n$ such that $u_n \in A_M$ for all $n \in \mathbb{N}$. We assume that there exists $u \in C^0([0,1])$ such that (u_n) converges uniformly to u in $C^0([0,1])$. Show that $(u_n')_n$ converges uniformly to a limit $f \in C^0([0,1])$. Show that $u \in C^1([0,1])$ and that $u' = f$.
- Show that A_M is relatively compact in $C^1([0,1])$.

Solution:

We have that $0 < u'(t) \leq 1 + 1 = 2$.With the mean value inequality and the preceding result, we have that

$$|u(t)| \leq |u(t) - u(0)| + |u(0)| \leq 2|t| + M \leq M + 2$$

for all $t \in [0,1]$. So $\|u\|_\infty \leq M + 2$ and $|u(t) - u(s)| \leq 2|t-s|$ for all $t, s \in [0,1]$. It then follows from Ascoli's Theorem that A_M is relatively compact in $C^0([0,1])$. Since $u_n'(t) = t + \frac{1}{1+u_n(t)^2}$ for all $t \in [0,1]$, then $u_n' \to f$ uniformly where $f(t) = t + \frac{1}{1+u(t)^2}$ for all $t \in [0,1]$. It follows from a classical theorem that in this situation, then $u \in C^1$ and $u' = f$. However, we can perform a direct proof. For $n \in \mathbb{N}$ and $t \in [0,1]$, we have that

$$u_n(t) = u_n(0) + \int_0^t \left(s + \frac{1}{1+u_n(s)^2}\right) ds.$$

Since (u_n) converges uniformly to u, passing to the limit yields

$$u(t) = u(0) + \int_0^t \left(s + \frac{1}{1+u(s)^2}\right) ds = u(0) + \int_0^t f(s)\, ds \text{ for all } t \in [0,1].$$

Since f is continuous, then $u \in C^1([0,1])$ and $u' = f$. Let $(u_n)_n$ be a sequence of A_M. Since A_M is relatively compact in $C^0([0,1])$, there exists a subsequence $(u_{j(n)})_n$ and $u \in C^0([0,1])$ such that $(u_{j(n)})_n$ converges to u uniformly. The preceding question then yields $u \in C^1$ such that $u_{j(n)} \to u$

and $u'_{j(n)} \to u'$ uniformly. Therefore $\|u_n - u\|_{C^1} \to 0$ as $n \to +\infty$. So A_M is compact in C^1.

□

5. Solution of the exercises of Chapter 3

Solution of Exercise 7: Assume by contradiction that there exists a subsequence $(f_{j(n)})_n$ that converges in $C^0([0,1])$, that is it converges uniformly in $[0,1]$ to some function f. Note that for all $x \in [0,1]$, $f_n(x) \to 0$ as $n \to +\infty$. Since uniform convergence implies pointwise convergence, then $f(x) = 0$ for all $x \in [0,1]$. Therefore $\|f_{j(n)}\|_\infty = \|f_{j(n)} - f\|_\infty \to 0$ as $n \to +\infty$. Since $\|f_{j(n)}\|_\infty = 1$ for all $n \in \mathbb{N}$, we get a contradiction. □

Solution of Exercise 8: We argue by contradiction and we assume that

$$\exists \epsilon_0 > 0, \text{ s.t. for any } \alpha > 0, \exists x, y \in X\, \exists f \in \mathcal{F} \text{ s.t. } d_X(x,y) < \alpha_\epsilon \text{ and } d_Y(f(x), f(y) > \epsilon_0.$$

We take $\alpha := \frac{1}{n}$, $n \geq 1$, and we then get the existence of $x_n, y_n \in X$ and $f_n \in \mathcal{F}$ such that

$$d_X(x_n, y_n) < \frac{1}{n} \to 0 \text{ and } d_Y(f_n(x_n), f_n(y_n)) \geq \epsilon_0.$$

Since (X, d) is compact, we then get the existence of subsequences $(x_{j(n)}), (y_{j(n)})$ that converge to $x_\infty, y_\infty \in X$ respectively. Since $d_X(x_n, y_n) \to 0$ as $n \to +\infty$, we get that $x_\infty = y_\infty$. Since $\mathcal{F}$ is equicontinuous, there exists $\alpha = \alpha_{x_\infty} > 0$ depending on x_∞ such that

$$\text{for all } x \in X,\ \{d_X(x, x_\infty) < \alpha \Rightarrow d_Y(f(x), f(x_\infty) < \frac{\epsilon_0}{3} \text{ for all } f \in \mathcal{F}\}$$

Therefore, for $n \geq n_0$ large enough, $d_X(x_{j(n)}, x_\infty) < \alpha$ and $d_X(y_{j(n)}, x_\infty) < \alpha$, and then

$$d_Y(f_{j(n)}(x_{j(n)}), f_{j(n)}(x_\infty) < \frac{\epsilon_0}{3} \text{ and } d_Y(f_{j(n)}(y_{j(n)}), f_{j(n)}(x_\infty) < \frac{\epsilon_0}{3}.$$

Then for $n \geq n_0$, we have that

$$\begin{aligned} d_Y(f_{j(n)}(x_{j(n)}), f_{j(n)}(y_{j(n)})) &\leq d_Y(f_{j(n)}(x_{j(n)}), f_{j(n)}(x_\infty)) + d_Y(f_{j(n)}(x_\infty), f_{j(n)}(y_{j(n)})) \\ &< \frac{\epsilon_0}{3} + \frac{\epsilon_0}{3} = \frac{2\epsilon_0}{3}, \end{aligned}$$

contradicting $d_Y(f_{j(n)}(x_{j(n)}), f_{j(n)}(y_{j(n)})) > \epsilon_0$. This proves uniform equicontinuity. □

Solution of Exercise 9: For $n \in \mathbb{N}$, we have that

$$\begin{aligned} j(n+1) &= j_0 \circ ... j_n(j_{n+1}(n+1)) \geq j_0 \circ ... j_n(n+1) \text{ since} j_{n+1}(n+1) \geq n+1 \\ &> j_0 \circ ... j_n(n) = j(n) \text{ since } j_0, ..., j_n \nearrow \end{aligned}$$

So j is increasing. Fix $k \in \mathbb{N}$. For $n > k$, we have that

$$j(n) = j_0 \circ .. \circ j_k(j_{k+1} \circ ... \circ j_n(n)) = j_0 \circ .. \circ j_k(\Phi_{k+1}(n))$$

where $\Phi_k(n) := j_{k+1} \circ ... \circ j_n(n)$ for all $n > k$. As above, Φ_k is increasing, and then $(f_{j(n)}(x_k))_{n>k}$ is a subsequence of $(f_{j_0 \circ \cdots \circ j_k(n)}(x_k))_n$, so it converges. □

Solution of Exercise 10:
Step 1: We have already proved that there exists $(x_k)_{k\in\mathbb{N}}$ such that $x_k \in [0,1]$ for all k and $\{x_k / k \in \mathbb{N}\}$ is dense in $[0,1]$.

Step 2: Cantor's diagonal argument. We claim that there exists a subsequence $(f_{j(n)})_n$ such that for all $k \in \mathbb{N}$, then $(f_{j(n)}(x_k))_n$ converges when $n \to +\infty$.

Proof: Note that we have that $\|f_n(x)\| \leq M$ for all $n \in \mathbb{N}$ and all $x \in [0,1]$. We claim that there exists $j_0, j_1, ... : \mathbb{N} \to \mathbb{N} \nearrow$ such that for any $k \in \mathbb{N}$,

$$(f_{j_0 \circ \cdots j_k(n)}(x_k))_n \text{ converges as } n \to +\infty. \tag{7}$$

We prove this claim by induction:

- We have that $\|f_n(x_0)\| \leq M$ for all $n \in \mathbb{N}$. Since E is finite-dimensional, it follows from Proposition 3.2 that there exists a subsequence $(f_{j_0(n)}(x_0))_n$ that is convergent in E.
- Similarly, we have that $(f_{j_0(n)}(x_1))_n$ is real-valued and bounded by M. Then there exists a subsequence $(f_{j_0 \circ j_1(n)}(x_1))_n$ that is convergent in E.
- We fix $L \in \mathbb{N}$ and we assume that we have constructed $j_0, ..., j_L$ such that for all $l \in \{0, ..., L\}$, $(f_{j_0 \circ \cdots \circ j_k(n)}(x_l))_n$ converges in E. Again, since $(f_{j_0 \circ \cdots \circ j_L(n)}(x_{L+1}))_n$ is valued and bounded in E, there exists $n \mapsto j_{L+1}(n)$ such that $(f_{j_0 \circ \cdots \circ j_{L+1}(n)}(x_{L+1}))_n$ converges in E.
- By induction, this yields (7) (see the remark in the preceding proof)

We define $j(n) := j_0 \circ j_1 \circ \cdots \circ j_n(n)$ for all $n \in \mathbb{N}$. As one checks, $j : \mathbb{N} \to \mathbb{N}$ is increasing and for any $k \in \mathbb{N}$, $(f_{j(n)}(x_k))_n$ is a subsequence of $(f_{j_0 \circ \cdots \circ j_k(n)}(x_k))_n$ for $n \geq k$. So it converges and this ends Step 2. $\square$

Step 4 (Step 3 is skipped): Uniform convergence. We claim that the sequence $(f_{j(n)})_n$ converges uniformly in $C^0([0,1], E)$ as $n \to +\infty$.

Proof: We are going to prove that $(f_{j(n)})_n$ is a Cauchy sequence in $C^0([0,1], E)$. We fix $\epsilon > 0$. Since $\{x_k / \, k \in \mathbb{N}\}$ is dense in $[0,1]$, we have that

$$[0,1] = \bigcup_{k \in \mathbb{N}} B\left(x_k, \left(\frac{\epsilon}{3K}\right)^{1/\alpha}\right) = \bigcup_{k \in \mathbb{N}} \left]x_k - \left(\frac{\epsilon}{3K}\right)^{1/\alpha}, x_k + \left(\frac{\epsilon}{3K}\right)^{1/\alpha}\right[\cap [0,1].$$

Since $[0,1]$ is compact, we then get that there exists $k_1, ..., k_N \in \mathbb{N}$ such that

$$[0,1] = \bigcup_{i=1,\dots,N} \left]x_{k_i} - \left(\frac{\epsilon}{3K}\right)^{1/\alpha}, x_{k_i} + \left(\frac{\epsilon}{3K}\right)^{1/\alpha}\right[\cap [0,1].$$

We fix $x \in [0,1]$. Then there exists $i \in \{1, ..., N\}$ such that $|x - x_{k_i}| < \left(\frac{\epsilon}{3K}\right)^{1/\alpha}$. We take $p, q \in \mathbb{N}$. We have that

$$\begin{aligned} \|f_{j(p)}(x) - f_{j(q)}(x)\| &\leq \|f_{j(p)}(x) - f_{j(p)}(x_k)\| + \|f_{j(p)}(x_k) - f_{j(q)}(x_k)\| \\ &\quad + \|f_{j(p)}(x_k) - f_{j(q)}(x)\| \\ &\leq 2K|x - x_k|^\alpha + \|f_{j(p)}(x_k) - f_{j(q)}(x_k)\| \end{aligned}$$

Since $(f_{j(n)}(x_{k_i}))_n$ converges (see Step 2), it is a Cauchy sequence, so there exists $n_0(k_i, \epsilon) \in \mathbb{N}$ such that for $p, q \geq n_0(k_i, \epsilon)$, we have that $\|f_{j(p)}(x_{k_i}) - f_{j(q)}(x_{k_i})\| < \frac{\epsilon}{3}$. Therefore, we have that

$$\|f_{j(p)}(x) - f_{j(q)}(x)\| < 2 \times \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \text{ for } p, q \geq n_0(k_i, \epsilon).$$

We then take $n_0(\epsilon) := \max\{n_0(k_1, \epsilon), ..., n_0(k_N, \epsilon)\}$, so that

$$|f_{j(p)}(x) - f_{j(q)}(x)| < 2 \times \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \text{ for } p, q \geq n_0(\epsilon),$$

which is valid for all $x \in [0,1]$, so $\|f_{j(p)} - f_{j(q)}\|_\infty < \epsilon$ for $p, q \geq n_0(\epsilon)$. So it is a Cauchy sequence in the complete space $C^0([0,1], E)$. Therefore it converges to some function $f \in C^0([0,1], E)$. This proves Step 4. $\square$

As one checks, $f \in \mathcal{F}$, so this yield the compactness of $\mathcal{F}$. This ends the proof of Theorem 3.4 in this case. $\square$

Solution of Exercise 11: We take a sequence $(f_n)_n$ such that $f_n \in \mathcal{F}$ for all $n \in \mathbb{N}$. We are going to prove that it has a limit in $C^0(X, Y)$ up to a subsequence.

Step 1: separability. For any $N \in \mathbb{N}$, we have that $X = \bigcup_{x \in X} B(x, \frac{1}{N+1})$. Since X is compact, there exists $x_N^1, ..., x_N^{k_N} \in X$ such that

$$X = \bigcup_{i=1}^{k_N} B(x_N^i, \frac{1}{N+1}).$$

Define $D := \{x_N^i / \, N \in \mathbb{N} \text{ and } i = 1, ..., k_N\}$. Then D is countable as the countable union of finite sets. We claim that D is dense in X: we fix $x \in X$ and $\epsilon > 0$. Take $N \in \mathbb{N}$ such that $\frac{1}{N+1} < \epsilon$. It follows from the definition of the x_N^i's that there exists $i_0 \in \{1, ..., k_N\}$ such that $x \in B(x_N^{i_0}, \frac{1}{N+1})$, and then $d(x, x_N^{i_0}) < \frac{1}{N+1} < \epsilon$ with $x_N^{i_0} \in D$. So D is dense in (X, d). For convenience, we write $D := \{x_k / \, k \in \mathbb{N}\}$.

Step 2: Cantor's diagonal argument. We claim that there exists a subsequence $(f_{j(n)})_n$ such that for all $k \in \mathbb{N}$, then $(f_{j(n)}(x_k))_n$ converges when $n \to +\infty$.

Proof: We claim that there exists $j_0, j_1, ... : \mathbb{N} \to \mathbb{N} \nearrow$ such that for any $k \in \mathbb{N}$,

$$(f_{j_0 \circ \cdots j_k(n)}(x_k))_n \text{ converges as } n \to +\infty. \tag{8}$$

We prove this claim by induction:

- Since $\{f_n(x_0) / \, n \in \mathbb{N}\}$ is relatively compact in Y (hypothesis of the theorem), then there exists a subsequence $(f_{j_0(n)}(x_0))_n$ that is convergent in Y.
- Similarly, we have that $\{f_{j_0(n)}(x_1) / \, n \in \mathbb{N}\}$ is relatively compact in Y. Then there exists a subsequence $(f_{j_0 \circ j_1(n)}(x_1))_n$ that is convergent in Y.
- We fix $L \in \mathbb{N}$ and we assume that we have constructed $j_0, ..., j_L$ such that for all $l \in \{0, ..., L\}$, $(f_{j_0 \circ \cdots \circ j_k(n)}(x_l))_n$ converges in Y. Again, since $(f_{j_0 \circ \cdots \circ j_L(n)}(x_{L+1}))_n$ is relatively cpompact in Y, then there exists $n \mapsto j_{L+1}(n)$ such that $(f_{j_0 \circ \cdots \circ j_{L+1}(n)}(x_{L+1}))_n$ converges in E.
- By induction, this yields (8) (see the remark in the preceding proof)

We define $j(n) := j_0 \circ j_1 \circ \cdots \circ j_n(n)$ for all $n \in \mathbb{N}$. As earlier, $j : \mathbb{N} \to \mathbb{N}$ is increasing and for any $k \in \mathbb{N}$, $(f_{j(n)}(x_k))_n$ is a subsequence of $(f_{j_0 \circ \cdots \circ j_k(n)}(x_k))_n$ for $n \geq k$. So it converges and this ends Step 2. $\square$

Step 4 (Step 3 is skipped): Uniform convergence. We claim that the sequence $(f_{j(n)})_n$ converges uniformly in $C^0(X, Y)$ as $n \to +\infty$.

Proof: We are going to prove that $(f_{j(n)})_n$ is a Cauchy sequence in $C^0([0,1], E)$. We fix $\epsilon > 0$. Since $\mathcal{F}$ is uniformly equicontinuous, there exists $\delta_\epsilon > 0$ such that for all $x, y \in X$ and all $f \in \mathcal{F}$,

$$d_X(x, y) < \delta_\epsilon \Rightarrow d_Y(f(x), f(y)) < \frac{\epsilon}{3}$$

Since $\{x_k /\, k \in \mathbb{N}\}$ is dense in X, we have that

$$X = \bigcup_{k\in\mathbb{N}} B\left(x_k, \delta_\epsilon\right).$$

Since X is compact, we then get that there exists $k_1, ..., k_N \in \mathbb{N}$ such that

$$X = \bigcup_{i=1,\ldots,N} B\left(x_{k_i}, \delta_\epsilon\right).$$

We fix $x \in X$. Then there exists $i \in \{1, ..., N\}$ such that $d_X(x, x_{k_i}) < \delta_\epsilon$. We take $p, q \in \mathbb{N}$. We have that

$$\begin{array}{rcl} d_Y(f_{j(p)}(x), f_{j(q)}(x)) & \leq & d_Y(f_{j(p)}(x), f_{j(p)}(x_k)) + d_Y(f_{j(p)}(x_k), f_{j(q)}(x_k)) \\ & & +d_Y(f_{j(p)}(x_k), f_{j(q)}(x)) \\ & \leq & 2 \times \dfrac{\epsilon}{3} + d_Y(f_{j(p)}(x_k), f_{j(q)}(x_k)) \text{ since } d_X(x, x_{k_i}) < \delta_\epsilon \end{array}$$

Since $(f_{j(n)}(x_{k_i}))_n$ converges (see Step 2), it is a Cauchy sequence, so there exists $n_0(k_i, \epsilon) \in \mathbb{N}$ such that for $p, q \geq n_0(k_i, \epsilon)$, we have that $d_Y(f_{j(p)}(x_{k_i}), f_{j(q)}(x_{k_i})) < \frac{\epsilon}{3}$. Therefore, we have that

$$d_Y(f_{j(p)}(x) - f_{j(q)}(x)) < 2 \times \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \text{ for } p, q \geq n_0(k_i, \epsilon).$$

We then take $n_0(\epsilon) := \max\{n_0(k_1, \epsilon), ..., n_0(k_N, \epsilon)\}$, so that

$$d_Y(f_{j(p)}(x), f_{j(q)}(x)) < \epsilon \text{ for } p, q \geq n_0(\epsilon),$$

which is valid for all $x \in X$, so $d_{C^0}(f_{j(p)}, f_{j(q)}) < \epsilon$ for $p, q \geq n_0(\epsilon)$. So it is a Cauchy sequence in the complete space $C^0(X, Y)$. Therefore it converges to some function $f \in C^0(X, Y)$. This proves Step 4.

As mentioned above, since we work on a metric space, Step 4 yields the compactness of $\mathcal{F}$. $\square$

Part 2

Baire spaces and Convexity

CHAPTER 4

Baire's lemma and applications

1. The lemma and a few corollaries

1.1. Statement.

THEOREM 4.1 (Baire's Lemma). *Let (X, d) be a complete space. Let $(\mathcal{O}_n)_{n\in\mathbb{N}}$ be a countable family of open dense subsets of X. Then $\cap_{n\in\mathbb{N}} O_n$ is dense in X.*

Remark: countable is essential!!!! For instance, for any $x \in \mathbb{R}$, define $I_x := \mathbb{R} \setminus \{x\}$, which is open and dense in $\mathbb{R}$. But $\cap_{x\in\mathbb{R}} I_x = \emptyset$.

1.2. Version with closed sets.

THEOREM 4.2 (Baire's Lemma version 2). *Let (X, d) be a complete space. Let $(F_n)_{n\in\mathbb{N}}$ be a countable family of closed sets of X. We assume that Int$(F_n) = \emptyset$ for all $n \in \mathbb{N}$ (the interior is empty). Then $\cup_{n\in\mathbb{N}} F_n$ has empty interior.*

PROOF. We define $\mathcal{O}_n := X \setminus F_n$ for any $n \in \mathbb{N}$. Then $\overline{\mathcal{O}_n} = \overline{X \setminus F_n} = X \setminus \mathrm{Int}(F_n) = X$. Then $\mathcal{O}_n$ is open and dense for all $n \in \mathbb{N}$, and Baire's Lemma yields that $\cap_{n\in\mathbb{N}} O_n$ is dense in X. So

$$X = \overline{\cap_{n\in\mathbb{N}} \mathcal{O}_n} = \overline{\cap_{n\in\mathbb{N}} (X \setminus F_n)} = \overline{X \setminus \cup_{n\in\mathbb{N}} F_n} = X \setminus \mathrm{Int}\left(\cup_{n\in\mathbb{N}} F_n\right),$$

and then $\mathrm{Int}\left(\cup_{n\in\mathbb{N}} F_n\right) = \emptyset$. $\square$

1.3. A very useful and common corollary.

Corollary 4.3. *Let (X, d) be a complete space. Let $(F_n)_{n\in\mathbb{N}}$ be a countable family of closed sets of X such that $\cup_{n\in\mathbb{N}} F_n$ has nonempty interior. Then there exists $n_0 \in \mathbb{N}$ such that Int$(F_{n_0}) \neq \emptyset$.*

2. Proof of Baire's Lemma

Let (X, d) and $(\mathcal{O}_n)_n$ be as in the statement of Theorem 4.1, and define $\mathcal{O} := \cap_{n\in\mathbb{N}} \mathcal{O}_n$. To prove that $\mathcal{O}$ is dense, we are taking $U \in X$ an open set and we are going to prove that $\mathcal{O} \cap U \neq \emptyset$.

Step 1: construction of balls included in each other. Since $\mathcal{O}_0$ is dense and U is open, then $\mathcal{O}_0 \cap U \neq$. Moreover $\mathcal{O}_0 \cap U$ is open as a finite intersection of open sets. So there exists $x_0 \in X$ and $\rho_0 > 0$ such that

$$B(x_0, \rho_0) \subset \mathcal{O}_0 \cap U.$$

We define

$$r_0 := \min\left\{\frac{\rho_0}{2}, 1\right\}.$$

To construct the next point, for the same reasons as above, we have that $\mathcal{O}_1 \cap B(x_0, r_0)$ is open and nonempty, so there exists $x_1 \in X$ and $\rho_1 > 0$ such that

$$B(x_1, \rho_1) \subset \mathcal{O}_1 \cap B(x_0, r_0).$$

We define

$$r_1 := \min\left\{\frac{\rho_1}{2}, \frac{1}{2}\right\}.$$

We construct the following terms via an induction. Given $k \geq 1$, we set the assertion

$$\mathbf{H}_k := \left\{ \begin{array}{c} \exists x_1, ..., x_k \in X,\ \rho_1, ..., \rho_k > 0 \text{ s.t.} \\ B(x_i, \rho_i) \subset \mathcal{O}_i \cap B(x_{i-1}, r_{i-1}) \text{ and } r_i := \min\left\{\frac{\rho_i}{2}, \frac{1}{2^i}\right\} \\ \text{for } 1 \leq i \leq k \end{array} \right\}$$

It follows from the above construction that $\mathbf{H}_1$ holds. Given $k \geq 1$, we assume that $\mathbf{H}_k$ holds, and we take $x_1, ..., x_k$ and $\rho_1, ...\rho_k$ as in the assertion. Since $\mathcal{O}_{k+1} \cap B(x_k, r_k)$ is open (finite intersection) and nonempty (density), then there exists $x_{k+1} \in X$ and $\rho_{k+1} > 0$ such that $B(x_{k+1}, \rho_{k+1}) \subset \mathcal{O}_{k+1} \cap B(x_k, r_k)$. Then $\mathbf{H}_{k+1}$ holds. The induction then yields that $\mathbf{H}_k$ is true for all k.

Then, (see Cantor's diagonal process , there is a little point to clarify here, see the footnote) we get a sequence $(x_k)_{k\in\mathbb{N}} \in X$ and $(\rho_k)_k > 0$ such that for all k

$$B(x_k, \rho_k) \subset \mathcal{O}_k \cap B(x_{k-1}, r_{k-1}) \text{ and } r_k := \min\left\{\frac{\rho_k}{2}, \frac{1}{2^k}\right\} \text{ for all } k \geq 1$$

Step 2: We claim that $(x_k)_k$ is a Cauchy sequence in X.

Proof: Since $r_k < \rho_k$ for all $k \geq 0$, we have that $B(x_k, r_k) \subset B(x_{k-1}, r_{k-1})$ for all $k \geq 1$. So by induction, we have that $B(x_{k+p}, r_{k+p}) \subset B(x_k, r_k)$ for all $k, p \in \mathbb{N}$, and then $x_{k+p} \in B(x_k, r_k)$ for all $k, p \in \mathbb{N}$, which yields

$$d(x_{k+p}, x_k) < r_k \leq \frac{1}{2^k} \text{ for all } k, p \in \mathbb{N}.$$

Then $(x_k)_k$ is a Cauchy sequence in (X, d). □

Since (X, d) is complete, then $(x_k)_k$ converges, and we let $x \in X$ be its limit as $k \to +\infty$.

Step 3: We claim that $x \in B(x_k, \rho_k)$ for all $k \in \mathbb{N}$.

Proof: We have noticed that $x_{k+p} \in B(x_k, r_k)$. Passing to the limit $p \to +\infty$, we get that $x \in \overline{B(x_k, r_k)} \subset \bar{B}(x_k, r_k)$ for all $k \in \mathbb{N}$ (Watch out: in general, the two sets are not equal). Since $r_k < \rho_k$, we then have that $x \in B(x_k, \rho_k)$ for all $k \in \mathbb{N}$. □

Step 4: Conclusion of the proof. It follows from the definition of the balls that $B(x_k, \rho_k) \subset \mathcal{O}_k$ for all $k \in \mathbb{N}$, so $x \in \mathcal{O}_k$ for all $k \in \mathbb{N}$ and then $x \in \mathcal{O}$. Moreover, $x \in B(x_0, \rho_0) \subset U$, so $x \in \mathcal{O} \cap U$ and then $\mathcal{O} \cap U \neq \emptyset$. This is valid for any U nonempty and open in X, so $\mathcal{O}$ is dense in (X, d). This ends the proof of Baire's Lemma.

3. Application 1: Banach-Steinhaus Theorem

3.1. Statement.

THEOREM 4.4 (Banach-Steinhaus theorem, or "uniform boundedness" theorem). *Let* $(E, \|.\|_E)$ *and* $(F, \|.\|_F)$ *be two normed spaces. Let* $(T_i)_{i\in I}$ *be a family such that* $T_i : E \to F$ *is linear and continuous for all* $i \in I$ *(*$T_i \in L_c(E, F)$ *for all* $i \in I$*). We assume that* $(E, \|\cdot\|_E)$ *is a Banach space abd that*

$$\textit{for any } x \in E, \textit{ then } \{T_i(x)/\, i \in I\} \textit{ is bounded in } F.$$

Then there exists $M > 0$ *such that*

$$\|T_i(x)\|_F \le M\|x\|_E \textit{ for all } x \in E \textit{ and } i \in I.$$

If you are more used to norms for spaces of linear functions, the conclusion can be read as $\|T_i\|_{E\to F} \le M$ for all $i \in I$, so there is a uniform bound for all the norms of the T_i's.

3.2. Proof of the Banach-Steinhaus Theorem. We let $(E, \|.\|_E)$, $(F, \|.\|_F)$ and $(T_i)_{i\in I}$ as in the statement of the Theorem. Given $N \in \mathbb{N}$, define

$$F_N := \{x \in E/\, \|T_i(x)\|_F \le N \text{ for all } i \in I\} = \cap_{i\in I} T_i^{-1}(\overline{B_F}(0, N)).$$

Step 1: We claim that $\cup_{N\in\mathbb{N}} F_n = E$.

Proof: Fix $x \in E$. Since $\{T_i(x)/\, i \in I\}$ is bounded in F, there exists $M_x > 0$ such that $\|T_i(x)\| \le M_x$ for all $i \in I$. So, taking $N_x \in \mathbb{N}$ such that $N > M_x$, we have that $\|T_i(x)\|_F \le N_x$ for all $i \in I$, so $x \in F_{N_x} \subset \cup_{N\in\mathbb{N}} F_n$. Then $E \subset \cup_{N\in\mathbb{N}} F_n$. The reverse inequality is clear. □

Step 2: We claim that F_N is closed for all $N \in \mathbb{N}$.

Proof: Indeed, since T_i is continuous for all i, then F_N is the intersection of the preimages of closed sets by continuous functions, so it is continuous. □

Step 3: Application of Baire's Lemma and conclusion.

Since $(E, \|\cdot\|_E)$ is complete and $\cup_{N\in\mathbb{N}} F_n = E$ has a nonempty interior, then the "closed" version of the Baire's Lemma (Theorem 4.2) yields the existence of $N_0 \in \mathbb{N}$ such that $\text{Int}(F_{N_0}) \ne \emptyset$. So there exists $x_0 \in E$ and $r > 0$ such that

$$B(x_0, r) \subset F_{N_0}.$$

Let $z \in E$ be such that $\|z\|_E = 1$. Then

$$x_0 + \frac{r}{2} z \in B(x_0, r) \subset F_{N_0} \;\Rightarrow\; \left\|T_i(x_0 + \frac{r}{2} z)\right\|_F \le N_0 \text{ for all } i.$$

Therefore, since $x_0 \in F_{N_0}$, we have that

$$\|T_i(z)\|_F \le \frac{2}{r} \left(\|T_i(x_0)\|_F + N_0\right) \le \frac{4N_0}{r} \text{ for all } i \in I.$$

Then, with $M := \frac{4N_0}{r}$, we have that $\|T_i(z)\|_F \le M$ for all $\|z\|_E = 1$, and then $\|T_i(x)\|_F \le M\|x\|_E$ for all $x \in E$ and $i \in I$. This ends the proof of the Banach-Steinhaus Theorem.

3.3. An interesting corollary. You have been taught that the uniform limit of continous functions is continuous, but that this does not hold for pointwise limits. For linear functions on a Banach space, this is working:

Corollary 4.5. *Let $(E, \|.\|_E)$ and $(F, \|.\|_F)$ be two normed spaces such that $(E, \|\cdot\|_E)$ is a Banach space. Let $(T_n)_{n\in\mathbb{N}}$ be a sequence for functions such that $T_n : E \to F$ is linear and continuous for all $n \in \mathbb{N}$. Assume that there exists $T : E \to F$ such that for all $x \in E$, $(T_n(x))_n$ converges and*

$$\lim_{n\to+\infty} T_n(x) = T(x).$$

Then $T : E \to F$ is linear and continuous.

Exercise 12: Prove this corollary.

Exercise 13: Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach spaces. We endow $E\times F$ with the norm $\|(x, y)\| := \|x\|_E + \|y\|_F$. Let $B : E \times F \to \mathbb{R}$ be a bilinear form.

- Arguing as in the characterization of continuity for linear functions, prove that B is continuous iff there exists $M > 0$ such that $|B(x, y)| \le M\|x\|_E\|y\|_F$ for all $x \in E$ and $y \in F$.
- Arguing as in the proof of Exercise 12, prove that B is continuous if and only if

$$\left\{ \begin{array}{cccc} B(x_0,\cdot): & F & \to & \mathbb{R} \\ & y & \mapsto & B(x_0, y) \end{array} \right\} \text{ and } \left\{ \begin{array}{cccc} B(\cdot, y_0): & E & \to & \mathbb{R} \\ & x & \mapsto & B(x, y_0) \end{array} \right\}$$

are continuous for all $x_0 \in E$ and $y_0 \in F$.

4. Application 2: Open mapping theorem

4.1. Statement.

THEOREM 4.6 (Open mapping Theorem). *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach spaces. Let $T : E \to F$ be a function such that*

- *T is linear*
- *T is continuous*
- *T is onto (that is $T(E) = F$).*

Then there exists $c > 0$ such that $B_F(0, c) \subset T(B_E(0, 1))$.

4.2. Corollaries.

Corollary 4.7. *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach spaces. Let $T : E \to F$ be a linear continuous function such that $T(E) = F$. Then the image by T of any open subset of E is an open subset of F.*

Exercise 14: Prove this corollary.

Corollary 4.8. *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach spaces. Let $T : E \to F$ be a linear bijection. Assume that T is continuous. Then T^{-1} is continuous. In other words, T is an homeomorphism.*

Exercise 15: Prove this corollary by using Corollary 4.7.

4.3. Proof of the Open Mapping Theorem as an exercise.

(1) For any $n \in \mathbb{N}$, we set $F_n := \overline{T(B_E(0,n))}$. Show that $F = \bigcup_{n\in\mathbb{N}} F_n$. Then show that there exists $c > 0$ such that
$$B_F(0,2c) \subset \overline{T(B_E(0,1))}.$$
(2) Fix $y \in B_F(0,c)$. Show that there exists $x_1 \in B_E(0,1)$ such that
$$\|2y - T(x_1)\|_F < c.$$
(3) Using an induction, show that there exists a sequence $(x_n)_{n\geq 1}$ such that $x_n \in B_E(0,1)$ for all $n \geq 1$ and
$$\left\| 2^n \left(y - T\left(\sum_{i=1}^{n} \frac{x_i}{2^i} \right) \right) \right\|_F < c \text{ for all } n \geq 1.$$
(4) Show that there exists $x \in B_E(0,1)$ such that $y = T(x)$.
(5) Conclude.

4.4. The closed graph Theorem.

THEOREM 4.9. *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach spaces. Let $T : E \to F$ be a linear map. Then the two following assertions are equivalent:*

- *T is continuous*
- *$G(T) := \{(x, T(x)/\, x \in E\}$ is closed in $E \times F$ (with the product topology).*

Proof: When T is continuous, one gets that $G(T)$ is closed by using the characterization of open sets via sequences. We now assume that $G(T)$ is closed and we prove that T is continuous.

- We define $\|x\| := \|x\|_E + \|T(x)\|_F$ for all $x \in E$.
- Prove that $\|\cdot\|$ is a norm on E and that $(E, \|\cdot\|)$ is a Banach space.
- Show that $Id_E : (E, \|\cdot\|) \to (E, \|\cdot\|_E)$ is linear and continuous.
- Show that the map Id_E above is a homeomorphism and deduce that T is continuous.

5. Application 3: Existence of nowhere differentiable continuous fonctions

The purpose of this section is to prove the following theorem:

THEOREM 4.10. *There exists real-valued continuous functions on $[0,1]$ that are nowhere differentiable. More precisely, the set of such functions is dense in $C^0([0,1])$ for the norm $\|\cdot\|_\infty$.*

The proof goes as follows. For $k \in \mathbb{N}$, define
$$O_k := \{f \in C([0,1]) \mid \forall x \in [0,1],\ \exists y \in [0,1] \text{ such that } |f(x) - f(y)| > k|x-y|\}.$$

1. Show that O_k is open in $(C([0,1]), \|\cdot\|_\infty)$ for all $k \in \mathbb{N}$. *Hint:* prove that $F_k := C([0,1]) \setminus O_k$ is closed.

2. Given $n \geq 1$, we define the function $f_n : [0,1] \to \mathbb{R}$ by:

- $f_n(\frac{k}{n}) = 0$ pour tout $k = 0, ..., n$
- $f_n(\frac{k+1/2}{n}) = 1$ pour tout $k = 0, ..., n-1$
- f_n is continuous and affine in $[\frac{k}{n}, \frac{k+1/2}{n}]$ and $[\frac{k+1/2}{n}, \frac{k+1}{n}]$ for suitable k.

Draw the function f_n. Show that for all $x \in [0,1]$, there exists $y \in [0,1]$ such that $|f_n(x) - f_n(y)| > n|x-y|$.

3. Using that Lipschitz-continuous functions are dense in $C([0,1])$ (why?), show that O_k is dense in $C([0,1])$ for all $k \in \mathbb{N}$.

4. Using Baire's Lemma, Prove Theorem 4.10

Remarks:

- This proves the existence of nowhere-differentiable functions, but there is no explicit expression. There are explicit examples, like the Weierstrass example $\sum_n a^n \sin(b^n x)$ for suitable $a, b > 0$.
- When you have to prove the existence of something "exotic" or "strange", think of using Baire's Lemma: a dense set is nonempty.

6. Solution of the exercises of Chapter 4

Solution of Exercise 12: The linearity of T is a direct consequence of the pointwise convergence. Fix $x \in E$. Since $(T_n(x))_n$ converges, it is bounded in F. It then follows from Theorem 4.4 that there exists $M > 0$ such that $\|T_n(x)\|_F \leq M\|x\|_E$ for all $n \in \mathbb{N}$ and $x \in E$. Passing to the limit $n \to +\infty$ yields $\|T(x)\|_F \leq M\|x\|_E$ for all $x \in E$. Then we get the continuity of the linear function T. $\square$

Solution of Exercise 13: • Assume that B is continuous. Then $0 \in B^{-1}((-1,1))$ open in $E \times F$, so there exists $r_1, r_2 > 0$ such $B_E(0, r_1) \times B_F(0, r_2) \subset B^{-1}((-1,1))$. Take $x \in E$ and $y \in F$ that are $\neq 0$. Then $\left(r_1 \frac{x}{2\|x\|_E}, r_2 \frac{y}{2\|y\|_F}\right) \in B_E(0, r_1) \times B_F(0, r_2)$ and then

$$\left|\left(r_1 \frac{x}{2\|x\|_E}, r_2 \frac{y}{2\|y\|_F}\right)\right| < 1 \;\Rightarrow\; |B(x,y)| < \frac{4}{r_1 r_2}\|x\|_E \|y\|_F.$$

Conversely, assume the existence of M as in the statement. Given $(x_0, y_0), (x, y) \in E \times F$, we have that

$$\begin{aligned} |B(x,y) - B(x_0,y_0)| &\leq |B(x,y) - B(x_0,y)| + |B(x_0,y) - B(x_0,y_0)| \\ &\leq M\|x - x_0\|_E \|y\|_F + M\|x_0\|_E \|y - y_0\|_F. \end{aligned}$$

By letting $(x,y) \to (x_0,y_0)$, we get the continuity of B.

• Set $I := \{x \in E \text{ such that } \|x\|_E = 1\}$ and define $T_x := B(x, \cdot)$ for all $x \in I$. Then for all $x_0 \in I$, we have that $T_{x_0} = B(x_0, \cdot) : F \to \mathbb{R}$ is linear and continuous. Now fix $y_0 \in F$ and take $x \in I$: we have that $T_x(y_0) = B(x, y_0) = B(\cdot, y_0)(x)$. Since $B(\cdot, y_0)$ is continuous, then there exists $M_{y_0} > 0$ such that $|B(\cdot, y_0)(x)| \leq M_{y_0}\|x\|_E = M_{y_0}$, so

$$\sup\{|T_x(y_0)| / \, x \in I\} \leq M_{y_0} < \infty.$$

It then follows from the Banach-Steinhaus Theorem 4.4 that there exists $M > 0$ such that $|T_x(y)| \leq M\|y\|_F$ for all $x \in I$ and all $y \in F$. Now for any $x \in E \setminus \{0\}$ and any $y \in F$, we have that

$$|B(x,y)| = \|x\|_E \left| B\left(\frac{x}{\|x\|_E}, y\right)\right| = \|x\|_E \left| T_{\frac{x}{\|x\|_E}}(y)\right| \leq M\|x\|_E\|y\|_F$$

which yields the continuity of B. $\square$

Solution of Exercise 14: Let $\mathcal{O} \subset E$ be open. If $T(\mathcal{O}) = \emptyset$, it is open. Assume that there exists $y_0 \in T(\mathcal{O})$. Then there exists $x_0 \in \mathcal{O}$ such that $T(x_0) = y_0$.

Since $\mathcal{O}$ is open, then there exists $r > 0$ such that $B_E(x_0, r) \subset \mathcal{O}$. We take $c > 0$ as in Theorem 4.6. We claim that $B(y_0, cr) \subset T(\mathcal{O})$. We prove the claim. Take $y \in B(y_0, cr)$. Then there exists $z \in F$ such that $\|z\|_F < c$ and $y = y_0 + rz$. Since $z \in B_F(0, c)$, there exists $x \in B_E(0, 1)$ such that $z = T(x)$, and then $y = y_0 + rT(x) = T(x_0 + rx) \subset T(B_E(x_0, r)) \subset T(\mathcal{O})$. This proves the claim. So $T(\mathcal{O})$ is open and the Theorem is proved. □

Solution of Exercise 15: Let $\mathcal{O} \subset E$ be any open subset in E. To prove that $T^{-1} : F \to E$ is continuous, it is enough to prove that $(T^{-1})^{-1}(\mathcal{O})$ is open in F. Since $(T^{-1})^{-1}(\mathcal{O}) = T(\mathcal{O})$ is open due to Corollary 4.7, we then get that T^{-1} is continuous. □

Solution of the Proof of the Open Mapping Theorem:

(1) Take $y \in F$. Since T is onto, there exists $x \in E$ such that $T(x) = y$. With $n \in \mathbb{N}$ such that $n > \|x\|_E$, we have that $x \in T(B_E(0, n)) \subset F_n \subset \bigcup_{n\in\mathbb{N}} F_n$. So $F \subset \bigcup_{n\in\mathbb{N}} F_n$. The converse is clear. Since F is complete, the closed version of Baire's Lemma (Theorem 4.2) yields the existence of $n_0 \in \mathbb{N}$ such that F_{n_0} has a nonempty interior.

So take $y_0 \in F$ and $r > 0$ such that $B_F(y_0, r) \subset F_{n_0} = \overline{T(B_E(0, n_0))}$. We claim that $B_F(0, \frac{r}{2n_0}) \subset \overline{T(B_E(0, 1)}$. We prove the claim. Since $y_0 \in B_F(y_0, r) \subset F_{n_0}$, there existe $(x_p) \in E$ such that $\|x_p\|_E < n_0$ and $T(x_p) \to y_0$ as $p \to +\infty$. Take $x \in B_E(0, \frac{r}{2n_0})$: then $y_0 + 2n_0 x \in B_F(y_0, r) \subset F_{n_0}$, there existe $(\tilde{x}_p) \in E$ such that $\|\tilde{x}_p\|_E < n_0$ and $T(\tilde{x}_p) \to y_0 + 2n_0 x$ as $p \to +\infty$. Then $T(\frac{\tilde{x}_p - x_p}{2n_0}) \to x$ as $n \to +\infty$ and $\|\frac{\tilde{x}_p - x_p}{2n_0}\|_E < \frac{n_0 + n_0}{2n_0} = 1$. This proves the claim. We conclude with $c = \frac{r}{4n_0}$.

(2) Since $2y \in B_F(0, 2c)$ and $c > 0$, the existence of x follows from the definition of the closure.

(3) The existence of x_1 was in the preceding quetsion. Assume that $x_1, ..., x_n$ are constructed. Then

$$2^{n+1}\left(y - T\left(\sum_{i=1}^{n} \frac{x_i}{2^i}\right)\right) \in B_F(0, 2c) \subset \overline{T(B_E(0, 1))},$$

and since $c > 0$, there exists $x_{n+1} \in B_E(0, 1)$ such that

$$\left\| 2^{n+1}\left(y - T\left(\sum_{i=1}^{n} \frac{x_i}{2^i}\right)\right) - T(x_{n+1}) \right\|_F < c$$

And $x_{n+1} \in E$ satisfies the requirements. Then the induction holds for all $n \geq 1$. Here again (see the footnote before), there is an argument to get the full result.

(4) With $(x_n)_n$ defined above, we set

$$X_n := \sum_{i=1}^{n} \frac{x_i}{2^i},$$

so that $\|y - T(X_n)\|_F < 2^{-n}c$ for all $n \geq 1$. Since $\|x_i\|_E < 1$ for all i, we get that for any $n \geq 2$

$$\|X_n\|_E \leq \sum_{i=1}^{n} \left\|\frac{x_i}{2^i}\right\|_E \leq \frac{\|x_1\|_E}{2} + \sum_{i=2}^{n} \frac{1}{2^i} < \frac{\|x_1\|_E}{2} + \frac{1}{2} = \frac{\|x_1\|_E + 1}{2} < +\infty.$$

Then the series is normally converging in a Banach spaces E, so there exists $x \in E$ such that $X_n \to x$ as $n \to +\infty$. It follows from the above inequality that $\|x\|_E \leq \frac{\|x_1\|_E+1}{2} < 1$, so $x \in B_E(0,1)$. Since T is continuous, we get that $T(x) = y$.

(5) So $y = T(x) \in T(B_E(0,1))$. Then $B_F(0,c) \subset T(B_E(0,1))$.

Solution of the proof of Theorem 4.10:
1. Set $F_k := C([0,1]) \setminus O_k$. Let us prove that F_k is closed. We take a sequence $(f_n)_n$ and $f \in C([0,1])$ such that $f_n \in F_k$ for all n and $f_n \to f$ uniformly in $[0,1]$. Since $f_n \notin O_k$, we have that

$$\exists x_n \in [0,1] \text{ such that for all } y \in [0,1],\ |f_n(x_n) - f_n(y)| \leq k|x_n - y|$$

Since $[0,1]$ is compact, there exists a subsequence $(x_{j(n)})$ that converges to $x_\infty \in [0,1]$. We have that

$$|f_{j(n)}(x_{j(n)}) - f_{j(n)}(y)| \leq k|x_n - y| \text{ for all } y \in [0,1]. \tag{9}$$

We write that $|f_{j(n)}(x_{j(n)}) - f(x_\infty)| \leq |f_{j(n)}(x_{j(n)}) - f(x_{j(n)})| + |f(x_{j(n)}) - f(x_\infty)| \leq \|f_{j(n)} - f\|_\infty + |f(x_{j(n)}) - f(x_\infty)|$. Since $(f_{j(n)})$ converges uniformly to f and f is continuous, we then get that $f_{j(n)}(x_{j(n)}) \to f(x_\infty)$ as $n \to +\infty$. Passing to the limit $n \to +\infty$ in (9), we get that

$$|f(x_\infty) - f(y)| \leq k|x_\infty - y| \text{ for all } y \in [0,1]$$

and then $f \in F_k$. So F_k is closed and O_k is open.

2. Since the function is affine with coefficient $\pm 2n$, we get that for $x, y \in [\frac{k}{n}, \frac{k+1/2}{n}]$ or $x, y \in [\frac{k+1/2}{n}, \frac{k+1}{n}]$, then $|f_n(x) - f_n(y)| = 2n|x-y|$. We then get the conclusion.

3. Polynomial are dense in $C([0,1])$ and polynomials are Lipschitz continuous on the compact $[0,1]$, which yields the density in $C([0,1])$. Fix $f \in C([0,1])$ that is $M-$Lipschitz, $k \geq 1$ and $\epsilon > 0$. Take $n \in \mathbb{N}$ to be fixed later. We have that $f + \epsilon f_n \in C([0,1])$ and $\|f - (f - \epsilon f_n)\|_\infty = \epsilon$.

We claim that for n large enough, then $f + \epsilon f_n \in O_k$. For $x, y \in [0,1]$, we have that

$$|(f+\epsilon f_n)(x) - (f+\epsilon f_n)(y)| = |f(x) - f(y) + \epsilon(f_n(x) - f_n(y))| \geq \epsilon|f_n(x) - f_n(y)| - M|x-y|.$$

It follows from **2.** that for all $x \in [0,1]$, there exists $y \in [0,1]$ such that $|f_n(x) - f_n(y)| > n|x-y|$. Therefore

$$|(f+\epsilon f_n)(x) - (f+\epsilon f_n)(y)| > (\epsilon n - M)|x-y|.$$

We choose n such that $\epsilon n - M > k$, so that $|(f+\epsilon f_n)(x) - (f+\epsilon f_n)(y)| > k|x-y|$, and then $f + \epsilon f_n \in O_k$. Therefore O_k is dense in the set of Lipschitz continuous functions that is itself dense in $C([0,1])$. This proves the result.

