

Gravitation and Electricity

by H. Weyl in Zurich

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According to Riemann [1] geometry is based on the following two facts:

1. *Space is a three-dimensional continuum*, the manifold of its points is therefore represented in a smooth manner by the values of three coordinates x_1, x_2, x_3 .

2. (*Pythagorean Theorem*) The square of the distance between two infinitesimally separated points

$$P = (x_1, x_2, x_3) \quad \text{and} \quad P' = (x_1 + dx_1, x_2 + dx_2, x_3 + dx_3) \quad (1)$$

is (in any coordinate system) a quadratic form in the relative coordinates dx_i :

$$ds^2 = \sum_{ik} g_{ik} dx_i dx_k \quad (g_{ik} = g_{ki}) \quad (2)$$

We express the second fact briefly by saying: the space is a *metrical* continuum. In the spirit of modern local physics we take the pythagorean theorem to be strictly valid only in the infinitesimal limit.

Special relativity leads to the insight that *time* should be included as a fourth coordinate x_0 on the same footing as the three space-coordinates, and thus the stage for physical events, *the world*, is a *four-dimensional, metrical continuum*. The quadratic form (2) that defines the world-geometry is not positive-definite as in the case of three-dimensional geometry, but has positive-index 3. Riemann already expressed the idea that the metric should be regarded as something physically meaningful since it manifests itself as an effective force for material bodies, in centrifugal forces for example, and that one should therefore take into account that it interacts with matter; whereas previously all geometers and philosophers believed that the metric was an intrinsic property of the space, independent of the matter contained in it. It was on the basis of this idea, for which the possibility of fulfillment was not available to Riemann, that in our time Einstein (independently of Riemann) erected the grandiose structure of general relativity. According to Einstein the phenomena of gravitation can be attributed to the world-metric, and the laws through by which matter and metric interact are nothing but the laws of gravitation; the g_{ik} in (2) are the components of the gravitational potential.—Whereas the the gravitational potentials

are the components of an invariant quadratic differential form, *electromagnetic phenomena* are controlled by a four-potential, whose components ϕ_i are the components of an invariant *linear* differential form $\sum \phi_i dx_i$. However, both phenomena, gravitation and electricity, have remained completely isolated from one another up to now.

From recent publications of Levi-Civita [2], Hessenberg [3] and the author [4] it has become evident that a natural formulation of Riemannian geometry is based on the concept of infinitesimal parallel-transfer. If P and P' are two points connected by a curve then one can transfer a vector from P to P' along the curve keeping it parallel to itself. However, the transfer of the vector from P to P' is, in general, not integrable i.e. the vector that is obtained at P' depends on the path. Integrability holds only for Euclidean ('gravitation-free') geometry. — But in the Riemannian geometry described above there is contained a residual element of rigid geometry—with no good reason, as far as I can see; it is due only to the accidental development of Riemannian geometry from Euclidean geometry. The metric (2) allows the magnitudes of two vectors to be compared, not only at the same point, but at any two arbitrarily separated points. *A true infinitesimal geometry should, however, recognize only a principle for transferring the magnitude of a vector to an infinitesimally close point* and then, on transfer to an arbitrarily distant point, the integrability of the magnitude of a vector is no more to be expected than the integrability of its direction. On the removal of this inconsistency there appears a geometry that, surprisingly, when applied to the world, *explains not only the gravitational phenomena but also the electrical*. According to the resultant theory both spring from the same source, indeed *in general one cannot separate gravitation and electromagnetism in an arbitrary manner*. In this theory *all physical quantities have a world-geometrical meaning; the action appears from the beginning as a pure number. it leads to an essentially unique universal law; it even allows us to understand in a certain sense why the world is four-dimensional*. —I shall first sketch the construction of the corrected Riemannian geometry without any reference to physics; the physical application will then suggest itself.

