

The problem of space in the light of relativity: the views of H. Weyl and E. Cartan

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Abstract

Starting from a short review of the “classical” space problem in the sense of the 19th century (Helmholtz – Lie – Klein) it is discussed how the challenges posed by special and general relativity to the classical analysis were taken up by Hermann Weyl and Elie Cartan. Both mathematicians reconsidered the space problem from the point of view of transformations operating in the infinitesimal neighbourhoods of a manifold (spacetime). In a short outlook we survey further developments in mathematics and physics of the second half of the 20th century, in which core ideas of Weyl’s and/or Cartan’s analysis of the space problem were further investigated (mathematics) or incorporated into basic theories (physics).

1. Introduction

Once Euclidean geometry was no longer considered the only possibility for describing physical space, the question arose how the structure of space, or of spacetime, could be characterized in terms of symmetry structures on manifolds. The question was posed and discussed by Riemann (1854), taken up anew by Helmholtz at the end of the 1860s with ingenious conceptual insight but mathematically quite vague, even from the standpoint of contemporary mathematics. Helmholtz’s argument was refined by Lie (1886ff.) and, independently, by Killing (1885ff.). It acquired a prominent place in the broader discourse on mathematics and reality at the turn to the 20th century. In these considerations the possibility of “freely moving” rigid bodies as measuring devices was crucial. We shall speak of the *classical* analysis of the *problem of space* (PoS).

At the beginning of the 20th century special and general relativity changed the conceptual scene and the interface with empirical reality drastically. Special relativity brought in the inseparable union of space and time and un-

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dermined the classical stipulation of “rigidity”. The general theory complicated things even further, because now the classical assumption of metrical transformations in finite regions, or even globally, lost any ground. The arising questions was under discussion among physicists, e.g. (Einstein 1921), philosophers (Schlick 1919, Carnap 1922, Reichenbach 1920, Reichenbach 1928), and mathematicians, in particular H. Weyl and E. Cartan. Weyl and Cartan had particular interest in the question of how to reformulate the older criteria of the choice of space structure in the light of differential geometry and the general theory of relativity (*modern* or *relativistic* PoS). They chose different roads and arrived at different conclusions, but there was an overlap and interchange between the two authors.

Our paper analyzes their respective ways of posing the question and their path toward an answer (section 4 and 5).¹ Before we do so, we review the classical problem of space and sketch how it was undermined by the two relativity theories (sections 2 and 3). In a short survey (section 6) we follow more recent developments by which Weyl’s and Cartan’s infinitesimal geometric refinement of the analysis of the PoS fanned out in mathematics and physics of the second half of the 20th century.²

2. The classical problem of space

Hermann Helmholtz (after 1883 *von* Helmholtz) based his analysis of the problem of space about the middle of the 1860s on “the demand of free mobility for rigid shapes without change of form in all parts of space”³. Twenty years earlier he had reflected on basic features of the concepts of space and time in a manuscript on “Allgemeine Naturbegriffe” (general concepts of nature) which was published posthumously by Leo Koenigsberger (Helmholtz 1845/1903). In these notes Helmholtz understood *space* as the *relationship* for ordering different *objects*. In these reflections “objects” were understood in a material sense, and space had to be specified such that “all kinds of changing matter distribution” could still be taken into account.⁴ In his later investigations such changes were restricted to rigid motions and abstraction was made from the material nature of bodies. Rigidity was now analyzed in terms of geometric (more precisely metric) properties.

He was not the only one to consider rigid motions as a crucial new feature in the foundations of geometry. Already Riemann had mentioned them

¹For this aspect see also (Scholz 2004b, Scholz 2012a).

²A similar question regarding the spread of Klein’s view in physics during the 20th century is being surveyed in (Goenner 2013).

³“Forderung einer unbedingt freien Beweglichkeit in sich fester Figuren ohne Formänderung in allen Theilen des Raumes” (Helmholtz 1868b, 614)

⁴“Ferner muss der Begriff des Raumes auch noch so bestimmt werden, dass er alle möglichen Aenderungen der Materie umfassen könne, die hier offenbar nur insoweit in Betracht kommen, als sie Aenderungen der Raumverhältnisse, d.i. Bewegungen sind.” (Helmholtz 1845/1903, 134)

among different possibilities for specifying the structure of spaces of constant curvature among all possible (Riemannian) manifolds. Of course he did not use the wording of free mobility; he rather circumscribed the same idea by postulating an “independence of spatial figures (Körper) of their position” (Riemann 1867, 283). In France, Hoüel emphasized the role of rigid motions as an additional foundational feature in (Hoüel 1867).⁵ That was a year before Helmholtz gave his famous talk at the Heidelberg Medizinische Verein (Helmholtz 1868a), from which his longer article in the *Göttinger Nachrichten* resulted (Helmholtz 1868b).