4. Since $(C([0,1]), \|\cdot\|_\infty)$ is complete, it follows from Baire's Lemma that $O := \cap_k O_k$ is dense in $C([0,1])$. Let $f \in O$ be a function. We claim that f is nowhere differentiable. We prove the claim by contradiction and we assume that there exists $x_0 \in [0,1]$ such that f is differentiable at x_0, so

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0),$$

and then there exists $\alpha > 0$ such that for all $x \in [0,1] \cap [x_0 - \alpha, x_0 + \alpha]$, $|f(x) - f(x_0)| \leq (1 + |f'(x_0)|)|x - x_0|$. For $x \in [0,1]$ such that $|x - x_0| > \alpha$, we have that

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| \leq \frac{2\|f\|_\infty}{\alpha}.$$

Therefore, there exists $K > 0$ such that $|f(x) - f(x_0)| \leq K|x - x_0|$ for all $x \in [0,1]$. So for $k > K$, $f \notin O_k$, contradicting $f \in O$. So f is differentiable nowhere. Then $O \subset \{\text{diff. nowhere fcts}\} \subset C([0,1])$, and since O is dense, so is the set of nowhere differentiable function. $\square$

CHAPTER 5

Hahn-Banach's Theorems and applications

1. Analytic form

1.1. Statement.

THEOREM 5.1 (Hahn-Banach, Analytic form). *Let $(E, \|\cdot\|)$ be a normed space and let $p : E \to \mathbb{R}$ be such that*

- $p(\lambda \cdot x) = \lambda p(x)$ *for all* $x \in E$ *and* $\lambda > 0$
- $p(x + y) \leq p(x) + p(y)$ *for all* $x, y \in E$.

Let $F \subset E$ be a linear subspace of E and let $f : F \to \mathbb{R}$ linear be such that $f(x) \leq p(x)$ for all $x \in F$. Then there exists $\tilde{f} : E \to \mathbb{R}$ linear be such that

- $\tilde{f}_{|F} = f$
- $\tilde{f}(x) \leq p(x)$ *for all* $x \in E$.

1.2. Proof in finite dimension. The proof in its full generality uses Zorn's Lemma. However, in finite dimension, Theorem 5.1 is not trivial and the proof is a good step before tackling the general case.

Let E, F, p and f as in the statement of Theorem 5.1 and assume that E is finite dimensional. Define

$$d := \max \left\{ \dim(G) / \, F \subset G \subset E \text{ and } \exists g : G \to \mathbb{R} \text{ linear s.t. } \left\{ \begin{array}{c} g_{|F} = f; \\ g(x) \leq p(x) \text{ for all } x \in G \end{array} \right\} \right\}$$

Exercise 16:

(1) Prove that d is well-defined. What happens if $d = \dim(E)$?

(2) We assume that $d < \dim(E)$ and we let $F \subset G \subset E$ linear such that $d = \dim(G)$, $g : G \to \mathbb{R}$ such that $g_{|F} = f$ and $g(x) \leq p(x)$ for all $x \in G$. Since $G \subset E$ and $G \neq E$, we fix $x_0 \in E \setminus G$ and we let $\alpha > 0$ to be fixed later. Define

$$\begin{array}{rccc} h: & G \oplus \mathbb{R}x_0 & \to & \mathbb{R} \\ & x + tx_0 & \to & g(x) + t\alpha \end{array}$$

(3) Prove that h is well-defined and that it is an extension of f.

(4) By considering the cases $t > 0$ and $t < 0$, prove that $h(z) \leq p(z)$ for all $z \in G \oplus \mathbb{R}x_0$ if and only if

$$g(y) - p(y - x_0) \leq \alpha \leq p(x + x_0) - g(x) \text{ for all } x, y \in G. \tag{10}$$

(5) Prove that for all $x, y \in G$, we have that $g(y) - p(y - x_0) \leq p(x + x_0) - g(x)$.

(6) Prove that there exists $\alpha \in \mathbb{R}$ such that (10) holds.

(7) Get a contradiction and prove the Theorem.

1.3. Proof via Zorn's Lemma. We need to find a maximal element, and Zorn's Lemma is a convenient tool for this:

DEFINITION 1.1. *We say that an ordered set $(X, \leq)$ is inductive if every subset of X that is totally ordered has an upper-bound in X.*

THEOREM 5.2 (Zorn's Lemma). *Every nonempty inductive set has a maximal element.*

This is equivalent to the Axiom of Choice. In order to prove the Hahn-Banach Theorem, we introduce

$$X := \left\{(G,g)/\, F \subset G \subset E \text{ and } \exists g : G \to \mathbb{R} \text{ linear s.t. } \left\{ \begin{array}{c} g_{|F} = f; \\ g(x) \leq p(x) \text{ for all } x \in G \end{array} \right\}\right\}$$

On X, we define the following order: given $(G,g),(H,h) \in X$, we say that $(G,g) \preceq (H,h)$ if $G \subset H$ and $h_{|G} = g$.

Exercise 17 (Proof of Theorem 5.1)**:**

1. Prove that $(X, \preceq)$ is an ordered set.
2. Prove that $(X, \preceq)$ is inductive.
3. It then follows from Zorn's Lemma that there exists $F \subset G \subset E$, $g : G \to \mathbb{R}$ linear such that $g(x) \leq p(x)$ for all $x \in G$ such that (G,g) is a maximal element of X.
4. Arguing as in the finite-dimensional case, prove that if $G \neq E$, then g can be extended.
5. Conclude.

1.4. Corollaries. Here and in the sequel, we adopt the following notation. For $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|)_F$ two normed spaces and $f : E \to F$ linear, we define

$$\|f\|_{E\to F} := \sup_{x \in E\setminus\{0\}} \frac{\|f(x)\|_F}{\|x\|_E}.$$

Proposition 5.3. *Let $(E, \|\cdot\|)$ be a normed space and let $F \subset E$ be a linear subspace of E. Let $f \in F'$ be a linear continuous function. Then there exists an extension $\tilde{f} : E \to \mathbb{R}$ such that*

- *$\tilde{f} \in E'$ is linear and continuous*
- *$\tilde{f}_{|F} = f$*
- *$\|\tilde{f}\|_{E\to\mathbb{R}} = \|f\|_{F\to\mathbb{R}}$.*

Exercise 18: Prove this proposition by using a suitable function p depending on $\|\cdot\|$.

Proposition 5.4. *Let $(E, \|\cdot\|)$ be a normed space, $E \neq \{0\}$. Then for all $x \in E$, there exists $f \in E'$ such that $f(x) = \|x\|$ and $\|f\|_{E\to\mathbb{R}} = 1$. In particular,*

$$\|x\|_E := \sup_{f\in E',\, f\not\equiv 0} \frac{|f(x)|}{\|f\|_{E\to\mathbb{R}}},$$

and the supremum is achieved.

Exercise 19: Prove Proposition 5.4 by constructing an extension of a linear function defined on $\mathbb{R}x$.

2. Geometric form

2.1. Separation.

DEFINITION 2.1. *Let E be a vector space. A set $H \subset E$ is a hyperplane if $H = f^{-1}(\{\alpha\})$ for some $\alpha \in \mathbb{R}$, $f : E \to \mathbb{R}$ linear and $f \not\equiv 0$*

Proposition 5.5. *Let $(E, \| \cdot |_E)$ be a normed space. Let $f : E \to \mathbb{R}$ be linear and α be a real number. Then $H = f^{-1}(\{\alpha\})$ is closed iff f is continuous.*

PROOF. If f is continuous, then H is closed as the preimage of a closed set via a continuous map. Conversely, assume that H is closed and assume by contradiction that f is not continuous. Then it is not continuous at 0, and then

$$\exists \epsilon_0 > 0 \text{ s.t. for all } \nu > 0 \text{ there exists } x \in E \text{ s.t. } \|x\|_E < \nu \text{ and } |f(x)| > \epsilon_0.$$

For simplicity, assume that $\alpha = 0$. By taking $\nu := 1/n$ for $n \in \mathbb{N}$, $n \geq 1$, we get that there exists $x_n \in E$ such that $x_n \to 0$ as $n \to +\infty$ and $|f(x_n)| \geq \epsilon_0$. Define $y_n := \frac{1}{f(x_n)} x_n$ for $n \geq 1$. We then have that $y_n \to 0$ as $n \to +\infty$ and $f(y_n) = 1$ for all $n \geq 1$. Since $y_n - y_1 \in \text{Ker } f$ is closed, we get that $-y_1 \in \text{Ker } f$ and $f(y_1) = 0$, which is a contradiction. So f is continuous. □

Exercise 20: Prove the result for $\alpha \in \mathbb{R}$ arbitrary.

DEFINITION 2.2. *Let E be a vector space and let $A, B \subset E$ be two subsets. We say that a hyperplane H separates A and B if there exists $f : E \to \mathbb{R}$, $f \not\equiv 0$, linear and $\alpha \in \mathbb{R}$ such that*

$$f(a) \leq \alpha \leq f(b) \text{ for all } a \in A \text{ and } b \in B$$

and $H = f^{-1}(\{\alpha\})$. We say that H separates strictly A and B if, in addition to the above assertion, there exists $\epsilon > 0$ such that

$$f(a) \leq \alpha - \epsilon < \alpha + \epsilon \leq f(b) \text{ for all } a \in A \text{ and } b \in B$$

2.2. Statements.

THEOREM 5.6 (Hahn-Banach, Geometric form 1). *Let $(E, \| \cdot \|_E)$ be a normed space. Consider $A, B \subset E$ be such that*

- *A, B are convex and nonempty*
- *$A \cap B = \emptyset$*
- *A is open.*

Then A and B are separated by a closed hyperplane.

THEOREM 5.7 (Hahn-Banach, Geometric form 2). *Let $(E, \| \cdot \|_E)$ be a normed space. Consider $A, B \subset E$ be such that*

- *A, B are convex and nonempty*
- *$A \cap B = \emptyset$*
- *A is compact and B is open.*

Then A and B are strictly separated by a closed hyperplane.

2.3. The gauge.

Definition 2.3. *Let $(E, \|\cdot\|_E)$ be a normed space. Let $C \subset E$ be such that*

- *C is open and nonempty*
- *C is convex.*

Assume that $0 \in C$. Define the gauge as

$$p(x) := \inf\left\{\alpha > 0 / \frac{x}{\alpha} \in C\right\} \text{ for all } x \in E.$$

Proposition 5.8. *Let $(E, \|\cdot\|_E)$ be a normed space. Let $C \subset E$ be a nonempty open convex of E such that $0 \in C$ and $p : E \to \mathbb{R}$ be its gauge. Then p satisfies the following:*

- *There exists $M > 0$ such that $0 \leq p(x) \leq M\|x\|_E$ for all $x \in E$*
- *$p(\lambda \cdot x) = \lambda p(x)$ for all $x \in E$ and $\lambda > 0$*
- *$p(x+y) \leq p(x) + p(y)$ for all $x, y \in E$.*

Moreover,

$$C = \{x \in E \text{ such that } p(x) < 1\}. \tag{11}$$

Exercise 21: The proof goes through several steps. Given $x \in E$, define

$$I_x := \{\alpha > 0 \text{ such that } \alpha^{-1}x \in C\}.$$

Fix $\delta > 0$ be such that $B_E(0, \delta) \subset C$.

(1) Prove that for any $x \in E$, then I_x is an open interval and that there exists $\alpha_x \geq 0$ such that $I_x = (\alpha_x, +\infty)$.
(2) Give a relation between α_x and δ for all $x \in E$.
(3) What is the relation between $p(x)$ and α_x?
(4) For the triangle inequality, use that for any $x, y \in E$ and $\epsilon > 0$, then $\left(\frac{x}{p(x)+\epsilon}, \frac{y}{p(y)+\epsilon}\right) \in C$ and use the convexity of C with a suitable barycenter.
(5) Prove the proposition.

2.4. Proof of the first geometric form.

The proof is using the analytic form and the gauge. We start with a Lemma:

Lemma 5.9. *Let $(E, \|\cdot\|_E)$ be a normed space and let C be a nonempty open convex set. Assume that there exists $x_0 \in E \setminus C$. Then there exists $f \in E'$ such that $f(x) < f(x_0)$ for all $x \in C$. In particular $f \not\equiv 0$.*

Exercise 22 (Proof of the Lemma)**:**

(1) We first assume that $0 \in C$ and we introduce the gauge p of C. For any $t \in \mathbb{R}$, define $g(tx_0) := t$. Prove that $g(x) \leq p(x)$ for all $x \in \mathbb{R}x_0$ and use the Hahn-Banach Theorem to get the existence of f.
(2) Deal with the general case by introducing $\tilde{C} := C - y_0$ for some $y_0 \in C$.

Now here goes the proof of the geometric version. We let A, B as in the statement of Theorem 5.6. Then

- $C = A - B = \{a - b / \, a \in A \text{ and } b \in B\}$ is open, convex, nonempty and $0 \notin C$
- It follows from the Lemma that there exists $f \in E'$ such that $f(x) < f(0) = 0$ for all $x \in C$.
- The conclusion holds by taking $\alpha := \sup\{f(a) / \, a \in A\}$.

Exercise 23: Prove Theorem 5.6 by following the structure above.

Solution to Exercise 23:

- Since A, B are nonempty, then so is C. Fix $x_0 = a_0 - b_0 \in A - B$ with $a_0 \in A$ and $b_0 \in B$. Since A is open, let $r > 0$ be such that $B(a_0, r) \subset A$. We claim that $B(x_0, r) \subset C$: indeed, take $x \in B(x_0, r)$, we then get that $x + b_0 \in B(a_0, r) \subset A$, so $x_0 = (x + b_0) - b_0 \in A - B$ and then $B(x_0, r) \subset C$. So C is open. Convexity is a direct consequence of the convexity of A and B. If $0 \in C$, then there exists $a \in A$ and $b \in B$ such that $0 = b - a \in A - B$, so $a = b \in A \cap B = \emptyset$, which is absurd. So $0 \notin C$.
-
- Take $a \in A$ and $b \in B$. We have that $a - b \in C$ and then $f(a - b) < 0$ so $f(a) < f(b)$. Fix $b_0 \in B$ (nonempty), so $f(a) < f(b_0)$ for all $a \in A$, and then $\alpha := \sup\{f(a)/\, a \in A\}$ is well defined and $f(a) \leq \alpha$ for all $a \in A$. For any $b \in B$, we have that $f(a) < f(b)$, so $f(b)$ is an upper bound of $\{f(a)/a \in A\}$, so $f(b) \geq \alpha$ for all $b \in B$. This proves the Theorem.

$\square$

2.5. Proof of the second geometric form. Let A, B be as in the statement of Theorem 5.7. We claim that there exists $\delta > 0$ such that $\|a - b\| \geq \delta$ for all $a \in A$ and $b \in B$. Assume this is not true: then there exists $(a_n)_n \in A^{\mathbb{N}}$ and $(b_n)_n \in B^{\mathbb{N}}$ such that $\|a_n - b_n\| 0$ as $n \to +\infty$. Since A is compact, there exists a subsequence $(a_{j(n)})$ that converges to $a \in A$. Then $(b_{j(n)})_n$ converges also to a, and since B is closed, then $a \in B$. So $a \in A \cap B = \emptyset$: absurd: this yielsd the conclusion.

Define $A_1 := \{a + z/\, a \in A \text{ and } z \in B(0, \delta/3)\}$ and $B_1 := \{b + z/\, b \in B \text{ and } z \in B(0, \delta/3)\}$. As above, since $B(0, \delta/3)$ is open, then A_1, B_1 are open. They are convex since $A, B, B(0, \delta/3)$ are convex. With the definition of δ, then $A_1 \cap B_1 = \emptyset$. It follows from the geometric Theorem 5.6 that there exists $f \in E'$, $f \not\equiv 0$, such that $f(a_1) \leq \alpha \leq f(b_1)$ for all $a_1 \in A_1$ and $b_1 \in B_1$. Since $f \not\equiv 0$, take $z \in B(0, \delta/3)$ be such that $f(z) > 0$. Then

$$f(a + z) \leq \alpha \leq f(b - z) \text{ for all } a \in A \text{ and } b \in B.$$

And then $f(a) \leq \alpha - f(z)$ and $f(b) \geq \alpha + f(z)$ for all $a \in A$ and $b \in B$. Since $f(z) > 0$, we then get the strict separation

2.6. A corollary.

Corollary 5.10. *Let $(E, \|\cdot\|_E)$ be a normed space and let $F \subset E$ be a subspace such that $\overline{F} \neq E$. Then there exists $f \in E'$ such that $f \not\equiv 0$ and $f(x) = 0$ for all $x \in F$.*

Solution: Choose $x_0 \in E \setminus \overline{F}$. We apply Theorem 5.7 to $\overline{F}$ (which is closed and convex) and $\{x_0\}$. So there exists $f \in E'$, $f \not\equiv 0$, $\alpha \in \mathbb{R}$ and $\epsilon > 0$ such that

$$f(x) \leq \alpha - \epsilon \text{ for all } x \in \overline{F} \text{ and } f(x_0) > \alpha + \epsilon.$$

Fix $t > 0$. Since F is a vector space, then for any $x \in F$, we have that $tx \in F$ and then $tf(x) = f(tx) \leq \alpha - \epsilon$. Letting $t \to 0$ and $t \to +\infty$ yields $\alpha \geq \epsilon$ and $f(x) \leq 0$ for all $x \in F$. Since $-x \in F$ for all $x \in F$, we then have that $-f(x) = f(-x) \leq 0$ for all $x \in F$, and then $f(x) = 0$ for all $x \in F$. In addition, $f(x_0) > \alpha + \epsilon \geq 2\epsilon > 0$, so $f \not\equiv 0$ and the corollary is proved.

3. Solution of the exercises of Chapter 5

Solution of Exercise 16:

(1) This is well-defined since E is finite-dimensional. If $d = \dim(E)$, then there exists $g : E \to \mathbb{R}$ that is an extension of f satisfying $g(x) \le p(x)$ for all $x \in E$. So the Theorem is proved.

(2)

(3) Since $x_0 \neq 0$, the expression of $\mathbb{R}x_0$ makes sense. For $x \in G$, $h(x) = h(x + 0 \cdot x_0) = g(x) + 0 \cdot \alpha = g(x)$. It is an extension of g, and then of f. Linearity is not a problem.

(4) What we need is that $g(x)+t\alpha \le p(x+tx_0)$ for all $x \in G$ and $t \in \mathbb{R}$. This is true for $t = 0$ since $g(x) \le p(x)$ for all $x \in G$.

Let us deal with the case $t > 0$. Then

$$\begin{aligned}\{g(x) + t\alpha \le p(x + tx_0)\, \forall x \in G,\, t > 0\} &\Leftrightarrow \left\{tg\left(\frac{x}{t}\right) + t\alpha \le tp\left(\frac{x}{t} + x_0\right)\, \forall x \in G,\, t > 0\right\}\\ &\Leftrightarrow \left\{g\left(\frac{x}{t}\right) + \alpha \le p\left(\frac{x}{t} + x_0\right)\, \forall x \in G,\, t > 0\right\}\\ &\Leftrightarrow \{g(y) + \alpha \le p(y + x_0) \text{ for all } y \in G\}\end{aligned}$$

For the case $t < 0$, we write $s = -t > 0$ and then

$$\begin{aligned}\{g(x) + t\alpha \le p(x + tx_0)\, \forall x \in G,\, t < 0\} &\Leftrightarrow \{g(x) - s\alpha \le p(x - sx_0)\, \forall x \in G,\, s > 0\}\\ &\Leftrightarrow \left\{sg\left(\frac{x}{s}\right) - \alpha \le sp\left(\frac{x}{s} - x_0\right)\, \forall x \in G,\, s > 0\right\}\\ &\Leftrightarrow \left\{g\left(\frac{x}{s}\right) - \alpha \le p\left(\frac{x}{s} - x_0\right)\, \forall x \in G,\, s > 0\right\}\\ &\Leftrightarrow \{g(y) - \alpha \le p(y - x_0) \text{ for all } y \in G\}\end{aligned}$$

So the claim is proved.

(5) For $x, y \in G$, we have that $g(x + y) \le p(x + y)$. With the triangle inequality, we then get that

$$\begin{aligned}g(x) + g(y) &= g(x + y) \le p(x + y) = p(x + x_0 + y - x_0)\\ &\le p(x + x_0) + p(y - x_0).\end{aligned}$$

Which yields the desired inequality.

(6) Define

$$\alpha := \sup_{x \in G}\{g(x) - p(x - x_0)\}.$$

The above question yields $g(x) - p(x - x_0) \le p(0 + x_0) - g(0) = p(x_0)$ for all $x \in G$. So the set is nonempty and bounded from above: this yields the existence of $\alpha \in \mathbb{R}$ such that $g(x) - p(x - x_0) \le \alpha$ for all $x \in G$. Fix now $y \in G$: we have that $g(x) - p(x - x_0) \le p(y + x_0) - g(y)$ for all $x \in G$, so $\alpha \le p(y + x_0) - g(y)$. Then this choice of α works.

(7) The function h is a linear extension of f on a space $H = G \oplus \mathbb{R}x_0$ of dimension $d + 1$, and it satisfies $h(z) \le p(z)$ for all $z \in H$. This contradicts the maximality of d for the dimension of such a space. So our initial hypothesis $d < dim(E)$ is wrong, so $d < dim(E)$, and, see the first question, this proves the theorem in finite dimension.

□

Solution of Exercise 17:

(1) Take $(G, g), (H, h), (K, k) \in X$.
   - We have that $(G, g) \preceq (G, g)$, there is no difficulty
   - Assume that $(G, g) \preceq (H, h)$ and $(H, h) \preceq (G, g)$. In particular $G \subset H \subset G$, so $G = H$. Since $h_{|G} = g$, then $h = g$ and then $(G, g) = (H, h)$.
   - Assume that $(G, g) \preceq (H, h)$ and $(H, h) \preceq (K, k)$. Then $G \subset H \subset K$, so $G \subset K$. Take $x \in G$, so $x \in H$ and $g(x) = h(x)$ since $h_{|G} = g$. Since $x \in H$, we have that $h(x) = k(x)$ since $k_{|H} = h$. So $g(x) = k(x)$ for all $g \in G$ and then $k_{|G} = g$, and $(G, g) \preceq (K, k)$.

   Then $(X, \preceq)$ is an ordered set.

(2) Let $A \subset X$ be a nonempty totally ordered subset of X. We write $A = \{(G_a, g_a)/\, a \in A\}$ for convenience. Define $G := \bigcup_{a \in A} G_a$. It is nomepty since A is nonempty. We define

$$\begin{cases} g: & G & \to & \mathbb{R} \\ & x & \mapsto & g_a(x) \text{ if } g \in G_a \end{cases}$$

   To make this definition consistent, we must prove that for any $x \in G$, $g(x)$ is independent of the choice of a. Take $a, b \in A$ be such that $x \in G_a$ and $x \in G_b$. Since A is totally ordered, either $(G_a, g_a) \preceq (G_b, g_b)$ or $(G_b, g_b) \preceq (G_a, g_a)$. If $(G_a, g_a) \preceq (G_b, g_b)$, then since $x \in G_a$, we have that $g_a(x) = (g_b)_{|G_a}(x) = g_b(x)$. This is similar if the reverse inequality holds. So $g_a(x)$ is independent of the choice of a and the definition makes sense.

   We claim that G is a linear space and that g is linear. Take $x, y \in G$ and $\lambda, \mu \in \mathbb{R}$. Then there exists $a, b \in A$ such that $x \in G_a$ and $y \in G_b$. Since A is ordered, we can assume that $(G_a, g_a) \preceq (G_b, g_b)$, so $G_a \subset G_b$. Then $x, y \in G_b$ which is linear, so $\lambda x + \mu y \in G_b \subset G$, and then G is linear. Moreover, $g_b(\lambda x + \mu y) = \lambda g_b(x) + \mu g_b(y)$ and then $g(\lambda x + \mu y) = \lambda g(x) + \mu g(y)$, so g is linear.

   Fix $x \in F$. Take $a \in A \neq \emptyset$, so that $x \in F \subset G_a \subset G$ and then $g(x) = g_a(x) = f(x)$, so $F \subset G$ and $g_{|G} = f$.

   Fix $x \in G$. Then there exists $a \in A$ such that $x \in G_a$. Then $g_a(x) \leq p(x)$, and then $g(x) \leq p(x)$ for all $x \in G$. These points prove that $(G, g) \in X$.

   Take $a \in A$. Then for $x \in G_a$, $G_a \subset G$ and we have that $g(x) = g_a(x)$. So $(G_a, g_a) \preceq (G, g)$. Then A admits an upper bound.

(3)

(4) This is exactly what was performed in the finite-dimensional case.

(5) So $G = E$ and $g : E \to \mathbb{R}$ is an extension that satisfies the conclusion of the theorem.

$\square$

Solution of Exercise 18 Since f is continuous, then $|f(x)| \leq M\|x\|_F = M\|x\|_E$ with $M = \|f\|_{F \to \mathbb{R}}$ for all $x \in F$. Define $p(x) = M\|x\|_E$. We apply the Hahn-Banach theorem to f and p, which yields the existence of $\tilde{f} : E \to \mathbb{R}$ linear such that

$$\tilde{f}(x) \leq p(x) = M\|x\|_E \text{ for all } x \in E \text{ and } \tilde{f}_F = f.$$

So we have that $-\tilde{(f)}(x) = \tilde{f}(-x) \leq p(-x) = M\| - x\|_E = M\|x\|_E$ for all $x \in E$, and then $|\tilde{f}(x)| \leq m\|x\|_E$ for all $x \in E$. Therefore $\tilde{f}$ is continuous and $\|\tilde{f}\|_{E \to \mathbb{R}} \leq$

$M = \|f\|_{F\to\mathbb{R}}$. For $x \in F$, we have that $|\tilde{f}(x)| = |f(x)|$, so we get that $\|\tilde{f}\|_{E\to\mathbb{R}} \geq \|f\|_{F\to\mathbb{R}}$. So we have equality. $\square$

Solution of Exercise 19: Fix $x \in E$. For $f \in E'$, we have that $|f(x)| \leq \|f\|_{E\to\mathbb{R}}\|x\|_E$, so the supremum exists and

$$\sup_{f\in E',\, f\not\equiv 0} \frac{|f(x)|}{\|f\|_{E\to\mathbb{R}}} \leq \|x\|_E.$$

Let us prove the existence of $f \in E'$ nonzero such that $|f(x)| = \|f\|_{E\to\mathbb{R}}\|x\|_E$, which will porve equality an dthe the supremum is achieved.

If $x \neq 0$, define $g(tx) = t\|x\|$ for all $t \in \mathbb{R}$, which defines $g : \mathbb{R}x \to \mathbb{R}$ as a linear map. Moreover $|g(y)| \leq \|y\|$ for all $y \in \mathbb{R}x$. We apply the Hahn-Banach Theorem with $p(y) = \|y\|$ for $y \in E$, which yields $f : E \to \mathbb{R}$ linear such that $f(x) = g(x) = \|x\|$ and $f(y) \geq \|y\|$ for all $y \in E$. Applying this to $-y$ yields $-f(y) = f(-y) \leq \|-y\| = \|y\|$, and then $|f(y)| \leq \|y\|$ for all $y \in E$, so f is continuous and $\|f\|_{E\to\mathbb{R}} \leq 1$. Since $\frac{|f(x)|}{\|x\|} = 1$, we then get that $\|f\|_{E\to\mathbb{R}} = 1$ and $|f(x)| = \|x\|$. Then the supremum is achieved.

If $x = 0$, take some $x_0 \neq 0$ such that $x_0 \in E$, and $f \in E'$ such that $|f(x_0)| = \|x_0\| \neq 0$. So $f \not\equiv 0$ and the supremum for $x = 0$ is zero, and it is achieved. $\square$

Solution of Exercise 20: if $H \neq \emptyset$, take $z_0 \in H$, that is $f(z_0) = \alpha$. Take the sequence (y_n) as above. Then $y_n - y_1 + z_0 \in H$ that is closed, so $-y_1 + z_0 \in H$, and then $f(-y_1 + z_0) = \alpha$, so $f(y_1) = 0$, which is a contradiction. If $H = \emptyset$ (which is closed), then $f \equiv 0$ (otherwise f is surjective), and then its is continuous. In all cases f is continuous. $\square$

Solution of Exercise 21:

(1) We have that $I_x := f_x^{-1}(C)$ where

$$\begin{cases} f_x : & (0,+\infty) & \to & H \\ & \alpha & \mapsto & \alpha^{-1}x \end{cases}$$

Since f_x is continuous and C is open, then $I_x = f_x^{-1}(C)$ is open in $(0,+\infty)$, which is open in $\mathbb{R}$, so I_x is open in $\mathbb{R}$. Assume that $\alpha \in I_x$ and take $\beta > \alpha$. Then we write

$$\frac{x}{\beta} = \left(1 - \frac{\alpha}{\beta}\right)0 + \frac{\alpha}{\beta}\cdot\frac{x}{\alpha} \in C \text{ since } 0 < \frac{\alpha}{\beta} < 1 \text{ and } 0, \frac{x}{\alpha} \in C,$$

so $\beta \in I_x$. This proves that I_x is convex and unbounded on the right, which yields the existence of $\alpha_x \geq 0$.

(2) Fix $x \in E \setminus \{0\}$. Then $\frac{\delta}{2}\frac{x}{\|x\|} \in B(0,\delta) \subset E$, so $2\delta^{-1}\|x\| \in I_x$, and then $\alpha_x \leq 2\delta^{-1}\|x\|$.

(3) $p(x) = \alpha_x \leq 2\delta^{-1}\|x\|$.

(4) Since $p(x) + \epsilon > p(x) = \alpha_x$, then $\frac{x}{p(x)+\epsilon} \in C$, and similarly $\frac{y}{p(y)+\epsilon} \in C$. We write that

$$\frac{x+y}{p(x)+p(y)+2\epsilon} = \frac{p(x)+\epsilon}{p(x)+p(y)+2\epsilon}\frac{x}{p(x)+\epsilon} + \frac{p(y)+\epsilon}{p(x)+p(y)+2\epsilon}\frac{y}{p(y)+\epsilon} \in C$$

since C is convex. So $p(x) + p(y) + 2\epsilon \geq \alpha_{x+y} = p(x+y)$ for all $\epsilon > 0$. Letting $\epsilon \to 0$ yields $p(x) + p(y) \leq p(x) + p(y)$.

(5) The three items are consequences of the definition and the questions above. We prove (11):

(⊂): we take $x \in C$. So $x/1 \in C$ and $1 \in I_x = (\alpha_x, +\infty)$, so $p(x) = \alpha_x < 1$.

(⊃): we take $x \in E$ such that $\alpha_x = p(x) < 1$. Since $I_x = (\alpha_x, +\infty)$, we have that $1 \in I_x$ and then $x \in C$.

□

Solution of Exercise 22:

(1) Note that $p(x_0) \geq 1$ since $x_0 \notin C$. The inequality is clear for $t \leq 0$. For $t > 0$, we have that $g(tx_0) = t \leq p(x_0) = p(tx_0)$. It follows from Theorem 5.1 that there exists $f : H \to \mathbb{R}$ linear such that $f(x) \leq p(x)$ for all $x \in E$ and $f(x) = g(x)$ for all $x \in \mathbb{R}x_0$. So $f(x_0) = g(x_0) = 1$. For $x \in C$, then $f(x) \leq p(x) < 1$. Moreover, for $x \in E$, $f(x) \leq p(x) \leq M\|x\|$: doing the same for $-x$ yields $|f(x)| \leq M\|x\|$ for all $x \in E$, and then f is continuous.

(2) We have that $x_0 - y_0 \in E \setminus \tilde{C}$ and $0 \in \tilde{C}$. So there exists $f \in E'$ such that $f(x) < f(x_0 - y_0)$ for all $x \in \tilde{C}$. So for all $x \in C$, $x - y_0 \in \tilde{C}$ and $f(x - y_0) < f(x_0 - y_0)$ and then $f(x) < f(x_0)$.

□

CHAPTER 6

Intermission: ODEs with continuous coefficients via Zorn's Lemma. A version of Arzela-Peano's theorem

The Cauchy-Lipschitz existence Theorem 16.5 requires the coefficient to have a certain Lipschitz regularity. For instance, in Exercise 3, you proved the following:

THEOREM 6.1. *Let $f \in C^0(\mathbb{R}, \mathbb{R})$ be such that there exists $K > 0$ such that*

$$|f(x) - f(y)| \leq K|x - y| \text{ for all } x, y \in \mathbb{R}. \tag{12}$$

Then there exists $\delta > 0$ such that for all $x_0 \in \mathbb{R}$, there exists $u \in C^1([-\delta, \delta], \mathbb{R})$ such that

$$\begin{cases} u'(t) = f(u(t)) & \text{for all } t \in [-\delta, \delta] \\ u(0) = x_0 \end{cases}$$

Moreover, such a solution u is unique.

In the proof, you got an explicit value of δ, for instance $\delta = \frac{1}{2K}$ is fine. Then, if your function is not Lipschitz continuous, this would yield $K = \infty$ and then $\delta = 0$... so the method does not work here.

In this chapter, we are going to deal with continuous functions only. We already know that there is no uniqueness: the function

$$\begin{cases} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} 0 & \text{if } x \leq 0 \\ x^2 & \text{if } x \geq 0 \end{cases} \end{cases}$$

is a solution to $y' = 2\sqrt{|y|}$, but so is the null function. They both vanish at 0: so there is no uniqueness.

Here, we are going to show a particular case of the Arzela-Peano Theorem:

THEOREM 6.2 (see Arzela-Peano). *Let $f \in C^0(\mathbb{R})$ be a continuous bounded function in $\mathbb{R}$ and let $t_0, x_0 \in \mathbb{R}$ be parameters. Then there exists $u \in C^1(\mathbb{R})$ such that:*

$$\begin{cases} u'(t) = f(u(t)) & \text{for all } t \in \mathbb{R} \\ u(t_0) = x_0 \end{cases} \tag{13}$$

Here are the steps of the proof. We take such a function f and we take $M > 0$ be such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$. For simplicity we take $t_0 = 0$.

Step 1: regularization. We approximate f by a C^1-function. The standard way is to take a mollifier $(\rho_n)_n$ (we say "approximation of the unity") such that

- $\rho_n \in C^1(\mathbb{R})$ for all $n \in \mathbb{N}$,

- $\rho_n \geq 0$ for all $n \in \mathbb{N}$,
- Supp $\rho_n \subset]-1/(n+1), 1/(n+1)[$ for all $n \in \mathbb{N}$,
- $\int_{\mathbb{R}} \rho_n \, dx = 1$ for all $n \in \mathbb{N}^*$.

and we take the convolution $f_n := f \star \rho_n$ for all $n \in \mathbb{N}$. It follows from classical approximation theorems (see undergraduate classes and Theorem 12.4) that

- $f_n \in C^1(\mathbb{R})$ for all $n \in \mathbb{N}$,
- For any $R > 0$, $(f_n)_n$ converges to f uniformly on $[-R, R]$ as $n \to +\infty$.
- For all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we have that $|f_n(x)| \leq M$.

Step 2: application of the Cauchy-Lipschitz Theorem 16.5. Here we satisfy the hypothesis of the Theorem, which yields the existence of solutions to the approximate problem that are defined around 0. Since f_n is bounded for all $n \in \mathbb{N}$, then extension theorems for ODEs show that our solution can be extended to $\mathbb{R}$. Therefore, there exists $u_n \in C^1(\mathbb{R})$ unique such that

$$\left\{ \begin{array}{ll} u_n'(t) = f_n(u_n(t)) & \text{for all } t \in \mathbb{R} \\ u_n(t_0) = x_0 & \end{array} \right.$$

Step 3: Ascoli's Theorem. Fix $R > 0$.

- With Ascoli's Theorem, we get that $\{u_n / \, n \in \mathbb{N}\}$ is relatively compact in $C^0([-R, R])$.
- Letting $u \in C^0([-R, R])$ be the limit of a subsequence, we get that $u \in C^1([-R, R])$ and
$$\left\{ \begin{array}{ll} u'(t) = f(u(t)) & \text{for all } t \in [-R, R] \\ u(t_0) = x_0 & \end{array} \right.$$
Be Careful: u is not necessarily unique!!!!

Step 4: Application of Zorn's Lemma. Define the set

$$X := \left\{ (u, R) / \, R \in (0, +\infty] \text{ and } u \in C^1(]-R, R[) \text{ s.t. } \left\{ \begin{array}{ll} u'(t) = f(u(t)) & \forall t \in]-R, R[\\ u(t_0) = x_0 & \end{array} \right. \right\}$$

(It is important to tolerate $R = +\infty$). On X, define the order:

$$(u, R) \preceq (v, S) \text{ if } R \leq S \text{ and } v_{]-R,R[} = u.$$

Then

- $(X, \preceq)$ is ordered, non-empty, and it is inductive.
- Zorn's Lemma yields a maximal element (u, R) avec $0 < R \leq +\infty$.
- Assume that $R < +\infty$. Then $u(t)$ has a finite limite (say l) as $t \to R$. *Hint:* first show that $(u(t_i))_i$ is a Cauchy sequence for all $(t_i)_i \in]-R, R[^{\mathbb{N}}$ going to R. This is classical in ODE theory.
- Assume that $R < +\infty$. Then there exists $v \in C^1(]R-1, R+1[)$ such that $v(R) = l$ et $v'(t) = f(v(t))$ for all $t \in]R-1, R+1[$.
- Assume that $R < +\infty$. Define $\tilde{u} :]-R, R+1[\to \mathbb{R}$ par $\tilde{u}(t) = u(t)$ if $t < R$ and $v(t)$ if $t \geq R$. Then $\tilde{u} \in C^1(]-R, R+1[)$ is a soltion to $\tilde{u}' = f(\tilde{u})$.
- Assume that $R < +\infty$. Arguing similarly for $-R$, we get that there is an extension on $]-R-1, R+1[$. This is a contradiction with the hypothesis $R < +\infty$.
- Then $R = +\infty$ and we get a solution to (13).

Remarks.

- When $f \in C^1(\mathbb{R})$, the uniqueness in ODEs yields that $(X, \preceq)$ is totally ordered.
- The local existence of solutions to $u' = f(u)$ is valid when f is continuous, but not necessarily bounded. It is more intricate, see Quéffelec-Zuily [5].

Part 3

Continuous linear functions

CHAPTER 7

The method of continuity

This chapter is strongly inspired by Part II, Chapter 2, Exercise 9 in the book by Gonnord and Tosel [3].

1. Continuity with respect to the parameter

1.1. The space $Gl_c(E, F)$.

Definition 1.1. *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two normed spaces. We define*

$$Gl_c(E, F) := \{u : E \to F \text{ linear, continuous, bijective and } u^{-1} \text{ is continuous}\}.$$

In other words, this is the set of linear homomorphisms. This set might be empty.

Remarks:

- If E and F are finite-dimensional, the continuity is given for free.
- If E and F are finite-dimensional, but not of the same dimension, then $Gl_c(E, F) = \emptyset$.
- It follows from Corollary 4.8 that when E, F are Banach spaces, the continuity of u^{-1} is automatic if u is continuous.

Let us recall the following classical result:

Proposition 7.1. *Let $(E, \|\cdot\|_E)$ be a Banach space. We endow $L_c(E) = L_c(E, E)$ with the norm $\|\cdot\|_{E\to E}$. Then $B(Id_E, 1) \subset Gl_c(E)$. As a consequence, $Gl_c(E)$ is open in $L_c(E)$.*

1.2. Statement.

Theorem 7.2. *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be Banach spaces. Let $A \in C^0([0, 1], L_c(E, F))$ be a continuous function such that*

(1) *There exists $c > 0$ such that*

$$\|A(\lambda)[x]\|_F \geq c\|x\|_E \text{ for all } x \in E \text{ and all } \lambda \in [0, 1], \tag{14}$$

(2) *$A(0) \in Gl_c(E, F)$.*

Then $A(\lambda) \in Gl_c(E, F)$ for all $\lambda \in [0, 1]$.

In other words, the invertibility is preserved along $[0, 1]$. This is telling you that

- If you can pass continuously from a problem to another problem,
- if you have a good control on potential solutions to the continuum of problems

then if one of the problem can be solved, then the other can be too.

1.3. The proof. For simplicity, we first assume that $E = F$ and that $A(0) = Id_E$.

- By using Proposition 7.1, show that there exists $\epsilon_0 > 0$ such that $A(\lambda \in Gl_c(E)$ for all $\lambda \in [0, \epsilon_0)$.
- By using Proposition 7.1 and (14), show that there exists $\alpha > 0$ such that for any $\lambda \in [0,1]$ such that $A(\lambda) \in Gl_c(E)$, then $A(\mu) \in Gl_c(E)$ for all $\mu \in [0,1]$ such that $|\lambda - \mu| < \alpha$. The difficulty is that α is independent of the choice of λ.
- Define

  $$\lambda_0 := \sup\{\lambda \in [0,1] \text{ such that } A(\mu) \in Gl_c(E) \text{ for all } \mu \in [0,\lambda]\}.$$

  Using the two preceding steps, show that $\lambda_0 > 0$, and then that $\lambda_0 = 1$ (argue by contradiction).
- Conclude.

You can then tackle the general case $E \neq F$. The detailed proof is at the end of this chapter.

2. Application: resolution of an ODE with Dirichlet boundary conditions

Let $a, \omega \in C^0([0,1])$ be continuous functions. We assume that $\omega(t) > 0$ for all $t \in [0,1]$. We fix $y \in C^0([0,1])$. Our objective is to prove that there exists a unique solution $x \in C^2([0,1])$ to

$$\begin{cases} -x''(t) + a(t)x'(t) + \omega(t)x(t) = y(t) & \text{for all } t \in [0,1] \\ x(0) = x(1) = 0 \end{cases} \tag{15}$$

To solve this problem, we are going to introduce a parameter that will transform (15) into an equation with constant coefficient, which can be solved. For $\lambda \in [0,1]$, we consider the equation

$$\begin{cases} -x''(t) + a(\lambda t)x'(t) + \omega(\lambda t)x(t) = y(t) & \text{for all } t \in [0,1] \\ x(0) = x(1) = 0 \end{cases} \qquad (E_\lambda).$$

Define $X := \{x \in C^2([0,1]) \text{ such that } x(0) = x(1) = 0\}$. and we define $A(\lambda) : X \to C^0([0,1])$ such that for $x \in C^2([0,1])$, we have that $A(\lambda)(x) := \varphi$ where

$$\varphi(t) := -x''(t) + a(\lambda t)x'(t) + \omega(\lambda t)x(t) \text{ for all } t \in [0,1].$$

On $X \subset C^2([0,1])$, we define the norm

$$\|u\|_{C^2} := \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty \text{ for all } u \in X.$$

The solutions to the questions are at the end of this chapter.

Step 1: Analysis of $A(\lambda)$:

1. Show that for all $\lambda \in [0,1]$, $A(\lambda) : (X, \|\cdot\|_{C^2}) \to (C^0([0,1]), \|\cdot\|_\infty)$ is continuous.

2. Show that $A : [0,1] \to L_c(X, C^0([0,1]))$ is continuous (with the suitable norms).

Step 2: Control of the solutions to (E_λ). We let $x \in C^2([0,1])$ be a solution to (E_λ).

3. We assume that there exists $0 < t_0 < 1$ such that x has a minimum at t_0 on $[0,1]$. What can you say of $x'(t_0)$ and $x''(t_0)$? Show that there exists $\gamma > 0$ independent of x, y, λ such that $\|x\|_\infty \leq \gamma \|y\|_\infty$.

4. We set $u := x'$. Show that there exists $t_1 \in (0,1)$ such that $u(t_1) = 0$. Let $\mathcal{A}$ be a primitive of $t \mapsto -a(\lambda t)$ sur $[0,1]$. What is the differential of $t \mapsto (u(t)e^{\mathcal{A}(t)})$? Show then that there exists $B > 0$ independent of x, y, λ such that $\|x'\|_\infty \leq B\|y\|_\infty$.

5. Show that there exists $M > 0$ independent of x, y, λ such that $\|x\|_{C^2} \leq M\|y\|_\infty$.

6. Show that for all $\lambda \in [0,1]$ and all $x \in X$, then $\|A(\lambda)(x)\|_\infty \geq M^{-1}\|x\|_{C^2}$.

Step 3: Analysis of $A(0)$: We set $a_0 := a(0)$ and $\omega_0 := \omega(0) > 0$. We consider the system

$$\left\{\begin{array}{ll} -x''(t) + a_0 x'(t) + \omega_0 x(t) = y(t) & \text{for all } t \in [0,1] \\ x(0) = x(1) = 0 & \end{array}\right. \qquad (E_0)$$

7. Assume that $y \equiv 0$. Determine $r_1 > 0 > r_2$ such that $t \mapsto e^{rt}$ is a solution to (E_0) iff $r \in \{r_1, r_2\}$.

8. Show that the function $x_0 \in C^2([0,1])$ defined as

$$x_0(t) := -e^{r_1 t}\int_0^t \frac{y(s)e^{-r_1 s}}{r_1 - r_2}\, ds + e^{r_2 t}\int_0^t \frac{y(s)e^{-r_2 s}}{r_1 - r_2}\, ds$$

is a solution to $-x_0'' + a_0 x_0' + \omega_0 x_0 = y$.

9. Determine the unique solution to (E_0).

10. Show that $A(0) : (X, \|\cdot\|_{C^2}) \to (C^0([0,1]), \|\cdot\|_\infty)$ is a homeomorphism.

11. Show that $A(\lambda)$ is invertible for all $\lambda \in [0,1]$. Show that (15) has a unique solution.

How to get a particular solution? The expression in Question 8 arises from the method of "variation of the parameters". Here is this method in dimension > 1.

- Write (E_0) as a 2×2 system $Z'(t) = AZ(t) + B(t)$;
- Give two functions Z_1, Z_2 linearly independent that are solutions to $X' = AX$ (use the resolution of 2nd order nondegenerate equations).
- Set $Z(t) = \lambda(t)Z_1(t) + \mu(t)Z_2(t)$ for $t \in [0,1]$ and show that $Z' = AZ + B$ iff $\dot{\lambda}Z_1 + \dot{\mu}Z_2 = B$
- Compute $\dot{\lambda}$ and $\dot{\mu}$.
- Find an explicit solution to (15).

3. Solutions of the exercises of Chapter 7

Detailed proof of Theorem 7.2: We assume that $E = F$ and $A(0) = Id_E$.

- Since A is continuous at 0, there exists $\epsilon_0 > 0$ such that for all $\lambda \in [0,1]$, if $|\lambda - 0| < \epsilon_0$, then $\|A(\lambda) - A(0)\|_{E\to E} < 1$. Since $A(0) = Id_E$, for any $\lambda \in [0, \epsilon_0)$, $A(\lambda) \in B(Id_E, 1) \subset Gl_c(E)$ by Proposition 7.1.
- Let us first prove the following claim: for any $\lambda \in [0,1]$ such that $A(\lambda) \in Gl_c(E)$, then $\|A(\lambda)^{-1}\|_{E\to E} \leq c^{-1}$. We prove the claim: take $y \in E$ and define $x := A(\lambda)^{-1}(y)$, it follows from (14) that

  $$c\|A(\lambda)^{-1}(y)\|_E = c\|x\|_E \leq \|A(\lambda)[x]\|_E = \|y\|_E,$$

  which yields the claim.

  Now take $\lambda, \mu \in [0,1]$ such that $A(\lambda) \in Gl_c(E)$. We have that

  $$A(\mu) = A(\lambda)\left(Id_E + A(\lambda)^{-1}(A(\mu) - A(\lambda))\right) := A(\lambda)(Id_E + H)$$

with

$$\|H\|_{E\to E} \leq \|A(\lambda)^{-1}\|_{E\to E}\|A(\mu) - A(\lambda)\|_{E\to E} \leq c^{-1}\|A(\mu) - A(\lambda)\|_{E\to E}.$$

Since $A : [0,1] \to L_c(E)$ is continuous and $[0,1]$ is compact, A is uniformly continuous and there exists $\alpha > 0$ such that for any $a, b \in [0,1]$, $|a-b| < \alpha \Rightarrow \|A(a) - A(b)\|_{E\to E} < c$. Therefore, if $|\lambda - \mu| < \alpha$, then $\|H\|_{E\to E} < 1$, and then $Id_E + H \in Gl_c(E)$ (this is Proposition 7.1), and since $A(\lambda) \in Gl_c(E)$, we get that $A(\mu) \in Gl_c(E)$.

