Appendix A: Methods for Ordinary Differential Equations (ODEs)

We consider the initial value problem:

$$\begin{cases} \dot{y} = f(t, y(t)) \quad , \quad t \in [0,T] = \Omega & (1a) \\ y(t=0) = y_0 & (1b) \end{cases}$$

often called a Cauchy problem.

I) Building of basic numerical schemes

the basic methods used in CFD (Euler schemes, Crank-Nicolson method) will be built here by use of numerical quadrature.

First, the time domain $\Omega = [0,T]$ is subdivided as $\{t_n = n\,\Delta t \,,\ n = 0, \dots, N\}$

where $\Delta t = \frac{T}{N}$ is the time-step.

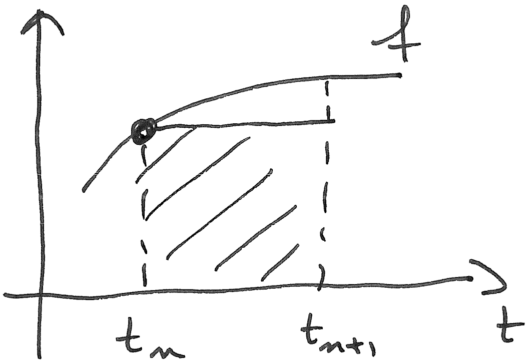
The EDO (1a) is first integrated in $[t_n, t_{n+1}]$ to yield:

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(t, y(t))\,dt \quad (2)$$

~~There~~ No discrete approximation has yet been performed. Now, the RHS of (2) will be approximated with numerical quadrature based on Lagrange polynomials.

I-1) Forward Euler method (FE1)

The simplest method steams form the approximation of (2) by using values at time t_n:



$$\int_{t_n}^{t_{n+1}} f\,dt \overset{?}{=} \Delta t\, f(t_n, y^n)$$

where we used the compact notation:

$$\underbrace{y^n}_{\text{approx. solution}} \overset{?}{=} \underbrace{y(t_n)}_{\text{exact solution}}$$

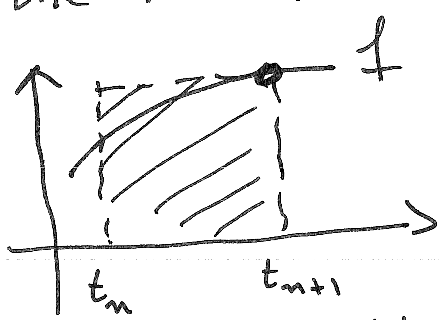
The forward Euler scheme reads then:

$$\frac{y^{n+1} - y^n}{\Delta t} = f(t_n, y^n) \qquad (3)$$

Its acronym is FE 1, because we shall see later that the method is first order in time.

I-2) Backward Euler Method (BE1)

If the RHS of (2) is approximated with end point t_{n+1}:



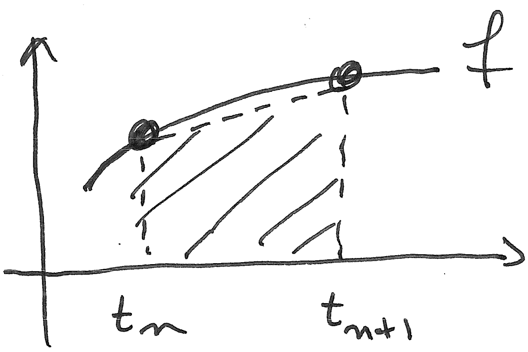
$$\int_{t_n}^{t_{n+1}} f\,dt \overset{?}{=} \Delta t\, f(t_{n+1}, y^{n+1})$$

We obtaint the backward Euler method (BE1):

$$\frac{y^{n+1} - y^n}{\Delta t} = f(t_{n+1}, y^{n+1}) \qquad (4)$$

I-3] Trapezoidal method

The RHS of (2) can be approximated by the Trapezoidal method (use of Lagrange polynomial of order 2)