In a given coordinate system the coordinates dx_i of a point P' relative to an infinitesimally close point P are the components of the *infinitesimal translation* $\overrightarrow{PP'}$ — see (1). The change from one coordinate-system to another is expressed by the continuous transformation:

$$x_i = x_i(x_1^* x_2^* \dots x_n^*) \quad (i = 1, 2, \dots, n)$$

which determines the connection between the coordinates of the same point in the different systems. For the components dx_i and dx_i^* of the same infinitesimal translation of the point P we then have the linear transformation

$$dx_i = \sum_k \alpha_{ik} dx_k^* \quad (3)$$

in which the α_{ik} are the values of the derivatives $\partial x_i / \partial x_k^*$ at the point P . A (contravariant) vector at the point P has n numbers ξ^i as components in every coordinate system, and on transforming the coordinates, these numbers transform in the same way as the infinitesimal translations in (3). I shall call the set of all vectors at P the *vector-space* at P . It is 1. *linear or affine* i.e. it is invariant with respect to the multiplication of a vector by a number and the addition of two vectors, and 2. *metrical*: by means of the symmetric bilinear form (2) an invariant scalar product

$$\chi \cdot \eta = \eta \cdot \chi = \sum g_{ik} \chi^i \eta^k$$

is defined for each pair of vectors χ, η . However, according to our point of view *this form is determined only up to an arbitrary positive proportionality-factor*. If the manifold is described by the coordinates x_i only the ratios of the components g_{ik} are determined by the metric at P . Physically also, only the ratios of the g_{ik} have a direct physical meaning. For a given point P the neighbouring points P' which can receive light-signals from P satisfy the equation

$$\sum_{ik} g_{ik} dx_i dx_k = 0$$

For the purpose of analytical representation we have to 1. choose a coordinate system and 2. in each point P determine the arbitrary proportionality-factor of the g_{ik} . Correspondingly each formula must have a double-invariance: 1. it must be *invariant with respect to arbitrary smooth coordinate transformations* 2. it must remain unchanged *when the g_{ik} are replaced by λg_{ik} where λ is an arbitrary smooth function of position*. Our theory is characterized by the appearance of this second invariance property.

An affine or linear map of the vector space at the point P onto the vector space at the point P^* is defined as the map $A \rightarrow A^*$ such that $\alpha \chi \rightarrow \alpha \chi^*$ and $\chi + \eta \rightarrow \chi^* + \eta^*$, where α is an arbitrary number. In particular the map is said to be a *similarity* map if the inner-product $\chi^* \cdot \eta^*$ is proportional to the inner-product $\chi \cdot \eta$ for all pairs of vectors χ and η . (Only this concept of similar maps has an objective meaning in our context; the previous theory allowed one to introduce the sharper concept of *congruent* maps.) The *parallel-transfer of a vector* at P to a neighbouring point P' is defined by the following two axioms:

1. The parallel transfer of the vectors at P to vectors at P' defines a similarity map.
2. If $\underline{P_1}$ and $\underline{P_2}$ are two neighbouring points to P and if the infinitesimal vectors $\underline{PP_2}$ and $\underline{PP_1}$ become $\underline{P_1P_{12}}$ and $\underline{P_2P_{21}}$, on parallel-transfer to P_2 and P_1 respectively then P_{12} and P_{21} coincide (commutativity).

The part of the first axiom that says that the parallel-transfer is an affine transformation of the vector space from P to P' is expressed analytically as

follows: the vector ξ^i at $P = (x_1 x_2 \dots x_n)$ is transferred to the vector

$$\xi^i + d\xi^i \quad \text{at } P' = (x_1 + dx_1, x_2 + dx_2, \dots, x_n + dx_n)$$

whose components are linear in ξ^i :

$$d\xi^i = - \sum_r d\gamma_r^i \xi^r \quad (4)$$

The second axiom requires that the $d\gamma_r^i$ are linear differential forms:

$$d\gamma_r^i = \sum_s \Gamma_{rs}^i dx_s,$$

whose coefficients have the symmetry property

$$\Gamma_{sr}^i = \Gamma_{rs}^i \quad (5)$$

If two vectors ξ^i, η^i at P are parallel-transferred to the vectors $\xi^i + d\xi^i, \eta^i + d\eta^i$ at P' the part of axiom 1 that goes beyond affinity to include similarity requires that