- The first step shows that $\lambda_0 \geq \epsilon_0 > 0$. It follows from the definition of λ_0 that $A(\lambda) \in Gl_c(E)$ for all $0 \leq \lambda < \lambda_0$. Take $\beta := \min\{\frac{\lambda_0}{2}, \frac{\alpha}{3}\}$. So $\lambda_0 - \beta \in [0, \lambda_0)$ and then $A(\lambda_0 - \beta) \in Gl_c(E)$. Take $\mu \in [0,1]$ such that $\lambda_0 \leq \mu \leq \lambda_0 + \beta$: we have that $|\mu - (\lambda_0 - \beta)| \leq 2\beta < \alpha$, so $A(\mu) \in Gl_c(E)$. We then have proved that $A(\lambda) \in Gl_c(E)$ for all $\lambda \in [0,1] \cap [0, \lambda_0 + \beta]$. If $\lambda_0 < 1$, we then get a contradiction.
- So $\lambda_0 = 1$ and $A(\lambda) \in Gl_c(E)$ for all $\lambda \in [0,1] \cap [0, 1+\beta] = [0,1]$, which proves the Theorem.

For the general case $E \neq F$, define $\tilde{A}(\lambda) := A(0)^{-1}A(\lambda)$ for $\lambda \in [0,1]$. As one checks, $\tilde{A} \in C^0([0,1], L_c(E))$ satisfies the hypothesis of Theorem 7.2 and $\tilde{A}(0) = Id_E$, so the preceding case yields $\tilde{A}(\lambda) \in Gl_c(E)$ for all $\lambda \in [0,1]$. Theorem 7.2 follows by writing $A(\lambda) = A(0) \cdot \tilde{A}(\lambda)$. □

Solution to "Resolution of an ODE with Dirichlet boundary conditions":
1. For $\lambda \in [0,1]$ and $x \in X$, we have that

$$\begin{aligned}\|A(\lambda)[x]\|_\infty &= \|x'' + a(\lambda\cdot)x' + \omega(\lambda\cdot)x\|_\infty \\ &\leq \|x''\|_\infty + \|a\|_\infty\|x'\|_\infty + \|\omega\|_\infty\|x\|_\infty \\ &\leq \max\{1, \|a\|_\infty, \|\omega\|_\infty\}\|x\|_{C^2}\end{aligned}$$

Since $A(\lambda)$ is linear, this yields the continuity.

2. For $\lambda, \mu \in [0,1]$ and $x \in X$, we have that

$$\begin{aligned}\|(A(\lambda) - A(\mu))[x]\|_\infty &= \|(a(\lambda\cdot) - a(\mu\cdot))x' + (\omega(\lambda\cdot) - \omega(\mu\cdot))x\|_\infty \\ &\leq \|a(\lambda\cdot) - a(\mu\cdot)\|_\infty \cdot \|x'\|_\infty + \|\omega(\lambda\cdot) - \omega(\mu\cdot)\|_\infty \cdot \|x\|_\infty \\ &\leq (\|a(\lambda\cdot) - a(\mu\cdot)\|_\infty + \|\omega(\lambda\cdot) - \omega(\mu\cdot)\|_\infty) \cdot \|x\|_{C^2}\end{aligned}$$

Therefore

$$\|(A(\lambda) - A(\mu))\|_{X\to C^0} \leq \|a(\lambda\cdot) - a(\mu\cdot)\|_\infty + \|\omega(\lambda\cdot) - \omega(\mu\cdot)\|_\infty$$

Fix $\epsilon > 0$. Since $a, \omega \in C^0([0,1])$ and $[0,1]$ is compact, they are uniformly continuous and there exists $\delta_\epsilon > 0$ such that for all $t, s \in [0,1]$.

$$|t-s| < \delta_\epsilon \Rightarrow \{|a(t) - a(s)| < \frac{\epsilon}{and|\omega(t) - \omega(s)| < \frac{\epsilon}{2}\}.$$

So, for $|\lambda - \mu| < \delta_\epsilon$, we have that $|\lambda t - \mu t| \leq |\lambda - \mu| < \delta_\epsilon$ for all $t \in [0,1]$ and then $|a(\lambda t) - a(\mu t)| < \frac{\epsilon}{2}$ for all $t \in [0,1]$, so $\|a(\lambda\cdot) - a(\mu\cdot)\|_\infty \leq \frac{\epsilon}{2}$. This is similar for ω, and then

$$|\lambda - \mu| < \delta_\epsilon \Rightarrow \|(A(\lambda) - A(\mu))\|_{X\to C^0} \leq \epsilon,$$

so $A : [0,1] \to L_c(X, C^0)$ is continuous (and even uniformly continuous since δ_ϵ is independent of the choice of λ, μ).

3. Since t_0 is an interior point, we have that $x'(t_0) = 0$ and $x''(t_0) \geq 0$. So we have that

$$\omega(\lambda t_0)x(t_0) \geq -x''(t_0) + a(\lambda t)x'(t_0) + \omega(\lambda t_0)x(t_0) = y(t_0) \geq -\|y\|_\infty$$

Since ω is continuous on the compact set $[0,1]$ and positive, there exists $\gamma > 0$ such that $\omega(t) \geq \frac{1}{\gamma}$ for all $t \in [0,1]$, and we get that

$$x(t_0) \geq -\frac{\|y\|_\infty}{\gamma}.$$

If $t_0 \in \{0,1\}$, this inequality still holds since $x(0) = x(1) = 0$. So in any case, $\min_{[0,1]} x \geq -\gamma^{-1}\|y\|_\infty$. Taking $-x$ instead of x, we get that $\max_{[0,1]} x \leq \gamma^{-1}\|y\|_\infty$. Therefore $\|x\|_\infty \leq \gamma\|y\|_\infty$.

4. The existence of t_1 is a consequence of Rolle's Lemma since $x(0) = x(1) = 0$. The map $t \mapsto K(t) := u(t)e^{\mathcal{A}(t)}$ is differentiable and for all $t \in [0,1]$, we have that

$$K'(t) = (u' + \mathcal{A}'u)e^{\mathcal{A}(t)} = (x'' - a(\lambda\cdot)x')e^{\mathcal{A}(t)} = (\omega(\lambda\cdot)x - y)e^{\mathcal{A}(t)}$$

and therefore

$$|K'(t)| \leq (\|\omega\|_\infty\|x\|_\infty + \|y\|_\infty)e^{\|a\|_\infty} \leq (\|\omega\|_\infty\gamma + 1)e^{\|a\|_\infty}\|y\|_\infty$$

for all $t \in [0,1]$. We then get that

$$|u(t)e^{\mathcal{A}(t)}| = |K(t)| = |K(t) - K(t_1)| \leq (\|\omega\|_\infty\gamma + 1)e^{\|a\|_\infty}\|y\|_\infty \cdot |t - t_1| \leq (\|\omega\|_\infty\gamma + 1)e^{\|a\|_\infty}\|y\|_\infty$$

for all $t \in [0,1]$, and then

$$|x'(t)| = |u(t)| \leq (\|\omega\|_\infty\gamma + 1)e^{\|a\|_\infty}\|y\|_\infty e^{-\mathcal{A}(t)} \leq (\|\omega\|_\infty\gamma + 1)e^{2\|a\|_\infty}\|y\|_\infty$$

So we get the conclusion with $B := (\|\omega\|_\infty\gamma + 1)e^{2\|a\|_\infty}$.

5. We write $x'' = a(\lambda\cdot)x' + \omega(\lambda\cdot)x - y$, therefore, using questions **3.** and **4.**, we get

$$\|x''\|_\infty \leq \|a\|_\infty\|x'\|_\infty + \|\omega\|_\infty\|x\|_\infty + \|y\|_\infty \leq (\|a\|_\infty B + \|\omega\|_\infty\gamma + 1)\,\|y\|_\infty.$$

So with the definition of the norm and again questions **3.** and **4.**, we get

$$\|x\|_{C^2} \leq (\|a\|_\infty B + \|\omega\|_\infty\gamma + 1 + B + \gamma)\,\|y\|_\infty,$$

which yields a possible value for M.

6. Letting $y := A(\lambda)(x)$, this is just rewriting question **5.**.

7. We have a solution iff $-r^2 + a_0 r + \omega_0 = 0$. The discriminant is $\Delta = a_0^2 + 4\omega_0 > 0$ since $\omega_0 > 0$. The zeroes are then

$$r_1 := \frac{a_0 - \sqrt{a_0^2 + 4\omega_0}}{2} < 0 \text{ and } r_2 := \frac{a_0 + \sqrt{a_0^2 + 4\omega_0}}{2} > 0.$$

8. This is a direct computation.

9. We add a solution to the homogeneous equation to x_0: we write $x(t) := x_0(t) + Ae^{r_1 t} + Be^{r_2 t}$ for $t \in [0,1]$ where $A, B \in \mathbb{R}$ are to be determined. Linearity yields $-x'' + a_0 x' + \omega_0 x = y$. We have that

$$x(0) = x_0(0) + A + B \text{ and } x(1) = x_0(1) + Ae^{r_1} + Be^{r_2}$$

So to get $x(0) = x(1) = 0$, we have a system to solve and we get

$$A = \frac{-x_0(1) + x_0(0)e^{r_2}}{e^{r_1} - e^{r_2}} \text{ and } B = \frac{+x_0(1) - x_0(0)e^{r_1}}{e^{r_1} - e^{r_2}}.$$

So we get a solution to (E_0). It is unique due to the control of question **5.**.

10. We have proved that $A(0)$ is linear, continuous and is a bijection. Question **6.** yields $\|A(0)^{-1}(y)\|_{C^2} \leq M\|y\|_\infty$ for all $y \in C^0([0,1])$, which yields the continuity of $A(0)$, so $A(0)$ is a homeomorphism. An alternative is to use Corollary 4.8 since $(X, \|\cdot\|_{C^2})$ and $(C^0([0,1]), \|\cdot\|_\infty)$ are Banach spaces.

11. The function $A : [0,1] \to L_c(X, C^0([0,1]))$ satisfies the hypothesis of Theorem 7.2, so $A(\lambda) \in Gl_c(X, C^0([0,1]))$ for all $\lambda \in [0,1]$, so $A(1)$ is invertible. Given $y \in C^0([0,1])$, $x := A(1)^{-1}(y)$ is a solution to (15), and therefore the unique solution since $A(1)$ is injective.

CHAPTER 8

Hilbert spaces

1. Review of a few classical definitions and facts

1.1. Definition.

DEFINITION 1.1. *A scalar product on a real vector space H is a map $\langle\cdot,\cdot\rangle : H\times H\to\mathbb{R}$ such that*

- *$\langle\cdot,\cdot\rangle$ is bilinear*
- *$\langle\cdot,\cdot\rangle$ is symmetric*
- *for all $x\in H$, we have that $\langle x,x\rangle\geq 0$, and $=0$ iff $x=0$*

Then $x\mapsto\|x\|:=\sqrt{\langle x,x\rangle}$ is a norm on H. A vector space $(H,\langle\cdot,\cdot\rangle)$ with such a scalar product is called a <u>*pre-Hilbert space*</u>*. A* <u>*Hilbert space*</u> *is a pre-Hilbert that is complete for the norm $\|\cdot\|$.*

Here are some examples:

- $\mathbb{R}^n$, $n\geq 1$, endowed with the canonical scalar product $\langle x,y\rangle_{\mathbb{R}^n}:=\sum_{i=1}^n x_iy_i$. It is a Hilbert space (completeness is immediate in finite dimension).
- $l^2(\mathbb{N})=\{(x_n)_n/\ \sum_n x_n^2<\infty\}$ and $\langle x,y\rangle_{l^2}:=\sum_{n\in\mathbb{N}}x_ny_n$
- Given Ω an open set of $\mathbb{R}^n$, we define the L^2-scalar product $\langle u,v\rangle_{L^2}:=\int_\Omega uv\,dx$ for $u,v\in L^2(\Omega)$. Then $(L^2(\Omega),\langle\cdot,\cdot\rangle_{L^2})$ is a Hilbert space.
- $(C^0([0,1]),\langle\cdot,\cdot\rangle_{L^2})$ is a pre-Hilbert space, but not a Hilbert space.

1.2. Inequalities and equalities.

Proposition 8.1. *Let $(H,\langle\cdot,\cdot\rangle)$ be a pre-Hilbert space. Then for all $x,y\in H$, we have that*

$$|\langle x,y\rangle|\leq\|x\|\cdot\|y\|.$$

Moreover, equality holds iff x and y are not linearly independent.

Proposition 8.2. *Let $(H,\langle\cdot,\cdot\rangle)$ be a pre-Hilbert space. Then for all $x,y\in H$, we have that*

$$\left\|\frac{x+y}{2}\right\|^2+\left\|\frac{x-y}{2}\right\|^2=\frac{\|x\|^2+\|y\|^2}{2}\tag{16}$$

Proposition 8.3. *Let $(H,\|\cdot\|)$ be a normed space. We assume that (16) holds for all $x,y\in H$. Then H is a pre-Hilbert space and $\|\cdot\|$ is the norm associated to the scalar product.*

1.3. Orthogonality.

DEFINITION 1.2. *Let $(H,\langle\cdot,\cdot\rangle)$ be a pre-Hilbert space. For $A\subset H$, we define*

$$A^\perp:=\{x\in H\text{ such that }\langle x,a\rangle=0\text{ for all }a\in A\}.$$

The space $A^\perp$ is a closed linear space.

Proposition 8.4. *Let $(H, \langle\cdot,\cdot\rangle)$ be a pre-Hilbert space. For $A \subset H$, the space $A^\perp$ is a closed linear space and $A \cap A^\perp = \{0\}$.*

2. The projection theorem

2.1. Statements.

THEOREM 8.5 (Projection theorem). *Let $(H, \langle\cdot,\cdot\rangle)$ be a pre-Hilbert space . Let $C \subset H$ be a nonempty convex and complete space. Then for all $x \in H$, there exists $x_C \in C$ unique such that*

$$\|x - x_C\| = \inf\{\|x - y\| / \, y \in C\}. \tag{17}$$

We say that x_C is the orthogonal projection of x on C. Moreover given $z \in H$, we have the equivalence:

- $z = x_C$
- $\langle x - z, y - z\rangle \leq 0$ *for all* $y \in C$.

Proposition 8.6. *Let $(H, \langle\cdot,\cdot\rangle)$ be a pre-Hilbert space . Let $C \subset H$ be a nonempty convex and complete space. For $x \in C$, we let $\pi_C(x)$ be its orthogonal projection. Then $\pi_C : H \to H$ is $1-$Lipschitz.*

THEOREM 8.7 (Projection theorem for Hilbert spaces). *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Let $C \subset H$ be a nonempty convex and closed space. Then for all $x \in H$, there exists $x_C \in C$ unique such that*

$$\|x - x_C\| = \inf\{\|x - y\| / \, y \in C\}. \tag{18}$$

We say that x_C is the orthogonal projection of x on C. Moreover given $z \in H$, we have the equivalence:

- $z = x_C$
- $\langle x - z, y - z\rangle \leq 0$ *for all* $y \in C$.

2.2. Proof.

See undergraduate class.

2.3. Consequences.

Corollary 8.8. *Let F be be a closed linear space of a Hilbert space $(H, \langle\cdot,\cdot\rangle)$. Then $H = F \oplus F^\perp$. In particular F admits a supplement and π_F is linear and symmetric.*

PROOF. For any $x \in E$, let $x_F = \pi_F(x)$ its projection on F. Fix $y \in F$. Then $\epsilon y + x_F \in F$ for all $\epsilon \in \mathbb{R}$ and then $\epsilon\langle x - x_F, y\rangle = \langle x - x_F, (x_F + \epsilon y) - x_F\rangle \leq 0$ for all $\epsilon \in \mathbb{R}$ and $y \in F$, and then $\langle x - x_F, y\rangle = 0$ for all $y \in F$, so that $x - x_F \in F^\perp$. Then $x \in F \oplus F^\perp$. The characterization of x_F then reads $z = \pi_F(x)$ iff $x - z \in F^\perp$. This yields the linearity of π_F. $\square$

Another common consequence is the following:

Corollary 8.9. *Let F be be linear space of a Hilbert space $(H, \langle\cdot,\cdot\rangle)$. Then $H = \overline{F}$ iff $F^\perp = \emptyset$.*

THEOREM 8.10 (Riesz representation Theorem). *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space . Then for all $\varphi \in H'$ (continuous linear map $\varphi : H \to \mathbb{R}$), there exists $x \in H$ unique such that $\varphi(z) = \langle x, z\rangle$ for all $z \in H$. In other words, the map*

$$\begin{cases} i: & H & \to & H' \\ & x & \mapsto & (z \mapsto \langle x, z\rangle) \end{cases}$$

is an isometry with the suitable norms.

PROOF. Fix $\varphi \in H'$ with $\varphi \not\equiv 0$ (otherwise take $x = 0$). Since Ker φ is closed, then $H = \text{Ker } \varphi \oplus (\text{Ker } \varphi)^{\perp}$. Since Ker φ is a hyperplane, then the supplement $(\text{Ker } \varphi)^{\perp}$ is one-dimensional. Let $x_0 \in H$ be such that $(\text{Ker } \varphi)^{\perp} = \mathbb{R}x_0$, $x_0 \neq 0$ and $\varphi(x_0) \neq 0$. Take $u \in H$ and define $u' = u - \frac{\langle u, x_0 \rangle}{\|x_0\|^2} x_0$. Then $\langle u', x_0 \rangle = 0$ and then $u' \in (\mathbb{R}x_0)^{\perp} = \text{Ker } \varphi$, so $\varphi(u') = 0$, and then $\varphi(u) - \frac{\langle u, x_0 \rangle}{\|x_0\|^2} \varphi(x_0) = 0$. So with $x := \frac{\varphi(x_0)}{\|x_0\|^2} x_0$, we have that $\varphi(u) = \langle u, x \rangle$ for all $u \in H$. This proves the theorem. $\square$

3. Weak compactness of the unit ball

3.1. A new topology on Hilbert spaces.

Proposition 8.11. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Then there exists a topology on H, called the "weak topology", such that for any sequence $(x_n)_n$ in H and $x \in H$, then the two following assertions are equivalent:*

- *$(x_n)_n$ converges to x in the weak topology*
- *For all $\varphi \in E'$, then $\lim_{n \to +\infty} \varphi(x_n) = \varphi(x)$*

Vocabulary and notations: the topology of normed space of $(H, \| \cdot \|)$ is called the strong topology. Given a sequence $(x_n)_n$ such that $x_n \in H$ for all n and $x \in H$, we will write

- $x_n \to x$ as $n \to +\infty$ when $(x_n)_n$ to x for the strong topology (we say that "$(x_n)_n$ converges strongly to x"
- $x_n \rightharpoonup x$ as $n \to +\infty$ when $(x_n)_n$ to x for the weak topology (we say that "$(x_n)_n$ converges weakly to x"

It follows from the representation Theorem 8.10 that for $(x_n)_n \in H$ and $x \in H$,

$$\{x_n \rightharpoonup x \text{ as } n \to +\infty\} \Leftrightarrow \{ \text{ for any } z \in H,\ \langle x_n, z \rangle \to \langle x, z \rangle \text{ as } n \to +\infty\} \tag{19}$$

3.2. Properties.

Proposition 8.12. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Let $(x_n)_n$ be such that $x_n \in H$ for all $n \in \mathbb{N}$ and $x \in H$. Then*

- *When it exists, the weak limit is unique*
- *Strong convergence $\Rightarrow$ weak convergence: that is $\{x_n \to x\} \Rightarrow \{x_n \rightharpoonup x\}$ as $n \to +\infty$*
- *If $x_n \rightharpoonup x$, then $\|x\| \leq \liminf_{n \to +\infty} \|x_n\|$*
- *If $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$ strongly as $n \to +\infty$*
- *In finite dimension, weak and strong convergence are equivalent (the two topologies coincide).*

PROOF.

- Assume that $(x_n)_n$ converges to x and $x' \in H$. Fix $z \in H$. Then $\langle x_n, z \rangle \to \langle x, z \rangle$ and $\langle x', z \rangle$ as $n \to +\infty$, so $\langle x, z \rangle = \langle x', z \rangle$ for all $z \in H$. Taking $z = x - x'$ yields $x = x'$. This proves uniqueness.
- Take $(x_n)_n$ that converges strongly to x. Fix $\varphi \in H'$. Since φ is continuous for the strong topology, then $\varphi(x_n) \to \varphi(x)$, so $x_n \rightharpoonup x$ as $n \to +\infty$.
- For $n \in \mathbb{N}$, we have that

$$\begin{aligned} \|x_n\|^2 &= \|x_n - x + x\|^2 = \|x_n - x\|^2 + \langle x_n - x, x \rangle + \|x\|^2 \\ &\geq \langle x_n - x, x \rangle + \|x\|^2 \end{aligned}$$

Since $x_n \rightharpoonup x$, then $\langle x_n - x, x \rangle \to 0$ as $n \to +\infty$. This yields the result.

- It follows from the above analysis that when $x_n \rightharpoonup x$, we have that $\|x_n\|^2 = \|x_n - x\|^2 + \|x\|^2 + o(1)$ as $n \to +\infty$. Then, if $\|x_n\| \to \|x\|$, we get that $\|x_n - x\| \to 0$ as $n \to +\infty$, which yields strong convergence.
- Let $\{e_1, ..., e_N\}$ be an orthonormal basis of H. Then for all $n \in \mathbb{N}$, we have that $x_n = \sum_{i=1}^N \langle x_n, e_i\rangle e_i$. If $x_n \rightharpoonup x$, then $\langle x_n, e_i\rangle \to \langle x, e_i\rangle$ for all $i = 1, ..., N$, and then $x_n \to \sum_{i=1}^N \langle x, e_i\rangle e_i = x$ strongly.

□

3.3. An example of a weakly- but not strongly-convergence sequence. It follows from the latest assertion that we need to work in infinite dimension to find such an example. We are working in the Hilbert space $(L^2(\mathbb{R}), \langle\cdot,\cdot\rangle_{L^2})$.
Fix $\theta \in C_c^\infty(\mathbb{R})$ such that $\theta \not\equiv 0$. Define

$$\theta_n(x) := \sqrt{n}\theta(nx) \text{ for all } x \in \mathbb{R} \text{ and } n \in \mathbb{N},\ n \geq 1.$$

We claim that

$$\theta_n \rightharpoonup 0 \text{ as } n \to +\infty \text{ but does not converge for the strong topology.}$$

We prove the claim. The first remark is that for any $n \geq 1$, with a change of variables, we have that

$$\|\theta_n\|_2^2 = \int_{\mathbb{R}} \theta_n^2(x)\,dx = \int_{\mathbb{R}} n\theta^2(nx)\,dx = \int_{\mathbb{R}} \theta^2(x)\,dx = \|\theta\|_2^2 \tag{20}$$

Let us prove the weak convergence of (θ_n) to 0. Let $R > 0$ be such that $\theta(x) = 0$ for all $|x| \geq R$. Fix $f \in L^2(\mathbb{R})$. Then

$$\begin{aligned}
|\langle\theta_n, f\rangle_{L^2}| &= \left|\int_{\mathbb{R}} \theta_n(x) f(x)\,dx\right| = \left|\int_{-R/n}^{R/n} \theta_n(x) f(x)\,dx\right| \\
&\leq \sqrt{\int_{-R/n}^{R/n} \theta_n(x)^2\,dx}\sqrt{\int_{-R/n}^{R/n} f(x)^2\,dx} \\
&\leq \|\theta_n\|_2 \sqrt{\int_{\mathbb{R}} \mathbf{1}_{[-R/n,R/n]} f(x)^2\,dx} = \|\theta\|_2 \sqrt{\int_{\mathbb{R}} \mathbf{1}_{[-R/n,R/n]} f(x)^2\,dx}
\end{aligned}$$

Since $f \in L^2(\mathbb{R})$, it follows from Lebesgue's convergence Theorem that

$$\lim_{n\to+\infty} \langle\theta_n, f\rangle_{L^2} = 0$$

for all $f \in L^2(\mathbb{R})$. Therefore, with Riesz's representation theorem, we get that $\theta_n \rightharpoonup 0$ as $n \to +\infty$. Assume that $(\theta_n)_n$ has a strong limite, say Θ. Then the convergence holds in the weak sense (see the preceding proposition), and by uniqueness, we have that $\Theta = 0$, so $\theta_n \to 0$ strongly as $n \to \infty$, and then $\|\theta_n\|_2 \to 0$ as $n \to +\infty$, contradicting (20). Then (θ_n) admits no strong limit.

3.4. Main theorem.

3.4.1. *Statement.*

THEOREM 8.13. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space . Then $\overline{B_H(0,1)} = \{x \in H$ such that $\|x\| \leq 1\}$ is compact for the weak topology. In particular, for any bounded sequence $(x_n)_n$ such that $x_n \in H$ and $\|x_n\| \leq M$ for all $n \in \mathbb{N}$ ($M \geq 0$ is independent of n), there exists $x \in H$ and a subsequence $(x_{j(n)})_n$ such that*

$$x_{j(n)} \rightharpoonup x \text{ weakly as } n \to +\infty.$$

3.4.2. *Proof.* The abstract proof can be found in Brezis [1]. Here, we are just proving that from a bounded sequence, we can extract a weakly converging subsequence when H is separable (that is admits a countable dense subspace). Note that, this is equivalent to compactness only when the topology arises from a metric space, which indeed is the case when H is separable. See discussions on this issue below.

The proof we perform is extremely close to the proof of Ascoli's Theorem, and it is even simpler since we will not require uniform convergence.

So assume that H is separable, that is there exists $(z_k)_{k\in\mathbb{N}} \in H$ such that $H = \overline{\{z_k/\, k \in \mathbb{N}\}}$. We take a sequence $(x_n)_{n\in\mathbb{N}}$ such that there exists $M > 0$ such that

$$x_n \in H \text{ and } \|x_n\| \leq M \text{ for all } n \in \mathbb{N}.$$

Step 1: transfer to H'. For any $n \in \mathbb{N}$, we define

$$f_n(z) := \langle x_n, z\rangle \text{ for all } z \in H.$$

We have that $f_n \in H'$ for all $n \in \mathbb{N}$ and that $|f_n(z)| \leq M\|z\|$ for all $n \in \mathbb{N}$ and all $z \in H$. In particular, $|f_n(z) - f_n(z')| \leq M\|z - z'\|$ for all $n \in \mathbb{N}$, $z, z' \in H$. So this is some uniform equicontinuity!!!

Step 2: Cantor's diagonal process. Fix $k \in \mathbb{N}$. Then the sequence $(f_n(z_k))_n$ is in $[-M, M]$, and so it is bounded in $\mathbb{R}$. So a subsequence converges, given $k \in \mathbb{N}$. Performing Cantor's Diagonal procedure as in the proof of Ascoli's Theorem 3.5, we get the existence of a subsequence $(f_{j(n)})$ such that

$$\text{for all } k \in \mathbb{N},\ (f_{j(n)}(z_k))_n \text{ converges as } n \to +\infty.$$

Step 3: Pointwise convergence. We claim that there exists $f : H \to \mathbb{R}$ such that for any $z \in H$, $(f_{j(n)}(z))_n$ converges to $f(z)$.

We prove the claim. Fix $z \in H$ and $k \in \mathbb{N}$. We have that

$$\begin{array}{rcl} |f_{j(p)}(z) - f_{j(q)}(z)| & \leq & |f_{j(p)}(z) - f_{j(p)}(zk)| + |f_{j(p)}(z_k) - f_{j(q)}(z_k)| + |f_{j(q)}(z_k) - f_{j(q)}(z)| \\ & \leq & 2M\|z - z_k\| + |f_{j(p)}(z_k) - f_{j(q)}(z_k)| \end{array}$$

We argue again as in the proof of Ascoli's Theorem. Fix $\epsilon > 0$. Fix $k \in \mathbb{N}$ such that $\|z - z_k\| < \frac{\epsilon}{3(M+1)}$. Since $(f_{j(n)}(z_k))_n$ converges, it is a Cauchy sequence and there exists $n_0 = n_0(k, \epsilon) = n_0(z, \epsilon) \in \mathbb{N}$ such that $|f_{j(p)}(z_k) - f_{j(q)}(z_k)| < \frac{\epsilon}{3}$ for all $p, q \neq n_0$. So

$$|f_{j(p)}(z) - f_{j(q)}(z)| < \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \text{ for all } p, q \geq n_0$$

and then $(f_{j(n)}(z))_n$ is a Cauchy sequence in $\mathbb{R}$, so it converges. Let $f(z)$ be its limit.

Step 4: continuity of f. We claim that $f \in H'$.

For $z, z' \in H$ and $\lambda, \mu \in \mathbb{R}$, we have that $f_{j(n)}(\lambda z + \mu z') = \lambda f_{j(n)}(z) + \mu f_{j(n)}(z')$ for all $n \in \mathbb{N}$. Letting $n \to +\infty$ and using the poitnwise convergence of Step 3 yields the linearity of f. Moreover, we have that $|f_{j(n)}(z)| \leq M\|z\|$ for all $n \in \mathbb{N}$ and $z \in H$. Letting $n \to +\infty$ yields $|f(z)| \leq M\|z\|$ for all $z \in H$. Since f is linear, this yields continuity.

Step 5: Existence of x and conclusion.

Since $f \in H'$ and H is a Hilbert space, it follows from Riesz representation theorem 8.10 that there exists $x \in H$ such that $f(z) = \langle x, z\rangle$ for all $z \in H$. Then, for any $z \in H$, we have that

$$\langle x_{j(n)}, z\rangle = f_{j(n)}(z) \to f(z) = \langle x, z\rangle \text{ as } n \to +\infty,$$

and then $x_{j(n)} \rightharpoonup x$ as $n \to +\infty$. Since $\|x\|^2 = |f(x)| \leq M\|x\|$, we get that $\|x\| \leq M$. □

3.5. Applications.

Proposition 8.14 (Riesz Theorem revisited)**.** *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Fix $f \in H'$. Then there exists $x \in H$ such that $f(z) = \langle x, z\rangle$ for all $z \in H$.*

Of course this is the classical representation theorem. However it is of interest to write it in a variational setting and to prove it via a minimization argument. Define

$$J(z) := \frac{\|z\|^2}{2} - f(z) \text{ for all } z \in H.$$

Step 0: The variational setting. Let us prove that $J \in C^1(H)$. Fix $z_0 \in H$ and $h \in H$. We have that

$$\begin{aligned} J(z_0 + h) &= \frac{\|z_0 + h\|^2}{2} - f(z_0 + h) = \frac{\|z_0\|^2 + 2\langle z_0, h\rangle + \|h\|^2}{2} - f(z_0) - f(h) \\ &= J(z_0) + \langle z_0, h\rangle - f(h) + \frac{\|h\|^2}{2} \end{aligned}$$

Since $h \mapsto \langle z_0, h\rangle - f(h)$ is linear and continuous and $\lim_{h\to 0} \frac{\|h\|^2}{\|h\|} = 0$, then J is differentiable at z_0 and $dJ_{z_0}[h] = \langle z_0, h\rangle - f(h)$ for all $h \in H$.
We prove that dJ is C^0, where

$$\left\{ \begin{array}{cccc} dJ: & H & \to & H' \\ & z & \mapsto & dJ_z \end{array} \right.$$

Take $z_1, z_2 \in H$ and $h \in H$. We have that

$$|(dJ_{z_1} - dJ_{z_2})[h]| = |\langle z_1, h\rangle - f(h) - (\langle z_2, h\rangle - f(h))| = |\langle z_1 - z_2, h\rangle| \leq \|z_1 - z_2\| \cdot \|h\|,$$

and then

$$\|dJ_{z_1} - dJ_{z_2}\|_{H\to\mathbb{R}} = \sup_{h\neq 0} \frac{|(dJ_{z_1} - dJ_{z_2})[h]|}{\|h\|} \leq \|z_1 - z_2\|.$$

Then dJ is Lipschitz, so it is continuous and $J \in C^1(H)$.
Then we have the following: for any $x \in H$, we have that

$$dJ_x = 0 \Leftrightarrow \langle x, h\rangle = f(h) \text{ for all } h \in H.$$

So our problem enjoys a *variational formulation* , that is $dJ_x = 0$. In order to find a critical point, we are going to minimize J.

Step 1: we claim that J has a lower bound.
We prove the claim. Indeed, since f is continuous, we have that

$$J(z) \geq \frac{\|z\|^2}{2} - \|f\|_{H\to\mathbb{R}}\|z\| = \frac{1}{2}\left(\|z\| - \|f\|_{H\to\mathbb{R}}\right)^2 - \frac{\|f\|^2_{H\to\mathbb{R}}}{2} \geq -\frac{\|f\|^2_{H\to\mathbb{R}}}{2} \tag{21}$$

for all $z \in H$. So J has a lower bound.

Define

$$m := \inf\{J(z)/\, z \in H\} \in \mathbb{R}.$$

Let $(x_n)_n \in H$ be a minimizing subsequence, that is

$$\lim_{n\to+\infty} J(x_n) = m. \tag{22}$$

Step 2: We claim that $(x_n)_n$ is bounded.

We prove the claim. It follows from (21) that

$$m + o(1) = J(x_n) \geq \frac{1}{2}\left(\|x_n\| - \|f\|_{H\to\mathbb{R}}\right)^2 - \frac{\|f\|^2_{H\to\mathbb{R}}}{2},$$

and then $(\|x_n\|)_n$ is bounded in $\mathbb{R}$, which yields boundedness of $(x_n)_n$.

Step 3: there exists $x \in H$ and a subsequence $x_{j(n)} \rightharpoonup x$ weakly in H.

Since $(x_n)_n$ is bounded, this is a consequence of the weak compactness of the unit ball of Theorem 8.13.

Step 4: We claim that x minimizes J.

We prove the claim. We set $\theta_n := x_{j(n)} - x$. As one checks, we have that

$$J(x_{j(n)}) = J(x+\theta_n) = J(x) + \frac{\|\theta_n\|^2}{2} + \langle \theta_n, x\rangle - f(\theta_n).$$

With (22) and $\theta_n \rightharpoonup 0$, we get that

$$m + o(1) = J(x) + \frac{\|\theta_n\|^2}{2} \text{ as } n \to +\infty. \tag{23}$$

Then $m + o(1) \geq J(x)$, and then $m \geq J(x)$. Since $J(x) \geq m$ (this comes from the definition of the infimum), we then get $J(x) = m$. So the infimum is achieved by $x \in H$. Moreover, with (23), we then get that $\|x_{j(n)} - x\| = \|\theta_n\| \to 0$ as $n \to +\infty$, so $x_{j(n)} \to x$ strongly as $n \to +\infty$. □

Step 5: We claim that $f(z) = \langle x, z\rangle$ for all $z \in H$.

We prove the claim. For $z \in H$ and $t \in \mathbb{R}$, we have that

$$J(x+tz) = \frac{\|x+tz\|^2}{2} - f(x+tz) = J(x) + t\left(\langle x, z\rangle - f(z)\right) + t^2\frac{\|z\|^2}{2}.$$

Since this is minimized by $m = J(x)$, we the get that $\langle x, z\rangle - f(z) = 0$, and the Theorem is proved. □

Remark: This proof is only made as an illustration. Indeed, we reprove the representation theorem by using the weak compactness of the unit ball.... that is proved via the representation theorem!

Exercise 24: Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space . Let $F : H \to \mathbb{R}$ be such that

- $F \in C^1(H)$
- F is bounded on H
- For all $(z_n)_{n\in\mathbb{N}}$ and $z \in H$ such that $z_n \rightharpoonup z$ weakly, then
$$\lim_{n\to+\infty} F(z_n) = F(z).$$

We set

$$J(x) := \frac{1}{2}\|x\|^2 - F(x) \text{ for all } x \in H.$$

1. Show that $m := \inf\{J(x)/\, x \in H\}$ exists. We let $(x_n)_{n\in\mathbb{N}}$ be a minimizing sequence for J, that is $x_n \in H$ for all $n \in \mathbb{N}$ and $\lim_{n\to+\infty} J(x_n) = m$.

2. Show that there exists $x \in H$ and a subsequence $(x_{j(n)})$ that goes weakly to x as $n \to \infty$.

3. We set $\theta_n := x_{j(n)} - x$ for all $n \in \mathbb{N}$. Show that there exists a sequence (t_n) such that $J(x_{j(n)}) = J(x) + t_n + o(1)$ as $n \to +\infty$ with $t_n \geq 0$ for all $n \in \mathbb{N}$.

4. Show that $J(x) = m$.

5. Show that there exists $\varphi : H \to H$ continuous such that for all $x, y \in H$, we have that $dF_x(y) = \langle \varphi(x), y \rangle$. Here, dF_x is the differential of F at x.

6. Show that $x = \varphi(x)$.

4. Problem: Minimization of convex functions

This section is devoted to the proof of the following result:

THEOREM 8.15. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and take $\varphi : H \to \mathbb{R}$. We assume that*

- *φ is convex;*
- *φ is continuous for the strong topology;*
- *$\lim_{\|x\| \to +\infty} \varphi(x) = +\infty$ in the following sense: for all $A > 0$, there exists $R > 0$ such that*

$$\text{for all } x \in H,\ \{\|x\| > R \Rightarrow\ \varphi(x) > A\}.$$

Then φ has a minimizer on H, that is there exists $x_0 \in H$ such that $\varphi(x_0) = \inf\{\varphi(x)/\, x \in H\}$.

Indeed, this results hold if φ is only lower-semi-continuous, see the remark below. The proof is going through several step, each of them being an exercise.

Exercise 25: Let $C \subset H$, $C \neq H$, be a nonempty convex set that is closed for the strong topology. Fix $x \in H \setminus C$. Show that there exists $\vec{u} \in H$ and $\alpha \in \mathbb{R}$ such that

$$\langle z, \vec{u} \rangle \leq \alpha \text{ for all } z \in C \text{ et } \langle x, \vec{u} \rangle > \alpha.$$

Hint: A possibility is to use directly a geometric version of Hahn-Banach's theorem. An alternative is to use the Hilbert structure: consider the projection of x on C and use the characterization of the projection via the scalar product. And draw a picture to help you!

Exercise 26: Let C be a nonempty convex set of H that is closed for the strong topology. Take $(x_n)_n$ a sequence in C and $x \in H$ such that $x_n \rightharpoonup x$ in the weak sense. Show that $x \in C$. In some sense, this means that C is closed for the weak topology.

Hint: use the previous exercise.

Exercise 27: Let $(H, \langle \cdot, \rangle)$ be a Hilbert space . Let $\varphi : H \to \mathbb{R}$ be a continuous (for the strong topology) convex function. Let $(x_n)_n$ be such that $x_n \in H$ for all $n \in \mathbb{N}$ and fix $x \in H$. We assume that $x_n \rightharpoonup x$ weakly as $n \to +\infty$. Show that

$$\varphi(x) \leq \liminf_{n \to +\infty} \varphi(x_n).$$

Hint: argue by contradiction and assume that the result does not hold. Then show that $\exists \epsilon > 0$ and a subsequence such that $\varphi(x_{j(n)}) \leq \varphi(x) - \epsilon$ for all $n \in \mathbb{N}$. Define $C := \varphi^{-1}\left((-\infty, \varphi(x) - \epsilon]\right)$ and use the result of the preceding exercise to get a contradiction.

Exercise 28: By following the minimization method for the proof of Theorem 8.14, use the preceding exercise to prove Theorem 8.15.

Remark: As one checks, the conclusions of Exercises 27 and 28 hold if instead of requiring the continuity of φ, we only require *lower-semi-continuity*, that is for all $k \in \mathbb{R}$, $\varphi^{-1}((-\infty, k])$ is closed in H for the strong topology.

5. The Lax-Milgram Theorem

5.1. Statement.

THEOREM 8.16. *[Lax-Milgram] Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space . Let $a : H \times H \to \mathbb{R}$ be a continuous bilinear form (not necessarily symmetric!). We assume that a is coercive, that is $\exists \alpha > 0$ such that*

$$a(u,u) \geq \alpha \|u\|^2 \text{ for all } u \in H.$$

Let $f \in H'$ be a continuous linear form and let $C \subset H$ be a closed nonempty convex set. Then there exists a unique element $u_C \in C$ such that

$$a(u_C, v - u_C) \geq f(v - u_C) \text{ pour tout } v \in C. \tag{E_C}$$

When C is the full space H, the theorem takes a more friendly form:

THEOREM 8.17. *[Lax-Milgram, version $C = H$] Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Let $a : H \times H \to \mathbb{R}$ be a continuous bilinear form (not necessarily symmetric!). We assume that a is coercive, that is $\exists \alpha > 0$ such that*

$$a(u,u) \geq \alpha \|u\|^2 \text{ for all } u \in H.$$

Let $f \in H'$ be a continuous linear form. Then there exists a unique element $u_H \in C$ such that

$$a(u_H, v) = f(v) \text{ for all } v \in H. \tag{E_H}$$

The applications will be in the Part about Sobolev spaces .

5.2. Proof of Theorem 8.17: the case $C = H$. We consider the hypothesis of Theorem 8.17. The proof goes through five steps.

Step 1: We claim that there exists $A : H \to H$ such that

$$a(u,v) = \langle A(u), v\rangle \text{ for all } u, v \in H.$$

Proof of the claim: Given $u \in H$, the map $a(u,\cdot) : H \to \mathbb{R}$ is linear and continuous. It then follows from Riesz's representation theorem that there exists $A(u) \in H$ such that $a(u,\cdot) = \langle A(u), \cdot\rangle$. This proves the claim.

Step 2: We claim that $A : H \to H$ is linear and continuous.

Proof of the claim: for $u, v, w \in H$ and $\lambda, \mu \in \mathbb{R}$, using the bilinearity of a, we have that

$$\begin{aligned}\langle A(\lambda u + \mu v), w\rangle &= a(\lambda u + \mu v, w) = \lambda a(u,w) + \mu a(v,w)\\ &= \lambda\langle A(u), w\rangle + \mu\langle A(v), w\rangle = \langle \lambda A(u) + \mu A(v), w\rangle\end{aligned}$$

Since this is valid for all $w \in H$, we then get that $A(\lambda u + \mu v) = \lambda A(u) + \mu A(v)$. This proves the linearity.

Since a is continuous, there exists $M > 0$ such that $|a(u,v)| \leq M\|u\| \cdot \|v\|$ for all $u, v \in H$, and then

$$|\langle A(u), v\rangle| \leq M\|u\| \cdot \|v\|.$$

Taking $v = A(u)$, we get that $\|A(u)\|^2 \leq M\|u\| \cdot \|A(u)\|$, and then

$$\|A(u)\| \leq M\|u\| \text{ for all } u \in H. \tag{24}$$

This proves the continuity. The claim is proved.

Step 3: We claim that there exists $x_0 \in H$ such that solving (E_H) amounts to finding $u_H \in H$ such that $A(u_H) = x_0$.

Proof of the claim: Since $f \in H'$, Riesz's representation theorem yields $x_0 \in H$ such that $f(v) = \langle x_0, v\rangle$ for all $v \in H$. Then for any $u_H \in H$, u_H solves (E_H) iff $\langle A(u_H), v\rangle = a(u_H, v) = f(v) = \langle x_0, v\rangle$ for all $v \in H$, which is equivalent to $A(u_H) = x_0$. This proves the claim.

For $\rho > 0$ (that will be fixed later), we define

$$\begin{array}{rccc} T: & H & \to & H \\ & u & \mapsto & u - \rho(A(u) - x_0) \end{array}$$

Step 4: We claim that for some suitable $\rho > 0$, T is a contracting map.

Proof of the claim: Let u_1, u_2 be two elements of H. Using the coercivity of a and (24), we have that

$$\begin{array}{rcl} \|T(u_1) - T(u_2)\|^2 & = & \|u_1 - u_2 - \rho A(u_1 - u_2)\|^2 \\ & = & \|u_1 - u_2\|^2 - 2\rho\langle u_1 - u_2, A(u_1 - u_2)\rangle + \rho^2\|A(u_1 - u_2)\|^2 \\ & \leq & \|u_1 - u_2\|^2 - 2\rho\|u_1 - u_2\|^2 + M^2\rho^2\|u_1 - u_2\|^2 \\ & \leq & \left(1 - 2\alpha\rho + M^2\rho^2\right)\|u_1 - u_2\|^2 \end{array}$$

Taking $\rho \in \left(0, \frac{M^2}{\alpha}\right)$, we have that $k := \sqrt{1 - 2\alpha\rho + M^2\rho^2} < 1$ and T is a contracting map. This proves the claim.

Step 5: We claim that there exists $u_H \in H$ such that $A(u_H) = x_0$.

Proof of the claim: Since $T : H \to H$ is contracting and $(H, \|\cdot\|)$ is complete, this is a consequence of Picard's/Banach's fixed point theorem. This proves the claim.

As mentioned in Step 3, the existence of such u_H is equivalent to solve (E_H). Moreover, u_H is unique due to he uniqueness in Picard's/Banach's Theorem. □

5.3. The general case (Theorem 8.16).

Exercise 29: Taking inspiration in the proof of the case $C = H$, show that there exists $x_0 \in H$ such that for all $\rho > 0$, solving (E_C) amounts to show the existence of $u_C \in H$ such that

$$u_C = \pi_C(u_C - \rho(A(u_C) - x_0)),$$

where π_C is the orthogonal projection on C. Conclude with a fixed-point theorem.

6. Solution of the exercises of Chapter 8

Solution of Exercise 24:

1. Let $M \in \mathbb{R}$ be such that $|F(x)| \leq M$ for all $x \in H$. So for any $x \in H$, we have that $J(x) \geq \|x\|^2/2 - M \geq -M$, so it is bounded from below. Moreover H is nonempty, so $m \in \mathbb{R}$ is defined.

2. Since $1 > 0$, there exists $n_0 \in \mathbb{N}$ such that $J(x_n) \leq m + 1$ for all $n \geq n_0$, so

$$\|x_n\|^2 \leq 2(m + 1 + F(x_n)) \leq 2(m + 1 + M) \text{ for all } n \geq n_0.$$

So $(x_n)_n$ is bounded.

3. Define $\theta_n := x_{j(n)} - x$. We have that

$$\begin{aligned} J(x_{j(n)}) &= \frac{\|x+\theta_n\|^2}{2} - F(x_{j(n)}) = \frac{\|x\|^2}{2} + \langle x, \theta_n \rangle + \frac{\|\theta_n\|^2}{2} - F(x_{j(n)}) \\ &= J(x) + \frac{\|\theta_n\|^2}{2} + F(x) - F(x_{j(n)}) + \langle x, \theta_n \rangle \end{aligned}$$

Since $\theta_n \rightharpoonup 0$, then $\langle x, \theta_n \rangle \to 0$ as $n \to +\infty$. Since $x_{j(n)} \rightharpoonup x$, then $F(x_{j(n)}) \to F(x)$ as $n \to +\infty$. Therefore, defining $t_n := \frac{\|\theta_n\|^2}{2}$, we get the desired result.

4. we have that $m + o(1) = J(x_{j(n)}) = J(x) + t_n + o(1) \geq J(x) + o(1)$. Passing to the limit yields $m \geq J(x)$. The definition of m yields $m \leq J(x)$, so $m = J(x)$.

5. Fix $x \in H$. Then $dF_x : H \to \mathbb{R}$ is linear and continuous, so Riesz's representation theorem yields the existence of $\varphi(x) \in H$ such that $dF_x(y) = \langle \varphi(x), y \rangle$ for all $y \in H$. Take $x, x' \in H$. We have that

$$\begin{aligned} \|\varphi(x) - \varphi(x')\|^2 &= \langle \varphi(x) - \varphi(x'), \varphi(x) - \varphi(x') \rangle \\ &= (dF_x - dF_{x'})(\varphi(x) - \varphi(x')) \leq \|dF_x - dF_{x'}\|_{H\to\mathbb{R}} \|\varphi(x) - \varphi(x')\| \end{aligned}$$

and then

$$\|\varphi(x) - \varphi(x')\| \leq \|dF_x - dF_{x'}\|_{H\to\mathbb{R}}$$

Note, but it is of no use here, that the reverse inequality holds since $(dF_x - dF_{x'})(y) = \langle \varphi(x) - \varphi(x'), y \rangle \leq \|\varphi(x) - \varphi(x')\| \cdot |y\|$ for all $y \in H$.

Since $x \mapsto dF_x$ is continuous, we get that $x \mapsto \varphi(x)$ is also continuous.

6. Since $J \in C^1(H)$ is minimum at x, then $dJ_x = 0$. So for any $y \in H$, we have that

$$dJ_x(y) = \langle x, y \rangle - dF_x(y) = \langle x, y \rangle - \langle \varphi(x), y \rangle = \langle x - \varphi(x), y \rangle$$

for all $y \in H$. So $x - \varphi(x) = 0$ and the conclusion follows. □

Solution of Exercise 25: Since C is convex, close and nonempty and $\{x\}$ is convex, compact, nonempty and $C \cap \{x\} = \emptyset$, it follows from Theorem 5.7 that there exists $\alpha \in \mathbb{R}$, $\epsilon > 0$ and $f \in H'$ such that $f(z) \leq \alpha - \epsilon < \alpha + \epsilon \leq f(x)$ for all $z \in C$. Since $f \in H'$, Riesz's Theorem yields the existence of $\vec{u}$ such that $f(y) = \langle \vec{u}, y \rangle$ for all $y \in H$. We get the conclusion.