Peculiar for Helmholtz’s approach was that the concept of *free mobility* stood at the center of his analysis. He established a set of axioms for free mobility from which, so he claimed, one could derive the infinitesimal expression for distances by a (non-degenerate positive definite) differential form of second order, similar to what Riemann had proposed in the more general approach of his inaugural lecture. In this way he achieved a justification of Riemann’s choice for what Weyl would later call the “Pythagorean nature” of the metric (Helmholtz 1868b, §4). In Helmholtz’s analysis the Riemannian metric had to be of constant curvature in order to satisfy the axioms of free mobility. In contrast to Riemann, Helmholtz erroneously concluded from constant curvature and the assumption of infinity that his space had to be Euclidean (Helmholtz 1868a), (Helmholtz 1868b, §4). He was reminded of the possibility of non-Euclidean space structure by Beltrami in a letter from 24. 04. 1869. The story of this premature conclusion and its correction has been told at several places; also the problematics in Helmholtz’s statement of his axioms for free mobility.⁶

Under the influence of the rising study of transformation groups and their role in geometry by S. Lie and F. Klein, Helmholtz’s analysis was received and generalized by mathematicians in different ways. One may even conceive of Klein’s idea to characterize a geometry by a manifold and a transformation group (*Hauptgruppe*) as kindred in spirit to Helmholtz’s more specific enterprise, even though Klein did not establish a direct and explicit link to Helmholtz’s analysis of the PoS. He did not even discuss rigid motions in his *Erlanger Programm*. We shall see that Klein’s view played a crucial role for E. Cartan’s adaptation of the analysis of space to the new situation after the advent of general relativity.

Sophus Lie, on the other hand, continued along the lines opened by Helmholtz. He first sketched how Helmholtz’s vaguely formulated conditions of free mobility could be restated in terms of infinitesimal transformation groups (Lie 1886/87, Lie/Engel 1893); later he investigated it in more detail

⁵“Nous demanderons qu’une figure invariable de form puisse être transportée d’une manière quelconque dans son plan ou dans l’espace”, (Hoüel 1867, 7) here quoted from (Henke 2010, 198ff.).

⁶(Koenigsberger 1901–1903, Volkert 1996, Boi e.a. 1998, Merker 2010).

in collaboration with F. Engel (Lie/Engel 1893).⁷ Helmholtz did not clearly distinguish between finite and infinitesimal motions. Lie realized that for his goal the interplay of finite and infinitesimal motions played a crucial role. He (and Engel) defined *free mobility* by a group G operating transitively on a three-dimensional manifold, such that (1) for any two points P and P' and any two “linear elements” passing through them the group allows to transform these into another, while still “one continuous movement” is possible, i.e., one degree of freedom is still open. (2) Only if one fixes two “surface elements” passing through the two line elements, the respective transformation is uniquely determined (Lie/Engel 1893, §98).⁸ With that more precise definition and with his general theory of continuous transformation groups as background, Lie (supported by Engel) was able to demonstrate that the space is Riemannian with constant curvature, if free mobility in continuous spaces of dimension $2 \leq n \leq 4$ is assumed.

Independently Wilhelm Killing started to rethink Helmholtz’s analysis under the influence of Weierstrass’ approach to mathematics (Killing 1880, Killing 1885). That led him to a mathematical reconstruction of the Riemannian metric from infinitesimal motions and an extended study of *space forms*, a term formed by him, originally with reference to the work of Clifford and Klein who first took global topological features into account (which he himself did not).⁹ For Killing’s peculiar combination of infinitesimal considerations (Hawkins) with those of the global structure of the geometries (Volkert) it was a lucky circumstance that he did not yet dispose of an explicit concept of transformation groups. He rather characterized the structure morphisms (to use modernized descriptive terminology) of space forms in terms of “motions of a rigid body”. In hindsight this seems to have come down to considering a groupoid of local transformations containing a neighbourhood of unity of a Lie group and its infinitesimal transformations (Lie algebra). In the following the historical terminology “infinitesimal group” and the present one (“Lie algebra”) will be used indiscriminately.¹⁰

For the next generation, and already at the life time of Killing, it was Lie’s point of view which became dominant. According to Hawkins it was essentially through E. Cartan’s work that the part of Killing’s contribution relating to infinitesimal transformations entered the mainstream of transformation group studies (Hawkins 1980, 290). All in all, the classical space

⁷Cf. (Merker 2010).