$$\sum_{ik} (g_{ik} + dg_{ik})(\xi^i + d\xi^i)(\eta^k + d\eta^k) \quad \text{and} \quad \sum_{ik} g_{ik} \xi^i \eta^k$$

are proportional. If we call the proportionality factor, which is infinitesimally close to unity, $(1 + d\phi)$ and define the lowering of indices in the usual manner as

$$a_i = \sum_k g_{ik} a^k$$

we then have

$$dg_{ik} - (d\gamma_{ki} + d\gamma_{ik}) = g_{ik} d\phi \quad (6)$$

From this it follows that $d\phi$ is a linear differential form:

$$d\phi = \sum_i \phi_i dx_i \quad (7)$$

If it is known, then the quantities Γ are determined by equation (6) or

$$\Gamma_{i,kr} + \Gamma_{k,ir} = \frac{\partial g_{ik}}{\partial x_r} - g_{ik} \phi_r$$

and the symmetry property (5). *The metrical connection of the space depends not only on the quadratic form (2) (which is determined only up to a proportionality factor) but on the linear form (7).* If, without changing coordinates,

we replace g_{ik} by λg_{ik} the quantities $d\gamma_k^i$ remain unchanged, the $d\gamma_{ik}$ acquire a factor λ and dg_{ik} becomes $\lambda dg_{ik} + g_{ik}d\lambda$. Equation (6) then shows that $d\phi$ becomes

$$d\phi + \frac{d\lambda}{\lambda} = d\phi + d(\ln \lambda).$$

For the linear form $\phi_i dx_i$ the arbitrariness takes the form of an *additive total differential* rather than a proportionality factor that would be determined by a choice of scale. For the analytic representation of the geometry the forms

$$g_{ik} dx_i dx_k \quad \phi_i dx_i \quad (8)$$

are on the same footing as

$$\lambda g_{ik} dx_i dx_k \quad \text{and} \quad \phi_i dx_i + d(\ln \lambda), \quad (9)$$

where λ is an arbitrary function of position. *The invariant quantity is therefore the anti-symmetric tensor with components*

$$F_{ik} = \frac{\partial \phi_i}{\partial x_k} - \frac{\partial \phi_k}{\partial x_i} \quad (10)$$

i.e. the form

$$F_{ik} dx_i \delta x_k = \frac{1}{2} F_{ik} \Delta x_{ik},$$

which depends bilinearly on two arbitrary translations dx and δx at the point P or, more precisely, on the surface-element

$$\Delta x_{ik} = dx_i \delta x_k - dx_k \delta x_i$$

determined by these two translations. The special case for which the magnitude of a vector at an arbitrary initial point can be parallel-transferred throughout the space in a path-independent manner appears when the g_{ik} can be chosen in such a way that the ϕ_i vanish. The Γ_{rs}^i are then nothing but the Christoffel 3-index symbols. The necessary and sufficient condition for this to be the case is the vanishing of the tensor F_{ik} .

Accordingly, it is very suggestive to interpret ϕ_i as the electromagnetic potential and the tensor F as the *electromagnetic field*. Indeed, the absence of an electromagnetic field is the condition for the validity of Einstein's gravitational theory. If one accepts this interpretation one sees that electromagnetic quantities are such that their characterization by numbers in a given coordinate system is independent of the scale. In this theory one must adopt a new approach to the question of scales and dimensions. Previously one spoke of a

tensor being of second rank when, *after making an arbitrary choice of scale*, it was represented in every coordinate system by a matrix a_{ik} whose entries were the coefficients of an invariant bilinear form of two arbitrary independent infinitesimal translations