The projection theorem yields an alternative. Define $x_C \in C$ as the projection of x on C: this makes sense since C is closed, convex and nonempty in the Hilbert space H. Then $\langle z - x_C, x - x_C \rangle \leq 0$ for all $z \in C$. Set $\alpha := \langle x_C, x - x_C \rangle$ and $\vec{u} := x - x_C$. We then have that $\langle z, \vec{u} \rangle \leq \alpha$ for all $z \in C$. Moreover, $\langle x, \vec{u} \rangle - \alpha = \langle x - x_C, x - x_C \rangle = \|x - x_C\|^2 > 0$ since $x \neq x_C$ (otherwise $x \in C$, contradiction). □

Solution of Exercise 26: Assume by contradiction that $x \notin C$ and take $\alpha \in \mathbb{R}$ and $\vec{u}$ as in Exercise 25. Then $\langle x_n, \vec{u} \rangle \alpha$ for all n and $\langle x, \vec{u} \rangle > \alpha$. Since $x_n \rightharpoonup x$, we have that $\langle x_n, \vec{u} \rangle \to \langle x, \vec{u} \rangle$, and then $\langle x, \vec{u} \rangle \geq \alpha$, contradicting the construction of α and $\vec{u}$. So $x \in C$. □

Solution of Exercise 27: Assume that $\liminf_{n\to+\infty} \varphi(x_n) < \varphi(x)$ (note that the liminf might be $-\infty$). Then

$$\lim_{N\to+\infty} \inf\{\varphi(x_n)/n \geq N\} < \varphi(x),$$

so $\exists\epsilon > 0$ such that

$$\lim_{N\to+\infty} \inf\{\varphi(x_n)/n \geq N\} < \varphi(x) - \epsilon$$

then, since the sequence $\inf\{\varphi(x_n)/n \geq N\}$ is increasing, we have that

$$\inf\{\varphi(x_n)/n \geq N\} < \varphi(x) - \epsilon \text{ for all } N \in \mathbb{N}$$

so there exists a subsequence $(\varphi(x_{j(n)}))_n$ such that

$$\varphi(x_{j(n)}) < \varphi(x) - \epsilon \text{ for all } n \in \mathbb{N}.$$

So $x_{j(n)} \in C := \varphi^{-1}\left((-\infty, \varphi(x) - \epsilon]\right)$. Since φ is continuous, then C is closed for the strong topology as the pre-image of a closed intervalle. Moreover, as one checks, since φ is convex, then C is convex. Since $x_{j(n)} \rightharpoonup x$, it the follows from Exercise 26 that $x \in C$, which rewrites $\varphi(x) \leq \varphi(x) - \epsilon$, contradicting $\epsilon > 0$. Therefore $\liminf_{n\to+\infty} \varphi(x_n) \geq \varphi(x)$: in particular, it is $> -\infty$. □

Solution of Exercise 28: Define

$$m := \inf\{\varphi(x)/\, x \in H\} \geq -\infty.$$

Be cautious: m might be $-\infty$. Take a sequence $(x_n)_n$ such that

$$x_n \in H \text{ for all } n \in \mathbb{N} \text{ and } \lim_{n\to+\infty} \varphi(x_n) = m.$$

We claim that $(x_n)_n$ is bounded. We prove the claim. We have that $m \leq \varphi(0) < \varphi(0) + 1$, so there exists n_0 such that for $n \geq n_0$, we have that $\varphi(x_n) < \varphi(0) + 1$. It follows from the properties of φ that there exists $R_0 > 0$ such that for all $z \in H$, $\{||z|| > R_0 \Rightarrow \varphi(z) \geq \varphi(0) + 1\}$. Therefore, $\|x_n\| \leq R_0$ for all $n \geq n_0$. So $(x_n)_n$ is bounded in H

It follows from the weak compactness of the unit ball that there exists a subsequence $(x_{j(n)})_n$ and $x \in H$ such that $x_{j(n)} \rightharpoonup x$ weakly. It then follows from Exercise 27 that $\varphi(x) \leq \liminf_{n\to+\infty} \varphi(x_n)$, and then $\varphi(x) \leq m$. In particular, $m > -\infty$. Since $m \leq \varphi(x)$ (definition of m), we have that $\varphi(x) = m$, so the infimum is achieved. □

Solution of Exercise 29: We take a, f, C as in the satement of the Theorem. We use Steps 1 and 2 of the above proof. Take $u_C \in C$ and fix $\rho > 0$. Then

$$\begin{aligned}
u_C \text{ solves } (E_C) \quad &\Leftrightarrow \quad \langle A(u_C), v - u_C\rangle \geq \langle x_0, v - u_C\rangle \text{ for all } v \in C\\
&\Leftrightarrow \quad \langle A(u_C) - x_0, v - u_C\rangle \geq 0 \text{ for all } v \in C\\
&\Leftrightarrow \quad \langle \rho(A(u_C) - x_0), v - u_C\rangle \geq 0 \text{ for all } v \in C\\
&\Leftrightarrow \quad \langle u_C - \rho(A(u_C) - x_0) - u_C, v - u_C\rangle \leq 0 \text{ for all } v \in C
\end{aligned}$$

Since C is a nonempty closed convex, it follows from the characterization of the projection in Theorem 8.7 that

$$u_C \text{ solves } (E_C) \quad \Leftrightarrow \quad u_C = \pi_C\left(u_C - \rho(A(u_C) - x_0)\right)$$

We define

$$\begin{array}{rccc} T: & C & \to & C \\ & u & \mapsto & \pi_C\left(u - \rho(A(u) - x_0)\right) \end{array}$$

For $u_1, u_2 \in H$, since π_C is 1-Lipschitz, we have that

$$\|T(u_1) - T(u_2)\| = \|\pi_C(X_1) - \pi_C(X_2)\| \leq \|X_1 - X_2\|$$

where $X_i = u_i - \rho(A(u_i) - x_0)$. Then, as in the case $C = H$, we get that for a suitable $\rho > 0$, then T is a contracting map. Since C is closed in H complete, then C is complete and Picard's/Banach's fixed point theorem yields the existence and uniqueness of the fixed point u_C, that is the existence and uniqueness of the solution to (E_C). $\square$

Part 4

Sobolev spaces and applications to elliptic PDEs

CHAPTER 9

Dimension 1 and main properties

1. Motivation

1.1. The problem. Take $f \in C^0([0,1])$. Is there $u \in C^2([0,1])$ such that

$$\left\{ \begin{array}{ll} -u'' + u = f & \text{in } [0,1] \\ u(0) = u(1) = 0 \end{array} \right. ? \tag{E}$$

1.2. Weak formulation of (E). Define

$$X := \{u \in C^1([0,1]) /\, u(0) = u(1) = 0\}.$$

Multiply (E) by $\varphi \in X$ and integrate by parts. We get that

$$\begin{aligned} \int_0^1 f\varphi\, dx &= \int_0^1 (-u'' + u)\varphi\, dx = -\int_0^1 u''\varphi\, dx + \int_0^1 u\varphi\, dx \\ &= -[u'\varphi]_0^1 + \int_0^1 u'\varphi'\, dx + \int_0^1 u\varphi\, dx = \int_0^1 (u'\varphi' + u\varphi)\, dx \end{aligned}$$

Conversely, assume that $u \in C^2([0,1])$ is such that $u(0) = u(1) = 0$ (that is $u \in X$) and

$$\int_0^1 (u'\varphi' + u\varphi)\, dx = \int_0^1 f\varphi\, dx \text{ for all } \varphi \in X.$$

Then, with the computations above, we get that $\int_0^1 (-u'' + u - f)\varphi\, dx = 0$ for all $\varphi \in X$. Since $-u'' + u - f \in C^0([0,1]) \subset L^2((0,1))$ and $C_c^1((0,1)) \subset X$ is dense in $L^2((0,1))$, we get that $-u'' + u - f = 0$, and then u is a solution to (E). We then have proved the following:

Let $u \in C^2([0,1])$. Then

(25)

$$u \text{ solves } (E) \Leftrightarrow \left\{ u \in X \text{ and } \int_0^1 (u'\varphi' + u\varphi)\, dx = \int_0^1 f\varphi\, dx \text{ for all } \varphi \in X \right\}$$

We will define rigorously the weak formulation, but keep in mind that it is essentially (25).

1.3. A (not successful) tentative to solve (E). Define $B(u,v) := \int_0^1 (u'v' + uv)\, dx$ for $u, v \in X$. Then (25) rewrites

$$B(u, \varphi) = T(\varphi) \text{ for all } \varphi \in X$$

where $T(\varphi) = \int_0^1 f\varphi\, dx$. It is natural to use Riesz's represention to find solutions to this equation, since B is a scalar product on X... **But** (X, B) is not a Hilbert space !!! It is not complete for the norm $\sqrt{B(\cdot,\cdot)}$. The space $X \subset C^1([0,1])$ is not flexible enough.

1.4. What would be a good space on which working? In order to define $B(u,v) = \int_0^1 (u'v' + uv)\,dx$, we need a space of functions u such that $u \in L^2((0,1))$, u' is defined and $u' \in L^2((0,1))$. This is going to be the definition of Sobolev spaces.

In the sequel, $I = (a,b)$ where $-\infty < a < b < +\infty$ is a bounded open interval.

2. The Sobolev space $H^1(I)$

2.1. The definition of the Sobolev space.

2.1.1. *A useful Lemma.* For $u, v \in C^1([a,b])$ such that u and v vanish on $\partial[a,b] = \{a,b\}$, we have that

$$\int_a^b u'v\,dx = [uv]_a^b - \int_a^b uv'\,dx = -\int_a^b uv'\,dx \tag{26}$$

2.1.2. *Definition.*

Definition 2.1 (1D Sobolev space). *We define*

$$H^1(I) := \{u \in L^2(I) / \exists v \in L^2(I) \text{ s.t. } \int_a^b u\varphi'\,dx = -\int_a^b v\varphi\,dx \,\forall \varphi \in C_c^1(I)\}$$

2.1.3. *Uniqueness of the derivative.*

Proposition 9.1. *Let $u \in H^1(I)$ be as in Definition 2.1. Then $v \in L^2(I)$ is unique. We then define $u' := v$: it is the "weak derivative" of u, or "derivative in the weak sense".*

Proof. Assume that $v, w \in L^2$ are possible. Then for all $\varphi \in C_c^1(I)$, we have that $\int_a^b v\varphi\,dx = \int_a^b w\varphi\,dx$ so $\int_a^b (v-w)\varphi\,dx = 0$. Since $v - w \in L^2(I)$ and $C_c^1(I)$ is dense in $L^2(I)$, then $\int_a^b (v-w)\varphi\,dx = 0$ for all $\varphi \in L^2(I)$. Taking $\varphi = v - w$ yields $v = w$ a.e. $\square$

2.2. Examples.

- $C^1([0,1]) \subset H^1(I)$. Moreover, the weak and strong (i.e. usual) derivatives coincide.
- $C^1((0,1)) \notin H^1(I)$!!! It is because of the integrability condition. For instance, $x \mapsto x^{-1}$ is in $C^1((0,1))$, but it is not in L^2.
- Define the function $u(x) := x_+ = \max\{0, x\}$ for $x \in (-1,1)$. Then $u \in H^1(-1,1)$ and $u' = \mathbf{1}_{(0,1)}$. This is a straightforward computation. Clearly $u \in L^2(-1,1)$: Take $\varphi \in C_c^1(-1,1)$, we have that

$$\begin{aligned} \int_{-1}^1 u\varphi'\,dx &= \int_0^1 x\varphi'(x)\,dx = [x\varphi(x)]_0^1 - \int_0^1 x'\varphi(x)\,dx \\ &= -\int_0^1 \varphi\,dx = -\int_{-1}^1 \mathbf{1}_{(0,1)}\varphi\,dx \end{aligned}$$

  Since $\mathbf{1}_{(0,1)} \in L^2(-1,1)$, we have that $u \in H^1(-1,1)$ and $u' = \mathbf{1}_{(0,1)}$.
- But $\mathbf{1}_{(0,1)} \notin H^1(-1,1)$!!! Assume it is, and let $f \in L^2(-1,1)$ be its derivative then. For any $\varphi \in C_c^1(-1,1)$, we would have that

$$\int_{-1}^1 f\varphi\,dx = -\int_{-1}^1 \mathbf{1}_{(0,1)}\varphi'\,dx = -\int_0^1 \varphi'\,dx = -(\varphi(1) - \varphi(0)) = \varphi(0)$$

This is impossible since, morally, on the left this makes sense for φ that is defined almost everywhere (so you can change the value at 0), and on the right, the value at 0 is essential.

Exercise 30: Show that $x \mapsto |x|$ is in $H^1(-1,1)$. Give an optimal condition on $\alpha \in \mathbb{R}$ to have that $x \mapsto x^{-\alpha} \in H^1(0,1)$.

3. Structure of $H^1(I)$

3.1. Linear structure.

Proposition 9.2. *The space $H^1(I)$ is a vector space and the derivation is linear. In other words, for any $u, v \in H^1(I)$ and $\lambda, \mu \in \mathbb{R}$, then $\lambda u + \mu v \in H^1(I)$ and*

$$(\lambda u + \mu v)' = \lambda u' + \mu v'.$$

3.2. Hilbert structure.

Definition 3.1. *For $u, v \in H^1(I)$, define the scalar product*

$$\langle u, v \rangle_{H^1} := \int_I (u'v' + uv)\, dx.$$

Proposition 9.3. *The space $(H^1(I), \langle \cdot, \cdot \rangle_{H^1})$ is a Hilbert space.*

Proof. We just need to prove that it is complete for the norm

$$u \mapsto \|u\|_{H^1} = \sqrt{\int_I ((u')^2 + u^2)\, dx}.$$

Let $(u_n)_n$ be a Cauchy sequence in $H^1(I)$. Then:

- (u_n) and (u_n') are Cauchy sequences in $L^2(I)$ that is complete
- Let $u, v \in L^2(I)$ be their respective limits in $L^2(I)$
- For any $\varphi \in C^1_c(I)$, we have that
$$\int_I u_n \varphi'\, dx = -\int_I u_n' \varphi\, dx.$$
Letting $n \to +\infty$ yields $u \in H^1(I)$ and $u' = v$.
- We have that $u_n \to u$ in $H^1(I)$.

So $H^1(I)$ is complete. □

4. Regularity and definition of $H^1_0(I)$

4.1. Vanishing derivative. We start with a natural, although not trivial, result:

Proposition 9.4. *Let $u \in H^1(I)$ be such that $u' = 0$. Then there exists $C \in \mathbb{R}$ such that $u(x) = C$ a.e.*

Proof. For convenience, we take $I = (0,1)$. Since $u' = 0$, then for any $\varphi \in C^1_c(0,1)$, we have that

$$\int_0^1 u\varphi'\, dx = 0.$$

We would like to have an arbitrary function $\psi \in C^1_c(0,1)$ in place of φ'. Is it possible? Well, if $\psi = \varphi'$, we would have that

$$\int_0^1 \psi\, dx = \int_0^1 \varphi'\, dx = \varphi(1) - \varphi(0) = 0.$$

So not any ψ can be written as φ' with $\varphi \in C_c^1(0,1)$.

Let $\theta \in C^1(\mathbb{R})$ be a function such that $\theta(x) = 0$ for $x < 1/3$ and $\theta(x) = 1$ for $x > 2/3$. For $\psi \in C_c^1(0,1)$, define

$$\varphi(x) := \int_0^x \psi(t)\,dt - \theta(x)\int_0^1 \psi(t)\,dt \text{ for all } x \in (0,1).$$

We claim that $\varphi \in C_c^1(0,1)$. We prove the claim:

- $\varphi \in C^1$ is fine
- Since ψ has compact support, then there exists $0 < \alpha < \beta < 1$ such that $\psi(x) = 0$ for $x < \alpha$ or $x > \beta$. With no loss of generality, we can assume that $\alpha < 1/3$ and $\beta > 2/3$.
- For $0 < x < \alpha$, we get that $\varphi(x) = 0$
- For $\beta < x < 1$, we get that

$$\varphi(x) = \int_0^x \psi(t)\,dt - 1\cdot\theta(x)\int_0^1 \psi(t)\,dt = \int_0^1 \psi(t)\,dt - \theta(x)\int_0^1 \psi(t)\,dt = 0.$$

- So $\varphi \in C_c^1(0,1)$

We are now in position to prove the proposition. Given $\psi C_c^1(0,1)$, we define $\varphi \in C_c^1(0,1)$ as above, and we get that

$$\begin{aligned}
0 &= \int_0^1 u\varphi'\,dx = \int_0^1 u(x)\left(\psi(x) - \theta'(x)\int_0^1 \psi(t)\,dt\right)dx \\
&= \int_0^1 u(x)\psi(x)\,dx - \left(\int_0^1 u\theta'\,dx\right)\left(\int_0^1 \psi\,dx\right) \\
&= \int_0^1 u(x)\psi(x)\,dx - \left(\int_0^1 C\psi\,dx\right) \text{ with } C = \int_0^1 u\theta'\,dx \\
&= \int_0^1 (u-C)\psi(x)\,dx
\end{aligned}$$

Since $u - C \in L^2(0,1)$ and $C_c^1(0,1)$ is dense in L^2, this is valid for any $\psi \in L^2(0,1)$. Taking $\psi = u - C$ yields $u - C = 0$ a.e. $\square$

From this, we have a regularity result that is specific to the 1D case.

4.2. Regularity.

THEOREM 9.5. *The space $H^1(I)$ is continuously embedded in $C^{0,1/2}(\bar{I})$. More precisely, for all $u \in H^1(I)$, there exists a unique $\bar{u} \in C^{0,1/2}(\bar{I})$ such that $u = \bar{u}$ a.e.*

PROOF. For convenience we take $I = (0,1)$. We would like to write an identity like $u(x) - u(y) = \int_x^y u'(t)\,dt$... but $u \notin C^1$!!! Define

$$\tilde{u}(x) := \int_0^x u'(t)\,dt \text{ for all } x \in [0,1].$$

The function $\tilde{u}$ is well defined.

We claim that $\tilde{u} \in C^{0,1/2}([0,1])$.

We prove the claim. For $x, y \in [0,1]$, with Cauchy-Schwartz inequality, we have that

$$|u(x) - u(y)| = \left| \int_x^y u'(t)\, dt \right| \leq \sqrt{|\int_x^y dt|} \sqrt{\int_x^y (u'(t))^2\, dt} \leq \|u'\|_2 \sqrt{|x-y|}$$

So $\tilde{u} \in C^{0,1/2}([0,1])$. Moreover, since $\tilde{u}(0) = 0$, we have that $\|u\|_\infty \leq \|u'\|_2$, and then

$$\|\tilde{u}|_{C^{0,1/2}} \leq 2\|u'\|_2.$$

We claim that $\tilde{u} \in H^1(0,1)$ and that $\tilde{u}' = u'$ a.e.

We prove the claim. Take $\varphi \in C_c^1(0,1)$. We have with Fubini's theorem that

$$\begin{aligned} \int_0^1 \tilde{u}\varphi'\, dx &= \int_0^1 \left(\int_0^x u'(t)\, dt \right) \varphi(x)\, dx \\ &= \int_0^1 \int_0^1 \mathbf{1}_{t<x} u'(t)\varphi'(x)\, dxdt = \int_0^1 u'(t) \left(\int_t^1 \varphi'(x)\, dx \right) dt \\ &= \int_0^1 u'(t) \left(\varphi(1) - \varphi(t)\right) dt = -\int_0^1 u'\varphi\, dt \end{aligned}$$

So $\tilde{u} \in H^1(0,1)$ and $\tilde{u}' = u'$. This proves the claim.

We are in position to conclude. We have that $(u - \tilde{u})' = 0$, so there exists $C \in \mathbb{R}$ such that $u - \tilde{u} = C$ a.e. Define $\bar{u} := \tilde{u} + C$. So $\bar{u} \in C^{0,1/2}([0,1])$ and $u(x) = \bar{u}(x)$ a.e.

Regarding uniqueness, if another function v is suitable, then $v(x) = \bar{u}(x)$ a.e., and then everywhere since they are continuous. Concerning continuity of $u \mapsto \bar{u}$, we have that

$$|C| \quad = \quad \|C\|_2 = \|u - \tilde{u}\|_2 \leq \|u\|_2 + \|\tilde{u}\|_2 \leq \|u\|_2 + 2\|u'\|_2 \leq 2\|u\|_{H^1}$$

Therefore

$$\|\bar{u}\|_{C^{0,1/2}} = \|\tilde{u} + C\|_{C^{0,1/2}} \leq 3\|u\|_{H^1},$$

and then $u \mapsto \bar{u}$ is continuous. □

4.3. A corollary.

Proposition 9.6. *Let $u, v \in H^1(I)$. Then $uv \in H^1(I)$ and $(uv)' = u'v + uv'$.*

Note that since $u, v \in L^\infty(I)$, then $uv, u'v, uv' \in L^2(I)$.

Exercise 31: Prove this Proposition when $v \in C^1(I)$.

4.4. The space $H_0^1(I)$. In the sequel, we will always assimilate $u \in H^1(I)$ to its representative $\bar{u} \in C^0$.

Definition 4.1. *Define*

$$H_0^1(a,b) := \{u \in H^1(a,b) \text{ such that } u(a) = u(b) = 0\},$$

where u is asimilated to its continuous representative.

Proposition 9.7. *The space $H_0^1(I)$ is a closed linear subspace of $H^1(I)$. In particular, it is a Hilbert space with the scalar product $\langle \cdot, \cdot \rangle_{H^1}$.*

Proof. Define

$$\begin{cases} T: & H^1(a,b) & \to & \mathbb{R}^2 \\ & u & \mapsto & (\bar{u}(a), \bar{u}(b)) \end{cases}$$

Then T is linear and, for any norm on $\mathbb{R}^2$, it is continuous since $u \mapsto \bar{u}$ is continuous. Since $H_0^1(a,b) = T^{-1}(0,0)$, it is both linear and closed. □

5. Solution of the exercises of Chapter 9

Solution of Exercise 30: Note that $x \mapsto |x| \in L^2(-1,1)$. Take $\varphi \in C_c^1(-1,1)$. We have that'

$$\begin{aligned} \int_{-1}^{1} |x|\varphi'(x)\,dx &= \int_{-1}^{0} (-x)\varphi'(x)\,dx + \int_0^1 x\varphi'(x)\,dx \\ &= [-x\varphi(x)]_{-1}^{0} - \int_{-1}^{0} (-x)'\varphi(x)\,dx + [x\varphi(x)]_0^1 - \int_0^1 x'\varphi(x)\,dx \\ &= 0 \times \varphi(0) + \varphi(-1) + \int_{-1}^{0} \varphi(x)\,dx + \varphi(1) - x\varphi(0) - \int_0^1 \varphi\,dx \\ &= -\int_{-1}^{1} g\varphi\,dx \text{ since } \varphi(-1) = \varphi(1) = 0 \end{aligned}$$

where $g(x) = -1$ for $x < 0$ and $g(x) = 1$ for $x > 0$. Since $g \in L^2(-1,1)$, we get that $x \mapsto |x| \in H^1(-1,1)$ and that $u' = g$.

Set $f(x) = x^{-\alpha}$ for $0 < x < 1$. Assume that $\alpha \neq 0$. Since f is C^1, the integration by parts formula works. We just need to check integrability. We have that $\int_0^1 f(x)^2\,dx = \int_0^1 x^{-2\alpha}\,dx < \infty$ iff $-2\alpha > -1$, that is $\alpha < 1/2$. We have that $\int_0^1 f'(x)^2\,dx = \alpha^2 \int_0^1 x^{-2\alpha-2}\,dx < \infty$ iff $-2\alpha - 2 > -1$, that is $\alpha < -1/2$. So for $\alpha \neq 0$, $f \in H^1(0,1)$ iff $\alpha < -1/2$. □

Solution of Exercise 31: Note that $|(uv)(x)| \leq \|u\|_\infty |v(x)|$ for a.e. $x \in I$, so since $u \in L^\infty$ and $v \in L^2(I)$, we get that $uv \in L^2(I)$. This is similar for $u'v$ and uv'. Take $\varphi \in C_c^1(I)$. Then $v\varphi \in C_c^1(I)$ and we have that

$$\begin{aligned} \int_I uv\varphi'\,dx &= \int_I u((v\varphi)' - v'\varphi)\,dx = \int_I u(v\varphi)'\,dx - \int_I uv'\varphi\,dx \\ &= -\int_I u'(v\varphi)\,dx - \int_I uv'\varphi\,dx = -\int_I (u'v + uv')\varphi\,dx \end{aligned}$$

Since $u'v + uv' \in L^2(I)$, we get the result. The general case $v \in H^1(I)$ uses the density of C^1-functions in $H^1(I)$. □

CHAPTER 10

Applications to 1D boundary value problems

This is the main point in Sobolev spaces

1. Weak solutions

Arguing as in the motivation of this chapter, we define properly weak solutions:

DEFINITION 1.1. *Fix $f \in L^2(0,1)$. We say that $u \in H_0^1(0,1)$ is a weak solution to*

$$\begin{cases} -u'' + u = f & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \qquad (E)$$

if

$$\int_0^1 (u'\varphi' + u\varphi)\, dx = \int_0^1 f\varphi\, dx \text{ for all } \varphi \in H_0^1(0,1) \qquad (E_w).$$

We say that (E_w) is the "weak formulation" of (E).

2. Application via Riesz's theorem

2.1. Main theorem.

THEOREM 10.1. *Let $f \in L^2(0,1)$. Then there exists a unique weak solution $u \in H_0^1(0,1)$ to (E). Moreover, if $f \in C^0([0,1])$, then $u \in C^2([0,1])$ is a classical solution to (E).*

So we have solved our problem. This is basic, but a good first example.

2.2. Proof of the existence of the weak solution. Define $T_f : H_0^1(0,1) \to \mathbb{R}$ as $T_f(\varphi) = \int_0^1 f\varphi\, dx$ for all $\varphi \in H_0^1(0,1)$. Then it is continuous on $H_0^1(0,1)$ with the norm $\|\cdot\|_{H^1}$. And $u \in H_0^1(0,1)$ is a weak solution to (E) iff

$$\langle u, \varphi\rangle_{H^1} = T_f(\varphi) \text{ for all } H_0^1(0,1).$$

Then existence and uniqueness follows immediately from Riesz representation theorem since $H_0^1(0,1)$ is a Hilbert space (remember, it is not the case for $C^1([0,1])$). This proves existence and uniqueness.

2.3. Proof of regularity. We assume that $f \in C^0([0,1])$. Since we have a weak solution, we have that

$$\int_0^1 u'\varphi'\, dx = \int_0^1 (f-u)\varphi\, dx \text{ for all } \varphi \in H_0^1(0,1).$$

Since $u \in H_0^1(0,1)$, it is continuous and $f - u \in C^0([0,1])$. So, there exists $w \in C^2([0,1])$ such that

$$-w'' = u - f \text{ in } [0,1] \text{ and } w(0) = w(1) = 0.$$

We then have that

$$\int_0^1 u'\varphi'\,dx = -\int_0^1 w''\varphi\,dx = \int_0^1 w'\varphi'\,dx \text{ for all } H_0^1(0,1).$$

And then $\int_0^1 (u'-w')\varphi'\,dx = 0$ for all $\varphi \in H_0^1(0,1)$. Setting $h := u' - w' \in L^2(0,1)$, this yields $h \in H^1(0,1)$ and $h' = 0$, so $h = C$ a.e for some $C \in \mathbb{R}$. So we have that

$$u' - w' = C \;\Rightarrow\; (u - w - Cx)' = 0 \text{ with } u - w - Cx \in H^1(0,1).$$

It then follows form our regularity result that there exists $D \in \mathbb{R}$ such that $u - w - Cx = D$ a.e $x \in (0,1)$, and then $u(x) = w(x) + Cx + D$ for a.e. $x \in (0,1)$. Since $w \in C^2$, then $u \in C^2([0,1])$ (or at least has a representative). We even have that $C = D = 0$ since u, w coincide at 0 and 1.

2.4. Proof of the existence of a classical solution. Since $u \in C^2([0,1])$ is a weak solution, arguing as in the motivation of Chapter 9, we get that it is a classical solution.

3. Further...

We are going to solve some problems of the same flavour

3.1. With a potential a.

THEOREM 10.2. *Let $f \in L^2(0,1)$ and $a \in L^\infty((0,1))$. We assume that there exists $c > 0$ such that*

$$\int_0^1 \left((u')^2 + a(x)u^2\right)\,dx \geq c\|u\|_{H_1}^2 \textit{ for all } u \in H_0^1(0,1).$$

Then there exists a unique weak solution $u \in H_0^1(0,1)$ to

$$\begin{cases} -u'' + a(x)u = f & \textit{in } (0,1) \\ u(0) = u(1) = 0. \end{cases} \tag{E_1}$$

Moreover, if $a, f \in C^0([0,1])$, then $u \in C^2([0,1])$ is a classical solution to (E_1).

PROOF. We say that $u \in H_0^1(0,1)$ is a weak solution to (E_1) if

$$\int_0^1 (u'\varphi' + a(x)u\varphi)\,dx = \int_0^1 f\varphi\,dx \text{ for all } \varphi \in H_0^1(0,1).$$

Define the bilinear form

$$B(u,v) := \int_0^1 (u'v' + a(x)uv)\,dx \text{ for } u, v \in H_0^1(0,1).$$

Then

- B is continuous on $H_0^1(0,1) \times H_0^1(0,1)$ (see Exercise 13)
- B is coercive
- The weak formulation of (E_1) reads $B(u,\varphi) = \int_0^1 f\varphi\,dx$ for all $\varphi \in H_0^1(0,1)$
- The Lax-Milgram theorem 8.17 applies and yields existence and uniqueness

An alternative is to consider B as a scalar product and to prove that $(H_0^1(0,1), B)$ is a Hilbert space. This amounts to prove completeness: this holds since coercivity and continuity yields that the norm associated to B is equivalent to $\|\cdot\|_{H^1}$.

Exercise 32: Prove that $(H_0^1(0,1), B)$ is a Hilbert space .

The regularity goes as in the proof of Theorem 10.1. $\square$

3.2. A non symmetric problem.

THEOREM 10.3. *Let $f \in L^2(0,1)$ and $\epsilon \in \mathbb{R}$. Then there exists $\epsilon_0 > 0$ such that if $|\epsilon| < \epsilon_0$, hen there exists a unique weak solution $u \in H_0^1(0,1)$ to*

$$\begin{cases} -u'' + \epsilon u' + u = f & \text{in } (0,1) \\ u(0) = u(1) = 0. \end{cases} \tag{E_2}$$

Moreover, if $a, f \in C^0([0,1])$, then $u \in C^2([0,1])$ is a classical solution to (E_1).

PROOF. We say that $u \in H_0^1(0,1)$ is a weak solution to (E_2) if

$$\int_0^1 (u'\varphi' + \epsilon u'\varphi + u\varphi)\, dx = \int_0^1 f\varphi\, dx \text{ for all } \varphi \in H_0^1(0,1).$$

Define the bilinear form

$$B(u,v) := \int_0^1 (u'v' + \epsilon u'v + uv)\ dx \text{ for } u, v \in H_0^1(0,1).$$

Then

- B is continuous on $H_0^1(0,1) \times H_0^1(0,1)$ (see Exercise 13)
- B is coercive when $|\epsilon| < \epsilon_0$ with $\epsilon_0 > 0$ to be determined
- The weak formulation of (E_2) reads $B(u,\varphi) = \int_0^1 f\varphi\, dx$ for all $\varphi \in H_0^1(0,1)$
- The Lax-Milgram theorem 8.17 applies and yields existence and uniqueness

Exercise 33: Prove all these points.

The regularity is taking a bit more efforts. Assume that $f \in C^0([0,1])$. Let $u \in H_0^1(0,1)$ be a weak solution to (E_2).
A "formal" argument. we would like to write that $u'' = g := \epsilon u' + u - f \in L^2(0,1)$. But we cannot argue as in (E) take a C^2-function such that its second derivative is an L^2-function. Let us take a "primitive" of g: as in the proof of Theorem 9.5, we define

$$v(x) := \int_0^x g(t)\, dt \text{ for } x \in [0,1].$$

Then $v \in H^1(0,1)$ and $v' = g$, so that

$$u'' = v' \ \Rightarrow\ (u' - v)' = 0\ \Rightarrow u' - v = C \in \mathbb{R}$$

(this last argument will have to be made rigorous). So $u' = v + C \in C^0([0,1])$. And then $g = \epsilon u' + u - f \in C^0([0,1])$ and we can argue as in the regularity of (E).

Rigorous proof of the regularity. Take $v \in H_0^1(0,1)$ as above, so that $v' = g$. Take $\varphi \in H_0^1(0,1)$: since u is a weak solution to (E_2), we have that

$$\int_0^1 u'\varphi'\, dx = \int_0^1 (f - \epsilon u' - u)\varphi\, dx = -\int_0^1 g\varphi\, dx.$$

Since $g = v'$, we then have that

$$\int_0^1 u'\varphi'\, dx = -\int_0^1 v'\varphi\, dx = \int_0^1 v\varphi'\, dx \ \Rightarrow\ \int_0^1 (u' - v)\varphi'\, dx = 0.$$

Set $U := u' - v$: we have that $U \in L^2(0,1)$ and $\int_0^1 U\varphi'\,dx = 0$ for all $\varphi \in H_0^1(0,1)$, that is $U \in H_0^1(0,1)$ and $U' = 0$. It then follows from the regularity Theorem 9.5 that $U = C$ a.e. for some $C \in \mathbb{R}$. So $u' = v + C \in C^0([0,1])$, and then $g \in C^0([0,1])$. Then we can argue as for (E) and get that $u \in C^2([0,1])$ (in the sense that it admits a C^2-representative).

Arguing as in the motivation of Chapter 9, we then get that u is a classical solution to (E_2). □

3.3. Non-homogenous boundary conditions. Here, we must re-think the notion of weak solution. Consider $a, f \in C^0([0,1])$ and assume that there exists $u \in C^2([0,1])$ solution to

$$\begin{cases} -u'' + a(x)u = f & \text{in } (0,1) \\ u(0) = \alpha \text{ and } u(1) = \beta. \end{cases} \tag{E_4}$$

Exercise 34: Prove that for all $\varphi \in C_c^1((0,1))$, we have that

$$\int_0^1 (u'\varphi' + a(x)u\varphi)\,dx = \int_0^1 f\varphi\,dx.$$

Since u does not vanish on the boundary, we work on the space H^1. We say that $u \in H^1(0,1)$ is a weak solution to (E_4) if $u(0) = \alpha$, $u(1) = \beta$ and

$$\int_0^1 (u'\varphi' + a(x)u\varphi)\,dx = \int_0^1 f\varphi\,dx \text{ for all } \varphi \in H_0^1(0,1).$$

Note that we take $\varphi \in H_0^1$, that is zero boundary condition.

THEOREM 10.4. *Let $f \in L^2(0,1)$ and $a \in L^\infty((0,1))$. Fix $\alpha, \beta \in \mathbb{R}$. We assume that there exists $c > 0$ such that*

$$\int_0^1 \left((u')^2 + a(x)u^2\right)\,dx \geq c\|u\|_{H_1}^2 \text{ for all } u \in H^1(0,1).$$

Then there exists a unique weak solution $u \in H^1(0,1)$ to (E_4). Moreover, if $a, f \in C^0([0,1])$, then $u \in C^2([0,1])$ is a classical solution to (E_4).

PROOF. To characterize the boundary condition, we are going to work on $C := \{u \in H^1(0,1)/\, u(0) = \alpha \text{ and } u(1) = \beta\}$.

Exercise 35: Show that C is nonempty, convex and closed.

Define the bilinear form

$$B(u,v) := \int_0^1 (u'v' + a(x)uv)\,dx \text{ for } u, v \in H^1(0,1).$$

Then

- B is continuous on $H^1(0,1) \times H^1(0,1)$
- B is coercive
- The weak formulation of (E_4) reads $B(u,\varphi) = \int_0^1 f\varphi\,dx$ for all $\varphi \in H_0^1(0,1)$ (and not $H^1(0,1)$)
- The Lax-Milgram theorem 8.17 applies and yields existence and uniqueness

Exercise 36: Apply the Lax-Milgram theorem to the convex C, the bilinear form B and the linear form T_f. Show that the solution $u \in H_0^1(0,1)$ you get is a weak solution to (E_4). Show that $u \in C^2([0,1])$ and it is a classical solution to (E_4).

An alternative proof is to make the change of function $\tilde{u}(x) = u(x) - a - (b-a)x$ for $x \in [0,1]$. Then $\tilde{u}$ and we rewrite (E_4) with $\tilde{u}$: this yields a problem like (E), and the existence +uniqueness hold. □

There are many other types of boundary conditions: Neumann ($u'(0) = u'(1) = 0$), mix Dirichlet-Neumann ($u(0) = 0$ and $u'(1) = 0$), homogenous or not.

3.4. A Neumann boundary problem. Consider $a, f \in C^0([0,1])$ and assume that there exists $u \in C^2([0,1])$ solution to

$$\begin{cases} -u'' + a(x)u = f & \text{in } (0,1) \\ u'(0) = u'(1) = 0. \end{cases} \tag{E_5}$$

Exercise 37: Prove that for all $\varphi \in C^1([0,1])$ (this is a different space!!!), we have that

$$\int_0^1 (u'\varphi' + a(x)u\varphi)\, dx = \int_0^1 f\varphi\, dx.$$

Since u does not vanish on the boundary, we work on the space H^1. We say that $u \in H^1(0,1)$ is a weak solution to (E_5) if

$$\int_0^1 (u'\varphi' + a(x)u\varphi)\, dx = \int_0^1 f\varphi\, dx \text{ for all } \varphi \in H^1(0,1).$$

Here we take H^1, and we do not impose any boundary condition. We will see that this is enough indeed.

Theorem 10.5. *Let $f \in L^2(0,1)$ and $a \in L^\infty((0,1))$. We assume that there exists $c > 0$ such that*

$$\int_0^1 \left((u')^2 + a(x)u^2\right)\, dx \geq c\|u\|_{H_1}^2 \text{ for all } u \in H^1(0,1).$$

Then there exists a unique weak solution $u \in H^1(0,1)$ to (E_5). Moreover, if $a, f \in C^0([0,1])$, then $u \in C^2([0,1])$ is a classical solution to (E_5).

Proof. It goes through the following steps, each being an exercise:

Exercise 38: Using Lax-Milgram's theorem, prove the existence of weak solutions.

From now on, we assume that $a, f \in C^0([0,1])$ and $u \in H^1(0,1)$ is a weak solution to (E_5).

Exercise 39: Prove that $u \in C^2([0,1])$.

Exercise 40: Take $\varphi \in C^1([0,1])$. Prove that

$$\int_0^1 (-u'' + au)\varphi\, dx = \int_0^1 f\varphi\, dx + (-u'(1)\varphi(1) + u'(0)\varphi(0))$$

Exercise 41: Taking some suitable φ above, prove that $-u'' + au = f$ in $[0,1]$.

Exercise 42: Prove that $u'(0) = u'(1) = 0$.

□

3.5. A mixed boundary condition problem.

THEOREM 10.6. *Let $a, f \in C^0([0,1])$ be two functions. We assume that there exists $c > 0$ such that*

$$\int_0^1 \left((u')^2 + a(x)u^2\right)\, dx \geq c\|u\|_{H_1}^2 \text{ for all } u \in H^1(0,1).$$

Then there exists a unique solution $u \in C^2([0,1])$ to

$$\begin{cases} -u'' + a(x)u = f & \text{in } (0,1) \\ u(0) = 0 \text{ and } u'(1) = 0. \end{cases}$$

PROOF. Do it yourself! □

4. A little disappointment...

The methods above are very classical, and they are mostly used for higher-dimensional problem. Sometimes, in 1D, the theory of ODEs is enough to get solutions.

Here is an example. Take $k, a, f \in C^0([0,1])$ and investigate the existence of solutions to

$$\begin{cases} -u'' + k(x)u' + a(x)u = f & \text{in } (0,1) \\ u(0) = u(1) = 0. \end{cases} \tag{E_5}$$

THEOREM 10.7. *In addition to the hypothesis above, assume that $a(x) > 0$ for all $x \in [0,1]$. Then there exists a unique solution $u \in C^2([0,1])$ to (E_5).*

PROOF. It follows from standard theory on ODEs that there exists $u_0 \in C^2([0,1])$ that is a solution to $-u_0'' + k(x)u_0' + a(x)u_0 = f$ in $[0,1]$. For instance, one can impose the value of $u_0(0)$ and $u_0'(0)$ to get existence and uniqueness.

We make the change of function $v := u - u_0$, so that

$$\begin{cases} -v'' + k(x)v' + a(x)v = 0 & \text{in } (0,1) \\ v(0) = -u_0(0) \text{ and } v(1) = -u_0(1). \end{cases} \tag{E_5'}$$

We define $\mathcal{S} := \{C^2 - \text{Solutions to } -u'' + k(x)u' + a(x)u = 0 \text{ in } [0,1]\}$. Linear theory for ODEs yields that $\mathcal{S}$ is 2-dimensional. Define

$$\begin{cases} T: & \mathcal{S} & \to & \mathbb{R}^2 \\ & v & \mapsto & (v(0), v(1)) \end{cases}$$

If we prove that T is a bijection, we are done since then $u = T^{-1}(-u_0(0), -u_0(1))$ is a solution to $(E_5)'$. Since T is linear and $\dim(\mathcal{S}) = \dim(\mathbb{R}^2)$, it is enough to prove that T is injective, that is

$$v \in C^2([0,1]) \text{ and } \left\{ \begin{array}{ll} -v'' + k(x)v' + a(x)v = 0 & \text{in } (0,1) \\ v(0) = v(1) = 0 \end{array} \right\} \Rightarrow v \equiv 0.$$

Which is exactly Step 2 in the application of the method of continuity in Chapter 7.

So T is bijective and we get a solution to $(E_5)'$, yielding a solution to (E) □

Disappointing, isn't it? It should not be for at least two reasons:

- The method works for a large class of non-continuous coefficients;
- This is an introduction to the PDEs in d-dimensions, for which the is no Cauchy-Lipschitz theorem.

5. Solution of the exercises of Chapter 10

Solution of Exercise 32: The bilinearity of B is clear. The definition of the constant c yields

$$B(u,u) = \int_0^1 \left((u')^2 + a(x)u^2\right)\, dx \geq c\|u\|_{H_1}^2 \text{ for all } u \in H_0^1(0,1).$$

Therefore, given $u \in H_0^1$, we have that $B(u,u) \geq 0$, with equality only if $u \equiv 0$. Therefore B is a scalar product. Moreover,

$$B(u,u) = \int_0^1 \left((u')^2 + a(x)u^2\right)\, dx \leq \int_0^1 (u')^2\, dx + \|a\|_\infty \int u^2\, dx \leq M\|u\|_{H^1}^2$$

for all $u \in H_0^1(0,1)$ with $M := 1 + \|a\|_\infty$. So we then have that

$$\sqrt{c}\|u\|_{H^1} \leq \sqrt{B(u,u)} \leq \sqrt{M}\|u\|_{H^1} \text{ for all } u \in H_0^1.$$

So the norms $\|\cdot\|_B = \sqrt{B(\cdot,\cdot)}$ and $\|\cdot\|_{H^1}$ are equivalent, and since $(H_0^1, \|\cdot\|_{H^1})$ is complete, so is $(H_0^1, \|\cdot\|_B)$. Then (H_0^1, B) is a Hilbert space. $\square$

Solution of Exercise 33:

- There is no difficulty to prove that B is bilinear. For $u, v \in H_0^1(0,1)$, we have that

$$\begin{aligned}
|B(u,v)| &= \left|\int_0^1 (u'v' + \epsilon u'v + uv)\, dx\right| \\
&\leq \int_0^1 |u'|\cdot|v'|\, dx + |\epsilon| \int_0^1 |u'|\cdot|v|\, dx + \int_0^1 |u|\cdot|v|\, dx \\
&\leq \|u'\|_2\|v'\|_2 + |\epsilon|\cdot\|u'\|_2\|v\|_2 + \|u\|_2\|v\|_2 \leq (1+|\epsilon|+1)\|u\|_{H^1}\|v\|_{H^1}
\end{aligned}$$

  So, according to the first point of Exercise 13, B is continuous.
- For $u \in H_0^1(0,1)$, we have that

$$B(u,u) = \int_0^1 (u')^2\, dx + \epsilon \int_0^1 u'u\, dx + \int_0^1 u^2\, dx$$

  With Cauchy-Schwarz inequality, and $ab \leq (a^2+b^2)/2$ for all $a, b \geq 0$, we have that

$$\left|\int_0^1 u'u\, dx\right| \leq \|u'\|_2\|u\|_2 \leq \frac{\|u'\|_2^2 + \|u\|_2^2}{2},$$

  and then

$$\begin{aligned}
B(u,u) &\geq \|u'\|_2^2 - |\epsilon| \frac{\|u'\|_2^2 + \|u\|_2^2}{2} + \|u\|_2^2 \\
&\geq \left(1 - \frac{|\epsilon|}{2}\right)\left(\|u'\|_2^2 + \|u\|_2^2\right) = \left(1 - \frac{|\epsilon|}{2}\right)\|u\|_{H^1}^2
\end{aligned}$$

  And then for $|\epsilon| < 2$, B is coercive.
- Clear.
- Since $T_f : H_0^1(0,1) \to \mathbb{R}$ is linear and continuous, then we apply the Theorem with $H = C = H_0^1(0,1)$, which yields existence and uniqueness of the weak solution.

□

Solution of Exercise 34: This is a direct consequence of the integration by parts

$$\int_0^1 (-u'' + a(x)u)\varphi \, dx = [-u'(1)\varphi(1) + u'(0)\varphi(0)] + \int_0^1 (u'\varphi' + a(x)u\varphi) \, dx.$$

□

Solution of Exercise 35: $C = T^{-1}(\alpha, \beta)$ with T continuous, so it is closed. It is also convex (easy to check). It is nonempty since $x \mapsto \alpha + (\beta - \alpha)x \in C$. □

Solution of Exercise 36: The continuity and coercivity of B are as above. We apply Lax-Milgram's theorem. Then there exists a unique $u_C \in C$ such that

$$B(u_C, v - u_C) \geq T_f(v - u_C) \text{ for all } v \in C.$$

The space C is not linear, but is has an affine structure. Take $\varphi \in H_0^1(0,1)$: we have that $u_C + \varphi \in C$, and then

$$B(u_C, \varphi) = B(u_C, (u_C + \varphi) - u_C) \geq T_f((u_C + \varphi) - u_C) = T_f(\varphi) \text{ for all } \varphi \in H_0^1(0,1).$$

Taking $-\varphi$ and using the bilinearity of B, we get that

$$-B(u_C, \varphi) = B(u_C, -\varphi) \geq T_f(-\varphi) = -T_f(\varphi)$$

and therefore $B(u_C, \varphi) = T_f(\varphi)$ for all $\varphi \in H_0^1(0,1)$, which says that u_C is a weak solution to (E_4). We can go backwards and show that there is an equivalence, which yields existence and uniqueness.

The regularity goes as in the proof of Theorem 10.1, and then $u_C \in C^2([0,1])$.