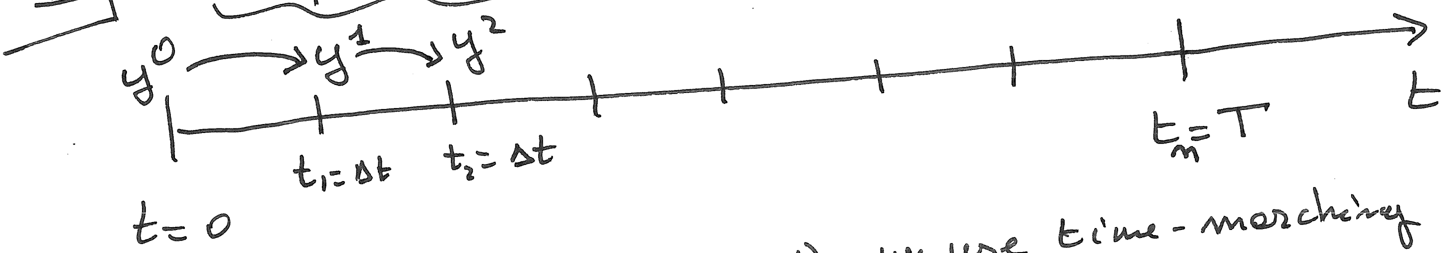
$$\int_{t_n}^{t_{n+1}} f\,dt \stackrel{?}{=} \frac{\Delta t}{2}\left[f(t_{n+1}, y^{n+1}) + f(t_n, y^n) \right]$$

The resulting trapezoidal method:

$$\frac{y^{n+1} - y^n}{\Delta t} = \frac{1}{2}\left[f(t_{n+1}, y^{n+1}) + f(t_n, y^n) \right] \quad (5)$$

is often called the Crank-Nicolson method (CN) when used for time integration of PDEs.

II] Use of time integration scheme. Implicit or explicit method



For ~~integrating~~ solving ODE (1), we use time-marching by starting with the initial condition at $t_0 = 0$:

$$y^0 = y(0),$$

Then, y^1 at $t = t_1$ is calculated with FE1, for example, written in its iterative form:

$$\underbrace{y^1}_{\text{Unknown at } t = t_1} = \underbrace{y^0 + \Delta t\, f(t_0, y^0)}_{\text{known at } t = t_1} \quad (6)$$

This expression is ready to be programed in any language, such as FORTRAN or MATLAB. Since there is no equation to solve to obtain y^1 from values at previous time-level, the FE1 scheme is called <u>explicit</u>. Furthermore, Eq. (6) involves the solution at only time levels t_0 and t_1, thus FE1 is called a <u>two-level</u> scheme.

By contrast, the BE1 scheme leads to the iterative equation:

$$y^1 - \Delta t\, f(t_1, y^1) = y^0, \qquad (7)$$

the solution of an equation is needed to obtain y^1, resulting in more computional work. The BE1 is called an <u>implicit scheme</u>, and is also a two-level scheme.

The CN (trapezoidal) scheme is an implicit two-level scheme.

III) Truncation error of time-integrator scheme.

The local truncation error of a scheme (LTE) is the difference between the numerical scheme and the ODE at time t_n. It can be summarized as:

$$\underset{\text{at } t_n}{\text{scheme}} = \underset{\text{at } t_n}{\text{ODE}} + \tau^n$$

where τ^n is the LTE at time t_n. It is usually performed by using Taylor expansion of the solution, i.e.:

$$y^{n+1} = y^n + \Delta t\, \dot{y}^n + \frac{\Delta t^2}{2} \ddot{y}^n + O(\Delta t^3) \qquad (8)$$

Let us calculate the LTE of backward-Euler scheme by introducing this Taylor expansion in (3):

$$\underbrace{\frac{y^{n+1}-y^n}{\Delta t} - f(t_n,y_n)}_{\text{FE1 scheme at } t_n} = \frac{y^n + \Delta t\,\dot{y}^n + \frac{\Delta t^2}{2}\ddot{y}^n + O(\Delta t^3) - y^n}{\Delta t} - f(t_n,y_n)$$

$$= \underbrace{\dot{y}^n - f(t_n,y_n)}_{\text{ODE at } t_n} + \tau^n$$

where the LTE is $\tau^n = \frac{\Delta t}{2}\ddot{y}^n + O(\Delta t^2)$.