$$a_{ik} dx_i \delta x_k \quad (11)$$

Here we talk of a tensor when, having fixed a coordinate system and *making a definite choice of the proportionality factor of the g_{ik}* , the components a_{ik} are uniquely determined and indeed are determined in such a way that the form (11) is invariant with respect to coordinate transformations, but a_{ik} changes to $\lambda^e a_{ik}$ when g_{ik} changes to λg_{ik} . We say then that the tensor has *weight e* or, if a 'scale' l is assigned to the line-element ds , that it has dimension l^{2e} . The absolute invariant tensors are only those of weight zero. The field-tensor with the components F_{ik} is of this kind. According to (10) it satisfies the first system of Maxwell equations

$$\frac{\partial F_{kl}}{\partial x_i} + \frac{\partial F_{li}}{\partial x_k} + \frac{\partial F_{ik}}{\partial x_l} = 0$$

Once the concept of parallel-transfer is defined the geometry and tensor calculus is easily deduced.

a) *Geodesics*. Given a point P and a vector at P , the geodesic originating at P in the direction of this vector is obtained by continuously parallel-transferring the vector in its own direction. The differential equation for the geodesic takes the form

$$\frac{d^2 x_i}{d\tau^2} + \Gamma_{rs}^i \frac{dx_r}{d\tau} \frac{dx_s}{d\tau} = 0$$

for a suitable choice of the parameter τ . (It cannot, of course, be interpreted as the line of shortest length since the concept of length along a curve is not meaningful.)

b) *Tensor Calculus*. For example, to obtain a tensor-field of rank 2 from a covariant tensor-field of rank 1 and weight zero and components f_i by differentiation, we take any vector ξ^i at the point P with coordinates x_i , construct the invariant $f_i \xi^i$ and compute its infinitesimal variation on parallel-transfer to a neighbouring point P' with coordinates $x_i + dx_i$. We obtain

$$\frac{\partial f_i}{\partial x_k} \xi^i dx_k + f_r d\xi^r = \left(\frac{\partial f_i}{\partial x_k} - \Gamma_{ik}^r f_r \right) \xi^i dx_k.$$

The quantities in brackets on the right-hand side are the components of a tensor of rank 2 and weight zero which has been derived from the field f in a fully invariant manner.

c) *Curvature*. To construct the analogue of the Riemann tensor consider the infinitesimal parallelogram consisting of the points P , P_1 , P_2 and $P_{12} = P_{21}$. Since the points P_{12} and P_{21} coincide, it makes sense to compute the difference between the vectors obtained at this point by taking any vector $\xi = \xi^i$ and parallel-transferring it to P_{12} via P_1 and P_2 respectively. For its components one obtains

$$\Delta \xi^i = R_j^i \xi^j, \quad (12)$$

where the R_j^i are independent of the vector ξ but depend linearly on the surface-element spanned by the two infinitesimal transfers $\overrightarrow{PP_1} = (dx_i)$ and $\overrightarrow{PP_2} = (\delta x_i)$:

$$R_j^i = R_{jkl}^i dx_k \delta x_l = \frac{1}{2} R_{jkl}^i \Delta x_{kl}.$$

The curvature components R_{jkl}^i , which depend only on the point P , have the following two symmetry properties: 1. they change sign on permutation of the last indices k and l ; 2. if one cyclically permutes the indices j, k, l and adds, the sum is zero. If the index i is lowered we obtain in R_{ijkl} the components of a covariant tensor of 4th rank and weight 1. One sees by inspection that R splits in an invariant manner into two parts

$$R_{jkl}^i = P_{jkl}^i - \frac{1}{2} \delta_j^i F_{kl} \quad \delta_j^i = 1 \quad (i = j) \quad \delta_j^i = 0 \quad (i \neq j), \quad (13)$$

where P_{ijkl} is anti-symmetric in the indices i and j as well as k and l . Whereas the equations $F_{ik} = 0$ characterize the absence of an electromagnetic field i.e. a space in which the transfer of magnitude is integrable, one sees from (13) that $P_{jkl}^i = 0$ are the invariant conditions for the absence of a gravitational field i.e. for the parallel transfer of directions to be integrable. Only in Euclidean space is there neither electromagnetism nor gravitation.