Let us now prove that $u = u_C$ is a solution to (E_4). This goes as in the motivation of Chapter 9. Integrating by parts as in the solution of Exercise 34 and using that we have a weak solution to (E_4), then for any $\varphi \in C_c^1((0,1)) \subset H_0^1(0,1)$, we have that

$$\int_0^1 (-u'' + a(x)u)\varphi \, dx = \int_0^1 (u'\varphi' + a(x)u\varphi) \, dx = B(u, \varphi) = T_f(\varphi) = \int_0^1 f\varphi \, dx.$$

Then, writing $h := -u'' + a(x)u - f \in C^0([0,1])$, we get that $\int_0^2 h\varphi \, dx = 0$ for all $\varphi \in C_c^1((0,1))$. This extends to any $\varphi \in L^2(0,1)$ by density, then taking $h = \varphi$ yields $h(x) = 0$ a.e. $x \in (0,1)$, and then $h \equiv 0$ since h is continuous. So $-u'' + a(x)u = f(x)$ in $[0,1]$. Moreover, since $u \in C$, we have that $u(0) = \alpha$ and $u(1) = \beta$. So we have got a solution to (E_4). □

Solution of Exercise 37: Here again, this is a consequence of the integration by parts

$$\int_0^1 (-u'' + a(x)u)\varphi \, dx = [-u'(1)\varphi(1) + u'(0)\varphi(0)] + \int_0^1 (u'\varphi' + a(x)u\varphi) \, dx.$$

The condition $u'(0) = u'(1) = 0$ allows to allow any boundary condition for φ. □

Solution of Exercise 38: As above, we get that B is continuous and coercive on the Hilbert space $(H^1(0,1), \langle \cdot, \cdot \rangle_{H^1})$ where

$$B(u, v) = \int_0^1 (u'v' + a(x)uv) \, dx \text{ for } u, v \in H^1(0,1).$$

Since $T_f : v \mapsto \int_0^1 fv \, dx$ is continuous on H^1, Lax-Milgram's theorem yields a unique weak solution to (E_5). □

Solution of Exercise 39: We argue as in the proof of Theorem 10.1. We have that

$$\int_0^1 u'\varphi'\,dx = \int_0^1 (f-u)\varphi\,dx \text{ for all } \varphi \in H^1(0,1).$$

Since $u \in H^1(0,1)$, it is continuous and $f-u \in C^0([0,1])$. So, there exists $w \in C^2([0,1])$ such that $-w'' = u-f$ in $[0,1]$. We then have that

$$\int_0^1 u'\varphi'\,dx = -\int_0^1 w''\varphi\,dx = \int_0^1 w'\varphi'\,dx \text{ for all } \varphi \in H^1(0,1).$$

And then $\int_0^1 (u'-w')\varphi'\,dx = 0$ for all $\varphi \in H^1(0,1)$, therfore for all $\varphi \in C_c^1(0,1)$. Setting $h := u'-w' \in L^2(0,1)$, this yields $h \in H^1(0,1)$ and $h'=0$, so $h=C$ a.e for some $C \in \mathbb{R}$. So we have that

$$u'-w' = C \;\Rightarrow\; (u-w-Cx)' = 0 \text{ with } u-w-Cx \in H^1(0,1).$$

It then follows form our regularity result that there exists $D \in \mathbb{R}$ such that $u-w-Cx = D$ a.e $x \in (0,1)$, and then $u(x) = w(x)+Cx+D$ for a.e. $x \in (0,1)$. Since $w \in C^2$, then $u \in C^2([0,1])$ (or at least has a representative). □

Solution of Exercise 40: The integration by parts of Exercise 37 yields

$$\int_0^1 (-u''+au)\varphi\,dx = \int_0^1 (u'v'+a(x)uv)\,dx + (-u'(1)\varphi(1)+u'(0)\varphi(0)),$$

and the definition of a weak solutions gives the result. □

Solution of Exercise 41: The identity of Exercise 40 yields

$$\int_0^1 (-u''+au)\varphi\,dx = \int_0^1 f\varphi\,dx$$

for all $\varphi \in C_c^1((0,1))$. Define $h := -u''+au-f \in C^0([0,1])$ so that $\int_0^1 h\varphi\,dx = 0$ for all $\varphi \in C_c^1((0,1))$. Since $h \in L^2(0,1)$ and $C_c^1(0,1)$ is dense in $L^2(0,1)$, we have that this identity holds for all $\varphi \in L^2(0,1)$. Taking $\varphi := h$, we get $h \equiv 0$ a.e, so $-u''+au = f$ in $[0,1]$ since it is continuous. □

Solution of Exercise 42: It follows from Exercises 40 and 41 that

$$-u'(1)\varphi(1)+u'(0)\varphi(0) = 0 \text{ for all } \varphi \in C^1([0,1]).$$

Taking $\varphi(x) = x$ for all $x \in [0,1]$ yields $u'(1)=0$. Taking $\varphi(x) = 1-x$ for all $x \in [0,1]$ yields $u'(0)=0$. We have then obtained a strong solution $u \in C^2([0,1])$ to the problem

$$\left\{\begin{array}{ll} -u''+a(x)u = f & \text{in } (0,1)\\ u'(0)=u'(1)=0. & \end{array}\right. \tag{E_5}$$

and Theorem 10.5 is proved. □

Part 5

The Fourier transform

Until now, we have developed tools to solve somes ODEs defined on a bounded set, namely $(0,1)$ or $[0,1]$. The definition of Sobolev spaces extends to the full real line $\mathbb{R}$, and many of the results above extend. At this point, we have two additional difficulties.

First, one must replace the notion of condition on the boundary. A simple example is the equation $u''=0$ on $\mathbb{R}$. The C^2-solutions are exactly the affine functions, so there is not uniqueness. However, if one imposes that the limit at $\pm\infty$ exists and vanishes, then 0 is the unique solution. We then have that

$$\left\{ \begin{array}{cc} u''=0 & \text{in } \mathbb{R} \\ \lim_{x\to\pm\infty} u(x)=0 & \end{array} \right\} \Rightarrow \; u\equiv 0.$$

So we get existence and uniqueness. Another boundary condition is an integrability condition: requiring $u\in L^1(\mathbb{R})$ implies that u is not too large at $\pm\infty$. As one checks, we have that

$$\left\{ \begin{array}{cc} u''=0 & \text{in } \mathbb{R} \\ u\in L^1(\mathbb{R}) & \end{array} \right\} \Rightarrow \; u\equiv 0. \tag{27}$$

This is going to be the kind of integrability condition that we will require now.

The second difficulty is thet we have made an intensive use of compactness of closed intervals like $[0,1]$ to get existence results. It will not work on $\mathbb{R}$. However, $\mathbb{R}$ has the structure of a group, more precisely a topological group (the sum and inverse are continuous). This is what will allow us to define the Fourier Transform via integration. The main feature here will be that derivation will be transformed into a product, and vice-versa. We will then have to prove some regularity. This is going to provide an effective method to solve ODEs on $\mathbb{R}$.

CHAPTER 11

The L^1–Fourier transform

1. What is it? What is its potential utility?

This section is pure Science-Fiction: everything here will be formal...

1.1. Formal definition.

DEFINITION 1.1. *When it makes sense, for $f:\mathbb{R}\to\mathbb{C}$, define the* <u>*Fourier transform*</u> *$\hat{f}:\mathbb{R}\to\mathbb{C}$ as*

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-i\xi x}\,dx \text{ for all } \xi\in\mathbb{R}. \tag{28}$$

1.2. If the following computations are justified... (And it will be our main issue), the Fourier transforms turns:

1.2.1. *The convolution product into a classical product.* When it makes sense, for $f,g:\mathbb{R}\to\mathbb{C}$, define

$$(f\star g)(x) = \int_{\mathbb{R}} f(x-t)g(t)\,dt \text{ for } x\in\mathbb{R}.$$

Then, we have that

$$\begin{aligned}
\widehat{f\star g}(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{x\in\mathbb{R}}(f\star g)(x)e^{-i\xi x}\,dx\\
&= \frac{1}{\sqrt{2\pi}}\int_{x\in\mathbb{R}}\left(\int_{t\in\mathbb{R}} f(x-t)g(t)\,dt\right)e^{-i\xi x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{x\in\mathbb{R}}\int_{t\in\mathbb{R}} f(x-t)g(t)e^{-i\xi x}\,dxdt\\
&= \frac{1}{\sqrt{2\pi}}\int_{y\in\mathbb{R}}\int_{t\in\mathbb{R}} f(y)g(t)e^{-i\xi(y+t)}\,dydt \text{ with } x\to y = x-t\\
&= \frac{1}{\sqrt{2\pi}}\left(\int_{y\in\mathbb{R}} f(y)e^{-i\xi y}\,dy\right)\left(\int_{t\in\mathbb{R}} g(t)e^{-i\xi t}\,dt\right) = \sqrt{2\pi}\hat{f}(\xi)\hat{g}(\xi)
\end{aligned}$$

and then

$$\boxed{\widehat{f\star g} = \sqrt{2\pi}\hat{f}\cdot\hat{g}.} \tag{29}$$

1.2.2. *Derivatives into the product by $i\xi$.*

$$\begin{aligned}
\widehat{f'}(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{df}{dx}(x)e^{-i\xi x}\,dx = \frac{1}{\sqrt{2\pi}}\left(\left[f(x)e^{-i\xi x}\right]_{-\infty}^{+\infty} - \int_{\mathbb{R}} f(x)\frac{d}{dx}(e^{-i\xi x})\,dx\right)\\
&= \frac{1}{\sqrt{2\pi}}i\xi\int_{\mathbb{R}} f(x)e^{-i\xi x}\,dx = i\xi\hat{f}(\xi)
\end{aligned}$$

and then

$$\boxed{\widehat{f'}(\xi) = i\xi\hat{f}(\xi)}$$

1.2.3. *The product by x into a derivative.*

$$\begin{aligned}\widehat{xf(x)}(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} xf(x)e^{-i\xi x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)\frac{1}{-i}\frac{d}{d\xi}\left(e^{-i\xi x}\right)\,dx \\ &= i\frac{d}{d\xi}\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-i\xi x}\,dx\right) = i\left(\hat{f}\right)'(\xi)\end{aligned}$$

and then

$$\boxed{\widehat{xf(x)} = i\left(\hat{f}\right)'(\xi).}$$

1.3. Application to an ODE. Take $f \in C^0(\mathbb{R})$.

$$\textbf{(Q1):}\ \text{is there } u \in C^2(\mathbb{R}) \text{ such that } -u'' + u = f? \tag{30}$$

The answer is yes: use the classical linear Cauchy-Lipschitz Theorem 16.6 to get a two-dimensional affine space of solutions to (30).

Let us solve the problem formally via the Fourier transform. If u is a solution to (30), then

$$\widehat{-u''+u} = \hat{f} \quad\Rightarrow\quad -(i\xi)^2\hat{u} + \hat{u} = \hat{f} \;\Rightarrow\; (1+|\xi|^2)\hat{u} = \hat{f}$$

and then

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{1+|\xi|^2}$$

Therefore, if we can inverse the Fourier transform (and we will!), we get a solution u, and it will be unique.

About uniqueness. So, if we can justify all the computations above, then we will get existence and uniqueness of solution. However, take any solution $u \in C^2(\mathbb{R})$ to (30), then for any $\lambda, \mu \in \mathbb{R}$, we have that $x \mapsto u(x) + \lambda e^x + \mu e^{-x}$ is also a solution to (30): so there is no uniqueness.

So what is wrong here? Defining the Fourier transform as in (28) requires an integrability condition that will force uniqueness, see for instance the example (27). What we will prove is that when $f \in L^1(\mathbb{R})$, there is a unique solution $u \in C^2(\mathbb{R})$ to (30) such that $u \in L^1(\mathbb{R})$.

Note that when $u \in L^1(\mathbb{R})$, adding $x \mapsto e^x$ kills the integrability at $+\infty$, and $x \mapsto e^{-x}$ kills the integrability at $-\infty$: so you do not preserve integrability by adding $x \mapsto \lambda e^x + \mu e^{-x}$ unless $\lambda = \mu = 0$. This is how the integrability condition will ensure uniqueness.

Let us now make all this mathematically rigorous.

2. Definition and a few classical properties

2.1. Definition.

Definition 2.1 (L^1–Fourier transform). *Let $f \in L^1(\mathbb{R}, \mathbb{C})$ be an integrable function. We define its L^1–Fourier transform $\hat{f} : \mathbb{R} \to \mathbb{C}$ as*

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-i\xi x}\,dx \text{ for all } \xi \in \mathbb{R}. \tag{31}$$

Note that this definition makes sense since for all $\xi \in \mathbb{R}$, then $x \mapsto |f(x)e^{-i\xi x}| = |f(x)| \in L^1(\mathbb{R})$.

2.2. First computations.

- Let us compute the transform of $\mathbf{1}_{[0,1]}$. This makes sens since $\mathbf{1}_{[0,1]} \in L^1(\mathbb{R})$. For $\xi \in \mathbb{R}$, we have that

$$\begin{aligned}\widehat{\mathbf{1}_{[0,1]}}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathbf{1}_{[0,1]}(x) e^{-i\xi x}\, dx = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-i\xi x}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \begin{cases} \int_0^1 dx = 1 & \text{if } \xi = 0 \\ \int_0^1 \frac{d}{dx}\left(\frac{e^{-ix\xi}}{-i\xi}\right) dx = \frac{e^{-i\xi}-1}{-i\xi} = e^{-i\frac{\xi}{2}} \frac{e^{i\frac{\xi}{2}} - e^{-i\frac{\xi}{2}}}{i\xi} & \text{if } \xi \neq 0 \end{cases}\end{aligned}$$

and therefore

$$\boxed{\widehat{\mathbf{1}_{[0,1]}}(\xi) = \begin{cases} \frac{1}{\sqrt{2\pi}} & \text{if } \xi = 0 \\ \frac{1}{\sqrt{2\pi}} e^{-i\frac{\xi}{2}} \frac{2\sin\left(\frac{\xi}{2}\right)}{\xi} & \text{if } \xi \neq 0 \end{cases}} \tag{32}$$

Observe that since $\lim_{X\to 0} \frac{\sin X}{X} = 1$, then $\widehat{\mathbf{1}_{[0,1]}}$ is continuous on $\mathbb{R}$. Therefore, we will often write

$$\widehat{\mathbf{1}_{[0,1]}}(\xi) = \frac{1}{\sqrt{2\pi}} e^{-i\frac{\xi}{2}} \frac{2\sin\left(\frac{\xi}{2}\right)}{\xi}$$

taking for granted that this expression is valid for $\xi \neq 0$ and extends continuously to $\xi = 0$.

- Let us compute the transform of $x \mapsto e^{-a|x|}$ for $a > 0$. This makes sens since $x \mapsto e^{-a|x|} \in L^1(\mathbb{R})$. For $\xi \in \mathbb{R}$, we have that

$$\widehat{f_a}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-a|x|} e^{-i\xi x}\, dx = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^0 e^{ax - i\xi x}\, dx + \int_0^{+\infty} e^{-ax-i\xi x}\, dx \right)$$

It is always dangerous to make some computation with $\int^\infty$, since it includes a limite at ∞. So let us play it safe and let us write the limite explicitly

$$\widehat{f_a}(\xi) = \frac{1}{\sqrt{2\pi}} \lim_{R\to+\infty} \left(\int_{-R}^0 e^{(a-i\xi)x}\, dx + \int_0^R e^{-(a+i\xi)x}\, dx \right),$$

which makes sense since $f_a \in L^1(\mathbb{R})$. We then have that

$$\begin{aligned}\widehat{f_a}(\xi) &= \frac{1}{\sqrt{2\pi}} \lim_{R\to+\infty} \left(\left[\frac{e^{(a-i\xi)x}}{a - i\xi} \right]_{-R}^0 + \left[\frac{e^{-(a+i\xi)x}}{-a-i\xi} \right]_0^R \right) \\ &= \frac{1}{\sqrt{2\pi}} \lim_{R\to+\infty} \left(\frac{1 - e^{-R(a-i\xi)}}{a - i\xi} + \frac{e^{-(a+i\xi)R} - 1}{-a-i\xi} \right)\end{aligned}$$

Since $|e^{-R(a-i\xi)}| = |e^{-R(a+i\xi)}| = e^{-aR} \to 0$ as $R \to +\infty$ since $a > 0$, we get that

$$\widehat{f_a}(\xi) = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a-i\xi} + \frac{-1}{-a-i\xi} \right) = \frac{1}{\sqrt{2\pi}} \frac{a + i\xi + a - i\xi}{(a-i\xi)(a+i\xi)}$$

and then

$$\boxed{\widehat{f_a}(\xi) = \frac{1}{\sqrt{2\pi}} \frac{2a}{a^2 + \xi^2}}$$

for all $\xi \in \mathbb{R}$.

Notation: Instead of writing rigorously

$$\widehat{x \mapsto f(x)}(\xi) = \hat{f}(\xi),$$

to avoid unnecessary notations, we will often write the Fourier transform in a more compact (and abusive!!) way as

$$\widehat{f(x)} = \hat{f}(\xi) \text{ for } \xi \in \mathbb{R}.$$

For instance, we will write

$$\widehat{\mathbf{1}_{[0,1]}} = \frac{1}{\sqrt{2\pi}} e^{-i\frac{\xi}{2}} \frac{2\sin\left(\frac{\xi}{2}\right)}{\xi} \text{ and } \widehat{e^{-a|x|}} = \frac{1}{\sqrt{2\pi}} \frac{2a}{a^2+\xi^2}. \tag{33}$$

If there is any ambiguity, or if you do not feel confortable with this notation, go back immediately to the rigorous formulation.

Exercise 43: For $a < b$ any real numbers, compute $\widehat{\mathbf{1}_{[a,b]}}$. What is the Fourier transform of $\mathbf{1}_{[-1,1]}$?

Exercise 44: Define the triangle function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \begin{cases} 1+x & \text{if } -1 \leq x \leq 0 \\ 1-x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Compute its Fourier transform if it is defined.

2.3. Regularity and behavior at ∞. In the examples above, we have that the two Fourier transforms are continuous, and that

$$\lim_{\xi \to \pm\infty} \widehat{\mathbf{1}_{[0,1]}}(\xi) = \lim_{\xi \to \pm\infty} \widehat{e^{-a|x|}}(\xi) = 0.$$

Indeed, this is a general fact:

THEOREM 11.1. *Let $f \in L^1(\mathbb{R}, \mathbb{C})$ be an integrable function. Then $\hat{f} : \mathbb{R} \to \mathbb{C}$ is continuous and $\lim_{\xi \to \pm\infty} \hat{f}(\xi) = 0$. Moreover*

$$\|\hat{f}\|_\infty \leq \frac{\|f\|_1}{\sqrt{2\pi}}.$$

PROOF. We first deal with continuity via the continuity of parameter dependant integrals. Define $F(\xi, x) := \frac{1}{\sqrt{2\pi}} f(x) e^{-i\xi x}$ for all $\xi \in \mathbb{R}$ and a.e. $x \in \mathbb{R}$, so that

$$\hat{f}(\xi) = \int_{\mathbb{R}} F(\xi, x)\, dx \text{ for all } \xi \in \mathbb{R}.$$

We have that:

- For all $\xi \in \mathbb{R}$, $F(\xi, \cdot) \in L^1(\mathbb{R})$
- For a.e. $x \in \mathbb{R}$, $\xi \mapsto F(\xi, x) = f(x)e^{-i\xi x}$ is continuous on $\mathbb{R}$ (this is just an exponential since x is fixed)
- For all $\xi \in \mathbb{R}$ and a.e. $x \in \mathbb{R}$, we have that $|F(\xi, x)| \leq g(x)$ where $g \in L^1(\mathbb{R})$ (here, we simply take $g(x) = |f(x)|$ for a.e. $x \in \mathbb{R}$)

It then follows from the continuity Theorem 16.8 that $\hat{f}$ is continuous.

We now deal with the limit at $\pm\infty$. Indeed, this is the so-called Riemann-Lebesgue Lemma. We prove it. First assume that $f \in C_c^1(\mathbb{R})$, and let $R > 0$ be such that $f(x) = 0$ for $|x| \geq R$. For $\xi \neq 0$, we then have that

$$\begin{aligned}\sqrt{2\pi}\hat{f}(\xi) &= \int_{\mathbb{R}} f(x)e^{-i\xi x}\,dx = \int_{-R}^{R} f(x)e^{-i\xi x}\,dx = \int_{-R}^{R} f(x)\left(\frac{e^{-i\xi x}}{-i\xi}\right)'\,dx \\ &= \left[f(x)\frac{e^{-i\xi x}}{-i\xi}\right]_{-R}^{R} - \frac{1}{-i\xi}\int_{-R}^{R} f'(x)e^{-i\xi x}\,dx = \frac{1}{i\xi}\int_{-R}^{R} f'(x)e^{-i\xi x}\,dx\end{aligned}$$

and then

$$|\hat{f}(\xi)| \leq \frac{2R\max_{x\in[-R,R]}|f'(x)|}{\sqrt{2\pi}|\xi|} \text{ for all } \xi \neq 0.$$

And then $\lim_{\xi\to\pm\infty}\hat{f}(\xi) = 0$ when $f \in C_c^1(\mathbb{R})$.

Let us now tackle the general case. Take $f \in L^1(\mathbb{R})$ and fix $\epsilon > 0$. Since $C_c^1(\mathbb{R})$ is dense in $L^1(\mathbb{R})$ for the L^1-norm, there exists $g \in C_c^1(\mathbb{R})$ such that $\|f-g\|_1 < \sqrt{2\pi}\frac{\epsilon}{2}$. For $\xi \in \mathbb{R}$, we then write

$$|\hat{f}(\xi)| \leq |\hat{f}(\xi) - \hat{g}(\xi)| + |\hat{g}(\xi)| \leq \frac{\|f-g\|_1}{\sqrt{2\pi}} + |\hat{g}(\xi)| < \frac{\epsilon}{2} + |\hat{g}(\xi)|.$$

Since $g \in C_c^1(\mathbb{R})$, we have that $\lim_{\xi\to\pm\infty}\hat{g}(\xi) = 0$. So let $R_\epsilon > 0$ be such that $|\hat{g}(\xi)| < \frac{\epsilon}{2}$ for $|\xi| > R_\epsilon$. We then have that

$$|\hat{f}(\xi)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for } |\xi| > R_\epsilon,$$

and then $\lim_{\xi\to\pm\infty}\hat{f}(\xi) = 0$, which concludes the proof. □

2.4. Algebraic operations.

2.4.1. *Linearity and continuity of $f \mapsto \hat{f}$.* We endow the space $C_0(\mathbb{R},\mathbb{C}) := \{f \in C^0(\mathbb{R},\mathbb{C}) \text{ s.t. } \lim_{x\to\pm\infty} f(x) = 0\}$ with the norm $\|f\|_\infty := \sup_{x\in\mathbb{R}}|f(x)|$ for $f \in C_0(\mathbb{R},\mathbb{C})$.

Exercise 45: Prove that $\|f\|_\infty < \infty$ for all $f \in C_0(\mathbb{R},\mathbb{C})$ and that $(C_0(\mathbb{R},\mathbb{C}),\|\cdot\|_\infty)$ is a Banach space.

Proposition 11.2. *For any $f,g \in L^1(\mathbb{R},\mathbb{C})$ and $\lambda,\mu \in \mathbb{R}$, we have that*

$$\widehat{\lambda f + \mu g} = \lambda\hat{f} + \mu\hat{g}.$$

Moreover, the map

$$\left\{\begin{array}{cccc} \hat{}: & (L^1(\mathbb{R},\mathbb{C}),\|\cdot\|_1) & \to & (C_0(\mathbb{R},\mathbb{C}),\|\cdot\|_\infty) \\ & f & \mapsto & \hat{f} \end{array}\right.$$

is linear and continuous.

PROOF. Linearity is clear and continuity comes from the inequality $|\hat{f}(\xi)| \leq \|f\|_1/\sqrt{2\pi}$ for all $f \in L^1(\mathbb{R},\mathbb{C})$. □

2.4.2. *Multiplication by an exponential, translation, dilation.*

Proposition 11.3. *Let $f \in L^1(\mathbb{R},\mathbb{C})$ be an integrable function. Then*

- $\widehat{f(x)e^{i\alpha x}} = \hat{f}(\xi-\alpha)$ *for all* $\alpha \in \mathbb{R}$
- $\widehat{f(x-\alpha)} = \hat{f}(\xi)e^{-i\alpha\xi}$ *for all* $\alpha \in \mathbb{R}$
- $\widehat{f(\lambda x)} = \frac{1}{\lambda}\hat{f}\left(\frac{\xi}{\lambda}\right)$ *for all* $\lambda > 0$

Exercise 46: Prove Proposition 11.3. Give a general expression for $\widehat{f(\lambda x)}$ for $\lambda \neq 0$ (so it must include $\lambda < 0$).

Exercise 47: Write $\mathbf{1}_{[a,b]}$ as a composition of $\mathbf{1}_{[0,1]}$ with dilations and translations. Use (33) and Proposition 11.3 to give an alternative proof of $\widehat{\mathbf{1}_{[a,b]}}$, and compare with Exercise 43.

2.5. Convolution product.

Proposition 11.4. *Let $f, g \in L^1(\mathbb{R},\mathbb{C})$. Then $f \star g(x)$ is defined for a.e. $x \in \mathbb{R}$ and $f \star g \in L^1(\mathbb{R},\mathbb{C})$. Moreover, $\widehat{f \star g} = \sqrt{2\pi}\hat{f}\cdot\hat{g}$.*

Proof. Since we deal with nonnegative functions, Fubini's Theorem yield

$$\begin{aligned}
\int_{x\in\mathbb{R}}\left(\int_{t\in\mathbb{R}}|f(x-t)g(t)|\,dt\right)dx &= \int_{t\in\mathbb{R}}\int_{x\in\mathbb{R}}|f(x-t)g(t)|\,dtdx = \int_{t\in\mathbb{R}}|g(t)|\left(\int_{x\in\mathbb{R}}|f(x-t)|\,dx\right)dt \\
&= \int_{t\in\mathbb{R}}|g(t)|\left(\int_{y\in\mathbb{R}}|f(y)|\,dy\right)dt \text{ with } y = x-t \\
&= \int_{t\in\mathbb{R}}|g(t)|\|f\|_1\,dt = \|f\|_1\|g\|_1 < \infty \qquad (34)
\end{aligned}$$

So $\int_{t\in\mathbb{R}}|f(x-t)g(t)|\,dt < +\infty$ for a.e. $x \in \mathbb{R}$, and then $f \star g(x)$ is defined for a.e. $x \in \mathbb{R}$. The estimates above yields $\|f \star g\|_1 \leq \|f\|_1\|g\|_1 < +\infty$, so $f \star g \in L^1(\mathbb{R},\mathbb{C})$. Then, using Fubini's theorem for integrable functions and a change of variable, we can make the proof of (29) rigorous.

Exercise 48: Rewrite the proof of (29) rigorously.

□

3. Derivation

3.1. A Lemma.

Lemma 11.5. *Let $f \in C^1(\mathbb{R},\mathbb{C})$ be such that $f' \in L^1(\mathbb{R},\mathbb{C})$. Then $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to+\infty} f(x)$ exist..*

Proof. Since we have no idea of the value of the limit, the proof is going to be a matter of completeness and Cauchy sequence. Take $x, y \in \mathbb{R}$ such that $x < y$. We have that

$$|f(y)-f(x)| = \left|\int_x^y f'(t)\,dt\right| \leq \int_x^y |f'(t)|\,dt \leq \int_x^\infty |f'(t)|\,dt.$$

Since $f' \in L^1(\mathbb{R},\mathbb{C})$, then $\lim_{x\to+\infty}\int_x^\infty |f'(t)|\,dt = 0$. Therefore, we have that

$$\forall\epsilon > 0,\ \exists A > 0 \text{ such that } \forall x, y \in \mathbb{R},\ \{x, y > A \Rightarrow |f(x)-f(y)| < \epsilon\} \qquad (35)$$

which is a "continuous" version of Cauchy sequence. We are going to prove that $\lim_{x\to+\infty} f(x)$ exists.

- Step 1: for all $(x_n)_n$ such that $x_n \in \mathbb{R}$ and $x_n \to +\infty$, then $(f(x_n))_n$ has a limit as $n \to \infty$.

  For this, we prove that $(f(x_n))_n$ is a Cauchy sequence. We fix $\epsilon > 0$ and we let $A > 0$ be such that (35) holds. We set $n_0 \in \mathbb{N}$ be such that $x_n > A$ for $n \geq n_0$. So $|f(x_p) - f(x_q)| < \epsilon$ for $p, q \geq n_0$. So $(f(x_n)_n)$ is a Cauchy sequence in $\mathbb{C}$, so it converges.
- Step 2: we claim that there exists $l \in \mathbb{R}$ such that for all $(x_n) \to +\infty$, $f(x_n) \to l$ as $n \to +\infty$.

  For this, we let $l := \lim_{n\to+\infty} f(n)$. This exists due to Step 1. We take a sequence $(x_n)_n$ such that $x_n \to +\infty$. It follows from Step 1 that there exists $l' \in \mathbb{R}$ such that $f(x_n) \to l'$ as $n \to +\infty$. For $\epsilon > 0$, let $A \in \mathbb{R}$ be such that (35) holds. Let $n_0 \in \mathbb{N}$ be such that $x_n > A$ and $n > A$ for $n > n_0$. Therefore $|f(x_n) - f(n)| < \epsilon$ for all $n > n_0$. Letting $n \to +\infty$ yields $|l' - l| \leq \epsilon$. Letting $\epsilon \to 0$ yields $l = l'$, and the claim is proved.
- Step 3: it follows from Step 2 and the characterization of limits by sequences that $\lim_{x\to+\infty} f(x)$ exists and its value is l. By taking $x \mapsto f(-x)$, we also get the existence of a limit as $x \to -\infty$.

$\square$

3.2. Fourier transform of the derivative.

Proposition 11.6. *Let $f \in C^1(\mathbb{R}, \mathbb{C})$ be such that $f, f' \in L^1(\mathbb{R}, \mathbb{C})$. Then*

$$\widehat{f'}(\xi) = i\xi \hat{f}(\xi) \text{ for all } \xi \in \mathbb{R}.$$

Proof. We first claim that

$$\lim_{x\to\pm\infty} f(x) = 0.$$

We prove the claim. It follows from Lemma 11.5 that there exists $l \in \mathbb{C}$ such that $\lim_{x\to+\infty} f(x) = l$. Assume that $l \neq 0$. Then $\lim_{x\to+\infty} |f(x)| = |l > |l|/2$, so there exists $A > 0$ such that $|f(x)| > |l|/2$ for all $x > A$, and then $\int_A^M |f(x)|\, dx \geq (M - A)|l|/2$ for all $M > A$. Letting $M \to \infty$ yields $\int_A^\infty |f(x)|\, dx = \infty$, contradicting $f \in L^1(\mathbb{R}, \mathbb{C})$. So $l = 0$. The result for $x \to -\infty$ is going via the map $x \mapsto f(-x)$. The claim is proved.

We are in proposition. As for the convolution product, we are going to make rigorous the formal proof of the first section. Take $\xi \in \mathbb{R}$ and, taking integral over $(-R, R)$ for safety (this is justified since $f' \in L^1(\mathbb{R}, \mathbb{C})$, we have that

$$\begin{aligned}
\widehat{f'}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f'(x) e^{-i\xi x}\, dx = \frac{1}{\sqrt{2\pi}} \lim_{R\to+\infty} \int_{-R}^{R} f'(x) e^{-i\xi x}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \lim_{R\to+\infty} \left(\left[f(x) e^{-i\xi x} \right]_{-R}^{R} - \int_{-R}^{R} f(x) (e^{-i\xi x})'\, dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \lim_{R\to+\infty} \left(\left[f(x) e^{-i\xi x} \right]_{-R}^{R} + i\xi \int_{-R}^{R} f(x) e^{-i\xi x}\, dx \right)
\end{aligned}$$

Since $f \in L^1(\mathbb{R}, \mathbb{C})$ and $\lim_{x\to\pm\infty} f(x) = 0$, we then get that

$$\widehat{f'}(\xi) = \frac{1}{\sqrt{2\pi}} i\xi \int_{-\infty}^{+\infty} f(x) e^{-i\xi x}\, dx = i\xi \hat{f}(\xi).$$

This proves the proposition. $\square$

3.3. Derivative of the Fourier transform.

Proposition 11.7. *Let $f \in L^1(\mathbb{R}, \mathbb{C})$ be such that $x \mapsto xf(x) \in L^1(\mathbb{R}, \mathbb{C})$. Then $\hat{f} \in C^1(\mathbb{R}, \mathbb{C})$ and*

$$\left(\hat{f}\right)'(\xi) = -i\widehat{x \mapsto xf}(x)(\xi) \text{ for all } \xi \in \mathbb{R}. \tag{36}$$

In a more compact notation,

$$\left(\hat{f}\right)' = -i\widehat{xf(x)} \tag{37}$$

Proof. We take the notations of the proof of Theorem 11.1. Define $F(\xi, x) := \frac{1}{\sqrt{2\pi}} f(x) e^{-i\xi x}$ for all $\xi \in \mathbb{R}$ and a.e. $x \in \mathbb{R}$, so that

$$\hat{f}(\xi) = \int_{\mathbb{R}} F(\xi, x)\, dx \text{ for all } \xi \in \mathbb{R}.$$

We have that:

- For all $\xi \in \mathbb{R}$, $F(\xi, \cdot) \in L^1(\mathbb{R})$
- For a.e. $x \in \mathbb{R}$, $\xi \mapsto F(\xi, x) = \frac{1}{\sqrt{2\pi}} f(x) e^{-i\xi x}$ is C^1 on $\mathbb{R}$ (this is just an exponential since x is fixed)
- For all $\xi \in \mathbb{R}$ and a.e. $x \in \mathbb{R}$, we have that $|\frac{\partial}{\partial \xi} F(\xi, x)| \leq h(x)$ where $h \in L^1(\mathbb{R})$. Here, we take $h(x) = |xf(x)|$ for a.e. $x \in \mathbb{R}$: we have that $h \in L^1(\mathbb{R})$ via our hypothesis.

It then follows from the differentiability Theorem 16.9 that $\hat{f}$ is differentiable and that

$$\begin{aligned}
\left(\hat{f}\right)'(\xi) &= \int_{\mathbb{R}} \frac{\partial}{\partial \xi} F(\xi, x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) \times (-ix) \times e^{-i\xi x}\, dx \\
&= -i(\widehat{x \mapsto xf}(x))(\xi) \text{ for all } \xi \in \mathbb{R}.
\end{aligned}$$

Moreover, since $x \mapsto xf(x) \in L^1(\mathbb{R}, \mathbb{C})$, its Fourier transform is continuous (Theorem 11.1), and then $\hat{f} \in C^1(\mathbb{R}, \mathbb{C})$. $\square$

3.4. Higher order derivatives.

Iterating the proofs above, we get the following Propositions:

Proposition 11.8. *Let $f : \mathbb{R} \to \mathbb{C}$ be a function and $n \in \mathbb{N}$ be an integer. Then*

$$x \mapsto (1 + |x|^n) f(x) \in L^1(\mathbb{R}, \mathbb{C}) \ \Rightarrow \hat{f} \in C^n(\mathbb{R}, \mathbb{C}).$$

In addition, for all $k = \{0, ..., n\}$, we have that

$$\left(\hat{f}\right)^{(k)} = (-i)^k \widehat{x^k f(x)}.$$

Proposition 11.9. *Let $f : \mathbb{R} \to \mathbb{C}$ be a function. Then*

$$\left\{ \begin{array}{ll} f \in C^n(\mathbb{R}, \mathbb{C}) & \\ f^{(k)} \in L^1(\mathbb{R}, \mathbb{C}) & \forall 0 \leq k \leq n \end{array} \right\} \Rightarrow \widehat{f^{(k)}} = (i\xi)^k \hat{f}(\xi) \ \forall 0 \leq k \leq n \text{ and } \xi \in \mathbb{R}.$$

Exercise 49: Prove these propositions.

Important remark: As a consequence of Proposition 11.9, if $f \in C^n(\mathbb{R},\mathbb{C})$ with $f^{(k)} \in L^1(\mathbb{R},\mathbb{C})$ for all $k \in \{0,...,n\}$, we get that

$$(1+|\xi|^n)|\hat{f}(\xi)| = |\hat{f}(\xi)| + |\widehat{f^{(n)}(\xi)}| \le \|f\|_{L^1} + \|f^{(n)}\|_{L^1} \tag{38}$$

for all $\xi \in \mathbb{R}$. This inequality and Proposition 11.8 show that there is (almost) a correspondance between regularity and behavior at infinity via the Fourier transform. For instance, mixing these two propositions and Theorem 11.1, taking $f \in L^1(\mathbb{R},\mathbb{C})$ and $n \in \mathbb{N}$, we have that

$$x \mapsto (1+|x|^n)f(x) \in L^1(\mathbb{R},\mathbb{C}) \ \Rightarrow \ f \in C^n(\mathbb{R},\mathbb{C})$$

and (38) yields

$$f \in C^n(\mathbb{R},\mathbb{C}) \ \Big(\text{and } f^{(k)} \in L^1\, \forall k \le n\Big) \ \Rightarrow (1+|\xi|^n)\hat{f}(\xi) = o(1) \text{ as } \xi \to \pm\infty.$$

This is not really a correspondance, since we have to make some additional hypothesis to go from $o(1)$ to L^1, but this is the idea to keep in mind:

Via the Fourier Transform, the more regularity we have, the more integrability we get, and vice-versa.

3.5. Application: Fourier transform of $x \mapsto e^{-ax^2}$.

Proposition 11.10. *For $a > 0$, we have that*

$$\widehat{e^{-ax^2}} = \frac{e^{-\frac{\xi^2}{4a}}}{\sqrt{2a}}.$$

In particular, taking $a = 1/2$, we have that $\widehat{e^{-\frac{x^2}{2}}} = e^{-\frac{\xi^2}{2}}$: the function $x \mapsto e^{-x^2/2}$ is invariant via the Fourier transform.

We prove this result in several steps. Define $f(x) := e^{-ax^2}$ for all $x \in \mathbb{R}$. We have that $f \in C^\infty(\mathbb{R})$ and

$$f'(x) = -2axf(x) \text{ for all } x \in \mathbb{R}.$$

Let us take the Fourier transform: this makes sense since $f, f' \in L^1(\mathbb{R},\mathbb{C})$ and $x \mapsto xf(x) \in L^1(\mathbb{R},\mathbb{C})$, yielding $\hat{f} \in L^1(\mathbb{R},\mathbb{C})$. We get with Propositions 11.6 and 11.7 that

$$i\xi\hat{f}(\xi) = \widehat{f'}(\xi) = -\widehat{2axf(x)} = -2a\widehat{xf(x)} = -2ai(\hat{f})'(\xi) \ \Rightarrow \ (\hat{f})'(\xi) = \frac{-\xi}{2a}\hat{f}(\xi)$$

for all $\xi \in \mathbb{R}$. Solving this linear ODE yields

$$\hat{f}(\xi) = \hat{f}(0)e^{-\frac{\xi^2}{4a}} \text{ for all } \xi \in \mathbb{R}.$$

We have that

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ax^2}\,dx = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{a}}\int_{\mathbb{R}} e^{-y^2}\,dy = \frac{1}{\sqrt{2a}}.$$

Therefore, we get that

$$\boxed{\widehat{e^{-ax^2}} = \frac{e^{-\frac{\xi^2}{4a}}}{\sqrt{2a}}}.$$

4. Solution of the exercises of Chapter 11

Solution of Exercise 43: This makes sense since $\mathbf{1}_{[a,b]} \in L^1(\mathbb{R})$. For $\xi \in \mathbb{R}$, we have that

$$\begin{aligned}
\widehat{\mathbf{1}_{[a,b]}}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathbf{1}_{[a,b]}(x) e^{-i\xi x}\, dx = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-i\xi x}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \begin{cases} \int_a^b dx = b-a & \text{if } \xi = 0 \\ \int_a^b \frac{d}{dx}\left(\frac{e^{-ix\xi}}{-i\xi}\right) dx = \frac{e^{-ib\xi}-e^{-ia\xi}}{-i\xi} = e^{-i\frac{a+b}{2}\xi} \frac{e^{i\frac{b-a}{2}\xi} - e^{-i\frac{b-a}{2}\xi}}{i\xi} & \text{if } \xi \neq 0 \end{cases}
\end{aligned}$$

and therefore

$$\widehat{\mathbf{1}_{[a,b]}}(\xi) = \begin{cases} \frac{b-a}{\sqrt{2\pi}} & \text{if } \xi = 0 \\ \frac{1}{\sqrt{2\pi}} e^{-i\frac{a+b}{2}\xi} \frac{2\sin\left(\frac{b-a}{2}\xi\right)}{\xi} & \text{if } \xi \neq 0 \end{cases} \tag{39}$$

In particular,

$$\widehat{\mathbf{1}_{[-1,1]}}(\xi) = \begin{cases} \frac{2}{\sqrt{2\pi}} & \text{if } \xi = 0 \\ \frac{1}{\sqrt{2\pi}} \frac{2\sin\xi}{\xi} & \text{if } \xi \neq 0 \end{cases} \tag{40}$$

□

Solution of Exercise 44: This makes sense since $f \in L^1(\mathbb{R})$. For $\xi \in \mathbb{R}$, we have that

$$\begin{aligned}
\sqrt{2\pi}\hat{f}(\xi) &= \int_{\mathbb{R}} f(x) e^{-i\xi x}\, dx = \int_{-1}^0 (1+x) e^{-i\xi x}\, dx + \int_0^1 (1-x) e^{-i\xi x}\, dx \\
&= \int_1^0 (1-x) e^{i\xi x} \times (-1)\, dx + \int_0^1 (1-x) e^{-i\xi x}\, dx \\
&= \int_0^1 (1-x) e^{i\xi x}\, dx + \int_0^1 (1-x) e^{-i\xi x}\, dx = \int_0^1 (1-x)(e^{i\xi x} + e^{-i\xi x})\, dx \\
&= 2\int_0^1 (1-x)\cos(\xi x)\, dx
\end{aligned}$$

When $\xi \not\equiv 0$, we get that

$$\sqrt{2\pi}\hat{f}(0) = 2\left[x - \frac{x^2}{2}\right]_0^1 = 1.$$

For $\xi \neq 0$, we have that

$$\begin{aligned}
\sqrt{2\pi}\hat{f}(\xi) &= 2\int_0^1 (1-x)\cos(\xi x)\, dx = 2\int_0^1 (1-x)\left(\frac{\sin(\xi x)}{\xi}\right)' dx \\
&= 2\left[(1-x)\left(\frac{\sin(\xi x)}{\xi}\right)\right]_0^1 - 2\int_0^1 (1-x)' \frac{\sin(\xi x)}{\xi}\, dx \\
&= 2\int_0^1 \frac{\sin(\xi x)}{\xi}\, dx = 2\int_0^1 \left(\frac{-\cos(\xi x)}{\xi^2}\right)' dx = 2\left[\frac{-\cos(\xi x)}{\xi^2}\right]_0^1 = 2\frac{-\cos(\xi)+1}{\xi^2} \\
&= 2\frac{2\sin(\xi/2)^2}{\xi^2} = \left(\frac{\sin\left(\frac{\xi}{2}\right)}{\frac{\xi}{2}}\right)^2
\end{aligned}$$

And then

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \begin{cases} 1 & \text{if } \xi = 0 \\ \left(\frac{\sin\left(\frac{\xi}{2}\right)}{\frac{\xi}{2}}\right)^2 & \text{if } \xi \neq 0 \end{cases}$$

□

Solution of Exercise 45: Since the limit is 0 at $\pm\infty$, there exists $R_0 > 0$ such that for $|x| \geq R_0$, we have that $|f(x)| < 1$. Set $M := \max_{[-R_0,R_0]} |f(x)|$, which is well-defined since f is continuous on the compact set $[-R_0, R_0]$. As one checks, we have that $|f(x)| \leq \max\{1, M\}$ for all $x \in \mathbb{R}$. So $\|f\|_\infty \leq \max\{1, M\} < +\infty$.

To prove that we have a Banach space, let us take a Cauchy sequence $(f_n)_n$ in the space $(C_0(\mathbb{R}, \mathbb{C}), \|\cdot\|_\infty)$. Since for all $p, q \in \mathbb{N}$ and for all $x \in \mathbb{R}$, we have that

$$|f_p(x) - f_q(x)| \leq \|f_p - f_q\|_\infty,$$

we then get that for all $x \in \mathbb{R}$, $(f_n(x))_n$ is a Cauchy sequence in $(\mathbb{C}, |.|)$ complete, so it has a limit, say $f(x) \in \mathbb{C}$. So $(f_n)_n$ has a pointwise limit.

Let us prove that the convergence is uniform. Fix $\epsilon > 0$ and let $n_0 \in \mathbb{N}$ such that $\|f_p - f_q\|_\infty < \epsilon$ for $p, q > n_0$. Given $x \in \mathbb{R}$, we have that $|f_p(x) - f_q(x)| \leq \|f_p - f_q\|_\infty < \epsilon$, so letting $q \to +\infty$ yields

$$|f_p(x) - f(x)| \leq \epsilon \text{ for all } x \in \mathbb{R} \text{ and all } p > n_0,$$

so $\|f_p - f\|_\infty \leq \epsilon$ for all $p > n_0$.Therefore $(f_n)_n$ converges uniformly to f, and then $f \in C^0(\mathbb{R}, \mathbb{C})$.