⁸The modernized description of a *simply transitive* operation on the *flag* manifold is very close to Lie/Engel’s characterization. Note, however, that these authors did not demand *simple* transitivity, but allowed underdetermination of the operation by the flag up to a discrete subgroup.

⁹For the first aspect see (Hawkins 2000, sect. 4.5) and (Hawkins 1980); the second one is discussed by K. Volkert (this volume).

¹⁰The terminology “Lie algebra” was introduced by Weyl in his lecture course on continuous groups 1933/34; see (Kosmann-Schwarzbach 2011, 32, fn. 12) and, in more detail, the commentary in (Eckes 2014).

problem had been stated and solved by Riemann, Helmholtz, Lie/Engel, Killing, Clifford, and Klein between 1851 and 1890. Its outcome showed the following:

- Quite general constraints on the automorphism group of spaces (free mobility in the sense of Helmholtz-Lie/Engel) were able to specify the type of metric (Riemannian) and its structural specification (constant curvature). The framework and the result of this analysis fitted well to the recent researches on the foundations of geometry.¹¹
- More general geometries, respectively space forms, came into sight if one allowed for more general transformation groups as automorphisms (Klein), which were no longer bound to physical interpretations, or for their infinitesimal versions (Killing).
- Still other aspects came to the fore, if one took topological characteristics in the large (space forms) into account (Clifford-Klein).

In the decades about the turn of the century the role of motion groups for the characterization of space became widely known in France, most prominently by talks and popular essays of H. Poincaré, e.g. (Poincaré 1898).¹² His view differed from Riemann's and Helmholtz's by taking sensual and psychological features of the formation of space perception much more serious than the latter, and by his conventionalist evaluation of the relationship between empirical evidence and conceptual structures.¹³

Also in physics the role of Euclidean motions attracted more and more attention. In a note written for the third volume (2nd edition) of Appell's *Traité de mécanique rationnelle*, E. and F. Cosserat demanded to consider action principles invariant under the group of Euclidean displacements for mechanics, in particular for elasticity theory, and even for (pre-relativistic) physics in general ("toute la physique théorique").¹⁴ By this remark they referred back to a proposal made by Helmholtz. Later it would come to play an important role for the theory of generalized deformable media (Brocato/Chatzis 2009) and for the infinitesimalization of geometric transformation structures in E. Cartan's work (see section 5).

¹¹For some of the more philosophically inclined participants in the discourse the outcome of the analysis of space seemed to go in hand with Leibniz's thoughts on the homogeneity of space. On *homogeneity* of space in Leibniz's philosophy see (de Risi 2007, 198ff.).

¹²Cf. (Gray 2013, 38–59).

¹³(Torretti 1984, Walter 2009).

¹⁴"... on peut observer que l'action, telle que Maupertuis l'a introduite à la Mécanique, est *invariante dans les groupe des déplacements euclidiens*. Ce même caractère se retrouve dans la statique des corps déformables, que repose sur la considération du ds^2 de l'espace. ... Suivant cette idée philosophique, toute la Mécanique classique et toute la physique théorique paraissent pouvoir se déduire de la notion unique d'*action euclidienne*." (Cosserat 1909a, 557f.). "Action euclidienne" was, of course, the abbreviated expression for an action invariant under the group of Euclidean motion; it stood at the center of the investigations of the Cosserat brothers.

On the other side of the Rhine, Hilbert generalized the classical PoS by passing over to a topological point of view. In two notes on the foundations of geometry, written in 1902 and 1903, he characterized the (real) plane by axioms for neighbourhoods, anticipating the later axiomatic characterization of (twodimensional) manifolds, and added two axioms for a purely topological characterization of the group of motions: (I) infinite cardinality of the isotropy group of every point, (II) 3-closedness of the transformation group (Hilbert 1902, Hilbert 1903*a*, Hilbert 1903*b*)¹⁵ Hilbert's initiative turned out to be of long-lasting influence on the topological transformation group approach to the foundations of geometry in the course of the 20th century.