The simplest invariant of a linear map like (12) that assigns a vector $\delta \xi$ to every ξ is the trace

$$\frac{1}{n} R_i^i.$$

For this we obtain from (13) the form

$$-\frac{1}{2} F_{ik} dx_i \delta x_k,$$

which we have already encountered. The simplest invariant that can be constructed from a tensor of the form $-F_{ik}/2$ is the square of its magnitude

$$L = \frac{1}{4} F_{ik} F^{ik}.$$

Since the tensor F has weight zero, L is clearly an invariant of weight -2 .

If g is the negative determinant of the g_{ik} and

$$d\omega = \sqrt{g} dx_0 dx_1 dx_2 dx_3 = \sqrt{g} dx$$

is the infinitesimal volume element, then, as is well-known, the Maxwell theory is determined by the electromagnetic Action, which is equal to the integral $\int L d\omega$ over an arbitrary volume of this simplest invariant, in such a way that for arbitrary variations of the g_{ik} and ϕ_i which vanish on the boundary we have

$$\delta \int L d\omega = \int (s^i \delta \phi_i + T^{ik} \delta g_{ik}) d\omega$$

where

$$s^i = \frac{1}{\sqrt{g}} \frac{\partial (\sqrt{g} F^{ik})}{\partial x_k}$$

are the left-hand side of the Maxwell equations (on the right-hand side of which is the electromagnetic current) and the T^{ik} are the components of the energy-momentum tensor of the electromagnetic field. Since L is an invariant of weight -2 and the volume element an invariant of weight $\frac{n}{2}$ the integral $\int L d\omega$ then has a meaning only when the dimension is $n = 4$. *Thus in our context the Maxwell equations are possible only in 4 dimensions.* But in four dimensions the electromagnetic action is a pure number. Its magnitude in CGS units can, of course, only be determined when a computation based on our theory is applied to a physical problem such as the electron.

Passing on from Geometry to Physics, we have to assume, following the example of Mie's theory [5], that the whole set of natural laws is based on a definite integral-invariant, the action

$$\int W d\omega = \int \mathcal{W} dx \quad (\mathcal{W} = W \sqrt{g})$$

in such a way that *the actual world is selected from the class of all possible worlds by the fact that the Action is extremal in every region with respect to the variations of the g_{ik} and ϕ_k which vanish on the boundary of that region.* W , the action-density, must be an invariant of weight -2 . *The action is in any case a pure number;* in this way our theory gives pride of place to that part of atomic theory that is the most fundamental according to modern ideas: the action. The simplest and most natural Ansatz that we can make for W is

$$W = R_{jkl}^i R_i^{jkl} = |R|^2 \quad (14)$$

According to (13) this can be written as

$$W = |P|^2 + 4L$$

(At most the factor 4, by which the second [electrical] term is added, could be open to doubt). But even without specifying the action there are some general conclusions that we can draw. We shall show that: *just as* according to the researches of Hilbert [6], Lorentz [7], Einstein [8], Klein [9] and the author [10] *the four conservation laws of matter* (of the energy-momentum tensor) *are connected with the invariance of the Action with respect to coordinate transformations*, expressed through four independent functions, the electromagnetic conservation law is connected with the new scale-invariance, expressed through a fifth arbitrary function. The manner in which the latter resembles the energy-momentum principle seems to me to be the strongest general argument in favour of the present theory—insofar as it is permissible to talk of justification in the context of pure speculation.