Let us prove that $f \in C_0(\mathbb{R}, \mathbb{C})$. Fix $\epsilon > 0$ and let $n_0 \in \mathbb{N}$ be such that $\|f_n - f\|_\infty < \epsilon/2$ for all $n \geq n_0$ (this is the uniform convergence. We then have that

$$|f(x)| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x)| < \frac{\epsilon}{2} + |f_{n_0}(x)| \text{ for all } x \in \mathbb{R}.$$

Since $f_{n_0} \in C_0(\mathbb{R}, \mathbb{C})$, there exists $R_0 > 0$ such that

$$|x| > R_0 \Rightarrow |f_{n_0}(x)| < \frac{\epsilon}{2}.$$

So for $|x| > R_0$, $|f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. And then $\lim_{x\to\pm\infty} f(x) = 0$, and then $f \in C_0(\mathbb{R}, \mathbb{C})$. This proves the completeness. □

Solution of Exercise 46: $x \mapsto f(\lambda x)$ is in L^1 for $\lambda \neq 0$. For $\lambda > 0$, we have that

$$\begin{aligned} \sqrt{2\pi}\widehat{f(\lambda x)} &= \int_{x=-\infty}^{x=+\infty} f(\lambda x) e^{-i\xi x}\, dx \\ &= \int_{y=-\infty}^{y=+\infty} f(y) e^{-i\xi \frac{y}{\lambda}} \frac{dy}{\lambda} \text{ with } x = \lambda y \\ &= \frac{1}{\lambda}\sqrt{2\pi}\hat{f}\left(\frac{\xi}{\lambda}\right) \end{aligned}$$

When $\lambda < 0$, the bounds of the integral are exchanged:

$$\begin{aligned}
\sqrt{2\pi}\widehat{f(\lambda x)} &= \int_{x=-\infty}^{x=+\infty} f(\lambda x)e^{-i\xi x}\,dx \\
&= \int_{y=+\infty}^{y=-\infty} f(y)e^{-i\xi\frac{y}{\lambda}}\,\frac{dy}{\lambda} \text{ with } x = \lambda y \\
&= \frac{1}{\lambda}(-1)\int_{-\infty}^{+\infty} f(y)e^{-i\xi\frac{y}{\lambda}}\,\frac{dy}{\lambda} \\
&= \frac{-1}{\lambda}\sqrt{2\pi}\hat{f}\left(\frac{\xi}{\lambda}\right) = \frac{1}{|\lambda|}\sqrt{2\pi}\hat{f}\left(\frac{\xi}{\lambda}\right)
\end{aligned}$$

So, in any case, for $\lambda \neq 0$, we have that

$$\widehat{f(\lambda x)} = \frac{1}{|\lambda|}\hat{f}\left(\frac{\xi}{\lambda}\right)$$

Via Lebesgue's integration theory, there is a uniform proof: set $\varphi(x) = x/\lambda$ for $x \in \mathbb{R}$ and $\lambda \neq 0$. We have that $\varphi(\mathbb{R}) = \mathbb{R}$, φ is a C^1–diffeomorphism and then the change of variable formula yields

$$\begin{aligned}
\sqrt{2\pi}\widehat{f(\lambda x)} &= \int_{\mathbb{R}} f(\lambda x)e^{-i\xi x}\,dx = \int_{\varphi(\mathbb{R})} f(\lambda x)e^{-i\xi x}\,dx \\
&= \int_{\mathbb{R}} f(\lambda\varphi(y))e^{-i\xi\varphi(y)}|\mathrm{Jac}(\varphi)(y)|\,dy = \int_{\mathbb{R}} f(y)e^{-i\xi\frac{y}{\lambda}}\left|\frac{1}{\lambda}\right|\,dy \\
&= \frac{1}{|\lambda|}\sqrt{2\pi}\hat{f}\left(\frac{\xi}{\lambda}\right)
\end{aligned}$$

$\square$

Solution of Exercise 47: For convenience, for $\lambda > 0$ and $\alpha \in \mathbb{R}$, we define

$$\delta_\lambda(x) := \lambda \cdot x \text{ and } \tau_\alpha(x) := x - \alpha \text{ for all } x \in \mathbb{R}.$$

As one checks, with $a < b$, for any $x \in \mathbb{R}$, we have that

$$\mathbf{1}_{[a,b]}(x) = \mathbf{1}_{[0,1]}\left(\frac{x-a}{b-a}\right) = \mathbf{1}_{[0,1]} \circ \delta_{\frac{1}{b-a}} \circ \tau_a(x),$$

Ler us compute the Fourier transform using Proposition 11.3 and (33):

$$\begin{aligned}
\widehat{\mathbf{1}_{[a,b]}} &= \mathbf{1}_{[0,1]}\widehat{\circ\,\delta_{\frac{1}{b-a}}} \circ \tau_a = e^{-ia\xi}\mathbf{1}_{[0,1]}\widehat{\circ\,\delta_{\frac{1}{b-a}}} \\
&= e^{-ia\xi}(b-a)\widehat{\mathbf{1}_{[0,1]}}((b-a)\xi) = e^{-ia\xi}(b-a)\cdot\frac{1}{\sqrt{2\pi}}e^{-i\frac{(b-a)\xi}{2}}\frac{2\sin\left(\frac{(b-a)\xi}{2}\right)}{(b-a)\xi} \\
&= \frac{1}{\sqrt{2\pi}}e^{-i\frac{(a+b)\xi}{2}}\frac{2\sin\left(\frac{(b-a)\xi}{2}\right)}{\xi}.
\end{aligned}$$

Which is what was obtained Exercise 43. We can also write

$$\mathbf{1}_{[a,b]}(x) = \mathbf{1}_{[0,1]}\left(\frac{x}{b-a} - \frac{a}{b-a}\right) = \mathbf{1}_{[0,1]} \circ \tau_{\frac{a}{b-a}} \circ \delta_{\frac{1}{b-a}}(x)$$

for all $x \in \mathbb{R}$. Then

$$\begin{aligned}
\widehat{\mathbf{1}_{[a,b]}} &= \widehat{\mathbf{1}_{[0,1]} \circ \tau_{\frac{a}{b-a}} \circ \delta_{\frac{1}{b-a}}} = (b-a)\widehat{\mathbf{1}_{[0,1]} \circ \tau_{\frac{a}{b-a}}}((b-a)\xi) \\
&= (b-a)e^{-i\frac{a}{b-a}(b-a)\xi}\widehat{\mathbf{1}_{[0,1]}}((b-a)\xi) \\
&= (b-a)e^{-ia\xi}\frac{1}{\sqrt{2\pi}}e^{-i\frac{(b-a)\xi}{2}}\frac{2\sin\left(\frac{(b-a)\xi}{2}\right)}{(b-a)\xi} \\
&= \frac{1}{\sqrt{2\pi}}e^{-i\frac{(a+b)\xi}{2}}\frac{2\sin\left(\frac{(b-a)\xi}{2}\right)}{\xi}.
\end{aligned}$$

Here again, this is the result we obtained with the other methods. □

Solution of Exercise 48: We fix $\xi \in \mathbb{R}$. Define

$$F(x,t) := f(x-t)g(t)e^{-i\xi x} \text{ for a.e. } x,t \in \mathbb{R}.$$

Since $f, g \in L^1(\mathbb{R},\mathbb{C})$, we have that $f \star g \in L^1(\mathbb{R},\mathbb{C})$ (Proposition 11.4) and

$$\begin{aligned}
\sqrt{2\pi}\widehat{f\star g}(\xi) &= \int_{x\in\mathbb{R}} (f\star g)(x)e^{-i\xi x}\,dx \\
&= \int_{x\in\mathbb{R}}\left(\int_{t\in\mathbb{R}} f(x-t)g(t)\,dt\right)e^{-i\xi x}\,dx = \int_{x\in\mathbb{R}}\left(\int_{t\in\mathbb{R}} F(x,t)\,dt\right)dx
\end{aligned}$$

It follows from the proof of Proposition 11.4 that $F \in L^1(\mathbb{R}^2)$, and we can apply Fubini's Theorem to get

$$\sqrt{2\pi}\widehat{f\star g}(\xi) = \int_{t\in\mathbb{R}}\left(\int_{x\in\mathbb{R}} F(x,t)\,dx\right)dt = \int_{t\in\mathbb{R}} g(t)\left(\int_{x\in\mathbb{R}} f(x-t)e^{-i\xi x})\,dx\right)dt$$

With the change of variable $y := x - t$ in the second integral, we get that

$$\begin{aligned}
\sqrt{2\pi}\widehat{f\star g}(\xi) &= \int_{t\in\mathbb{R}} g(t)\left(\int_{y\in\mathbb{R}} f(y)e^{-i\xi(y+t)})\,dy\right)dt \\
&= \int_{t\in\mathbb{R}} g(t)e^{-i\xi t}\left(\int_{y\in\mathbb{R}} f(y)e^{-i\xi y})\,dy\right)dt \\
&= \int_{t\in\mathbb{R}} g(t)e^{-i\xi t}\sqrt{2\pi}\hat{f}(\xi)\,dt = \sqrt{2\pi}^2\hat{g}(\xi)\cdot\hat{f}(\xi)
\end{aligned}$$

which is the expected result. □

Solution of Exercise 49: For Proposition 11.8, this is just an iteration of Proposition 11.7 and that $|x|^k \le 1 + |x|^n$ for all $0 \le k \le n$ and $x \in \mathbb{R}$. For instance, if $n \ge 2$, we have that $x \mapsto h(x) := xf(x) \in L^1$, so $\hat{f} \in C^1$ and $\hat{f}' = -i\hat{h}$. Since $x \mapsto xh(x) \in L^1$ too, then $\hat{h} \in C^1$ with $\hat{h}' = -i\widehat{x \mapsto xh(x)}$, so $\hat{f} \in C^2$ and $\hat{h}'' = (-i)^2\widehat{x \mapsto xh(x)} = (-i)^2\widehat{x \mapsto x^2 f(x)}$. And so on for the other derivatives.

For Proposition 11.9, this is also an iteration. For instance, when $n \ge 2$, we apply twice Proposition 11.6 to get that

$$\widehat{f''} = \widehat{(f')'} = (i\xi)\hat{f}'(\xi) = (i\xi)^2\hat{f}(\xi).$$

□

CHAPTER 12

The inversion theorem and application to ODEs

The central question is this chapter is the following

Q: Is a function uniquely determined by its Fourier transform? If yes, can one reconstruct a function from its Fourier transform?

1. The inversion theorem

1.1. The answer. The answer to this question is simple:

THEOREM 12.1 (Fourier inversion formula). *Let $f \in L^1(\mathbb{R}, \mathbb{C})$ be such that $\hat{f} \in L^1(\mathbb{R}, \mathbb{C})$. Then*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi}\, d\xi \text{ for a.e. } x \in \mathbb{R}. \tag{41}$$

However... be aware that there are hypothesis!!! This inversion formula is valid only if $f, \hat{f} \in L^1(\mathbb{R}, \mathbb{C})$. And not everywhere: only a.e....

Important! The power is $e^{ix\xi}$, not $e^{-ix\xi}$.

1.2. A few consequences.

Proposition 12.2. *Let $f \in L^1(\mathbb{R}, \mathbb{C})$ be such that $\hat{f} \equiv 0$ a.e. Then $f(x) = 0$ for a.e. $x \in \mathbb{R}$.*

PROOF. In this case, we have that $0 = \hat{f}(x)$ for all $x \in \mathbb{R}$ ($\hat{f}$ is continuous). So $\hat{f} \in L^1(\mathbb{R}, \mathbb{C})$ and the inversion formula (41) yields $f(x) = 0$ for a.e $x \in \mathbb{R}$. □

Proposition 12.3. *The Fourier Transform is injective from (or $1-1$) from $L^1(\mathbb{R}, \mathbb{C})$ into $C_0(\mathbb{R}, \mathbb{C})$.*

PROOF. The transform is linear, and its kernel is $\{0\}$ by the preceding proposition. So it is injective. □

2. Proof of the inversion theorem

2.1. Two useful classical theorems.

THEOREM 12.4 (Approximation of unity). *Let $(g_\epsilon)_{\epsilon>0}$ be such that $g_\epsilon \in L^1(\mathbb{R})$ for all $\epsilon > 0$. We assume that*

- *$g_\epsilon \geq 0$ for all $\epsilon > 0$;*
- *$\int_{\mathbb{R}} g_\epsilon(x)\, dx = 1$ for all $\epsilon > 0$;*
- *For all $\delta > 0$,*

$$\lim_{\epsilon \to 0} \int_{|x|>\delta} g_\epsilon(x)\, dx = 0.$$

Let $f \in L^p(\mathbb{R}, \mathbb{C})$ where $1 \leq p < +\infty$. Then $g_\epsilon \star f \in L^p(\mathbb{R}, \mathbb{C})$ and

$$\lim_{\epsilon \to 0} g_\epsilon \star f = f \text{ in } L^p(\mathbb{R}, \mathbb{C}).$$

Theorem 12.5 (Convergence in L^p "implies" convergence a.e.). *Let (X, μ) be a measured space. Let $(f_n)_n$ be a sequence such that $f_n \in L^p(X)$ for all n and assume that there exists $f \in L^p(X)$ such that $(f_n) \to f$ in L^p for some $1 \leq p \leq \infty$. Then there exists a subsequence $(f_{j(n)})_n$ such that $\lim_{n \to +\infty} f_{j(n)}(x) = f(x)$ for a.e. $x \in X$.*

2.2. An intermediate Lemma. Let $\varphi, \psi \in L^1(\mathbb{R}, \mathbb{C})$ be two integrable functions. Define

$$F(y, z) = \varphi(y)\psi(z)e^{-iyz} \text{ for a.e. } (y, z) \in \mathbb{R}^2.$$

1. Show that $F \in L^1(\mathbb{R}^2)$.

2. Show that

$$\int_{\mathbb{R}} \hat{\varphi}(z)\psi(z)\, dz = \int_{\mathbb{R}} \varphi(y)\hat{\psi}(y)\, dy \tag{42}$$

and that these integrals make sense.

2.3. Proof of Theorem 12.1. Let $f \in L^1(\mathbb{R}, \mathbb{C})$ be such that $\hat{f} \in L^1(\mathbb{R}, \mathbb{C})$. Define $h(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ for all $z \in \mathbb{R}$. It follows from 11.10 that

- $h \in C^\infty(\mathbb{R}) \cap L^1(\mathbb{R}, \mathbb{C})$
- $\hat{h} = h$ and $\int_{\mathbb{R}} h(z)\, dz = 1$

For $\epsilon > 0$, we define $h_\epsilon(z) := h(\epsilon z)$ for all $z \in \mathbb{R}$. We set $g_\epsilon = \widehat{h_\epsilon}$.

3. Show that $g_\epsilon(\xi) = \frac{1}{\epsilon} h\left(\frac{\xi}{\epsilon}\right)$ for all $\xi \in \mathbb{R}$.

4. Show that g_ϵ satisfies the hypothesis of Theorem 12.4.

We fix $x \in \mathbb{R}$.

5. Define $\varphi(y) := f(x - y)$ for all $y \in \mathbb{R}$. Show that

$$\hat{\varphi}(z) = e^{-ixz}\hat{f}(-z) \text{ for all } z \in \mathbb{R}.$$

6. Using (42) and the parity of h_ϵ, show that

$$\int_{\mathbb{R}} e^{ix\xi}\hat{f}(z)h_\epsilon(z)\, dz = f \star g_\epsilon(x)$$

7. Show that

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} e^{ixz}\hat{f}(z)h_\epsilon(z)\, dz = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixz}\hat{f}(z)\, dz$$

8. Show Theorem 12.1 by using Theorems 12.4 and 12.5. .

3. An application to the resolution of some ODEs

Let us take $f \in C^0(\mathbb{R})$. Is there $u \in C^2(\mathbb{R})$ such that

$$-u'' + u = f \text{ in } \mathbb{R}? \tag{43}$$

We are going to argue as in the begining of Chaper 11.

3.1. Step 1: Formal analysis. Here, the computations are made assuming that everything works. We have that

$$\widehat{-u''+u}=\hat{f} \quad\Rightarrow\quad -(i\xi)^2\hat{u}+\hat{u}=\hat{f} \;\Rightarrow\; (1+|\xi|^2)\hat{u}=\hat{f}$$

and then

$$\hat{u}(\xi)=\frac{\hat{f}(\xi)}{1+|\xi|^2}$$

With the inversion formula, this yields

$$u(x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{\hat{f}(\xi)}{1+|\xi|^2}e^{ix\xi}\,d\xi.$$

And this is going to be a good candidate to be a solution to our problem.

3.2. Step 2: Rigorous construction of u. We assume that

$$f\in L^1(\mathbb{R},\mathbb{C}) \tag{H_1}$$

and we define

$$u(x):=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{\hat{f}(\xi)}{1+|\xi|^2}e^{ix\xi}\,d\xi \text{ for all } x\in\mathbb{R}.$$

This makes sense: define $g:\xi\mapsto g(\xi):=\frac{\hat{f}(\xi)}{1+|\xi|^2}$, then

$$|g(\xi)|\le\frac{\|\hat{f}\|_\infty}{1+|\xi|^2}\in L^1(\mathbb{R},\mathbb{C}) \text{ since } 2>1 \text{ and } \hat{f}\in L^\infty.$$

So $u(x)$ is defined for all $x\in\mathbb{R}$. Define

$$F(x,\xi):=\frac{1}{\sqrt{2\pi}}\frac{\hat{f}(\xi)}{1+|\xi|^2}e^{ix\xi} \text{ for all } x,\xi\in\mathbb{R}$$

so that

$$u(x)=\int_{\mathbb{R}}F(x,\xi)\,d\xi \text{ for all } x\in\mathbb{R}.$$

Let us prove that u is continuous. We have that

- For all $x\in\mathbb{R}$, $F(x,\cdot)\in L^1(\mathbb{R})$
- For all $\xi\in X$, $x\mapsto F(x,\xi)$ is continuous
- For all $x,\xi\in\mathbb{R}$, we have that $|F(x,\xi)|\le|g(x)|$, and $g\in L^1(\mathbb{R},\mathbb{C})$

Therefore, it follows from Theorem 16.8 that $u\in C^0(\mathbb{R},\mathbb{C})$.

Remark: since g is even, with the change of variable, $\xi\mapsto-\xi$, we have that $u=\hat{g}$, and then u is continuous.

3.3. Step 3: Derivatives of u. Here, we use Theorem 16.9. Note that for all $\xi\in\mathbb{R}$, $x\mapsto F(x,\xi)\in C^\infty(\mathbb{R},\mathbb{C})$ and that

$$\frac{\partial F}{\partial x}(x,\xi)=\frac{1}{\sqrt{2\pi}}\frac{\hat{f}(\xi)}{1+|\xi|^2}i\xi e^{ix\xi} \text{ and } \frac{\partial^2 F}{\partial x^2}(x,\xi)=\frac{1}{\sqrt{2\pi}}\frac{\hat{f}(\xi)}{1+|\xi|^2}(i\xi)^2e^{ix\xi}$$

for all $x,\xi\in\mathbb{R}$. Let us find upper bounds:

$$\left|\frac{\partial F}{\partial x}(x,\xi)\right|\le\frac{1}{\sqrt{2\pi}}\frac{|\hat{f}(\xi)|}{1+|\xi|^2}|\xi|=\frac{1}{\sqrt{2\pi}}|\hat{f}(\xi)|\frac{|\xi|}{1+|\xi|^2}\le\frac{1}{\sqrt{2\pi}}|\hat{f}(\xi)|$$

for all $x, \xi \in \mathbb{R}$, and this upper-bound is independent of x. Similarly,

$$\left|\frac{\partial^2 F}{\partial x^2}(x,\xi)\right| \leq \frac{1}{\sqrt{2\pi}}\frac{|\hat{f}(\xi)|}{1+|\xi|^2}|\xi|^2 = \frac{1}{\sqrt{2\pi}}|\hat{f}(\xi)|\frac{|\xi|^2}{1+|\xi|^2} \leq \frac{1}{\sqrt{2\pi}}|\hat{f}(\xi)|$$

for all $x, \xi \in \mathbb{R}$, and this upper-bound is independent of x. In order to apply the theorem, we are assuming that

$$\hat{f} \in L^1(\mathbb{R}, \mathbb{C}) \tag{H_2}$$

Then, it follows from a double application of Theorem 16.9 that $u \in C^2(\mathbb{R}, \mathbb{C})$ and that

$$u'(x) = \int_{\mathbb{R}} \frac{\partial F}{\partial x}(x,\xi)\, d\xi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{\hat{f}(\xi)}{1+|\xi|^2} i\xi e^{ix\xi}\, d\xi \text{ for all } x \in \mathbb{R}.$$

$$u''(x) = \int_{\mathbb{R}} \frac{\partial^2 F}{\partial x^2}(x,\xi)\, d\xi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{\hat{f}(\xi)}{1+|\xi|^2} (i\xi)^2 e^{ix\xi}\, d\xi \text{ for all } x \in \mathbb{R}.$$

3.4. Step 4: u is a solution to (43). For $x \in \mathbb{R}$, we have that

$$\begin{aligned} -u''(x) + u(x) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{\hat{f}(\xi)}{1+|\xi|^2}\left(-(i\xi)^2+1\right) e^{ix\xi}\, d\xi \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{\hat{f}(\xi)}{1+|\xi|^2}\left(|\xi|^2+1\right) e^{ix\xi}\, d\xi \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi}\, d\xi \end{aligned}$$

Since $f, \hat{f} \in L^1(\mathbb{R}, \mathbb{C})$, the inversion formula yields

$$-u''(x) + u(x) = f(x) \text{ for a.e. } x \in \mathbb{R}.$$

Since $-u'' + u$ and f are continuous, we then get that the equality holds for all $x \in \mathbb{R}$, and then we have a solution to (43).

3.5. Step 5: About uniqueness. In our construction, $u = \hat{g}$ where $g \in L^1(\mathbb{R}, \mathbb{C})$, so

$$\lim_{x \to \pm\infty} u(x) = 0.$$

We claim that u is the only solution to (43) that goes to 0 at $\pm\infty$. Indeed, assume there is another such solution $v \in C^2(\mathbb{R}, \mathbb{C})$, then $w := u - v \in C^2$ is such that

$$-w'' + w = 0 \text{ in } \mathbb{R} \text{ and } \lim_{x \to \pm\infty} w(x) = 0.$$

So there exist $\lambda, \mu \in \mathbb{R}$ such that $w(x) = \lambda e^x + \mu e^{-x}$ for all $x \in \mathbb{R}$. If $\lambda \neq 0$, then $\lim_{x \to +\infty} |w(x)| = +\infty$, which is a contradiction. So $\lambda = 0$. If $\mu \neq 0$, then $\lim_{x \to -\infty} |w(x)| = +\infty$, which is a contradiction. So $w = 0$ and then $u = v$ which proves uniqueness.

We have proved the following theorem:

THEOREM 12.6. *Let $f \in C^0(\mathbb{R}, \mathbb{C})$ be such that $f, \hat{f} \in L^1(\mathbb{R}, \mathbb{C})$. Then there exists $u \in C^2(\mathbb{R}, \mathbb{C})$ such that*

$$\left\{ \begin{array}{cc} -u'' + u = f & \text{in } \mathbb{R} \\ \lim_{x \to \pm\infty} u(x) = 0 & \end{array} \right\}.$$

Moreover, such a solution is unique.

Exercise 50: Use the method above to prove that there is a unique solution $u \in C^2(\mathbb{R})$ to

$$\left\{ \begin{array}{ll} u'' + 2u' + u = e^{-|x|} & \text{in } \mathbb{R} \\ \lim_{x\to\pm\infty} u(x) = 0 & \end{array} \right\}. \tag{44}$$

4. Application to the heat equation

4.1. The context. Given $u_0 \in C^0(\mathbb{R}, \mathbb{C})$, we are investigating the existence of $u \in C^2((0, +\infty) \times \mathbb{R}) \cap C^0([0, +\infty) \times \mathbb{R})$ that is a solution to the heat equation

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 & \text{in } (0, +\infty) \times \mathbb{R} \\ u(0, x) = u_0(x) & \text{for all } x \in \mathbb{R}, \end{array} \right. \qquad (Heat)$$

It is much more classical to solve the heat equation on the circle via Fourier series. Here we present a method on $\mathbb{R}$, and we will see that the noncompactness yields some qualitative differences.

Here we have two variables, so this is a Partial Differential Equation (PDE) . We are going to perform the Fourier Transform with respect to the variable x (the "space" variable), leaving the time variable t unchanged. Then the PDE (in two variable) should be turned into an ODE (in the time variable).

4.2. Resolution of $(Heat)$. *Step 1: Formal resolution.* We perform the Fourier transform in the space variable, that is we define

$$\hat{u}(t, \xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(t, x) e^{-ix\xi}\, dx = \widehat{u(t, \cdot)}(\xi) \text{ for } t \geq 0 \text{ and } x \in \mathbb{R}. \tag{45}$$

Formally, we then have that

$$\widehat{\partial^2_{xx} u(t, \cdot)} = (i\xi)^2 \widehat{u(t, \cdot)} = -\xi^2 \hat{u}(t, \xi).$$

Differentiating (45) with respect to t, we get that

$$\frac{\partial}{\partial t} \hat{u}(t, \xi) = \frac{\partial}{\partial t} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(t, x) e^{-ix\xi}\, dx \right) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \partial_t u(t, x) e^{-ix\xi}\, dx = \widehat{\partial_t u(t, \cdot)}(\xi)$$

Then we rewrite $(Heat)$ as

$$0 = \hat{0} = \widehat{\partial_t u - \partial^2_{xx} u} = \partial_t \hat{u} + |\xi|^2 \hat{u}$$

so that

$$\partial_t \hat{u} + |\xi|^2 \hat{u} = 0 \text{ for } t \geq 0 \text{ and } \xi \in \mathbb{R}.$$

Freezing the space variable ξ, this is a linear first order ODE with respect to the time variable. And we can solve it: we get

$$\hat{u}(t, \xi) = \hat{u}(0, \xi) e^{-|\xi|^2 t}$$

With the initial condition $u(0, x) = u_0(x)$ for all $x \in \mathbb{R}$, we have that

$$\hat{u}(0, \xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(0, x) e^{-ix\xi}\, dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u_0(x) e^{-ix\xi}\, dx = \widehat{u_0}(\xi)$$

and then

$$\hat{u}(t, \xi) = \widehat{u_0}(\xi) e^{-|\xi|^2 t}.$$

With the inversion theorem, we then get the expression

$$u(t, x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{u}(t, \xi) e^{i\xi x}\, d\xi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{u_0}(\xi) e^{-|\xi|^2 t} e^{i\xi x}\, d\xi.$$

And this is going to be our candidate.
Step 2: Rigorous definition of $u(t,x)$ We assume that

$$u_0 \in L^1(\mathbb{R},\mathbb{C}) \text{ and } \widehat{u_0} \in L^1(\mathbb{R},\mathbb{C}).$$

We define

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{u_0}(\xi) e^{-|\xi|^2 t} e^{i\xi x}\, d\xi \text{ for all } t \geq 0 \text{ and } x \in \mathbb{R}.$$

This well-defined since, by defining

$$F(t,x,\xi) := \widehat{u_0}(\xi) e^{-|\xi|^2 t} e^{i\xi x} \text{ for } t,x,\xi \in \mathbb{R},$$

we have that $|F(t,x,\xi)| \leq |\widehat{u_0}(\xi)|$ for $t \geq 0$, $x,\xi \in \mathbb{R}$, and since $\widehat{u_0} \in L^1(\mathbb{R},\mathbb{C})$, then

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} F(t,x,\xi)\, d\xi \tag{46}$$

is defined for all $t \geq 0$ and $x \in \mathbb{R}$.
Step 3: Continuity of u *on* $[0,+\infty) \times \mathbb{R}$ Here again, this is a straightforward application of Theorem 16.8. We have that:

- For all $\xi \in \mathbb{R}$, $(t,x) \mapsto F(t,x,\xi) = \widehat{u_0}(\xi) e^{-|\xi|^2 t} e^{i\xi x}$ is continuous on $[0,+\infty) \times \mathbb{R}$ (no difficulty here).
- For all $(t,x) \in [0,+\infty) \times \mathbb{R}$, $\xi \mapsto F(t,x,\xi)$ is integrable on $\mathbb{R}$
- For all $(t,x) \in [0,+\infty) \times \mathbb{R}$ and $\xi \in \mathbb{R}$, we have that $|F(t,x\xi)| \leq |\widehat{u_0}(\xi)|$, with $\widehat{u_0} \in L^1(\mathbb{R},\mathbb{C})$, and this upper-bound is independent of t,x.

It then follows from Theorem 16.8 that

$$u \in C^0([0,+\infty) \times \mathbb{R}).$$

Step 4: Differentiability of u *on* $(0,+\infty) \times \mathbb{R}$ Here, we will have to exclude $t=0$. For all $\xi \in \mathbb{R}$, we have that $(t,\xi) \mapsto F(t,x,\xi)$ is C^2 (and even C^∞) and

$$\begin{aligned}
\frac{\partial F}{\partial x}(t,x,\xi) &= (i\xi)\widehat{u_0}(\xi) e^{-|\xi|^2 t} e^{i\xi x} \\
\frac{\partial^2 F}{\partial x^2}(t,x,\xi) &= (i\xi)^2\widehat{u_0}(\xi) e^{-|\xi|^2 t} e^{i\xi x} \\
\frac{\partial F}{\partial t}(t,x,\xi) &= -|\xi|^2\widehat{u_0}(\xi) e^{-|\xi|^2 t} e^{i\xi x} \\
\frac{\partial F^2}{\partial t^2}(t,x,\xi) &= |\xi|^4\widehat{u_0}(\xi) e^{-|\xi|^2 t} e^{i\xi x}
\end{aligned}$$

for all $t,x,\xi \in \mathbb{R}$. Therefore, we multiply by a polynomial that might break the integrability. However, it can be compensated by the exponential for positive t.

Let us take $a > 0$. Then for $t > a$, $x,\xi \in \mathbb{R}$, we have that

$$\begin{aligned}
\left|\frac{\partial F}{\partial x}(t,x,\xi)\right| &= |\xi| \cdot |\widehat{u_0}(\xi)| e^{-|\xi|^2 t} \leq |\xi| e^{-a|\xi|^2} |\widehat{u_0}(\xi)| \\
\left|\frac{\partial^2 F}{\partial x^2}(t,x,\xi)\right| &= |\xi|^2 |\widehat{u_0}(\xi)| e^{-|\xi|^2 t} \leq |\xi|^2 e^{-a|\xi|^2} |\widehat{u_0}(\xi)| \\
\left|\frac{\partial F}{\partial t}(t,x,\xi)\right| &= |\xi|^2 |\widehat{u_0}(\xi)| e^{-|\xi|^2 t} \leq |\xi|^2 e^{-a|\xi|^2} |\widehat{u_0}(\xi)| \\
\left|\frac{\partial F^2}{\partial t^2}(t,x,\xi)\right| &= |\xi|^4 |\widehat{u_0}(\xi)| e^{-|\xi|^2 t} \leq |\xi|^4 e^{-a|\xi|^2} |\widehat{u_0}(\xi)|
\end{aligned}$$

Since $a > 0$, we have that

$$\lim_{X \to +\infty} |X|^k e^{-aX^2} = 0 \text{ for all } k \in \mathbb{N}.$$

Therefore, there exists $M > 0$ such that

$$|X|^k e^{-aX^2} \leq M \text{ for all } k = 0, ..., 4 \text{ and } X \in \mathbb{R}.$$

Therefore, we get that

$$\left|\frac{\partial F}{\partial x}(t,x,\xi)\right| \leq M|\widehat{u_0}(\xi)| \; ; \; \left|\frac{\partial^2 F}{\partial x^2}(t,x,\xi)\right| \leq M|\widehat{u_0}(\xi)|$$

$$\left|\frac{\partial F}{\partial t}(t,x,\xi)\right| \leq M|\widehat{u_0}(\xi)| \; ; \; \left|\frac{\partial F^2}{\partial t^2}(t,x,\xi)\right| \leq M|\widehat{u_0}(\xi)|$$

for all $t > a$ and $x, \xi \in \mathbb{R}$.

We are now in position to prove differentiability of u. Given $a > 0$, we have that

- For all $t > a$ and all $x \in \mathbb{R}$, $\xi \mapsto F(t,x,\xi)$ is integrable
- For all $\xi \in \mathbb{R}$, then $(t,x) \mapsto F(t,x,\xi)$ is C^2 on $(a,+\infty) \times \mathbb{R}$
- All the derivatives ≤ 2 in the (t,x) variables are controled independently of (t,x) by an L^1-function

It the follows from a double application of Theorem 16.9 that $u \in C^1((a,+\infty) \times \mathbb{R})$ for all $a > 0$. And then $u \in C^2((0,+\infty) \times \mathbb{R})$.
Step 5: solution to $(Heat)$ For $t > 0$ and $x \in \mathbb{R}$, it follows from the preceding step that

$$\begin{aligned}
\partial_t u(t,x) - \partial_{xx} u(t,x) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\partial_t F(t,x,\xi) - \partial_{xx} F(t,x,\xi)\right) d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(-|\xi|^2 + |\xi|^2\right) \widehat{u_0}(\xi) e^{-|\xi|^2 t} e^{i\xi x} \, d\xi = 0
\end{aligned}$$

for all $t > 0$ and $\xi \in \mathbb{R}$. Moreover, for all $x \in \mathbb{R}$, we have that

$$u(0,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} F(0,x,\xi)\, d\xi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{u_0}(\xi) e^{i\xi x}\, d\xi$$

Since $u_0, \widehat{u_0} \in L^1(\mathbb{R}, \mathbb{C})$, the inversion formula yields $u(0,x) = u_0(x)$ for a.e. $x \in \mathbb{R}$. Since $u(0,\cdot)$ is continuous (Step 3) and u_0 too, we then get that $u(0,x) = u_0(x)$ for all $x \in \mathbb{R}$. So we have found a solution to $(Heat)$:

THEOREM 12.7. *Let $u_0 \in C^0(\mathbb{R}, \mathbb{C})$ be such that*

$$u_0 \in L^1(\mathbb{R}, \mathbb{C}) \text{ and } \widehat{u_0} \in L^1(\mathbb{R}, \mathbb{C}).$$

Then there exists a function $u \in C^2((0,+\infty) \times \mathbb{R}) \cap C^0([0,+\infty) \times \mathbb{R})$ to the problem

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 & \text{in } (0,+\infty) \times \mathbb{R} \\ u(0,x) = u_0(x) & \text{for all } x \in \mathbb{R}, \end{cases} \tag{$Heat$}$$

In order to prove uniqueness, one should impose some asymptotic behavior, essentially an integrability condition.

4.3. Dispersion of the heat. There is an interesting feature here. Take u as it was obtained here. Then we claim that

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}|u(t,x)|=0.$$

We prove the claim. Just write

$$u(t,x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}F(t,x,\xi)\widehat{u_0}(\xi)e^{-|\xi|^2t}e^{i\xi x}\,d\xi$$

for all $t>0$ and $x\in\mathbb{R}$, with $F(t,x,\xi)=\widehat{u_0}(\xi)e^{-|\xi|^2t}e^{i\xi x}$ so that $|F(t,x,\xi)|\leq|\widehat{u_0}(\xi)|$ for all $t,x,\xi\in\mathbb{R}$. Since

$$\lim_{t\to+\infty}F(t,x,\xi)=0 \text{ for all } \xi\neq 0 \text{ and } x\in\mathbb{R},$$

it follows from the Lebesgue's Theorem 16.7 (take any sequence (t_n,x_n) such that $t_n\to\infty$) that $u(t,x)\to 0$ as $t\to+\infty$ uniformly with respect to $x\in\mathbb{R}$.

What does this mean? It mean that whatever your initial condition is, the temperature at any point is going to decrease to 0. The energy that you put initially (in u_0) is going to scatter on the full line $\mathbb{R}$. This is a big difference with the heat equation on a circle: via Fourier series, one proves that on a circle, the heat is going to evolve, and asymptotically, it will be the same everywhere, corresponding to the average of u_0. To summarize:

compact domain $\Rightarrow$ *temperature is averaged with time*

noncompact domain $\Rightarrow$ *temperature vanishes everywhere*

4.4. About regularity. There is an important remark that is reminiscent in evolution equations: despite u_0 is only continuous (which could be very bad, see the examples of nowhere differentiable continuous functions), the solution to the heat equation is C^2 as soon as $t>0$. We say that the heat equation is immediately smoothing, whatever the initial condition is.

It can even be worse: we just need $\widehat{u_0}\in L^1(\mathbb{R},\mathbb{C})$ to prove C^2 for $t>0$. So if we can define the Fourier transform for something that is not L^1, we should be able to prove that the resulting solution is smooth for $t>0$. It is possible to define the Fourier transform for L^2-functions, and even for distributions, and then, one can prove that the heat equation is smoothing.

There is a possibility to extend the derivatives of u til $t=0$. For instance, take the derivative along x: we have that

$$\frac{\partial F}{\partial x}(t,x,\xi)=(i\xi)\widehat{u_0}(\xi)e^{-|\xi|^2t}e^{i\xi x},$$

so we have that

$$\left|\frac{\partial F}{\partial x}(t,x,\xi)\right|\leq|\xi\widehat{u_0}(\xi)|$$

for all $x,\xi\in\mathbb{R}$ and $t\geq 0$. So if $\xi\mapsto\xi\widehat{u_0}(\xi)$ is in L^1, we can get a uniform control for $t\geq 0$, and then apply the derivation theorem for $t\geq 0$. With Proposition 11.6, if $u_0\in C^1(\mathbb{R},\mathbb{C})$ and is L^1, we have that

$$\widehat{u_0'}(\xi)=i\xi\widehat{u_0}(\xi)$$

and then

$$\left|\frac{\partial F}{\partial x}(t,x,\xi)\right|\leq|\widehat{u_0'}(\xi)| \text{ for } t\geq 0 \text{ and } x,\xi\in\mathbb{R}.$$

And then, we can apply the derivation and continuity theorems. We then get the following

THEOREM 12.8. *Let $u_0 \in C^0(\mathbb{R}, \mathbb{C})$ be such that*

$$u_0 \in C^4(\mathbb{R}, \mathbb{C}),\ u_0^{(k)}, \widehat{u_0^{(k)}} \in L^1(\mathbb{R}, \mathbb{C}) \text{ for } k = 0, ..., 4.$$

Then there exists a function $u \in C^2((0, +\infty) \times \mathbb{R}) \cap C^0([0, +\infty) \times \mathbb{R})$ to the problem

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 & \text{in } (0, +\infty) \times \mathbb{R} \\ u(0, x) = u_0(x) & \text{for all } x \in \mathbb{R}, \end{array} \right. \qquad (Heat)$$

Moreover, the derivatives of order ≤ 2 extend continuously to $[0, +\infty) \times \mathbb{R}$.

Exercise 51: Prove this theorem.

5. Application to the wave equation

5.1. The context. Given $u_0, u_1 \in C^0(\mathbb{R}, \mathbb{C})$, we are investigating the existence of $u \in C^2((0, +\infty) \times \mathbb{R})$ with derivatives extending continuously to $[0, +\infty) \times \mathbb{R}$ that is a solution to the wave equation

$$\left\{ \begin{array}{ll} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 & \text{in } (0, +\infty) \times \mathbb{R} \\ u(0, x) = u_0(x) & \text{for all } x \in \mathbb{R}, \\ \partial_t u(0, x) = u_1(x) & \text{for all } x \in \mathbb{R}, \end{array} \right. \qquad (Wave)$$

This is another classical Partial Differential Equation (PDE), namely the model of hyperbolic PDEs. The difference with the heat equation is that we have two derivatives in time: but this will make a huge difference.

For convenience here, we assume that

$$u_1 \equiv 0$$

Step 1: Formal resolution. Here again, we perform the Fourier transform in the space variable, that is we define

$$\hat{u}(t, \xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(t, x) e^{-ix\xi}\, dx = \widehat{u(t, \cdot)}(\xi) \text{ for } t \geq 0 \text{ and } x \in \mathbb{R}.$$

Formally, we then have that

$$\widehat{\partial^2_{xx} u(t, \cdot)} = (i\xi)^2 \widehat{u(t, \cdot)} = -\xi^2 \hat{u}(t, \xi).$$

Differentiating twice with respect to t, we get that

$$\frac{\partial^2}{\partial t^2} \hat{u}(t, \xi) = \frac{\partial^2}{\partial t^2} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(t, x) e^{-ix\xi}\, dx \right) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \partial_{tt} u(t, x) e^{-ix\xi}\, dx = \widehat{\partial_{tt} u(t, \cdot)}(\xi)$$

Then we rewrite $(Wave)$ as

$$0 = \hat{0} = \widehat{\partial_{tt} u - \partial^2_{xx} u} = \partial_{tt} \hat{u} + |\xi|^2 \hat{u}$$

so that

$$\partial_{tt} \hat{u} + |\xi|^2 \hat{u} = 0 \text{ for } t \geq 0 \text{ and } \xi \in \mathbb{R}.$$

Freezing the space variable ξ, this is a linear second order ODE with respect to the time variable. And we can solve it: when $\xi \neq 0$ (everything is formal here), we get

$$\hat{u}(t, \xi) = A(\xi) \cos(\xi t) + B(\xi) \sin(\xi t)$$

With the initial condition $u(0,x)=u_0(x)$ and $\partial_t u(0,x)=0$ for all $x\in\mathbb{R}$, we have that

$$\hat{u}(0,\xi)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u(0,x)e^{-ix\xi}\,dx=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u_0(x)e^{-ix\xi}\,dx=\widehat{u_0}(\xi) \text{ and } \partial_t\hat{u}(0,x)=0$$

and then $A(\xi)=\widehat{u_0}(\xi)$ and $B(\xi)\xi=0$, so (formal...), $B(\xi)=0$

$$\hat{u}(t,\xi)=\widehat{u_0}(\xi)\cos(\xi t).$$

With the inversion theorem, we then get the expression

$$u(t,x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\hat{u}(t,\xi)e^{i\xi x}\,d\xi=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\widehat{u_0}(\xi)\cos(\xi t)e^{i\xi x}\,d\xi.$$

And this is going to be our candidate.

Step 2: Rigorous definition of $u(t,x)$ We assume that

$$u_0\in L^1(\mathbb{R},\mathbb{C}) \text{ and } \widehat{u_0}\in L^1(\mathbb{R},\mathbb{C}).$$

We define

$$u(t,x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\widehat{u_0}(\xi)\cos(\xi t)e^{i\xi x}\,d\xi \text{ for all } t\in\mathbb{R} \text{ and } x\in\mathbb{R}.$$

This well-defined since, by defining

$$F(t,x,\xi):=\widehat{u_0}(\xi)\cos(\xi t)e^{i\xi x} \text{ for } t,x,\xi\in\mathbb{R},$$

we have that $|F(t,x\xi)|\leq|\widehat{u_0}(\xi)|$ for $t\in\mathbb{R}$, $x,\xi\in\mathbb{R}$, and since $\widehat{u_0}\in L^1(\mathbb{R},\mathbb{C})$, then

$$u(t,x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}F(t,x,\xi)\,d\xi$$

is defined for all $t\in\mathbb{R}$ and $x\in\mathbb{R}$. Note that here, the definition makes sense for all $t\in\mathbb{R}$

Step 3: Continuity of u on $\mathbb{R}\times\mathbb{R}$ Here again, this is a straightforward application of Theorem 16.8. We have that:

- For all $\xi\in\mathbb{R}$, $(t,x)\mapsto F(t,x,\xi)=\widehat{u_0}(\xi)\cos(\xi t)e^{i\xi x}$ is continuous on $\mathbb{R}\times\mathbb{R}$ (no difficulty here).
- For all $(t,x)\in\mathbb{R}\times\mathbb{R}$, $\xi\mapsto F(t,x,\xi)$ is integrable on $\mathbb{R}$
- For all $(t,x)\in\mathbb{R}\times\mathbb{R}$ and $\xi\in\mathbb{R}$, we have that $|F(t,x\xi)|\leq|\widehat{u_0}(\xi)|$, with $\widehat{u_0}\in L^1(\mathbb{R},\mathbb{C})$, and this upper-bound is independent of t,x.

It then follows from Theorem 16.8 that

$$u\in C^0(\mathbb{R}\times\mathbb{R}).$$

Step 4: Differentiability of u on $\mathbb{R}\times\mathbb{R}$ For all $\xi\in\mathbb{R}$, we have that $(t,\xi)\mapsto F(t,x,\xi)$ is C^2 (and even C^∞) and

$$\begin{aligned}
\frac{\partial F}{\partial x}(t,x,\xi) &= (i\xi)\widehat{u_0}(\xi)\cos(\xi t)e^{i\xi x}\\
\frac{\partial^2 F}{\partial x^2}(t,x,\xi) &= (i\xi)^2\widehat{u_0}(\xi)\cos(\xi t)e^{i\xi x}\\
\frac{\partial F}{\partial t}(t,x,\xi) &= -\xi\widehat{u_0}(\xi)\sin(\xi t)e^{i\xi x}\\
\frac{\partial^2 F}{\partial t^2}(t,x,\xi) &= -\xi^2\widehat{u_0}(\xi)\cos(\xi t)e^{i\xi x}
\end{aligned}$$

for all $t, x, \xi \in \mathbb{R}$. Unlike the case of the heat equation, there is no way the cosine will compensate the polynomial. So we are going to argue as in Exercise 51. For $t \in \mathbb{R}$, $x, \xi \in \mathbb{R}$, we have that

$$\left|\frac{\partial F}{\partial x}(t,x,\xi)\right| \leq |\xi||\widehat{u_0}(\xi)| \; ; \; \left|\frac{\partial^2 F}{\partial x^2}(t,x,\xi)\right| \leq |\xi|^2|\widehat{u_0}(\xi)|$$
$$\left|\frac{\partial F}{\partial t}(t,x,\xi)\right| \leq |\xi||\widehat{u_0}(\xi)| \; ; \; \left|\frac{\partial^2 F}{\partial t^2}(t,x,\xi)\right| \leq |\xi|^2|\widehat{u_0}(\xi)|$$

With Proposition 11.6, if $u_0 \in C^2(\mathbb{R}, \mathbb{C})$ and $u_0', u_0'' \in L^1(\mathbb{R}, \mathbb{C})$, we have that

$$\widehat{u_0'}(\xi) = i\xi\widehat{u_0}(\xi) \text{ and } \widehat{u_0''}(\xi) = -\xi^2\widehat{u_0}(\xi)$$

and then

$$\left|\frac{\partial F}{\partial x}(t,x,\xi)\right| \leq |\widehat{u_0'}(\xi)| \; ; \; \left|\frac{\partial^2 F}{\partial x^2}(t,x,\xi)\right| \leq |\widehat{u_0''}(\xi)|$$
$$\left|\frac{\partial F}{\partial t}(t,x,\xi)\right| \leq |\widehat{u_0'}(\xi)| \; ; \; \left|\frac{\partial^2 F}{\partial t^2}(t,x,\xi)\right| \leq |\widehat{u_0''}(\xi)|$$

So assume that

$$\widehat{u_0'},\, \widehat{u_0''} \in L^1(\mathbb{R}, \mathbb{C}).$$

Then, we can apply the derivation theorem:

- For all $t \in \mathbb{R}$ and all $x \in \mathbb{R}$, $\xi \mapsto F(t, x, \xi)$ is integrable
- For all $\xi \in \mathbb{R}$, then $(t, x) \mapsto F(t, x, \xi)$ is C^2 on $\mathbb{R} \times \mathbb{R}$
- All the derivatives of order ≤ 2 in the (t, x) variables are controled independently of (t, x) by an L^1-function

It the follows from a double application of Theorem 16.9 that $u \in C^2(\mathbb{R} \times \mathbb{R}, \mathbb{C})$.
Step 5: solution to $(Wave)$ For $t \in \mathbb{R}$ and $x \in \mathbb{R}$, it follows from the preceding step that

$$\begin{aligned}
\partial_{tt}u(t,x) - \partial_{xx}u(t,x) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left(\partial_{tt}F(t,x,\xi) - \partial_{xx}F(t,x,\xi)\right) d\xi \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left(-|\xi|^2 + |\xi|^2\right) \widehat{u_0}(\xi)\cos(\xi t)e^{i\xi x}\, d\xi = 0
\end{aligned}$$

for all $t \in \mathbb{R}$ and $\xi \in \mathbb{R}$. Moreover, for all $x \in \mathbb{R}$, we have that

$$u(0,x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} F(0,x,\xi)\, d\xi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{u_0}(\xi)e^{i\xi x}\, d\xi$$

Since $u_0, \widehat{u_0} \in L^1(\mathbb{R}, \mathbb{C})$, the inversion formula yields $u(0, x) = u_0(x)$ for a.e. $x \in \mathbb{R}$. Since $u(0, \cdot)$ is continuous (Step 3) and u_0 too, we then get that $u(0, x) = u_0(x)$ for all $x \in \mathbb{R}$. Concerning the derivative at 0, we have that

$$\partial_t u(0,x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \partial_t F(0,x,\xi)\, d\xi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{u_0}(\xi)(-\xi)\sin(\xi \times 0)e^{i\xi x}\, d\xi = 0$$

for all $x \in \mathbb{R}$. So we have found a solution to $(Wave)$:

THEOREM 12.9. *Let $u_0 \in C^2(\mathbb{R}, \mathbb{C})$ be such that*

$$u_0^{(k)} \in L^1(\mathbb{R}, \mathbb{C}) \textit{ and } \widehat{u_0^{(k)}} \in L^1(\mathbb{R}, \mathbb{C}) \textit{ for all } k = 0, 1, 2.$$

Then there exists a function $u \in C^2(\mathbb{R} \times \mathbb{R})$ *to the wave equation*

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 & \text{in } (0,+\infty) \times \mathbb{R} \\ u(0,x) = u_0(x) & \text{for all } x \in \mathbb{R}, \\ \partial_t u(0,x) = 0 & \text{for all } x \in \mathbb{R} \end{cases} \qquad (Wave_0)$$

Exercise 52: Prove a similar result for the general case $u_1 \not\equiv 0$.

6. Comparison between the heat equation and the wave equation

6.1. About the hypothesis of the existence theorems. For the heat equation, we have proved in Theorem 12.7 that there is a solution $u \in C^2((0,+\infty) \times \mathbb{R}) \cap C^0([0,+\infty) \times \mathbb{R})$ to $(Heat)$ as soon as

$$u_0 \in C^0(\mathbb{R}) \cap L^1(\mathbb{R}) \text{ and } \hat{u_0} \in L^1(\mathbb{R}, \mathbb{C})$$

Moreover, as one checks, the same arguments as in the proof of Theorme 12.7 show that $u \in C^k((0,+\infty) \times \mathbb{R})$ for all $k \in \mathbb{N}$. Therefore, even with not much regulrity on the initial value u_0, we get the maximal regularity for solutions to the heat equation immediately when $t > 0$.

The situation is drastically different for the wave equation. In order to get C^2-regularity, we needed to impose a lot of regularity to u_0: C^2, all derivatives in L^1, and the Fourier transform of the derivatives all in L^1. Would we want to get solutions in C^k, $k > 2$, we would need to assume even more regularity and integrability for u_0. This is a stricking difference with the heat equation.

6.2. Is it reasonable to have such a difference of behavior? As good scientists, we must ask ourselves the following question:

"Were we not smart enough to get a better regularity of solutions to the wave equation?"

fortunately for us, the answer is "NO"!!! The heat equation is immediately regularizing, but the wave equation cannot decently be.

For instance, look at a wave in the sea (governed by... the wave equation). In a perfect world, a wave is supposed to keep the same shape, and to move on the sea: so, if the shape is odd, the shape will stay odd. In mathematical terms: if the initial shape is described by a non-smooth function, then time after time, the function describing the shape will stay non-smooth. We say that the singularities are preserved. A more dramatic example is a tsunami: initially, you have a wave that has more or less the shape of a wall, and, unfortunately, when the wave reaches the beach, it still has the same shape (in reality, this is a bit smoothened, but the consequence are still dramatic)

6.3. A classification of the PDEs.

7. Model of a flame

We want to answer here the following question:

Take a bar of iron with constant temperature, say 20 degrees Celsius. Put a flame at a point, and light it off. How does the heat evolve on the bar?