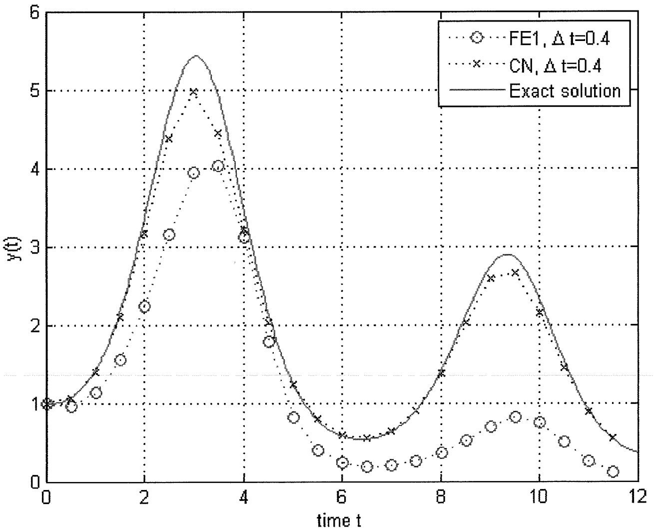
Hence, the FE1 scheme is first-order accurate.

In a similar fashion, we can show the the LTE of BE1 is:

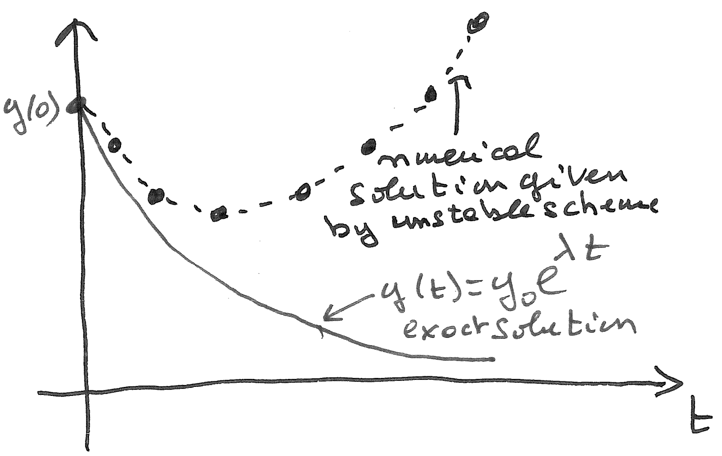
$$\tau^n = -\frac{\Delta t}{2}\ddot{y}^n + O(\Delta t^2)$$

and the LTE of CN scheme is:

$$\tau^n = \frac{\Delta t^2}{12}\dddot{y}^n + O(\Delta t^3).$$



IV Stability of a numerical scheme



The stability (also called absolute stability) is the property of the scheme to yield a numerical solution who does not diverge to ∞ when t is large.

Let us consider the model problem for $\lambda < 0$:

$$\begin{cases} \dot{y} = \lambda y, & t \in [0, +\infty[\\ y(0) = y_0 \end{cases} \qquad (9)$$

Its exact solution is:

$$y(t) = y_0 e^{\lambda t},$$

and since $\lambda = 0$ the solution decays with time ($\lim_{t\to\infty} y(t) = 0$).

When BE1 is applied to model problem (9), the iterative equation reads:

$$y^{n+1} = y^n + \Delta t \lambda y^n \Rightarrow y^n = \sigma y^{n-1} = \dots = (\sigma)^n y^0$$

where $\sigma = 1 + \lambda \Delta t$ is the <u>amplification factor</u> of the scheme. In order to have a solution that does not blow up in time, we must have:

$$|\sigma| < 1 \qquad (10)$$

which leads to $0 < \Delta t < -2/\lambda$.

Hence, the BE1 scheme is stable if we use a time-step that verifies the stability condition:

$$\Delta t < \frac{2}{-\lambda}. \qquad (11)$$

We say that BE1 is <u>conditionally</u> stable.

- If we use BE1 to solve model problem (9), the amplification factor is $\sigma = \dfrac{1}{1-\lambda\Delta t}$, and stability condition $|\sigma|<1$ leads to $|1-\lambda\Delta t|>1$, which is always verified for $\Delta t>0$. Hence, the BE1 is <u>stable with no conditions</u>.

- The amplification factor for CN scheme is

$$\sigma = \frac{1+\lambda\Delta t/2}{1-\lambda\Delta t/2}$$

and stability condition (10) is verified for all $\Delta t>0$. As the BE1 scheme, the CN scheme is <u>always stable</u>.

<u>Application:</u> We consider the model problem $\dot{y}+\frac{1}{2}y=0$. the stability condition of FE1 is $\Delta t<4$, while BE1 and CN computations are stable for all $\Delta t>0$. Fig. 1 shows computations with FE with various values of Δt, the one with $\Delta t=4.2$ is unstable. Fig. 2 shows that BE1 is stable for all values of Δt.