Rewriting Techniques for Programming and Proving

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A simple game

The rules of the game:

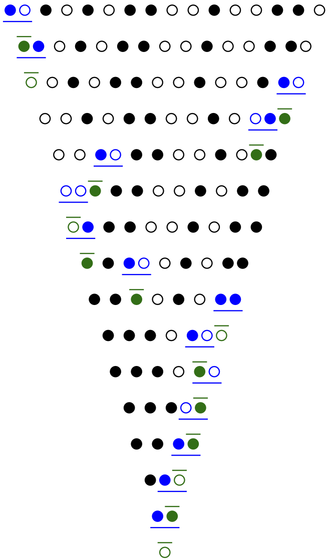
$$\begin{array}{ccc} \bullet\bullet & \longrightarrow & \circ \\ \circ\circ & \longrightarrow & \circ \\ \bullet\circ & \longrightarrow & \bullet \\ \circ\bullet & \longrightarrow & \bullet \end{array}$$

A starting point:

$$\bullet\ \circ\ \bullet\ \circ\ \bullet\ \circ\ \bullet\ \bullet\ \circ\ \circ\ \bullet\ \circ\ \circ\ \bullet\ \bullet\circ$$

Who puts the last white wins (for this starting point)

A simple game



Rules of the game

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●○ → ●

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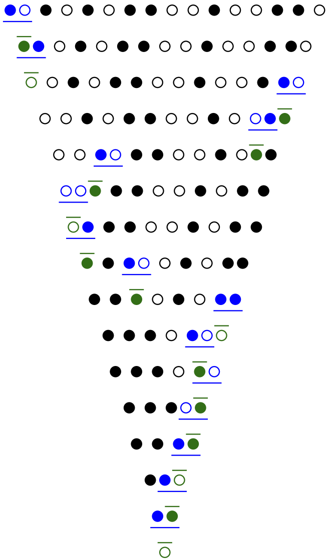

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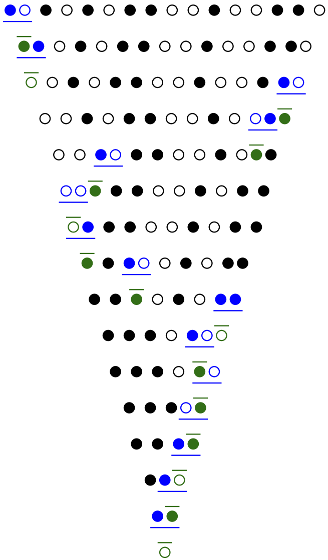

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Questions:

- May I always win?
- Does the game terminates?

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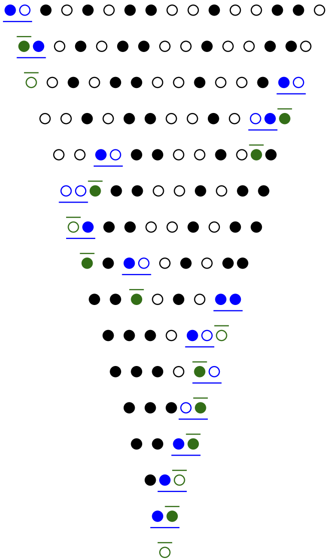

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Questions:

- May I always win?
- Does the game terminates?
- Do we always get the same result?

What are the basic operations that have been used?

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1– Matching

The data: $\bullet\bullet \quad \boxed{\bullet\circ}\ \circ\ \bullet\ \circ\ \bullet\bullet$

The rewrite rule: $\boxed{\bullet\circ} \to \boxed{\bullet}$

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2– Compute what should be substituted

The righthand side: $\boxed{\bullet}$

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The righthand side: $\boxed{\bullet}$

3– Replacement

The new generated data: $\bullet\ \bullet\ \boxed{\bullet}\ \circ\ \bullet\ \circ\ \bullet\bullet$

Addition in Peano arithmetic

Peano gives a meaning to addition by using the following axioms:

$$
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0 + x &= x \\
s(x) + y &= s(x + y)
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Is there a better result?

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- Is such a result unique?
- How to turn equations into rewrite rules?

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The new generated data: $\boxed{s(s(0)+s(s(0)))}$

Fibonacci

$$\begin{array}{lll} [\alpha] & \mathit{fib}(0) & \rightarrow & 1 \\ [\beta] & \mathit{fib}(1) & \rightarrow & 1 \\ [\gamma] & \mathit{fib}(n) & \rightarrow & \mathit{fib}(n-1) + \mathit{fib}(n-2) \end{array}$$

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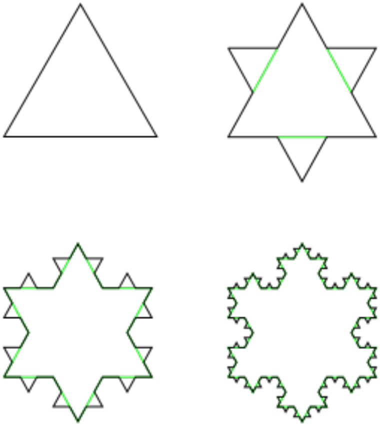
$$\boxed{\text{fib}(1)} + \mathit{fib}(0) + 1 \to \boxed{1} + \mathit{fib}(0) + 1$$

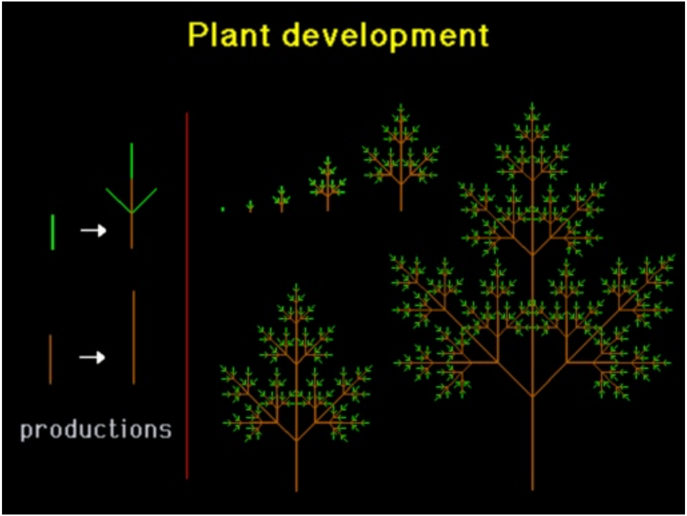
$$\dots$$

Finally $\mathit{fib}(3) = 3$, $\mathit{fib}(4) = 5$, ...

Fractal: Koch snowflake



Ecological Rewriting



http://algorithmicbotany.org/

Sorting by rewriting

```
rules for List
  X, Y : Nat ; L L' L'' : List;
   hd (X L) => X ;                         tl (X L) => L ;
   sort nil => nil .
   sort (L X L' Y L'') =>  sort (L Y L' X L'')  if Y < X .
end

sort (6 5 4 3 2 1) => ...
```

On what objects can rewriting act?

It can be defined on

- terms like $2 + \mathrm{i}(3)$ or XML documents
- strings like "What is rewriting?" (`sed` performs string rewriting)
- graphs
- sets
- multisets
- ...

References

- Rewriting, Solving, Proving
  Claude and Hélène Kirchner
- Term Rewriting System - Terese
  Marc Bezem, Jan Willem Klop, Roel de Vrijer
- Term Rewriting and All That
  Franz Baader and Tobias Nipkow