The difficulty we have is the initial condition: initially, the temperature is 20 outside the point, and very high at one point. If we modelize with with a function, it would be constant almost everywhere, so we would not see the influence of the flame. So we want something that is constant a.e., and that carries a lot of energy at a given point. For simplicity, we will require the temperature to be 0 everywhere except at one point.

Here is a first approximation. For $k \geq 1$, we define $f_k \in C^0(\mathbb{R})$ by

- $f_k(x) = 0$ for $|x| \geq 1/k$
- $f_k(0) = k$,
- f_k is affine on $\left[-\frac{1}{k}, 0\right]$ and on $\left[0, \frac{1}{n}\right]$.

This is a triangle. We have already seen one: denoting as f the triangle function of in Exercise 44, we have that

$$f_k(x) = kf(kx) \text{ for all } k \in \mathbb{N} \text{ and } x \in \mathbb{R}.$$

As one checks, the family $(f_k)_k$ has the following properties:

- $\lim_{k\to+\infty} f_k(0) = +\infty$,
- $\lim_{k\to+\infty} f_k(x) = 0$ for all $x \neq 0$,
- $$\|f_k\|_1 = \int_{\mathbb{R}} f_k(x)\, dx = 1 \text{ for all } k \in \mathbb{N}$$

So

$$\lim_{k\to+\infty} f_k(x) = \infty \cdot \mathbf{1}_0(x) \text{ for all } x \in \mathbb{R},$$

but this is not a handlable function, and it does not reflect that f_k carries some "energy", say the L^1-norm. The good point of view is to test it on functions:

Proposition 12.10. *For any $\varphi \in C_c^\infty(\mathbb{R})$, we have that*

$$\lim_{k\to+\infty} \int_{\mathbb{R}} f_k(x)\varphi(x) = \varphi(0).$$

When it makes sense, for a function $h \in L^1_{loc}(\mathbb{R})$(locally integrable, that is $f \in L^1(-R, R)$ for all $R > 0$), we define

$$T_h: \begin{array}{ccc} C_c^\infty(\mathbb{R}) & \to & \mathbb{R} \\ \varphi & \mapsto & \int_{\mathbb{R}} h\varphi\, dx \end{array} \quad \text{and} \quad \delta_0: \begin{array}{ccc} C_c^\infty(\mathbb{R}) & \to & \mathbb{R} \\ \varphi & \mapsto & \varphi(0) \end{array}$$

The above proposition then rewrites

$$\lim_{k\to+\infty} T_{f_k}(\varphi) = \delta_0(\varphi) \text{ for all } \varphi \in C_c^\infty(\mathbb{R}),$$

which is just a pointwise convergence. Here, δ_0 stands for the flame at the point 0. We prove the following theorem:

Theorem 12.11. *Let $(u_k)_k \in \mathbb{N}$ such that $u_k \in C^2((0, +\infty) \times \mathbb{R}) \cap C^0([0, +\infty) \times \mathbb{R})$ is the solution of the heat equation*

$$\begin{cases} \frac{\partial u_k}{\partial t} - \frac{\partial^2 u_k}{\partial x^2} = 0 & \text{in } (0, +\infty) \times \mathbb{R} \\ u_k(0, x) = f_k(x) & \text{for all } x \in \mathbb{R} \end{cases}$$

given by Theorem 12.7. Then there exists $u \in C^2((0, +\infty) \times \mathbb{R})$ such that for all $a > 0$, $u_k \to u$ in $C^2((a, +\infty) \times \mathbb{R})$. Moreover, for all $t > 0$, $u(t, \cdot) \in L^1_{loc}(\mathbb{R})$ and we have that

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 & \text{in } (0, +\infty) \times \mathbb{R} \\ \lim_{t\to 0} T_{u(t,\cdot)} = \delta_0 & \text{for the pointwise convergence.} \end{cases}$$

So, with a slight abuse of notation, we have an initial condition "$u(0,\cdot) = \delta_0$"

Note that since $f_k \in C^0(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\hat{f}_k = \hat{f}(k^{-1}\cdot) \in L^1(\mathbb{R}, \mathbb{C})$ since $\hat{f} \in L^1$ (see Exercise 44, then the existence of u_k follows from Theorem 12.7. The explicit expression is

$$u_k(x,t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f_k}(\xi) e^{-|\xi|^2 t} e^{i\xi x}\, d\xi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} F_k(x,t,\xi)\, d\xi$$

where

$$F_k(x,t,\xi) := \widehat{f}(\frac{\xi}{k}) e^{-|\xi|^2 t} e^{i\xi x}\, d\xi$$

Given $a > 0$, we have that

$$|F_k(x,t,\xi)| \leq \|\hat{f}\|_\infty e^{-a\xi^2} \in L^1 \text{ for all } k, x, \xi, t > a$$

$$\lim_{k\to+\infty} F_k(x,t,\xi) = \widehat{f}(0) e^{-|\xi|^2 t} e^{i\xi x} \text{ for all } x, \xi, t > 0$$

It then follows from Lebesgue's Theorem that

$$\lim_{k\to+\infty} u_k(x,t) = \frac{1}{\sqrt{2\pi}} \hat{f}(0) \int_{\mathbb{R}} e^{-|\xi|^2 t} e^{i\xi x}\, d\xi := u(x,t) \text{ for all } x, t \in \mathbb{R},\ t > a.$$

An application of the continuity and the differentiability theorems (we have a good estimate of the derivatives of F_k) then yields $u \in C^2((a,+\infty)\times\mathbb{R})$ for all $a > 0$ and therefore

$$u \in C^2((0,+\infty)\times\mathbb{R})$$

with

$$\begin{aligned}
\partial_t u(x,t) &= \frac{1}{\sqrt{2\pi}} \hat{f}(0) \int_{\mathbb{R}} (-\xi^2) \cdot e^{-|\xi|^2 t} e^{i\xi x}\, d\xi \\
\partial_{xx} u(x,t) &= \frac{1}{\sqrt{2\pi}} \hat{f}(0) \int_{\mathbb{R}} (i\xi)^2 e^{-|\xi|^2 t} e^{i\xi x}\, d\xi
\end{aligned}$$

so that

$$\partial_t u - \partial_{xx} u = 0 \text{ in } (0,+)\times\mathbb{R}.$$

In addition, due to the uniform control of F_k and its derivatives, the convergence of u_k to u holds uniformly in $C^2((a,+\infty)\times\mathbb{R})$ for all $a > 0$ (the details are left to the reader).

Let us now study the behavior when $t \to 0$. Since $u(t,\cdot) \in C^0(\mathbb{R})$ for all $t > 0$, then it is bounded on all compact set of $\mathbb{R}$, so $u(t,\cdot) \in L^1_{loc}(\mathbb{R})$ and $T_{u(t,\cdot)}$ makes sense. Let us take $\varphi \in C_c^\infty(\mathbb{R})$. We have that

$$T_{u(t,\cdot)}(\varphi) = \int_{\mathbb{R}} u(t,x)\varphi(x)\, dx = \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi}} \hat{f}(0) \int_{\mathbb{R}} e^{-|\xi|^2 t} e^{i\xi x}\, d\xi \right) \varphi(x)\, dx$$

Since $(x,t) \mapsto e^{-|\xi|^2 t} e^{i\xi x} \varphi(x) \in L^1(\mathbb{R}^2)$ for all $t > 0$(immediate computation), with Fubini's theorem, we get that

$$\begin{aligned}
\int_{\mathbb{R}} u(t,x)\varphi(x)\, dx &= \frac{1}{\sqrt{2\pi}} \hat{f}(0) \int_{\xi\in\mathbb{R}\mathbb{R}} \left(\int_{x\in\mathbb{R}} e^{-|\xi|^2 t} e^{i\xi x} \varphi(x)\, dx \right) d\xi \\
&= \frac{1}{\sqrt{2\pi}} \hat{f}(0) \int_{\xi\in\mathbb{R}} e^{-\xi^2 t} \left(\int_{x\in\mathbb{R}} e^{i\xi x} \varphi(x)\, dx \right) d\xi \\
&= \hat{f}(0) \int_{\xi\in\mathbb{R}} e^{-\xi^2 t} \hat{\varphi}(-\xi)\, d\xi
\end{aligned}$$

In order to let $t \to 0$, let us prove that $\hat{\varphi} \in L^1(\mathbb{R})$. Since φ has compact support, we have that $\varphi', \varphi'' \in L^1(\mathbb{R})$ and therefore

$$\widehat{\varphi'} = (i\xi)\hat{\varphi}(\xi) \text{ and } \widehat{\varphi''} = (i\xi)^2\hat{\varphi}(\xi)$$

for all $\xi \in \mathbb{R}$, and then

$$(1+\xi^2)|\hat{\varphi}(\xi)| = |\hat{\varphi}(\xi)| + |\widehat{\varphi''}(\xi)| \leq \|\hat{\varphi}\|_\infty + \|\widehat{\varphi''}\|_\infty \text{ for all } \xi \in \mathbb{R}$$

so there exists $M > 0$ such that $|\hat{\varphi}(\xi)| \leq \frac{M}{1+\xi^2}$ for all $\xi \in \mathbb{R}$, and then $\hat{\varphi} \in L^1(\mathbb{R})$. Therefore

$$|e^{-\xi^2 t}\hat{\varphi}(-\xi)| \leq |\hat{\varphi}(-\xi) \in L^1 \text{ independent of } t$$

and then, letting $t \to 0$. Lebesgue's theorem yields

$$\lim_{t\to 0}\int_{\mathbb{R}} u(t,x)\varphi(x)\,dx = \hat{f}(0)\int_{\xi\in\mathbb{R}}\hat{\varphi}(-\xi)\,d\xi$$

Since $\varphi, \hat{\varphi} \in L^1$ and $\varphi \in C^0(\mathbb{R})$, the inversion formula yields

$$\begin{aligned}\lim_{t\to 0}\int_{\mathbb{R}} u(t,x)\varphi(x)\,dx &= \frac{\int_{\mathbb{R}} f\,dx}{\sqrt{2\pi}}\int_{\xi\in\mathbb{R}}\hat{\varphi}(-\xi)\,d\xi \\ &= \frac{\int_{\mathbb{R}} f\,dx}{\sqrt{2\pi}}\int_{\xi\in\mathbb{R}}\hat{\varphi}(\xi)\,d\xi = (\int_{\mathbb{R}} f\,dx)\varphi(0) = \varphi(0).\end{aligned}$$

This concludes the proof of Theorem 12.11.

Let us now study the behavior when $t \to +\infty$. The explicit expression of $u(x,t)$ and Lebesgue's convergence theorem immediately yields

$$\lim_{t\to+\infty} u(t,x) = 0 \text{ uniformly for } x \in \mathbb{R}.$$

What does it mean? That when you put a flame on a bar, after you light it off, the temperature of the bar will go back to the normal : the heat will scatter everywhere on the compact domain. Here again, this is due to the non-compactness of the domain: a similar analysis on an annulus would result in a uniform repartition of the enrgy of the flame.

8. Solution of the exercises of Chapter 12

Solution of "An intermediate Lemma":

1. Since $|F| \geq 0$ is measurable, then Fubini's theorem yields

$$\begin{aligned}\int_{\mathbb{R}^2}|F(y,z)|\,dydz &= \int_{y\in\mathbb{R}}\left(\int_{z\in\mathbb{R}}|F(y,z)|\,dz\right)dy = \int_{y\in\mathbb{R}}|\varphi(y)|\left(\int_{z\in\mathbb{R}}|\psi(z)|\,dz\right)dy \\ &= \left(\int_{z\in\mathbb{R}}|\psi(z)|\,dz\right)\left(\int_{y\in\mathbb{R}}|\varphi(y)|\,dy\right) < \infty \text{ since } \varphi,\psi \in L^1(\mathbb{R}).\end{aligned}$$

Then $F \in L^1(\mathbb{R}^2)$.

2. Since $F \in L^1(\mathbb{R}^2, \mathbb{C})$, Fubini's theorem yields

$$\int_{y\in\mathbb{R}}\left(\int_{z\in\mathbb{R}}F(y,z)\,dz\right)dy = \int_{z\in\mathbb{R}}\left(\int_{y\in\mathbb{R}}F(y,z)\,dy\right)dz,$$

and all these integrals are defined. Therefore, the explicit expression of F yields

$$\int_{y\in\mathbb{R}}\varphi(y)\left(\int_{z\in\mathbb{R}}\psi(z)e^{-iyz}\,dz\right)dy = \int_{z\in\mathbb{R}}\psi(z)\left(\int_{y\in\mathbb{R}}\varphi(y)e^{-iyz}\,dy\right)dz.$$

Then the definition of the Fourier Transform yields

$$\int_{y\in\mathbb{R}} \varphi(y)\hat{\psi}(y)\,dy = \int_{z\in\mathbb{R}} \psi(z)\hat{\varphi}(z)\,dz.$$

All this makes sens via Fubini's theorem. Moreover, note that all these integrals make sense since $\varphi, \psi \in L^1(\mathbb{R},\mathbb{C})$ ans $\hat{\varphi}, \hat{\psi} \in L^\infty(\mathbb{R},\mathbb{C})$. □

Solution of "Proof of Theorem 12.1":

3. This is an application of Proposition 11.3 .

4. We have that $g_\epsilon \geq 0$. For $\delta \geq 0$, we have that

$$\int_{|z|\geq r} g_\epsilon(z)\,dz = \int_{|z|\geq r} \frac{1}{\epsilon} h\left(\frac{z}{\epsilon}\right)\,dz = \int_{|y|\geq \frac{r}{\epsilon}} h(y)\,dy.$$

If $\delta = 0$, then $\int_{\mathbb{R}} g_\epsilon(z)\,dz = \int_{\mathbb{R}} h(y)\,dy = 1$. For $\delta > 0$, we have that

$$\int_{|z|\geq\delta} g_\epsilon(z)\,dz = \int_{\mathbb{R}} \mathbf{1}_{|y|\geq\frac{\delta}{\epsilon}} h(y)\,dy \to 0 \text{ as } \epsilon \to 0 \text{ by the dominated convergence theorem 16.7.}$$

5. This is direct computations:

$$\sqrt{2\pi}\hat{\varphi}(z) = \int_{\mathbb{R}} f(x-y)e^{-iyz}\,dy = \int_{\mathbb{R}} f(t)e^{-i(x-t)z}\,dt = e^{-ixz}\int_{\mathbb{R}} f(t)e^{itz}\,dt = \sqrt{2\pi}\hat{f}(-z)$$

6. Since $f, h_\epsilon \in L^1(\mathbb{R},\mathbb{C})$, apply $(\star)$ to get

$$\begin{aligned}(f\star g_\epsilon)(x) &= \int_{\mathbb{R}} f(x-y)g_\epsilon(y)\,dy = \int_{\mathbb{R}} \varphi(y)\hat{h_\epsilon}(y) = \int_{\mathbb{R}} \hat{\varphi}(z)h_\epsilon(z)\,dz\\ &= \int_{\mathbb{R}} e^{-ixz}\hat{f}(-z)h_\epsilon(z)\,dz = \int_{\mathbb{R}} e^{ixz}\hat{f}(z)h_\epsilon(-z)\,dz,\end{aligned}$$

and we conclude since h_ϵ is even.

7. We have that $|e^{ixz}\hat{f}(z)h_\epsilon(z)| = |\hat{f}(z)h_\epsilon(z)| \leq |\hat{f}(z)|$ for all $\epsilon > 0$ et pp $z \in \mathbb{R}$. Since $\hat{f} \in L^1(\mathbb{R},\mathbb{C})$ and $\lim_{\epsilon\to 0} h_\epsilon(z) = 1$ for all $z \in \mathbb{R}$, the result follows from Lebesgue's dominated convergence theorem 16.7.

8. Since $f \in L^1(\mathbb{R},\mathbb{C})$, we have that $(f\star g_\epsilon) \to f$ in $L^1(\mathbb{R})$ as $\epsilon \to 0$, and then $(f\star g_{1/k}) \to f$ in $L^1(\mathbb{R},\mathbb{C})$ as $k \to +\infty$. Therefore, there exists a subsequence $(f\star g_{1/j(k)})_k$ such that $\lim_{k\to+\infty}(f\star g_{1/j(k)})(x) = f(x)$ for a.e. $x \in \mathbb{R}$. The conclusion follows from the questions above.

Solution of Exercise 50:

Step 1: Formal analysis With (33), we have that

$$\widehat{u'' + 2u' + u} = \widehat{e^{-|x|}} \quad\Rightarrow\quad (i\xi)^2\hat{u} + 2i\xi\hat{u} + \hat{u} = \sqrt{\frac{2}{\pi}}\frac{1}{1+|\xi|^2} \;\Rightarrow\; (-|\xi|^2 + 2i\xi + 1)\hat{u} = \sqrt{\frac{2}{\pi}}\frac{1}{1+|\xi|^2}$$

and then

$$\hat{u}(\xi) = \sqrt{\frac{2}{\pi}}\frac{1}{(-|\xi|^2 + 2i\xi + 1)(1+|\xi|^2)}$$

With the inversion formula, this yields

$$u(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{e^{ix\xi}}{(-|\xi|^2 + 2i\xi + 1)(1+|\xi|^2)}\,d\xi.$$

with the change of variable $\xi \to -\xi$, we get that

$$u(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{e^{-ix\xi}}{(-|\xi|^2 - 2i\xi + 1)(1+|\xi|^2)}\, d\xi.$$

And this is going to be a good candidate to be a solution to our problem.

Step 2: Rigorous construction of u Define

$$g(\xi) := \sqrt{\frac{2}{\pi}}\frac{1}{(-|\xi|^2 - 2i\xi + 1)(1+|\xi|^2)} \text{ for } \xi \in \mathbb{R}.$$

We have that

$$\begin{aligned} |g(\xi)| &= \sqrt{\frac{2}{\pi}}\frac{1}{|-|\xi|^2 - 2i\xi + 1| \times (1+|\xi|^2)} = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{(|\xi|^2-1)^2 + 4\xi^2} \times (1+|\xi|^2)} \\ &= \sqrt{\frac{2}{\pi}}\frac{1}{(1+|\xi|^2)^2} \text{ for all } \xi \in \mathbb{R}. \end{aligned}$$

So $g \in L^1(\mathbb{R}, \mathbb{C})$ and we define

$$u(x) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} g(\xi)e^{-ix\xi}\, d\xi = \hat{g}(x) \text{ for all } x \in \mathbb{R}.$$

Instead of reproving the continuity as above, let us use directly the results on Fourier transform. It follows from Theorem 11.1 that $u \in C^0(\mathbb{R}, \mathbb{C})$

Step 3: Derivatives of u Here we use twice Proposition 11.7 (or once Proposition 11.8 with $n = 2$): we have that

$$|\xi g(\xi)| = \sqrt{\frac{2}{\pi}}\frac{|\xi|}{(1+|\xi|^2)^2} \leq \sqrt{\frac{2}{\pi}}\frac{1}{(1+|\xi|^2)^{\frac{3}{2}}} \text{ for all } \xi \in \mathbb{R}.$$

and

$$|\xi^2 g(\xi)| = \sqrt{\frac{2}{\pi}}\frac{|\xi|^2}{(1+|\xi|^2)^2} \leq \sqrt{\frac{2}{\pi}}\frac{1}{1+|\xi|^2} \text{ for all } \xi \in \mathbb{R}.$$

Since $3 > 1$ and $2 > 1$, then $\xi \to\mapsto \xi g(\xi)$ and $\xi \to\mapsto \xi^2 g(\xi)$ are both in $L^1(\mathbb{R}, \mathbb{C})$, and then by Proposition 11.7 (or Proposition 11.8 with $n = 2$), we get that $u \in C^2(\mathbb{R}, \mathbb{C})$ and that

$$u'(x) = -\widehat{i\xi g(\xi)} = \frac{1}{\pi}\int_{\mathbb{R}} \frac{-i\xi e^{-ix\xi}}{(-|\xi|^2 - 2i\xi + 1)(1+|\xi|^2)}\, d\xi$$

and

$$u''(x) = -\widehat{\xi^2 g(\xi)} = \frac{1}{\pi}\int_{\mathbb{R}} \frac{-\xi^2 e^{-ix\xi}}{(-|\xi|^2 - 2i\xi + 1)(1+|\xi|^2)}\, d\xi$$

Step 4: u is a solution to (44) First, since $u = \hat{g}$ with $g \in L^1(\mathbb{R}, \mathbb{C})$, it follows from Theorem 11.1 that

$$\lim_{x \to \pm\infty} u(x) = 0.$$

For $x \in \mathbb{R}$, we have that

$$\begin{aligned} u''(x) + 2iu'(x) + u(x) &= \frac{1}{\pi}\int_{\mathbb{R}} \left(-\xi^2 - 2i - xi + 1\right) \frac{e^{-ix\xi}}{(-|\xi|^2 - 2i\xi + 1)(1+|\xi|^2)}\, d\xi \\ &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{e^{-ix\xi}}{(1+|\xi|^2)}\, d\xi = \frac{1}{\pi}\int_{\mathbb{R}} \frac{e^{ix\xi}}{(1+|\xi|^2)}\, d\xi \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{e^{-|x|}} e^{ix\xi}\, d\xi. \end{aligned}$$

Since $x \mapsto e^{-|x|}$ and $\widehat{e^{-|x|}}$ are both in $L^1(\mathbb{R}, \mathbb{C})$, the inversion formula yields

$$u''(x) + 2u'(x) + u(x) = e^{-|x|} \text{ for a.e. } x \in \mathbb{R}.$$

Since all functions are continuous, we get equality for all $x \in \mathbb{R}$, and then we have a solution to (44).

Step 5: Uniqueness. Assume that we have another function $v \in C^2(\mathbb{R}, \mathbb{C})$ that is a solution to (44). Define $w := u - v \in C^2(\mathbb{R}, \mathbb{C})$, so that

$$w'' + 2w' + w = 0 \text{ in } \mathbb{R} \text{ and } \lim_{x\to\pm\infty} w(x) = 0.$$

The equation $r^2 + 2r + 1 = 0$ has a double zero, which is -1, so there exist $\lambda, \mu \in \mathbb{R}$ such that $w(x) = (\lambda x + \mu)e^{-x}$ for all $x \in \mathbb{R}$. If $\lambda \neq 0$, then $\lim_{x\to-\infty} |w(x)| = +\infty$, which is a contradiction. So $\lambda = 0$. If $\mu \neq 0$, then $\lim_{x\to-\infty} |w(x)| = +\infty$, which is a contradiction. So $\lambda = \mu = 0$ and $w = 0$. Then $u = v$ which proves uniqueness. $\square$

Solution of Exercise 51: We follow the proof of the resolution of the heat equation above and we keep the same notations. We define $u(t, x)$ for $t \geq 0$ and $x \in \mathbb{R}$ as in (46). The expressions of the derivatives of F yield

$$\begin{aligned} \left|\frac{\partial F}{\partial x}(t,x,\xi)\right| &= \left|(i\xi)\widehat{u_0}(\xi)e^{-|\xi|^2 t}e^{i\xi x}\right| \leq |\xi\widehat{u_0}(\xi)| \\ \left|\frac{\partial^2 F}{\partial x^2}(t,x,\xi)\right| &= \left|(i\xi)^2\widehat{u_0}(\xi)e^{-|\xi|^2 t}e^{i\xi x}\right| \leq \left|\xi^2\widehat{u_0}(\xi)\right| \\ \left|\frac{\partial F}{\partial t}(t,x,\xi)\right| &= \left|-|\xi|^2\widehat{u_0}(\xi)e^{-|\xi|^2 t}e^{i\xi x}\right| \leq \left|\xi^2\widehat{u_0}(\xi)\right| \\ \left|\frac{\partial F^2}{\partial t^2}(t,x,\xi)\right| &= \left||\xi|^4\widehat{u_0}(\xi)e^{-|\xi|^2 t}e^{i\xi x}\right| \leq \left|\xi^4\widehat{u_0}(\xi)\right| \end{aligned}$$

With the hypothesis and Proposition 11.9, we have that for any $k = 0, ..., 4$,

$$\left|\xi^k\widehat{u_0}(\xi)\right| = \left|\widehat{u_0^{(k)}}\right| \in L^1(\mathbb{R})$$

We already now that $u \in C^2((0, +\infty) \times \mathbb{R})$ and

$$\partial_t u(t,x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \partial_t F(t,x,\xi)\, d\xi \text{ und } \partial_{tt} u(t,x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \partial_{tt} F(t,x,\xi)\, d\xi$$

for $t > 0$ and $x \in \mathbb{R}$. The continuity Theorem 16.8 yields continuous extensions to $t \geq 0$ of $\partial_t u$ and $\partial_{tt} u$. This is similar for the derivatives with respect to x. $\square$

Solution of Exercise 52: We follow the same procedure as the case $u_1 \equiv 0$.

Step 1: Formal analysis. We get as above that

$$\partial_{tt}\hat{u} + |\xi|^2\hat{u} = 0 \text{ for } t \geq 0 \text{ and } \xi \in \mathbb{R}.$$

$$\hat{u}(t,\xi) = A(\xi)\cos(\xi t) + B(\xi)\sin(\xi t)$$

at least for $\xi \neq 0$(formal...). With the initial condition $u(0,x) = u_0(x)$ and $\partial_t u(0,x) = u_1(x)$ for all $x \in \mathbb{R}$, we have that

$$\hat{u}(0,\xi) = \widehat{u_0}(\xi) \text{ and } \partial_t\hat{u}(0,x) = \widehat{u_1}(\xi)$$

and then $A(\xi) = \widehat{u_0}(\xi)$ and $B(\xi)\xi = \widehat{u_1}(\xi)$, so

$$\hat{u}(t,\xi) = \widehat{u_0}(\xi)\cos(\xi t) + \widehat{u_1}(\xi)\frac{\sin(\xi t)}{\xi}.$$

With the inversion theorem, we then get the expression

$$u(t,x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{u}(t,\xi)e^{i\xi x}\,d\xi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{u_0}(\xi)\cos(\xi t)e^{i\xi x}\,d\xi + \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{u_1}(\xi)\frac{\sin(\xi t)}{\xi}e^{i\xi x}\,d\xi.$$

And this is going to be our candidate.
Step 2: Rigorous definition of $u(t,x)$ We assume that

$$u_i \in L^1(\mathbb{R},\mathbb{C}) \text{ and } \widehat{u_i} \in L^1(\mathbb{R},\mathbb{C}) \text{ for } i = 0,1.$$

We define

$$u(t,x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{u_0}(\xi)\cos(\xi t)e^{i\xi x}\,d\xi + \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{u_1}(\xi)\frac{\sin(\xi t)}{\xi}e^{i\xi x}\,d\xi \text{ for all } t \in \mathbb{R} \text{ and } x \in \mathbb{R}.$$

The first term

$$v(t,x) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{u_0}(\xi)\cos(\xi t)e^{i\xi x}\,d\xi$$

has already been studied and is well-defined. We focus on the second one and we define

$$F_1(t,x,\xi) := \widehat{u_1}(\xi)\frac{\sin(\xi t)}{\xi}e^{i\xi x} \text{ for } t,x,\xi \in \mathbb{R},\ \xi \neq 0$$

, and, since $\lim_{X\to 0}\frac{\sin(X)}{X} = 1$, we set

$$F_1(t,x,0) := \widehat{u_1}(0)t \text{ for } t,x \in \mathbb{R}.$$

Noting that $|\sin(X)| \leq |X|$ for all $X \in \mathbb{R}$, we have that $|F_1(t,x,\xi)| \leq |\widehat{u_1}(\xi)|\cdot|t|$ for $t \in \mathbb{R}$, $x,\xi \in \mathbb{R}$, and since $\widehat{u_1} \in L^1(\mathbb{R},\mathbb{C})$, then

$$w(t,x) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} F_1(t,x,\xi)\,d\xi$$

is defined for all $t \in \mathbb{R}$ and $x \in \mathbb{R}$. So $u(t,x) = v(t,x) + w(t,x)$ is defined for all $t,x \in \mathbb{R}$.
Step 3: Continuity of u on $\mathbb{R}\times\mathbb{R}$. We focus only on v and we use that for all $x,t \in \mathbb{R}$ and $t \in]-R,R[$, we have that $|F_1(t,x,\xi)| \leq |\widehat{u_1}(\xi)|\cdot R$. The continuity of w on $]-R,R[\times\mathbb{R}$ (and then on $\mathbb{R}\times\mathbb{R}$) is a straightforward application of Theorem 16.8 as above. So u is continuous too.

Step 4: Differentiability of u on $\mathbb{R}\times\mathbb{R}$ Here again, we focus on w and F_1 since v has already been studied. For all $\xi\in\mathbb{R}$, we have that $(t,\xi)\mapsto F_1(t,x,\xi)$ is C^2 (and even C^∞) and

$$\begin{aligned}
\frac{\partial F_1}{\partial x}(t,x,\xi) &= (i\xi)\widehat{u_1}(\xi)\frac{\sin(\xi t)}{\xi}e^{i\xi x}\\
\frac{\partial^2 F_1}{\partial x^2}(t,x,\xi) &= (i\xi)^2\widehat{u_1}(\xi)\frac{\sin(\xi t)}{\xi}e^{i\xi x}\\
\frac{\partial F_1}{\partial t}(t,x,\xi) &= \widehat{u_1}(\xi)\frac{\xi\cos(\xi t)}{\xi}e^{i\xi x}\\
\frac{\partial^2 F_1}{\partial t^2}(t,x,\xi) &= \widehat{u_1}(\xi)\frac{-\xi^2\sin(\xi t)}{\xi}e^{i\xi x}
\end{aligned}$$

for all $t,x,\xi\in\mathbb{R}$. Then for $t\in\mathbb{R}$, $x,\xi\in\mathbb{R}$, we have that

$$\begin{aligned}
\left|\frac{\partial F_1}{\partial x}(t,x,\xi)\right| \leq |\widehat{u_1}(\xi)|\ ;\ \left|\frac{\partial^2 F_1}{\partial x^2}(t,x,\xi)\right| \leq |\xi|\cdot|\widehat{u_1}(\xi)|\\
\left|\frac{\partial F_1}{\partial t}(t,x,\xi)\right| \leq |\widehat{u_1}(\xi)|\ ;\ \left|\frac{\partial^2 F_1}{\partial t^2}(t,x,\xi)\right| \leq |\xi|\cdot|\widehat{u_1}(\xi)|
\end{aligned}$$

With Proposition 11.6, if $u_1\in C^1(\mathbb{R},\mathbb{C})$ and $u_1'\in L^1(\mathbb{R},\mathbb{C})$, we have that $\widehat{u_1'}(\xi)=i\xi\widehat{u_0}(\xi)$. Then

$$\begin{aligned}
\left|\frac{\partial F_1}{\partial x}(t,x,\xi)\right| \leq |\widehat{u_1}(\xi)|\ ;\ \left|\frac{\partial^2 F_1}{\partial x^2}(t,x,\xi)\right| \leq |\widehat{u_1'}(\xi)|\\
\left|\frac{\partial F_1}{\partial t}(t,x,\xi)\right| \leq |\widehat{u_1}(\xi)|\ ;\ \left|\frac{\partial^2 F_1}{\partial t^2}(t,x,\xi)\right| \leq |\widehat{u_1'}(\xi)|
\end{aligned}$$

So assume that $\widehat{u_1'}\in L^1(\mathbb{R},\mathbb{C})$. Then, we can apply the derivation theorem:

- For all $t\in\mathbb{R}$ and all $x\in\mathbb{R}$, $\xi\mapsto F_1(t,x,\xi)$ is integrable
- For all $\xi\in\mathbb{R}$, then $(t,x)\mapsto F_1(t,x,\xi)$ is C^2 on $\mathbb{R}\times\mathbb{R}$
- All the derivatives of order ≤ 2 in the (t,x) variables are controled independently of (t,x) by an L^1-function for $|t|<R$

It the follows from a double application of Theorem 16.9 that $w\in C^2(]-R,R[\times\mathbb{R},\mathbb{C})$ for all $R>0$, and then $w\in C^2(\mathbb{R}\times\mathbb{R},\mathbb{C})$ and so is $u=v+w$.

Step 5: solution to $(Wave)$ For $t\in\mathbb{R}$ and $x\in\mathbb{R}$, it follows from the preceding step that

$$\partial_{tt}u(t,x)-\partial_{xx}u(t,x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(\partial_{tt}F(t,x,\xi)-\partial_{xx}F(t,x,\xi)\right)d\xi=0$$

for all $t\in\mathbb{R}$ and $\xi\in\mathbb{R}$ via a direct computation. Moreover, for all $x\in\mathbb{R}$, we have that

$$u(0,x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}F(0,x,\xi)\,d\xi=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\widehat{u_0}(\xi)e^{i\xi x}\,d\xi$$

Since $u_0,\widehat{u_0}\in L^1(\mathbb{R},\mathbb{C})$, the inversion formula yields $u(0,x)=u_0(x)$ for a.e. $x\in\mathbb{R}$. Since $u(0,\cdot)$ is continuous (Step 3) and u_0 too, we then get that $u(0,x)=u_0(x)$ for all $x\in\mathbb{R}$. Concernign the derivative at 0, we have that

$$\partial_t u(0,x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\partial_t F(0,x,\xi)\,d\xi=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(\widehat{u_0}(\xi)(-\xi)\sin(\xi\times 0)+\widehat{u_1}(\xi)\cos(\xi\times 0))e^{i\xi x}\,d\xi=u_1(x)$$

for a.e. $x \in \mathbb{R}$ via the inversion formula. The continuity then yields equality everywhere. So we have found a solution to $(Wave)$:

THEOREM 12.12. *Let $u_0 \in C^2(\mathbb{R}, \mathbb{C})$ and $u_1 \in C^1(\mathbb{R}, \mathbb{C})$ be such that*

$$u_0^{(k)} \in L^1(\mathbb{R}, \mathbb{C}) \text{ and } \widehat{u_0^{(k)}} \in L^1(\mathbb{R}, \mathbb{C}) \text{ for all } k = 0, 1, 2.$$

$$u_1^{(k)} \in L^1(\mathbb{R}, \mathbb{C}) \text{ and } \widehat{u_1^{(k)}} \in L^1(\mathbb{R}, \mathbb{C}) \text{ for all } k = 0, 1.$$

Then there exists a function $u \in C^2(\mathbb{R} \times \mathbb{R})$ to the wave equation

$$\left\{ \begin{array}{ll} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 & \text{in } (0, +\infty) \times \mathbb{R} \\ u(0, x) = u_0(x) & \text{for all } x \in \mathbb{R}, \\ \partial_t u(0, x) = u_1(x) & \text{for all } x \in \mathbb{R} \end{array} \right. \tag{$Wave$}$$

□

CHAPTER 13

The L^2–Fourier transform

Something not convenient in the Fourier Transform is that it does not map the space on which it is defined (L^1) into itself. Think of fixed-point theorem, endomorphisms, reduction theory: all these require to work on a space that is mapped into itself. The key here is an L^2-structure that is hidden in the the Fourier transform.

1. Statement and examples

1.1. The extension theorem.

THEOREM 13.1 (Plancherel). *For any $f \in L^1(\mathbb{R},\mathbb{C}) \cap L^2(\mathbb{R},\mathbb{C})$, we have that*

$$\|\hat{f}\|_2 = \|f\|_2. \tag{47}$$

As a consequence, the Fourier transform $\hat{}$ extends to a bijective isometry $\mathcal{F}$: $L^2(\mathbb{R},\mathbb{C}) \to L^2(\mathbb{R},\mathbb{C})$. Moreover, for all $f \in L^2(\mathbb{R},\mathbb{C})$, we have that

$$\mathcal{F}(f) = \lim_{R\to+\infty} \widehat{f_R} \text{ in } L^2(\mathbb{R},\mathbb{C})$$

and

$$\mathcal{F}^{-1}(f) = \lim_{R\to+\infty} \widehat{G(f_R)} \text{ in } L^2(\mathbb{R},\mathbb{C})$$

where $f_R := \mathbf{1}_{[-R,R]}f$, $G(f_R)(x) = f_R(-x)$ for a.e. $x \in \mathbb{R}$ and

$$\widehat{f_R}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} f(\xi)e^{-ix\xi}\, d\xi \text{ for a.e. } \xi \in \mathbb{R}$$

and

$$\widehat{G(f_R)}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} f(\xi)e^{+ix\xi}\, d\xi \text{ for a.e. } \xi \in \mathbb{R}$$

1.2. Some examples. We know that $\widehat{e^{-|x|}} = \frac{1}{\sqrt{2\pi}}\frac{2}{1+|\xi|^2}$. Since $x \mapsto e^{-|x|} \in L^2$, then

$$\mathcal{F}\left(x \mapsto e^{-|x|}\right) = \frac{1}{\sqrt{2\pi}}\frac{2}{1+|\xi|^2} \text{ for a.e. } \xi \in \mathbb{R},$$

and then

$$\mathcal{F}^{-1}\left(x \mapsto \frac{1}{\sqrt{2\pi}}\frac{2}{1+|\xi|^2}\right) = e^{-|\xi|^2} \text{ for a.e. } \xi \in \mathbb{R}$$

Let us now compute the Fourier transform of $x \mapsto f(x) = \frac{1}{\sqrt{2\pi}}\frac{2}{1+|\xi|^2}$. Since $x \mapsto e^{-|x|}$ and $f = \widehat{x \mapsto e^{-|x|}}$ are both in L^1, the inversion formula yields

$$e^{-|x|} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(\xi)e^{ix\xi}\, d\xi \text{ for a.e. } x \in \mathbb{R}$$

With the change of variable $\xi \to -\xi$, since f is even, we get that

$$e^{-|x|} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(\xi) e^{-ix\xi}\, d\xi = \hat{f}(x) \text{ for a.e. } x \in \mathbb{R},$$

then $\hat{f} = (x \mapsto e^{-|x|})$ and since all thes functions are in L^2, we get that

$$\mathcal{F}\left(x \mapsto \frac{1}{\sqrt{2\pi}} \frac{2}{1+|x|^2}\right) = e^{-|\xi|}.$$

Since the functions involved here are both in L^1 and L^2, this is not a breakthrough...

Let us deal with $\mathbf{1}_{[-1,1]}$. We know that $\widehat{\mathbf{1}_{[-1,1]}} = \frac{1}{\sqrt{2\pi}} \frac{2\sin\xi}{\xi}$. All the functions are in L^2, so we have that

$$\mathcal{F}(\mathbf{1}_{[-1,1]}) = \frac{1}{\sqrt{2\pi}} \frac{2\sin\xi}{\xi} \text{ for a.e. } \xi \in \mathbb{R}.$$

Let us now compute the L^2−Fourier transform of $x \mapsto \frac{\sin x}{x}$: it is in L^2, but not in L^1. Note that the Fourier transform is "almost" its own inverse.

We claim that when $f \in L^1$ is even and $\hat{f} \in L^1$, then $\hat{f}$ is even and $\hat{\hat{f}} = f$.
We prove the claim. Concerning $\hat{f}$, with a change of variable, for all $\xi \in \mathbb{R}$, we have that

$$\hat{f}(-\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i(-\xi)x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(-x) e^{-i\xi x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\xi x}\, dx = \hat{f}(\xi).$$

So $\hat{f}$ is even. With the inversion formula and a change of variable, we have that for a.e. $x \in \mathbb{R}$,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{i\xi x}\, d\xi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(-\xi) e^{-i\xi x}\, d\xi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{-i\xi x}\, d\xi = \hat{\hat{f}}(x)$$

And the claim is proved.

Assume that $f \in L^2(\mathbb{R}, \mathbb{C})$ is even. We claim that $\mathcal{F}(f)$ is even and $\mathcal{F} \circ \mathcal{F}(f) = f$. We prove the claim by density. Indeed, for $f \in C_c^\infty(\mathbb{R}, \mathbb{C})$, it follows from Theorem that for any there is $C > 0$ such that $|\hat{f}(\xi)| \leq C(1+|\xi|^2)^{-1}$, so $\hat{f} \in L^1$. So if $f \in C_c^\infty(\mathbb{R}, \mathbb{C})$ is even, the result above yields

$$\mathcal{F} \circ \mathcal{F}(f) = \hat{\hat{f}} = f.$$

We now argue by density. Let $f \in L^2(\mathbb{R}, \mathbb{C})$ be an even L^2−function. Let $(f_n)_n \in C_c^\infty(\mathbb{R})$ be such that $f_n \to f$ in L^2 as $n \to +\infty$. Since f is odd, we also have that $\bar{f}_n := x \mapsto (f_n(x) + f_n(-x))/2 \to f$ in L^2 and $\bar{f}_n \in C_c^\infty(\mathbb{R}, \mathbb{C})$ is even. So $\mathcal{F} \circ \mathcal{F}(\bar{f}_n) = \bar{f}_n$ for all n, and letting $n \to \infty$ ($\mathcal{F}$ is continuous) yields the result.

We are now in position to conclude. Since $\mathbf{1}_{[-1,1]}$ is odd, we have that

$$\mathbf{1}_{[-1,1]} = \mathcal{F} \circ \mathcal{F}(\mathbf{1}_{[-1,1]}) = \mathcal{F}\left(\frac{1}{\sqrt{2\pi}} \frac{2\sin\xi}{\xi}\right)$$

and then

$$\mathcal{F}\left(\frac{\sin\xi}{\xi}\right) = \sqrt{\frac{\pi}{2}} \mathbf{1}_{[-1,1]}$$

Note that the Fourier transform here is not continuous: there is nothing surprising since we have taken a function in L^2, but not in L^1.

2. Proof of Theorem 13.1

The first point is to prove the Plancherel formula (47) for smooth compactly supported functions. We fix $f \in C_c^\infty(\mathbb{R}, \mathbb{C})$

Step 1: The Fourier transform $\hat{f}$ is defined and $\hat{f} \in L^p(\mathbb{R}, \mathbb{C})$ for all $p \geq 1$.

Proof of Step 1: It is a consequence of Proposition 11.9. Since $f \in C_c^\infty(\mathbb{R}, \mathbb{C})$, then $f^{(k)} \in L^1(\mathbb{R}, \mathbb{C})$ for all $k \in \mathbb{N}$, and it follows from (38) that for any $n \in \mathbb{N}$, there exists $C_n > 0$ such that

$$|\hat{f}(\xi)| \leq \frac{C_n}{1 + |\xi|^n} \text{ for all } \xi \in \mathbb{R}.$$

And then $\hat{f} \in L^p(\mathbb{R}, \mathbb{C})$ for all $1 \leq p \leq \infty$.

Step 2: Define

$$\tilde{f}(x) := \overline{f(-x)} \text{ for all } x \in \mathbb{R},$$

and

$$g(x) = (f \star \tilde{f})(x) = \int_{\mathbb{R}} f(x - y)\tilde{f}(y)\, dy \text{ for all } x \in \mathbb{R}.$$

We claim that

- this definition makes sense
- $g \in C^0(\mathbb{R}, \mathbb{C})$ and $g(0) = \|f\|_2^2$
- $|g(x)| \leq \|f\|_2$ for all $x \in \mathbb{R}$
- $g \in L^1(\mathbb{R}, \mathbb{C})$.

Proof of Step 2: We write that

$$g(x) = \int_{\mathbb{R}} F(x, y)\, dy \text{ for all } x \in \mathbb{R},$$

where $F(x, y) := f(x - y)\tilde{f}(y) = f(x - y)\overline{f(-y)}$ for $x, y \in \mathbb{R}$. Since $f \in C_c^\infty(\mathbb{R}, \mathbb{C})$, it is bounded and we have that $|F(x, y)| \leq \|f\|_\infty |f(-y)|$ for all $x, y \in \mathbb{R}$. Since $y \mapsto f(-y) \in L^1(\mathbb{R}, \mathbb{C})$ and $x \mapsto F(x, y)$ is continuous for all $y \in \mathbb{R}$, it follows from Theorem 16.8 that $g(x)$ is defined for all $x \in \mathbb{R}$ and that g is continuous. Moreover, with Cauchy-Schwarz inequality, and some changes of variables, we have that

$$\begin{aligned} |g(x)| &\leq \int_{\mathbb{R}} |f(x - y)| \cdot |\tilde{f}(y)|\, dy \leq \sqrt{\int_{\mathbb{R}} |f(x - y)|^2\, dy}\sqrt{\int_{\mathbb{R}} |\tilde{f}(y)|^2\, dy} \\ &\leq \sqrt{\int_{\mathbb{R}} |f(y)|^2\, dy}\sqrt{\int_{\mathbb{R}} |f(y)|^2\, dy} = \|f\|_2^2 \end{aligned}$$

therefore $g \in L^\infty(\mathbb{R}, \mathbb{C})$. Moreover, $g(0) = \int_{\mathbb{R}} f(-y)\overline{f(-y)}\, dy = \|f\|_2^2$. Concerning the L^1-norm, with (34), we get that

$$\int_{\mathbb{R}} |g(x)|\, dx \leq \|f\|_1 \|\tilde{f}\|_1 = \|f\|_1^2 < \infty,$$

and then $g \in L^1(\mathbb{R}, \mathbb{C})$.

Step 3: We claim that $\hat{g} = \sqrt{2\pi}|\hat{f}|^2$.

Proof: Since $g, f, \tilde{f} \in L^1(\mathbb{R}, \mathbb{C})$, the Fourier transform makes sense and Proposition 11.4 yields

$$\hat{g} = \widehat{f \star \tilde{f}} = \sqrt{2\pi}\hat{f} \cdot \hat{\tilde{f}}.$$

Now, for $\xi \in \mathbb{R}$, we have that

$$\begin{aligned}\widehat{\overline{f}}(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \overline{f(-y)}e^{-i\xi y}\,dy = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \overline{f(y)}e^{i\xi y}\,dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \overline{f(y)e^{-i\xi y}}\,dy = \overline{\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(y)e^{-i\xi y}\,dy} = \overline{\hat{f}(\xi)}\end{aligned}$$

and then $\hat{g}(\xi) = \sqrt{2\pi}\hat{f}(\xi)\overline{\hat{f}(\xi)} = \sqrt{2\pi}|\hat{f}(y)|^2$ for all $\xi \in \mathbb{R}$. This proves the claim.

Step 4: We conclude as in the proof of the inversion formula. In the proof of Theorem 12.1, we have defined

$$h_\epsilon(x) := \frac{1}{\sqrt{2\pi}}e^{-\epsilon^2\frac{x^2}{2}} \text{ for all } x \in \mathbb{R} \text{ and } \epsilon > 0.$$

Defining $g_\epsilon = \widehat{h_\epsilon}$ (this makes sense since $h_\epsilon \in L^1(\mathbb{R},\mathbb{C})$), recall that we have that $(g_\epsilon)_\epsilon$ is an approximation of unity in the sense of Theorem 12.4. Since $g, h_\epsilon \in L^1(\mathbb{R},\mathbb{C})$ for all $\epsilon > 0$, we have that (see (42)

$$g \star g_\epsilon(0) = \int_{\mathbb{R}} g(x)g_\epsilon(0)\,dx = \int_{\mathbb{R}} g(x)\widehat{h_\epsilon}(x)\,dx = \int_{\mathbb{R}} \hat{g}(x)h_\epsilon(x)\,dx.$$

Since g is continuous, it follows from Theorem 12.4 that $\lim_{\epsilon\to 0} g\star g_\epsilon(0) = g(0) = \|f\|_2^2$. Since for all $x \in \mathbb{R}$, $\lim_{\epsilon\to 0} h_\epsilon(x) = 1$ and $0 \leq \hat{g}h_\epsilon \leq \hat{g} \in L^1(\mathbb{R},\mathbb{C})$, it follows from the dominated convergence theorem 16.7

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}} \hat{g}(x)h_\epsilon(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{g}(x)\,dx = \int_{\mathbb{R}} |\hat{f}(x)|^2\,dx.$$

Therefore

$$\|f\|_2^2 = \|\hat{f}\|_2^2 \text{ for all } f \in C_c^\infty(\mathbb{R},\mathbb{C}).$$

This proves (47) for $f \in C_c^\infty(\mathbb{R},\mathbb{C})$.

So we have a map

$$\begin{cases} T: & C_c^\infty(\mathbb{R},\mathbb{C}) & \to & L^2(\mathbb{R},\mathbb{C}) \\ & f & \mapsto & \hat{f} \end{cases}$$

such that $\|T(f)\|_2 = \|f\|_2$ for all $f \in C_c^\infty(\mathbb{R},\mathbb{C})$.