The young modernist F. Hausdorff, interested in all aspects of the foundations of geometry, developed his own philosophical outlook on the problem of space trying to keep balance between empirical input in concept formation and mathematical liberty of theoretical generalization (characterized by himself as the view of "considered empiricism"). In his inaugural lecture Hausdorff put the discussion of the classical PoS in the wider perspective of general space concepts in the sense of differential geometry and the embryonic theory of point set topology (Hausdorff 1903).¹⁶

At that time, the study of electrodynamics and the motion of the recently detected electrons was already shifting physics in a different direction. Hausdorff was well aware of this development. He remarked that "the hypothesis that the so-called rigid bodies may be subject to certain fine deformations" had been stated in order to explain the Michelson interference experiments (Hausdorff 1903, 10). He even saw the possibility that empirical reasons might speak in favour of giving up homogeneity and isotropy of space. These were foresightful remarks; the whole conception of the relationship between mathematical spaces and physical geometry would soon be overturned by the two theories of relativity, special (1905ff.) and general (1915ff.).

3. New questions for the space problem raised by relativity

The rise of Einstein's theory of special relativity (STR) led beyond the classical frame of geometry, although at the outset Einstein did not intend to do so. In his famous article on *electrodynamics of moving bodies* he rather declared coordinate systems to be established "by means of rigid measuring rods and using the methods of Euclidean geometry" (Einstein 1905, 892). From Poincaré's perspective the situation seemed different. Although he was the first to understand Lorentz transformation in their general mathematical

¹⁵See (Strantzalos 2002) (difficult to access). A short description can be found in the historical commentaries in (Hausdorff 2001ff., **2**, 708ff.), more in volume **6** (in preparation). A transformation group is called *3-closed*, if for any sequence of its operations with images of a point triple (A, B, C) converging to the point triple (A', B', C') , there is a transformation g of the group, such that $g(A) = A', g(B) = B', g(C) = C'$.

¹⁶See (Hausdorff 2001ff., vol. **6** (to appear)).

form (in early 1905) his conventionalist philosophy of space led him to believe that the “Lorentz-Fitzgerald contraction” was a real obstacle for the “transport of rigid rulers upon which length measurement depends” (Walter 2009, 204).¹⁷ Even worse, the discussion of bodies in accelerated motion showed that rigidity in a material sense became precarious. The idealization of putting an extended body into motion instantaneously was, of course, incompatible with the finite speed of light and of causally transmitted signals more generally; moreover the unequal relative motion of different parts of bodies, like in the rotating disk, brought their own difficulties with it.

On the other hand, a more abstract mathematical understanding of “rigid motion” in the sense of metrical automorphism was only changed, not undermined, by special relativity, although now the orthogonal group had to be generalized to non-definite signature. That was most clearly shown by Minkowski’s conceptual integration of space and time in his four-dimensional “world geometry” (Minkowski 1908).¹⁸ The role of metrical automorphisms which characterized rigid body motions of the classical, Euclidean or non-Euclidean, geometries was taken over by Lorentz transformation (plus translations). Some physicists appreciated this view immediately, among them M. Born who considered not only rigid uniform motion but defined special types of rigid accelerated motions by proper time independent “deformation matrices” of a material substrate (Born 1909). But not all physicists gave due credit to this shift of affairs. Einstein, e.g., did not realize the conceptual innovative achievement of Minkowski’s four-dimensional representation of special relativity before 1912.

In a talk at London in the summer of the same year, Poincaré, distinguished between the groups of displacement of bodies (in the sense of Helmholtz-Lie) and the group underlying the “principle of relativity” for mechanics or electrodynamics. He discussed the *Galilei group* and the inhomogeneous Lorentz (later *Poincaré*) group as two distinct possibilities. Due to his conventionalist conviction he still argued in favour of the first one for the basic convention regarding spacetime and admitted the second only as a hypothesis of limited range for the study of electrodynamics and the moving electron (Walter 2009), (Gray 2013, chap. 6). Although from the point of view of the Minkowski/Einsteinians (and in hindsight) this distinction was unjustified, Poincaré indicated here in his peculiar way the possibility that the *physical automorphisms* of a dynamical theory (to take up here a terminology later introduced by Weyl) need not necessarily be identical to those of spacetime.¹⁹

¹⁷That this was not the case could be seen most clearly a little later by Minkowski’s four-dimensional spacetime (see below).