We set for an arbitrary variation which vanishes on the boundary

$$\delta \int \mathcal{W} dx = \int (\mathcal{W}^{ik} \delta g_{ik} + \mathbf{w}^i \delta \phi_i) dx \quad (\mathcal{W}^{ik} = \mathcal{W}^{ki}) \quad (15)$$

The field equations are then

$$\mathcal{W}^{ik} = 0 \quad \mathbf{w}^i = 0 \quad (16)$$

We can regard the first and second as the gravitational and the electromagnetic field equations respectively. The quantities defined by

$$\mathcal{W}_k^i = \sqrt{g} W_k^i, \quad \mathbf{w}^i = \sqrt{g} w^i$$

are the mixed (contravariant) components of a tensor of weight -2 and rank 2 (1) respectively. In the system of equations (16) there are five superfluous equations corresponding to the invariance properties. This is expressed by the following five identities that hold for the left-hand sides:

$$\frac{\partial \mathbf{w}^i}{\partial x_i} \equiv \mathcal{W}_i^i, \quad (17)$$

$$\frac{\partial \mathcal{W}_k^i}{\partial x_i} - \Gamma_{kr}^s \mathcal{W}_s^r \equiv \frac{1}{2} F_{ik} \mathbf{w}^i. \quad (18)$$

The first is a result of scale-invariance. For if, in the transition from (8) to (9) we take $\ln \lambda$ to be an infinitesimal function of position we obtain the variation

$$\delta g_{ik} = g_{ik} \delta \rho, \quad \delta \phi_i = \frac{\partial(\delta \rho)}{\partial x_i}.$$

For this (15) must vanish. If we express the invariance of the action with respect to coordinate transformations by an infinitesimal variation of the manifold [9][10] we obtain the identities

$$\left(\frac{\partial \mathcal{W}_k^i}{\partial x_i} - \frac{1}{2} \frac{\partial g_{rs}}{\partial x_k} \mathcal{W}^{rs} \right) + \frac{1}{2} \left(\frac{\partial \mathbf{w}^i}{\partial x_i} \phi_k - F_{ik} \mathbf{w}^i \right) \equiv 0,$$

which convert to (18) when $\frac{\partial \mathbf{w}^i}{\partial x_i}$ is replaced by $g_{rs} \mathcal{W}^{rs}$ according to (17). From the gravitational equations alone we obtain that

$$\frac{\partial \mathbf{w}^i}{\partial x_i} = 0 \quad (19)$$

and from the electromagnetic equations alone that

$$\frac{\partial \mathcal{W}_k}{\partial x_i} - \Gamma_{kr}^s \mathcal{W}_s^r = 0 \quad (20)$$

In Maxwell's theory $\mathbf{w}^i$ has the form

$$\mathbf{w}^i = \frac{\partial (\sqrt{g} F^{ik})}{\partial x_k} - \mathbf{s}^i \quad (\mathbf{s}^i = \sqrt{g} s^i),$$

where s^i is the four-current. Since the first part here identically satisfies (19) this yields the electromagnetic conservation law

$$\frac{1}{\sqrt{g}} \frac{\partial (\sqrt{g} s^i)}{\partial x_i} = 0.$$

In the same way $\mathcal{W}_k^i$ in Einstein's gravitational theory consists of two terms, the first of which identically satisfies equation (20), and the second of which is equal to the mixed energy momentum-tensor T_k^i multiplied by $\sqrt{g}$. In this way equation (20) leads to the four energy-momentum conservation equations. It is a completely analogous situation for our theory when we make the Ansatz (14) for the action. The five conservation laws can be eliminated from the field equations since they are obtained in two ways and thereby show that five of the field equations are superfluous.

For example for the Ansatz (14) the Maxwell equations read

$$\frac{1}{\sqrt{g}} \frac{\partial (\sqrt{g} F^{ik})}{\partial x_k} = s^i \quad \text{and} \quad s_i = \frac{1}{4} \left(R \phi_i + \frac{\partial R}{\partial x_i} \right). \quad (21)$$