Step 4: We claim that T extends continuously as a function $\mathcal{F} : L^2(\mathbb{R},\mathbb{C}) \to L^2(\mathbb{R},\mathbb{C})$.

Proof: Since T is an isometry, it is uniformly continuous. Moreover, $C_c^\infty(\mathbb{R},\mathbb{C})$ is dense in $(L^2(\mathbb{R},\mathbb{C}), \|\cdot\|_2)$ which is complete. The extension Theorem 16.2 then yields the extension of T to L^2.

Step 5: We claim that $\mathcal{F}$ is a bijection.

Proof: We are constructing its inverse via the inversion formula. For $f \in L^2(\mathbb{R},\mathbb{C})$, define $G(f) \in L^2(\mathbb{R},\mathbb{C})$ by $G(f)(x) = f(-x)$ for a.e. $x \in \mathbb{R}$. This is well defined. Moreover, $G : L^2(\mathbb{R},\mathbb{C}) \to L^2(\mathbb{R},\mathbb{C})$ is a linear isometry. Take $f \in C_c^\infty(\mathbb{R},\mathbb{C})$ so that $\hat{f} \in L^p(\mathbb{R},\mathbb{C})$ for all $p \geq 1$. We have with a change of variable that

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{f}(\xi)e^{ix\xi}\,d\xi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{f}(-\xi)e^{-ix\xi}\,d\xi = \widehat{G(\hat{f})}(x)$$

for a.e. $x \in \mathbb{R}$, and indeed for all $x \in \mathbb{R}$ since f is continuous. So with the extension theorem, we have that

$$f = \mathcal{F}\circ G\circ\mathcal{F}(f) \text{ for all } f \in C_c^\infty(\mathbb{R},\mathbb{C}).$$

Since $\mathcal{F}, G$ are continuous for the L^2-norm, by density, we get that $f = \mathcal{F}\circ G\circ\mathcal{F}(f)$ for all $f \in L^2(\mathbb{R},\mathbb{C})$, and then $Id_{L^2} = \mathcal{F}\circ G\circ\mathcal{F}$. Therefore $\mathcal{F}$ is a bijection and $\mathcal{F}^{-1} = \mathcal{F}\circ G$.

Step 5: Take $f \in L^2(\mathbb{R},\mathbb{C})$. Then for any $R > 0$, we have that $f_R := \mathbf{1}_{[-R,R]} f \in L^1(\mathbb{R},\mathbb{C}) \cap L^2(\mathbb{R},\mathbb{C})$. Moreover,

$$\lim_{R\to+\infty} f_R = f \text{ in } L^2(\mathbb{R},\mathbb{C}).$$

Therefore, since $\mathcal{F}$ is continuous, $\lim_{R\to+\infty}\mathcal{F}(f_R) = \mathcal{F}(f)$ in $L^2(\mathbb{R},\mathbb{C})$, but since $\mathcal{F}(f_R) = \hat{f}_R$, we get that $\lim_{R\to+\infty}\widehat{f_R} = \mathcal{F}(f)$ in $L^2(\mathbb{R},\mathbb{C})$.

3. An exercise: the linear homogenous Schrödinger equation

For $u_0 \in C^2(\mathbb{R},\mathbb{C})$, we investigate the existence of a solution $u \in C^2(\mathbb{R}\times\mathbb{R},\mathbb{C})$ to

$$\left\{\begin{array}{ll} i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0 & \text{in } \mathbb{R}\times\mathbb{R} \\ u(0,x) = u_0(x) & \text{for all } x \in \mathbb{R}. \end{array}\right. \tag{E}$$

Step 1. Solve (E) formally.

Assume that $u_0 \in C^4(\mathbb{R},\mathbb{C})$ and that

$$u_0^{(k)} \in L^1(\mathbb{R},\mathbb{C}) \text{ and } \widehat{u_0^{(k)}} \in L^1(\mathbb{R},\mathbb{C}) \text{ for all } k = 0,...,4.$$

Step 2. Show that the formal solution is a classical solution to (E).

Step 3. In addition to the above hypothesis, we assume that $u_0 \in L^2(\mathbb{R},\mathbb{C})$. Show that for all $t \in \mathbb{R}$, then $u(t,\cdot) \in L^2(\mathbb{R},\mathbb{C})$ and $\|u(t,\cdot)\|_{L^2(\mathbb{R},\mathbb{C})} = \|u_0\|_{L^2(\mathbb{R},\mathbb{C})}$ for all $t \in \mathbb{R}$.

Step 4. With the sames hypothesis as above, show that

$$\left\{\begin{array}{ccc} \mathbb{R} & \to & L^2(\mathbb{R}) \\ t & \mapsto & u(t,\cdot) \end{array}\right\} \text{ is continuous.}$$

Solution of "The linear homogenous Schrödinger equation":
Step 1. We perform the Fourier transform in the space variable, that is we define

$$\hat{u}(t,\xi) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u(t,x)e^{-ix\xi}\,dx = \widehat{u(t,\cdot)}(\xi) \text{ for } t \geq 0 \text{ and } x \in \mathbb{R}.$$

Formally, we then have that

$$\widehat{\partial^2_{xx}u(t,\cdot)} = (i\xi)^2\widehat{u(t,\cdot)} = -\xi^2\hat{u}(t,\xi).$$

Differentiating with respect to t, we get that

$$\frac{\partial}{\partial t}\hat{u}(t,\xi) = \widehat{\partial_t u(t,\cdot)}(\xi)$$

Then we rewrite (E) as

$$0 = \hat{0} = i\partial_t\widehat{u + \partial^2_{xx}}u = i\partial_t\hat{u} - |\xi|^2\hat{u}$$

so that

$$\partial_t\hat{u} + i|\xi|^2\hat{u} = 0 \text{ for } t \geq 0 \text{ and } \xi \in \mathbb{R}.$$

Freezing the space variable ξ, this is a linear first order ODE with respect to the time variable. And we can solve it: we get

$$\hat{u}(t,\xi) = \hat{u}(0,\xi)e^{-i|\xi|^2 t}$$

With the initial condition $u(0,x) = u_0(x)$ for all $x \in \mathbb{R}$, we have that

$$\hat{u}(0,\xi) = \widehat{u_0}(\xi)$$

and then

$$\hat{u}(t,\xi) = \widehat{u_0}(\xi)e^{-i|\xi|^2 t}.$$

With the inversion theorem, we then get the expression

$$u(t,x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{u}(t,\xi)e^{i\xi x}\,d\xi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{u_0}(\xi)e^{-i|\xi|^2 t}e^{i\xi x}\,d\xi.$$

And this is going to be our candidate.
Step 2. We define

$$u(t,x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{u_0}(\xi)e^{-i|\xi|^2 t}e^{i\xi x}\,d\xi \text{ for all } t \geq 0 \text{ and } x \in \mathbb{R}.$$

This well-defined since, by defining

$$F(t,x,\xi) := \widehat{u_0}(\xi)e^{-i|\xi|^2 t}e^{i\xi x} \text{ for } t,x,\xi \in \mathbb{R},$$

we have that $|F(t,x,\xi)| \leq |\widehat{u_0}(\xi)|$ for $t \geq 0$, $x,\xi \in \mathbb{R}$, and since $\widehat{u_0} \in L^1(\mathbb{R},\mathbb{C})$, then

$$u(t,x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} F(t,x,\xi)\,d\xi$$

is defined for all $t \geq 0$ and $x \in \mathbb{R}$. For all $\xi \in \mathbb{R}$, we have that $(t,\xi) \mapsto F(t,x,\xi)$ is C^2 (and even C^∞) and

$$\begin{aligned}
\frac{\partial F}{\partial x}(t,x,\xi) &= (i\xi)\widehat{u_0}(\xi)e^{-i|\xi|^2 t}e^{i\xi x}\\
\frac{\partial^2 F}{\partial x^2}(t,x,\xi) &= (i\xi)^2\widehat{u_0}(\xi)e^{-i|\xi|^2 t}e^{i\xi x}\\
\frac{\partial F}{\partial t}(t,x,\xi) &= -i|\xi|^2\widehat{u_0}(\xi)e^{-i|\xi|^2 t}e^{i\xi x}\\
\frac{\partial F^2}{\partial t^2}(t,x,\xi) &= -|\xi|^4\widehat{u_0}(\xi)e^{-i|\xi|^2 t}e^{i\xi x}
\end{aligned}$$

for all $t,x,\xi \in \mathbb{R}$. Since $u_0 \in C^4$ and the derivatives are in L^1, we get that

$$\left|\frac{\partial F}{\partial x}(t,x,\xi)\right| \leq |\xi\widehat{u_0}(\xi)| = |\widehat{u_0'}(\xi)|\ ;\ \left|\frac{\partial^2 F}{\partial x^2}(t,x,\xi)\right| \leq |\xi^2\widehat{u_0}(\xi)| = |\widehat{u_0''}(\xi)|$$

$$\left|\frac{\partial F}{\partial t}(t,x,\xi)\right| \leq |\xi^2\widehat{u_0}(\xi)| = |\widehat{u_0''}(\xi)|\ ;\ \left|\frac{\partial F^2}{\partial t^2}(t,x,\xi)\right| \leq |\xi^4\widehat{u_0}(\xi)| = |\widehat{u_0^{(4)}}(\xi)|$$

It follows from the hypothesis that $\widehat{u_0^{(k)}} \in L^1(\mathbb{R},\mathbb{C})$ for all $k = 0,...,4$. Then, we can apply the derivation theorem:

- For all $t \in \mathbb{R}$ and all $x \in \mathbb{R}$, $\xi \mapsto F(t,x,\xi)$ is integrable
- For all $\xi \in \mathbb{R}$, then $(t,x) \mapsto F(t,x,\xi)$ is C^2 on $\mathbb{R} \times \mathbb{R}$
- All the derivatives of order ≤ 2 in the (t,x) variables are controled independently of (t,x) by an L^1−function

It the follows from a double application of Theorem 16.9 that $u \in C^2(\mathbb{R}\times\mathbb{R},\mathbb{C})$.

Let us now check that u is a solution to (E). The derivation theorem yields

$$\begin{aligned} i\partial_t u(t,x) + \partial_{xx}u(t,x) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left(i\partial_t F(t,x,\xi) + \partial_{xx}F(t,x,\xi)\right)\,d\xi \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left(i\times(-i)|\xi|^2 - |\xi|^2\right)\widehat{u_0}(\xi)e^{-|\xi|^2 t}e^{i\xi x}\,d\xi = 0 \end{aligned}$$

for all $t,x\in\mathbb{R}$. Moreover, for all $x\in\mathbb{R}$, we have that

$$u(0,x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} F(0,x,\xi)\,d\xi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{u_0}(\xi)e^{i\xi x}\,d\xi$$

Since $u_0, \widehat{u_0}\in L^1(\mathbb{R},\mathbb{C})$, the inversion formula yields $u(0,x) = u_0(x)$ for a.e. $x\in\mathbb{R}$. Since $u(0,\cdot)$ is continuous and u_0 too, we then get that $u(0,x) = u_0(x)$ for all $x\in\mathbb{R}$. So we have found a solution to (E).

Step 3. Since $u_0\in L^2(\mathbb{R},\mathbb{C})\cap L^1(\mathbb{R},\mathbb{C})$, it follows from the Plancherel Theorem 13.1 that $\hat{u_0}\in L^2(\mathbb{R},\mathbb{C})$, and $\hat{u_0} = \mathcal{F}(u_0)$. Moreover,

$$|\hat{u_0}(\xi)e^{-i\xi^2 t}| = |\hat{u_0}(\xi)| \in L^2$$

and since $u(t,-\cdot) = \hat{u_0}\widehat{(\xi)e^{-i\xi^2 t}}$, we get that for all $t\in\mathbb{R}$, $u(t,\cdot)\in L^2(\mathbb{R},\mathbb{C})$. The Plancherel Formula then yields

$$\|u(t,\cdot)\|_2 = \|u(t,-\cdot)\|_2 = \|\hat{u_0}(\xi)e^{-i\xi^2 t}\|_2 = \|\hat{u_0}(\xi)e^{-i\xi^2 t}\|_2 = \|u_0\|_2.$$

Step 4. Let us take $t,t_0\in\mathbb{R}$. Via Plancherel's formula (47), we have that

$$\begin{aligned} \|u(t,\cdot) - u(t_0,\cdot)\|_2 &= \|u(t,-\cdot) - u(t_0,-\cdot)\|_2 \\ &= \|\hat{u_0}(\xi)e^{-i\xi^2 t} - \hat{u_0}(\xi)e^{-i\xi^2 t_0}\|_2 \\ &= \|\hat{u_0}(\xi)(e^{-i\xi^2 t} - e^{-i\xi^2 t_0})\|_2 \end{aligned}$$

We write

$$\|\hat{u_0}(\xi)(e^{-i\xi^2 t} - e^{-i\xi^2 t_0})\|_2^2 = \int_{\mathbb{R}} |\hat{u_0}(\xi)|^2 \times |e^{-i\xi^2 t} - e^{-i\xi^2 t_0}|^2\,d\xi.$$

Since $|e^{-i\xi^2 t} - e^{-i\xi^2 t_0}| \le 2$, $\hat{u_0}\in L^2(\mathbb{R},\mathbb{C})$ and $\lim_{t\to t_0} e^{-i\xi^2 t} = e^{-i\xi^2 t_0}$ for all $\xi\in\mathbb{R}$, Lebesgue's Theorem 16.6 yields

$$\lim_{t\to t_0}\int_{\mathbb{R}} |\hat{u_0}(\xi)|^2 \times |e^{-i\xi^2 t} - e^{-i\xi^2 t_0}|^2\,d\xi = 0$$

and then

$$\lim_{t\to t_0}\|u(t,\cdot) - u(t_0,\cdot)\|_2 = 0,$$

and then $t\mapsto u(t,\cdot)$ is continuous from $\mathbb{R}\to L^2$.

Part 6

A very short excursion in distributions

The theory of distributions is a generalization of the notion of functions: it is due to Laurent Schwartz (Fields medal in 1945).

Originally, a function is defined pontwisely by an element $f(x)$. Since the beginning of these lectures, we have changed the point of view by considering the function itself as an object on which to work to get solutions to problems via the fixed-point theorem or Hilbert methods, to name a few.

In the modelling of a flame for the heat equation, we have also met a family of functions that does not converge in the usual topologies, and we have constructed a new way to make it converge:

For $k \geq 1$, we define $f_k \in C^0(\mathbb{R})$ by

- $f_k(x) = 0$ for $|x| \geq 1/k$
- $f_k(0) = k$,
- f_k is affine on $\left[-\frac{1}{k}, 0\right]$ and on $\left[0, \frac{1}{n}\right]$.

Proposition 13.2. *For any $\varphi \in C_c^\infty(\mathbb{R})$, we have that*

$$\lim_{k\to+\infty} \int_{\mathbb{R}} f_k(x)\varphi(x) = \varphi(0).$$

When it makes sense, for a function $h \in L^1_{loc}(\mathbb{R})$(locally integrable, that is $f \in L^1(-R, R)$ for all $R > 0$), we define

$$T_h : \begin{array}{ccc} C_c^\infty(\mathbb{R}) & \to & \mathbb{R} \\ \varphi & \mapsto & \int_{\mathbb{R}} h\varphi\, dx \end{array} \quad \text{and} \quad \delta_0 : \begin{array}{ccc} C_c^\infty(\mathbb{R}) & \to & \mathbb{R} \\ \varphi & \mapsto & \varphi(0) \end{array}$$

The above proposition then rewrites

$$\lim_{k\to+\infty} T_{f_k}(\varphi) = \delta_0(\varphi) \text{ for all } \varphi \in C_c^\infty(\mathbb{R}),$$

which is just a pointwise convergence. This is a first example of distribution, and of convergence of distributions. This is also going to be a generalization of Sobolev spaces. And this is also going to be a efficient tool to solve PDEs (again).

CHAPTER 14

Distributions on $\mathbb{R}$

1. Definition, basic examples and structure

DEFINITION 1.1. *A distribution on* $\mathbb{R}$ *is a linear function* $T : C_c^\infty(\mathbb{R}) \to \mathbb{R}$ *such that for all* $R > 0$*, there exists* $k = k(R) \in \mathbb{N}$ *and* $C(R) > 0$ *such that*

$$|T(\varphi)| \leq C(R) \cdot \|\varphi\|_{C^k} \text{ for all } \varphi \in C_c^\infty(\mathbb{R}) \text{ such that } Supp(\varphi) \subset [-R, R]$$

Here, $\|\varphi\|_{C^k} := \sum_{i=0}^k \|\varphi^{(i)}\|_\infty = \sum_{i=0}^k \sup_{x\in\mathbb{R}} |\varphi^{(i)}(x)|$.
You can see that this is more or less a notion of continuous linear function, but we will not discuss topology here. Be aware that you will have some difficulties finding a "good" norm on $C_c^\infty(\mathbb{R})$! You have a family of (semi-)norms

$$\varphi \mapsto N_{k,R}(\varphi) = \max\{\sum_{i=0}^{k} |\varphi^{(i)}(x)| / \, x \in [-R, R]\}$$

but none of them is a norm on $C_c^\infty(\mathbb{R})$, and you have to take them all. If you are interested, look at the topology induced by a family of semi-norms (see Brezis [1]).

1.1. Notations and vocabulary.

- We will use the standard notation $\mathcal{D}(\mathbb{R}) := C_c^\infty(\mathbb{R})$
- We note $\mathcal{D}'(\mathbb{R})$ the set of distributions on $\mathbb{R}$ (more or less the dual of $\mathcal{D}(\mathbb{R})$.
- For $T \in \mathcal{D}'(\mathbb{R})$ and $\varphi \in \mathcal{D}(\mathbb{R})$, it is the custom to write

$$\langle T, \varphi \rangle := T(\varphi).$$

1.2. First examples. We have already seen many distributions without knowing they were...

(1) Given $f \in L^1_{loc}(\mathbb{R})$ (locally integrable, that is $f \in L^1(-R, R)$ for all $R > 0$), define

$$\left\{ \begin{array}{lccc} T_f : & \mathcal{D}(\mathbb{R}) & \to & \mathbb{R} \\ & \varphi & \mapsto & \int_{\mathbb{R}} f\varphi \, dx \end{array} \right\}$$

This is well-defined, linear, and for $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\operatorname{Supp} \varphi \subset [-R, R]$ with $R > 0$, we have that

$$|\langle T_f, \varphi \rangle| \leq \|f\|_{L^1(-R,R)} \cdot \|\varphi\|_\infty,$$

so this is a distribution.

(2) Given $a \in \mathbb{R}$, define the "Dirac mass at a" as

$$\left\{ \begin{array}{lccc} \delta_a : & \mathcal{D}(\mathbb{R}) & \to & \mathbb{R} \\ & \varphi & \mapsto & \varphi(a) \end{array} \right\}$$

This is also a distribution and $|\langle \delta_a, \varphi \rangle| \leq \|\varphi\|_\infty$ for all $\varphi \in \mathcal{D}(\mathbb{R})$.

Note thas these two basic examples make sense for $\varphi \in C_c^0(\mathbb{R})$ only, and even $C^0(\mathbb{R})$ for the Dirac mass.

Exercise 53: Are these functions on $\mathcal{D}(\mathbb{R})$ distributions?

$$\begin{array}{ll} (a)\, \varphi \mapsto \int_0^1 \varphi\, dx; & (b)\, \varphi \mapsto \int_0^\infty x^{17}\varphi\, dx; \\ (c)\, \varphi \mapsto \int_{\mathbb{R}} x^2\varphi\, dx; & (d)\, \varphi \mapsto \varphi(2) \times \int_{\mathbb{R}} \varphi\, dx; \\ (d)\, \varphi \mapsto \varphi(1)^2 + 2\varphi'(0); & (f)\, \varphi \mapsto \varphi'(1) + \int_{\mathbb{R}} \varphi\, dx. \end{array}$$

Hint: two of them are not...

1.3. Function or distribution?

Definition 1.2. *We say that a distribution $T \in \mathcal{D}'(\mathbb{R})$ is a* <u>*function*</u> *if there exists $f \in L^1_{loc}(\mathbb{R})$ such that $T = T_f$. We then write for short $T \in L^1_{loc}(\mathbb{R})$, or $T \in L^p$ if $f \in L^p$, etc.*

Proposition 14.1. *The Dirac mass is not a function. As a consequences, there are some distributions that are not functions.*

Proof. We argue by contradiction and we assume that there exists $f \in L^1_{loc}(\mathbb{R})$ such that $\delta_a = T_f$. For simplicity, assume that $a = 0$. Then

$$\int_{\mathbb{R}} f(x)\varphi(x)\, dx = \varphi(0) \text{ for all } \varphi \in \mathcal{D}(\mathbb{R}).$$

Observe that to define the left-hand-side, we only need φ to be defined a.e. But the right-hand-side require to have φ precisely defined at 0. The contradiction stems form this remark. Fix $\varphi \in \mathcal{D}(\mathbb{R})$: therefore $x \mapsto e^{-nx}\varphi(x) \in D$ for all $n \in \mathbb{N}$ and then

$$\int_{\mathbb{R}} f(x)e^{-nx}\varphi(x)\, dx = e^{-n\times 0}\varphi(0) = \varphi(0) \text{ for all } \in \mathbb{N}.$$

Letting $n \to +\infty$, Lebesgue's convergence Theorem (write the details) yields $\varphi(0) = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R})$, which is a contradiction. □

2. Limits

Definition 2.1. *Let $(T_n)_{n\in\mathbb{N}}$ be such that $T_n \in \mathcal{D}'(\mathbb{R})$ for all $n \in \mathbb{N}$. We say that $(T_n)_n$ converges to some $T \in \mathcal{D}'(\mathbb{R})$ in the distribution sense if*

$$\lim_{n\to+\infty} \langle T_n, \varphi\rangle = \langle T, \varphi\rangle \text{ for all } \varphi \in \mathcal{D}(\mathbb{R}).$$

This is simply the pointwise convergence!

Since we deal with linear continuous functions, remember Corollary 12 which is a consequence of The Banach-Steinhaus Theorem: under some hypothesis, a pointwise limit of continuous linear functions is also linear and continuous. A similar result holds her, but the proof require to use the topology associitaed to semi-norms. In the examples you will study, it will be easy to get that your limit is continuous.

2.1. First example of convergence. Let us go back to our initial example $(f_k)_k$. Proposition 13.2 can be written as "(T_{f_n}) converges to δ_0 in the sense of distribution". For short, we will often write "(f_k) converges to δ_0 in the distribution sense."

Exercise 54: For $n \geq 1$, define $g_n : \mathbb{R} \to \mathbb{R}$ as follows

$$g_n(x) := \begin{cases} -n^2 & \text{if } -\frac{1}{n} < x < 0 \\ n^2 & \text{if } 0 < x < \frac{1}{n} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that (g_n) converges to $\varphi \mapsto -\varphi'(0)$ in the distribution sense as $n \to +\infty$.

2.2. The principal value of x^{-1}. The function $x \mapsto x^{-1}$ is not in $L^1_{loc}(\mathbb{R})$. However, there is a way to define a distribution attached to it:

Definition 2.2. *For $\varphi \in \mathcal{D}'(\mathbb{R})$, we define*

$$\left\langle vp\left(\frac{1}{x}\right), \varphi\right\rangle := \lim_{n\to+\infty} \int_{|x|>\frac{1}{n}} \frac{\varphi(x)}{x}\, dx.$$

Then $vp\left(\frac{1}{x}\right)$ is well-defined and it is a distribution.

Proof. Fix $\varphi \in$ with Supp $\varphi \subset [-R, R]$. We write

$$\int_{|x|>\frac{1}{n}} \frac{\varphi(x)}{x}\, dx \quad = \quad \int_{R>|x|>\frac{1}{n}} \frac{\varphi(x)-\varphi(0)}{x}\, dx + \int_{R>|x|>\frac{1}{n}} \frac{\varphi(0)}{x}\, dx \tag{48}$$

Since $x \mapsto x^{-1}$ is odd, the last intergral is 0. Define the function

$$\psi(x) := \begin{cases} \frac{\varphi(x)-\varphi(0)}{x} & \text{if } x \neq 0 \\ \varphi'(0) & \text{if } x = 0. \end{cases}$$

As one checks, $\psi \in C^0(\mathbb{R})$ and the mean value inequality yields $|\psi(x)| \leq \|\varphi'\|_\infty$. So

$$\int_{|x|>\frac{1}{n}} \frac{\varphi(x)}{x}\, dx \quad = \quad \int_{-R}^{R} \mathbf{1}_{R>|x|>\frac{1}{n}} \psi(x)\, dx \tag{49}$$

and we Lebesgue's convergence theorem, we get that the right-had-side goes to $\int_{-R}^{R} \psi(x)\, dx$ as $n \to +\infty$. Therefore $vp(1/x)$ is defined, linear, and

$$|\langle \mathrm{vp}\left(\frac{1}{x}\right), \varphi\rangle| = |\int_{-R}^{R} \psi(x)\, d| \leq 2R\|\psi_\infty \leq 2R\|\varphi'\|_\infty$$

for all $\varphi \in \mathcal{D}(\mathbb{R})$ with support in $[-R, R]$. So it is a distribution. □

3. Product by a function

3.1. Investigation. We want to define the product gT when $g : \mathbb{R} \to \mathbb{R}$ is a function and $T \in \mathcal{D}'(\mathbb{R})$. We want that the product matches the standard product when T is a function, that is

$$gT_f = T_{gf} \text{ when } f \in L_{1\,loc}(\mathbb{R}).$$

what would it mean? For $\varphi \in \mathcal{D}(\mathbb{R})$, we would have

$$\langle gT_f, \varphi\rangle = \langle T_{gf}, \varphi\rangle = \int_{\mathbb{R}} fg\varphi\, dx = \langle T_f, g\varphi\rangle.$$

and this expression depends only on T_f. This yields to the following definition:

DEFINITION 3.1. *Let $T \in \mathcal{D}'(\mathbb{R})$ be a dtribution, and fix $g \in C^\infty(\mathbb{R})$ be a function. For any $\varphi \in \mathcal{D}(\mathbb{R})$, we define*

$$\langle gT, \varphi \rangle = \langle T, g\varphi \rangle.$$

Then gT is a distribution.

PROOF. For $\varphi \in \mathcal{D}(\mathbb{R}) = C_c^\infty(\mathbb{R})$, we have that $g\varphi \in C_c^\infty(\mathbb{R}) = \mathcal{D}(\mathbb{R})$, so $\langle gT, \varphi \rangle$ makes sense, and it is linear with respect to φ. We must now check the "continuity". Fix $R > 0$, $C = C(R)$ and $k = k(R)$ such that

$$|\langle T, \varphi \rangle| \leq C \|\varphi\|_{C^k} \text{ for all } \varphi \in \mathcal{D}(\mathbb{R}) \text{ such that } \mathrm{Supp}(\varphi) \subset [-R, R],$$

and then for $\varphi \in \mathcal{D}(\mathbb{R})$ with support in $[-R, R]$, we have that

$$|\langle gT, \varphi \rangle| \leq C \|g\varphi\|_{C^k}.$$

Via Leibnitz's formula, we have that $(g\varphi)^{(i)} = \sum_{p=0}^{i} C_i^p g^{(p)} \varphi^{(i-p)}$ for all $i \in \mathbb{N}$. Therefore, $\|(g\varphi)^{(i)}\|_\infty C(i,k) \|g\|_{C^k(-R,R)} \|\varphi\|_{C^k}$, and then,

$$|\langle gT, \varphi \rangle| \leq C \|g\|_{C^k(-R,R)} \|\varphi\|_{C^k}.$$

and so gT is a distribution. □

3.2. We have already seen examples for which $\langle T, \varphi \rangle$ makes sense even if φ is not in $\mathcal{D}(\mathbb{R})$. Therefore, $\langle T, g\varphi \rangle$ might sometimes makes sense even if $g \notin C^\infty(\mathbb{R})$. So, without giving a formal definition, we will sometimes write gT even in some nonsmooth cases.

Exercise 55: Compute $x\mathrm{vp}\left(\frac{1}{x}\right)$.

3.3. Resolution of $xT = 0$.

Proposition 14.2. *Let $T \in \mathcal{D}'(\mathbb{R})$ be such that $xT = 0$. Then there exists $\lambda \in \mathbb{R}$ such that $T = \lambda \delta_0$.*

PROOF. Assume that $xT = 0$. Then for all $\psi \in \mathcal{D}(\mathbb{R})$, we have that $\langle T, x\psi \rangle = 0$. Take $\varphi \in \mathcal{D}(\mathbb{R})$. Let us fix $\theta \in \mathcal{D}(\mathbb{R})$ be such that $\theta(0) = 1$ (this is always possible). Then $\varphi - \varphi(0)\theta \in \mathcal{D}(\mathbb{R})$ and vanishes at 0. Define

$$\psi(x) := \frac{\psi(x)}{x} \text{ if } x \neq 0, \text{ and } \varphi'(0) \text{ otherwise.}$$

As one checks, we have that $\psi(x) = \int_0^1 \varphi'(tx)\, dt$ for all $x \in \mathbb{R}$. With the differentiability under the integral, we then get that $\psi \in C^\infty(\mathbb{R})$, and also it has compact support, so $\psi \in \mathcal{D}(\mathbb{R})$ and

$$\varphi = \varphi(0)\theta + x\psi$$

Therefore,

$$\langle T, \varphi \rangle = \varphi(0)\langle T, \theta \rangle + \langle T, x\psi \rangle = \varphi(0)\langle T, \theta \rangle$$

and we get the result with $\lambda := \langle T, \theta \rangle$

□

Exercise 56: Solve $x^2 T = 0$ and $x(x-1)T = 0$.

4. Differentiability

This is the great point: every distrbution is differentiable.

4.1. Investigations and definition. We want to define the derivative T' when $T \in \mathcal{D}'(\mathbb{R})$. We want that the derivatives matches the standard derivative when T is a differentiable function, that is

$$(T_f)' = T_{f'} \text{ when } f \in L^1_{loc}(\mathbb{R}).$$

what would it mean? For $\varphi \in \mathcal{D}(\mathbb{R})$ with support in $[-R, R]$, we would have

$$\begin{aligned} \langle (T_f)', \varphi \rangle &= \langle T_{f'}, \varphi \rangle = \int_{\mathbb{R}} f' \varphi \, dx = \int_{-R}^{R} f' g \, dx = [fg]_{-R}^{R} - \int_{-R}^{R} f \varphi' \, dx \\ &= -\int_{\mathbb{R}} f \varphi' \, dx = -\langle T_f, \varphi' \rangle. \end{aligned}$$

Which yields to the definition:

Definition 4.1. *Let $T \in \mathcal{D}'(\mathbb{R})$ be a distribution. For $\varphi \in \mathcal{D}(\mathbb{R})$, we define*

$$\langle T', \varphi \rangle = -\langle T, \varphi' \rangle.$$

This is well-define and T' is a distribution.

Proof. Since $\varphi' \in \mathcal{D}(\mathbb{R})$, the definition makes sense and T' is linear. Moreover, if the support of φ is in $[-R, R]$, we have that

$$|\langle T', \varphi \rangle| = |\langle T, \varphi' \rangle| \leq C(R) \|\varphi'\|_{C^k} \leq C(R) \|\varphi\|_{C^{k+1}}$$

and so T' is a distribution. □

4.2. Some examples.

- Define $x_+ := \max\{x, 0\}$ for $x \in \mathbb{R}$. Then $T'_{x+} = T_H$ where

$$H(x) := \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

or, in short notation, $x'_+ = H$. The function H is called the Heaviside function.

Proof. The distributions T_{x_+}, T_H are well-defined since $x_+, H \in L^1_{loc}(\mathbb{R})$. Take $\varphi \in \mathcal{D}(\mathbb{R})$ with support in $[-R, R]$. We have that

$$\begin{aligned} \langle T'_{x^+}, \varphi \rangle &= -\langle T_{x^+}, \varphi' \rangle = -\int_{\mathbb{R}} x_+ \varphi' \, dx = -\int_0^R x \varphi'(x) \, dx \\ &= -[x\varphi(x)]_0^R + \int_0^R \varphi(x) \, dx = \int_{\mathbb{R}} H(x) \varphi(x) \, dx = \langle T_H, \varphi \rangle \end{aligned}$$

which proves the result. □

- We have that $T'_H = \delta_0$, or $H' = \delta_0$ for short.

Proof. Take $\varphi \in \mathcal{D}(\mathbb{R})$ with support in $[-R, R]$. We have that

$$\begin{aligned} \langle T'_H, \varphi \rangle &= -\langle T_H, \varphi' \rangle = -\int_{\mathbb{R}} H \varphi' \, dx = -\int_0^R \varphi'(x) \, dx \\ &= \varphi(0) - \varphi(R) = \varphi(0) = \langle \delta_0, \varphi \rangle \end{aligned}$$

□

- A straightforward application of the definition yields $\langle \delta'_0, \varphi \rangle = -\varphi'(0)$ for all $\varphi \in \mathcal{D}(\mathbb{R})$.

Exercise 57: Show that $(\ln|x|)' = \mathrm{vp}\left(\frac{1}{x}\right)$ in the distribution sense.

4.3. Product and derivative.

Proposition 14.3. *Let $T \in \mathcal{D}'(\mathbb{R})$ and $f \in C^\infty(\mathbb{R})$ be a distribution and a function. Then*
$$(fT)' = f'T + fT'.$$

Exercise 58: Prove this proposition.

Exercise 59: Define $g(x) = e^{-x_+}$ for all $x \in \mathbb{R}$. Show that $g' + Hg = 0$ in the distribution sense. (Remark: the product $Hg = HT_g$ makes sense even if H is not C^∞).

4.4. The jump formula.

Exercise 60: What is the distributional derivative of
$$x \mapsto \begin{cases} x^2 & \text{if } x < 0 \\ x+1 & \text{if } x \geq 0 \end{cases} ?$$
For $f, g \in C^1(\mathbb{R})$ and $a \in \mathbb{R}$, generalize to
$$x \mapsto \begin{cases} f(x) & \text{if } x < a \\ g(x) & \text{if } x \geq a \end{cases}$$

Proposition 14.4. *Let $f : \mathbb{R} \to \mathbb{R}$ that is piecewise C^1, that is there exists $a_0 < ... < a_N \in \mathbb{R}$ such that for all $i = 1, .., N$, there exists $g_i \in C^1([a_{i-1}, a_i])$ such that $f_{(a_{i-1},a_i)} = (g_i)_{(a_{i-1},a_i)}$. Then f' is defined a.e., it is in L^1 and*
$$(T_f)' = T_{f'} + \sum_{i=1}^N (f(a_i+) - f(a_i-))\delta_{a_i}.$$

The proof is straightforward.

4.5. Resolution of $T' = 0$. as for the Sobolev spaces!!!! For such $T \in \mathcal{D}(\mathbb{R})$, we have that
$$\langle T, \psi' \rangle = 0 \text{ for all } \psi \in \mathcal{D}(\mathbb{R}).$$

CHAPTER 15

Tempered distributions and Fourier transform

CHAPTER 16

Applications to differential equations

Epilogue

In this lecture, we have presented methods to find solutions to

- ODEs with Lipschitz coefficients and initial conditions (Cauchy-Lipschitz-Theorem, Exercise 3 and Theorem 16.5), a.k.a. $\{ u' = f(u)$ with $u(0)$ prescribed $\}$ or $\{u'' = f(u), u(0), u'(0)$ prescribed$\}$ and f is Lipschitz.
- ODEs with continuous coefficients and initial condition (Arzela-Peano's Theorem, see Theorem 6.1 of Chapter 6) a.k.a $u' = f(u)$, $u(0)$ prescribed and f is continuous.
- ODEs with boundary conditions on a compact domain (method of continuity, see Chapter 7, Sobolev spaces , see Chapter 10) a.k.a. $u'' = f(u)$ with $u(0) = u(1) = 0$.
- ODEs with boundary conditions on $\mathbb{R}$ (Fourier transform, see Chapter 12) a.k.a. $u'' = f(u)$ with $u(-\infty) = u(+\infty) = 0$.
- Evolution equations (heat equation, see see Chapter 12)

All these results were obtained by using abstract methods that mostly consider a function not as an object acting on a set, but as a variable on which to act. So instead of finding a function $x \mapsto u(x)$ satisfying an equation, we have investigated functions satisfying

$$\Phi(u) = 0 \text{ or } F'(u) = 0.$$

This lecture is only a glimpse at the abundance of methods that have been developed for such a task.

Here are a few references for the reader who is interested in going further in this direction. This list is nothing but exhaustive

• For more abstract Functional Analysis for graduate classes, see the classical monograph "Functional Analysis" by Rudin [6]

• For a Lecture in Functional Analysis oriented to the analysis of Partial Differential Equations, including Sobolev spaces of high dimensions, see the classical reference "Functional analysis, Sobolev spaces and partial differential equations" by Brezis [1]. It is also (and originally) available in French.

• Degree Theory is a a tool that allows to "count" the solutions to $\Phi(u) = 0$. This is applied to find solutions, and possibly many solutions, to differential equations. A great reference here is "Degree Theory" by Lloyd [4].

• When one writes a problem as $F'(u) = 0$, then solving it amounts to find critical points to F. In Chapter 8, a basic minimizing method is presented. Of course, there are many others, and the litterature more than abundant. To cite only one, a great classical reference for PDEs is "Variational Methods" by Struwe [8].

Appendix: definitions, undergraduate theorems and index

A. Topology

Definition 0.1 (Topological space). *Let X be a set. A* topology *on X is a set $\mathcal{T} \subset \mathcal{P}(X)$ such that*

- $\emptyset, X \in \mathcal{T}$
- *For any family $(A_i)_{i\in I}$ such that $A_i \in \mathcal{T}$, then $\bigcup_{i\in I} A_i \in \mathcal{T}$*
- *For any finite family $(A_i)_{i\in I}$ such that $A_i \in \mathcal{T}$ then $\bigcap_{i\in I} A_i \in \mathcal{T}$.*

We say that $(X, \mathcal{T})$ is a topological space. The elements of $\mathcal{T}$ are called open sets of the topology. Therefore, knowing a topology amounts to knowing its open sets. A closed set is the complement of an open set.

Definition 0.2 (Metric space). *Let X be a set. A distance on X is a function $d : X \times X \to \mathbb{R}$ such that*

- **Positivity** *$d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ iff $x = y$*
- **Symmetry** *$d(x, y) = d(y, x)$ for all $x, y \in X$*
- **Triangle inequality** *$d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.*

We then say that (X, d) is a metric space.

Definition 0.3 (Balls). *Let (X, d) be a metric space. For $x \in X$ and $r \geq 0$, we define*

$$B_d(x, r) := \{y \in X \text{ such that } d(x, y) < r\},$$

$$\bar{B}_d(x, r) := \{y \in X \text{ such that } d(x, y) \leq r\}.$$

Definition 0.4 (Topology on a metric space). *Let (X, d) be a metric space. Let $A \subset X$ be a subset of X. We say that A is an open set of (X, d) if*

- *Either $A = \emptyset$*
- *Or for any $x \in A$, there exists $r > 0$ such that $B_d(x, r) \subset A$.*

This defines a topology, called the topology induced by the metric (or topology of metric space).

Definition 0.5 (Product topology). *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $\Omega \subset X \times Y$. We say that Ω is an open subset for the product topology if either $\Omega = \emptyset$, or for all $(x, y) \in \Omega$, there exists $U \subset X$ and $V \subset Y$ open such that*

$$(x, y) \in U \times V \subset \Omega.$$

Proposition 16.1 (Product of metric spaces)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. Define $\delta : (X \times Y) \times (X \times Y) \to \mathbb{R}$ defined by*

$$\delta((x, y), (x', y')) = d_X(x, x') + d_Y(y, y') \text{ for all } (x, y), (x', y') \in X \times Y.$$

Then δ is a metric on $X \times Y$. Moreover, the topology of $(X \times Y, \delta)$ coincides with the product topology of (X, d_X) and (Y, d_Y).

Theorem 16.2. *Let (X, d_X) and (Y, d_Y) be two metric spaces. Take $D \subset X$ and a function $f : D \to Y$. We assume that*

- *D is dense in X*
- *$f : D \to Y$ is uniformly continuous for the metric induced by X*
- *(Y, d_Y) is complete.*

Then f extends uniquely as a uniformly continuous function $X \to Y$.

B. Differential Calculus

THEOREM 16.3 (Inverse functions theorem). *Inverse function theorem Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach spaces. Let $U \subset E$ and $V \subset F$ open subsets of respectively E and F. We take $f : U \to V$ and we assume that*

- $f \in C^1(U, V)$
- *There exists $x_0 \in U$ such that $df_{x_0} \in GL_c(E, F)$.*

Then there exists $U_0 \subset E$, $V_0 \subset F$ open sets such that $x_0 \in U_0 \subset U$, $f(x_0) \in V_0 \subset V$ and $f : U_0 \to V_0$ is a C^1-diffeomorphism. In addition, if $f \in C^k(U, V)$ for some $1 \leq k \leq \infty$, then $f : U_0 \to V_0$ is a C^k-diffeomorphism.

THEOREM 16.4 (Global Inverse functions theorem). *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach spaces. Let $U \subset E$ and $V \subset F$ open subsets of respectively E and F. We take $f : U \to V$ and we assume that*

- $f \in C^1(U, V)$
- *For all $x \in U$, we have that $df_x \in GL_c(E, F)$*
- *The function f is one-to-one.*

Then $W = f(U)$ is open in V and $f : U \to W$ is a C^1-diffeomorphism. In addition, if $f \in C^k(U, V)$ for some $1 \leq k \leq \infty$, then $f : U \to W$ is a C^k-diffeomorphism.

C. Cauchy-Lipschitz Theorem and consequences

DEFINITION 0.6. *Let (X, d_X), (Y, d_Y) and (Z, d_Z) be three metric spaces. Let $f \in C^0(X \times Y, Z)$ where $X \times Y$ is endowed with the product metric. We say that f is locally Lipschitz continuous in the second variable if for all $(x_0, y_0) \in X \times Y$, there exists $r > 0$ and $K > 0$ such that*

$$d_Z(f(x, y), f(x, y')) \leq K d_Y(y, y') \text{ for all } x \in B_X(x_0, r) \text{ and } y, y' \in B_Y(y_0, r).$$

THEOREM 16.5. *[Cauchy-Lipschitz Theorem] Let $U \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ and I be a nonempty interval of $\mathbb{R}$. Let $f \in C^0(I \times U, \mathbb{R}^n)$. We assume that f is locally Lipschitz continuous in the second variable. We fix $t_0 \in I$ and $x_0 \in U$. Then there exists $\delta > 0$ and $x \in C^1(]t_0 - \delta, t_0 + \delta[\cap I, U)$ such that*

$$\left\{ \begin{array}{ll} x'(t) = f(t, x(t)) & \text{in }]t_0 - \delta, t_0 + \delta[\cap I \\ x(t_0) = x_0. & \end{array} \right.$$

Moreover, such a solution is unique.

THEOREM 16.6. *[Linear Cauchy-Lipschitz Theorem] Let I be a nonempty interval of $\mathbb{R}$, let $A \in C^0(I, M_n(\mathbb{R}))$ and $B \in C^0(I, \mathbb{R}^n)$ be two continuous matrix or vector-valued functions. We fix $t_0 \in I$ and $X_0 \in \mathbb{R}^n$. Then there exists a unique function $X \in C^2(I, \mathbb{R}^n)$ such that*

$$\left\{ \begin{array}{ll} X'(t) = A(t)X(t) + B(t) & \text{in } I \\ X(t_0) = x_0. & \end{array} \right.$$

D. Integration

THEOREM 16.7. *[Lebesgue's convergence Theorem] Let (X, μ) be a meserured space. Let $(f_n)_n$ be such that $f_n \in L^1(X)$ for all $n \in \mathbb{N}$ and let $f : X \to \mathbb{R}$ defined a.e. such that*

- $\lim_{n\to+\infty} f_n(x) = f(x)$ *for a.e. $x \in X$*

- *There exists* $g \in L^1(X)$ *such that* $|f_n(x)| \leq g(x)$ *for all* $n \in \mathbb{N}$ *and a.e.* $x \in X$.

Then $f \in L^1(X)$ *and*

$$\lim_{n \to +\infty} \int_X f_n(x)\, d\mu(x) = \int_X f(x)\, d\mu(x).$$

Theorem 16.8. *Let* (Ω, d) *be a metric space and let* (X, μ) *be a mesured space. Let* $F : \Omega \times X \to \mathbb{R}$ *be a function such that*

- *For all* $\xi \in \Omega$, $F(\xi, \cdot) \in L^1(X)$
- *For a.e.* $x \in X$, $\xi \mapsto F(\xi, x)$ *is continuous on* Ω
- *There exists* $g \in L^1(X)$ *such that* $|F(\xi, x)| \leq g(x)$ *for all* $\xi \in \Omega$ *and a.e.* $x \in X$.

Then the function $\xi \mapsto \varphi(\xi) := \int_X F(\xi, x)\, d\mu(x)$ *is defined on* Ω. *Moreover, it is continuous on* Ω.

Theorem 16.9. *Let* Ω *be an open subset of* $\mathbb{R}^n$, $n \geq 1$ *and let* (X, μ) *be a mesured space. Let* $F : \Omega \times X \to \mathbb{R}$ *be a function such that*

- *For all* $\xi \in \Omega$, $F(\xi, \cdot) \in L^1(X)$
- *For a.e.* $x \in X$, $\xi \mapsto F(\xi, x)$ *is differentiable on* Ω
- *For all* $1 = 1, ..., n$, *there exists* $g_i \in L^1(X)$ *such that* $|\partial_i F(\xi, x)| \leq g_i(x)$ *for all* $\xi \in \Omega$ *and a.e.* $x \in X$

Then the function $\xi \mapsto \varphi(\xi) := \int_X F(\xi, x)\, d\mu(x)$ *is defined on* Ω. *Moreover, it is differentiable in all directions* Ω *and for all* $i = 1, ..., n$, *we have that*

$$\frac{\partial \varphi}{\partial \xi_i}(x) = \int_X \frac{\partial F}{\partial \xi_i}(\xi, x)\, d\mu(x) \text{ for all } \xi \in \Omega.$$

In addition, if $\xi \mapsto F(\xi, x)$ *is* C^1 *for all* $x \in X$, *then* $\varphi \in C^1(\Omega)$.

Theorem 16.10 (Fubini's Theorem). *Let* (X, μ_X) *and* (Y, μ_Y) *two measured spaces which are* $\sigma-$*finite. Endow* $X \times Y$ *with the product metric* $\mu_X \otimes \mu_Y$. *Take* $f : X \times Y \to \mathbb{R}$ *defined a.e. and measurable for* $\mu_X \otimes \mu_Y$. *Assume that*

- *Either* $f \geq 0$ *a.e. in* $X \times Y$,
- *Or* $f \in L^p(X \times Y, \mu_X \otimes \mu_Y)$,

then

$$\int_{X \times Y} f(x, y)\, d\mu_X \otimes \mu_Y(x, y) = \int_X \left(\int_Y f(x, y)\, d\mu_Y(y) \right) d\mu_X(x) = \int_Y \left(\int_X f(x, y)\, d\mu_X(x) \right) d\mu_Y(y)$$

with all these quantities making sense.

E. Miscellaneous

Theorem 16.11 (Dini's Theorem). *Let* (X, d) *be a compact metric space and let* $(f_n)_n$ *be a family of functions such that* $f_n \in C^0(X)$. *We assume that there exists* $f : X \to \mathbb{R}$ *such that*

- $(f_n(x))_n$ *converges to* $f(x)$ *for all* $x \in X$
- $f \in C^0(X, \mathbb{R})$
- $f_n(x) \leq f_{n+1}(x)$ *for all* $x \in X$ *and* $n \in \mathbb{N}$.

The $(f_n)_n$ *converges uniformly to* f *as* $n \to +\infty$, *that is* $\lim_{n \to +\infty} \|f_n - f\|_\infty = 0$.

THEOREM 16.12 (Mean Value Inequality). *Let* $(E, \|\cdot\|_E)$ *be a normed space (not necessarily complete). Let* $f : [a, b] \to E$ *be differentiable such that there exists* $M \geq 0$ *such that* $\|f'(x)\|_E \leq M$ *for all* $x \in [a, b]$. *Then*

$$\|f(b) - f(a)\|_E \leq M|b - a|.$$

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