¹⁸Cf. (Walter 2010).

¹⁹In his later reflections on the problem of space, Weyl posed the question which of the automorphism are of physical import, and which concern the underlying mathematical structure only (Weyl 1949, 87), a question difficult, perhaps even impossible

Other mathematicians seemed to be more impressed by the basic physical role attributed in special relativity to the Lorentz-Poincaré group. Already the year before Poincaré gave his London talk, G. Herglotz found that the invariance of the Lagrangian of a relativistically moving body under the full automorphism group of Minkowski space resulted in dynamical *conservation laws*, generalizing those known from classical mechanics:

Corresponding to the ten-dimensional group of “motions” in (x, y, z, t) -space with the metric

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2$$

10 theorems analogous to the principles of the center of inertia, of the area [conservation of linear momentum, respectively angular momentum, E.S.] and of energy in ordinary mechanics hold for the total body. (Herglotz 1911, 511)⁽ⁱ⁾

Although first insights on the relationship between conserved dynamical quantities and invariance under transformations in Euclidean geometry had been collected during the 19th century (Jacobi, Hamel), it was the challenge of relativity which led to reconsidering the role of “motions” in classical and in relativistic mechanics.²⁰ On the advice of Klein, F. Engel worked back from Herglotz’s relativistic conservation principles to classical ones, by letting the velocity of light $c \rightarrow \infty$. In a second step, documented in a letter written to Klein and published in *Göttinger Nachrichten* 1916, he derived energy conservation in Hamiltonian (classical) mechanics from time invariance, avoiding the variational methods of the Lagrangian approach (Kosmann-Schwarzbach 2011, 36), (Kastrup 1987).

In such a context and in Minkowski spacetime, Helmholtz-Lie’s flag transitivity, to put it in modernized terminology, lost its interest for researchers as a criterion of free mobility. It is unknown to me, whether an attempt to modify flag transitivity by distinguishing between flag elements with timelike and spacelike markers has been made by any author of the time. Probably

to decide but of interest to philosophy of science (see end of section 4). Here, we can consider Poincaré’s conventional characterization of the Galilei transformations as geometrical automorphisms, distinct from the physical transformations of electrodynamics (Lorentz/Poincaré group) as a distinction of analogous type.

²⁰In his 1842/43 lectures on mechanics in Königsberg Jacobi derived the principles of “center of inertia” and of “area” (i.e., conservation of linear and angular momentum) from the observation that the Lagrangian of classical mechanics is invariant under the change of orthogonal *coordinate* transformations (shift of the origin and rotation of coordinate axes) (Klein 1926, **2**, 56ff.). In 1897 I. Schütz derived energy conservation from the dynamical equations of point particle systems. In 1904 G. Hamel introduced Lie brackets into the study of “virtual displacements” in mechanics. Apparently, neither of them recognized energy conservation as a consequence of the time invariance of the Lagrangian (Kosmann-Schwarzbach 2011, 35).

nobody did; physicists were interested in dynamical questions, and mathematicians were soon to be attracted by the problems posed by the generalized theory of relativity (GR). This holds at least for the main authors considered here, H. Weyl and E. Cartan.

General relativity posed, of course, a much greater challenge to the characterization of the problem of space. Free mobility of finitely extended rigid figures became meaningless in the general case. It could be upheld, if at all, only in the infinitesimal. But even then one could opt for different strategies for posing the question in the new framework. Cartan's main goal was to integrate Klein's characterization of spaces (Erlangen program) with Riemann's approach to differential geometry, and to implement it on the level of the infinitesimal structures. The Göttingen mathematicians, including Weyl in his Zürich "outpost", preferred to stick to the Riemannian approach to differential geometry, enhanced by Ricci's and Levi-Civita's "absolute" differential calculus (i.e. vector and tensor analysis) and, in the case of Weyl, by a generalization to an even more general "purely infinitesimal" founded gauge geometry.