R denotes the invariant of weight -1 that is constructed from R_{jkl}^i by contracting i, k and j, l . The computation gives

$$R = R^* - \frac{3}{\sqrt{g}} \frac{\partial(\sqrt{g}\phi^i)}{\partial x_i} + \frac{3}{2}(\phi_i\phi^i),$$

where R^* denotes the Riemannian invariant constructed from the g^{ik} . In the static case, where the spacial components of the electromagnetic potential vanish and all quantities are independent of the time x_0 , we must have, according to (21)

$$R = R^* + \frac{3}{2}\phi_0\phi^0 = \text{const.}$$

But in a space-time region in which $R \neq 0$ one can quite generally, by suitable choice of the arbitrary scale, choose $R = \text{const} = \pm 1$. In time-dependent situations one must, however, expect to encounter surfaces where $R = 0$, which obviously play a singular role. R should not be used as an action since it is not of weight -2 (In Einstein's theory R^* is of this kind). This has the consequence that our theory leads to Maxwell's equations but not to Einstein's; instead of the latter we have fourth-order differential equations. But in fact it is not very probable that the Einstein gravitational field equations are strictly correct, particularly, since the gravitational constant contained in them is quite out of place with respect to the other natural constants, so that the gravitational radius of the mass and charge of an electron, for example, is of a completely different order of magnitude (about 10^{20} resp. 10^{40} times smaller) than the radius of the electron itself [11].

It was my intention to develop only the basis of the theory here. There arises the task of deriving the physical consequences of the Ansatz (14) and comparing them with experiment, in particular to see if they imply the existence of the electron and of other unexplained atomic phenomena. The problem is extraordinarily complicated from the mathematical point of view because it is out of the question to consider linear approximations. Since the neglect of non-linear terms in the interior of the electron is certainly not permissible, the linear equations obtained by neglecting them can have essentially only the trivial solution. I intend to return to these questions elsewhere.

Postscript. A Remark by Mr. A. Einstein Concerning the Above Work

If light-rays were the only means by which metrical relationships in the neighbourhood of a space-time point could be determined, there would indeed be an indeterminate factor left in the line-element ds (as well as in the g_{ik}). This ambiguity is removed, however, when measurements obtained through (infinitesimally small) rigid bodies and clocks are taken into account. A timelike

ds can be measured directly by a standard clock whose world-line is contained in ds .

Such a definition of the line-element ds would become illusory only if the assumptions concerning 'standard lengths' and 'standard clocks' was not valid in principle; this would be the case if the length of a standard rod (resp. speed of a standard clock) depended on its history. If this were really so in Nature, chemical elements with spectral-lines of definite frequency could not exist and the relative frequency of two neighbouring atoms of the same kind would be different in general. As this is not the case it seems to me that one cannot accept the basic hypothesis of this theory, whose depth and boldness every reader must nevertheless admire.

Author's reply

I thank Mr. Einstein for giving me the opportunity of answering immediately the objection that he raised. I do not believe, in fact, that he is correct. According to special relativity a rigid rod has always the same rest-length if it is at rest in an inertial frame, and, under the same circumstances, a standard clock has the same period in standard units (Michelson experiment, Doppler-effect). There is, however, no question of the clock measuring $\int ds$ when it is in arbitrary turbulent motion (as little as in thermodynamics an arbitrary fast and non-uniformly heated gas passes through only equilibrium states); it is certainly not the case when the clock (or atom) experiences the effect of a strongly varying electromagnetic field. In general relativity the most that one can say is: a clock at rest in a static gravitational field measures the integral $\int ds$ in the absence of an electromagnetic field. How a clock behaves in arbitrary motion in the common presence of arbitrary gravitational and electromagnetic fields can only be determined by the computation of the dynamics based on the physical laws. Because of this problematic behaviour of rods and clocks I have relied in my book *Raum-Zeit-Materie* only on the observation of light-signals for the measurement of the g_{ik} (P. 182ff.); in this way not only the ratios of these quantities but (by choice of a definite scale) even their absolute values can be determined so long as the Einstein theory is valid. The same conclusion has been reached independently by Kretschman [12].

According to the theory developed here, with a suitable choice of coordinates and the undetermined proportionality-factor, the quadratic form ds^2 is roughly the same as in special relativity, except in the interior of the atom, and the linear form is $= 0$ in the same approximation. In the case of no electromagnetic field (the linear form strictly $= 0$) ds^2 is exactly determined by the demand expressed in brackets (up to a constant proportionality-factor, which is also arbitrary in Einstein's theory; the same is true even for a static electromagnetic field). The most plausible assumption that can be made about a clock at rest in a static

field is that it measures the ds which is normalized in this way; this assumption [13] has to be justified by an explicit dynamical calculation in both Einstein's theory and mine. In any case an oscillating system of definite structure that remains in a definite static field will behave in a definite way (the influence of a possibly turbulent history will quickly dissipate); I do not believe that my theory is in contradiction with this experimental situation (which is confirmed by the existence of chemical elements for the atoms). It is to be observed that the mathematical ideal of vector-transfer, on which the construction of the geometry is based, has nothing to do with the real situation regarding the movement of a clock, which is determined by the equations of motion.

The geometry developed here is, it must be emphasized, the true infinitesimal geometry. It would be remarkable if in nature there was realized instead an illogical quasi-infinitesimal geometry, with an electromagnetic field attached to it. But of course I could be on a wild-goose chase with my whole concept; we are dealing here with pure speculation; comparison with experiment is an understood requirement. For this the consequences of the theory must be worked out; I am hoping for assistance in this difficult task.

- [1] B. Riemann, *Über die Hypothesen, welche der Geometrie zugrunde liegen*, Math. Werke 2.Aufl. (Leipzig 1892) Nr. 13, S.272.
- [2] T. Levi-Civita, *Rend. Circ. Mat. Palermo* **42** (1917).
- [3] G. Hessenberg, *Vectorielle Begründung der Differentialgeometrie*, Math. Ann. **78** (1917).
- [4] H. Weyl, *Raum-Zeit-Materie* (Berlin 1918) §14.
- [5] G. Mie, *Ann Phys.* **37**, **38**, **39** (1912/13). H. Weyl, *R-Z-M* §25.
- [6] D. Hilbert, *Die Grundlagen der Physik*, 1 Mitt. Gott. Nachr. 20 Nov. 1915.
- [7] H. A. Lorentz, in *Vier Abhandlungen in den Versl. Kgl. Akad. van Wetensch.*, Amsterdam 1915/16.
- [8] A. Einstein, *Berl. Ber.* (1916) 1111–1116.
- [9] F. Klein, *Gott. Nachr.* 25 Januar 1918.
- [10] H. Weyl, *Ann. Phys.* **54** (1917) 121–125.
- [11] H. Weyl, *Zur Gravitationstheorie*, *Ann. Phys.* (1917) 133.
- [12] E. Kretschman, *Über den Physikalischen Sinn der Relativitätspostulate*, *Ann. Phys.* **53** (1917) 575.
- [13] Part of whose experimental verification is still missing (red-shift of the spectral lines in the neighbourhood of large masses).

Postscript June 1955

This work was the beginning of the attempt to construct a 'unified field theory' which was taken up later by many others—without conspicuous success as far as I can see; as is well-known, Einstein himself was working at it until his death.

I completed the development of my theory in two papers [references omitted], further in the 4th and above all in the 5th edition of my book *Raum-Zeit-Materie*.

In this development I gave preference to another principle—first for formal reasons, then strengthened by an investigation of W. Pauli (Verh. dtsch. phys. Ges. 21 1919).

The strongest argument for my theory seems to be this, that gauge-invariance corresponds to the conservation of electric charge in the same way that coordinate-invariance corresponds to the conservation of energy and momentum. Later the quantum-theory introduced the Schrödinger-Dirac potential ψ of the electron-positron field; it carried with it an experimentally-based principle of gauge-invariance which guaranteed the conservation of charge, and connected the ψ with the electromagnetic potentials ϕ_i in the same way that my speculative theory had connected the gravitational potentials g_{ik} with the ϕ_i , and measured the ϕ_i in known atomic, rather than unknown cosmological, units. I have no doubt but that the correct context for the principle of gauge-invariance is here and not, as I believed in 1918, in the intertwining of electromagnetism and gravity. Compare in this context my Essay [reference omitted]: *Geometry and Physics*.