At Göttingen the question was being discussed, how the newly acquired insight into the group theoretic derivation of conservation principles (in special relativity and in classical mechanics) might be adapted to the general relativistic context. Motivated by F. Klein and D. Hilbert, Emmy Noether took up the question with great success. She found a general and deep solution to this problem in her now famous publication on *Invariant variational problems* (Noether 1918).²¹ Her result consisted essentially of two theorems: the first theorem generalized the derivation of conservation laws in classical and special relativistic mechanics; it stated the existence of conserved quantities to every generator of a finite dimensional Lie group under which a Lagrangian dynamics remains invariant (and vice versa). Her second theorem dealt with the infinite dimensional (infinitesimal) symmetries of a Lagrangian. It established differential identities ("Noether identities") from which, among others, vanishing divergence expressions could be derived. Their interpretation as physically meaningful conserved quantities (currents) was, however, a much more difficult task than in the case of special relativity or other fields. It soon led to controversial views. In particular, and most irritatingly for the protagonists, it turned out that energy-momentum conservation could not be generalized to the general relativistic case in a straightforward way²².

For studying energy-momentum conservation two infinitesimal geometric equivalents to translational invariance have been considered during the last century. In the first one, dominant in the literature on GR until today, infinitesimal diffeomorphisms were considered.²³ The second one developed

²¹For an English translation of Noether's 1918 article, a mathematical commentary and a study of its long delayed reception, see (Kosmann-Schwarzbach 2011).

²²(Rowe 1999, Brading 2003a, Brading 2002).

²³Historically, one considered (differentiable) coordinate changes. The "active" aspect

from Cartan’s approach in which point dependent translations are considered as part of the structure anyhow. The second view came to maturity only in the second half of the 20th century (see outlook). The diffeomorphism approach did not lead to an observer independent, covariantly conserved energy-momentum current; but the corresponding Noether equations (theorem II) are structurally contentful and turned out to be equivalent to the contracted Bianchi identities of the metric. Hilbert spoke of “uneigentliche Energieerhaltung” and considered such “improperness” as characteristic of GR (Brading/Ryckman 2008).

For fields satisfying the Einstein equations (in physicists idiom, for fields “on shell”), the Noether identities also imply vanishing of the covariant divergence of the energy-momentum tensor, $\nabla_j T_k^j = 0$. That was and is a valuable formal property, and could be considered as kind of consolation; but in general it does not integrate to a conserved energy-like quantity.²⁴ Proposals for adding additional non-covariant (i.e., non-tensorial) terms t_j^i (energy-momentum *pseudotensor*) derived from the gravitational potentials (g or Γ) were made by many physicists, among them already quite early Einstein himself (Einstein 1918), later among others (Landau 1951). None of these led to a convincing, unanimously accepted, physical interpretation.

In still another approach a so-called *canonical energy momentum* current t_{ij} , derived from the Noether identities, was considered directly. It is conserved if the fields satisfy the Einstein equations (“on shell”), but it lacks symmetry. Therefore t_{ij} could not be identified, without further ado, with the usual (the *dynamical*) energy-momentum tensor derived from varying the matter Lagrangian, $T_{ij} = \frac{\delta \mathcal{L}_m}{\delta g^{ij}}$. F. Belinfante and L. Rosenfeld made some progress along these lines.²⁵ That was, however, already more than twenty years after the rise of relativity. So the question whether infinitesimal coordinate changes (diffeomorphisms) or differently conceived infinitesimal transformations ought to be considered as physical morphisms or, rather, as morphisms of the embedding mathematical structure only (to express it again in Weyl’s later terminology) was wide open. It became a point of discussion among philosophically inclined experts in gravity theory.

(diffeomorphism) and the “passive” one (differentiable coordinate change) can be translated into another and lead to the same Noether identities.

²⁴In spite of that, $\nabla_j T_k^j = 0$ is often called “infinitesimal energy-momentum conservation”, which may appear misleading, if not commented further.

²⁵ These two authors managed to symmetrize t_{ij} by adding a term ∇s_{ij} in which s_{ij} was a linear combination of three intrinsic (spin) angular momentum terms, $T_{ij}^{(BR)} = t_{ij} + \nabla s_{ij}$. Moreover according to their analysis, the symmetrized tensor could be identified with the dynamical one, $T_{ij}^{(BR)} = T_{ij}$ (Rosenfeld 1940). I thank F. Hehle who did his best to explain these things, well known to any educated relativist, to me.

All in all, the rise of relativity, special and general, enriched the analysis of the PoS in many respects. It led to a deeper understanding of the physical symmetries of classical space (and time) and posed new problems. Some of them remain challenges until today.

4. Weyl's analysis of the PoS

Weyl turned towards general relativity in 1917,²⁶ Cartan a little later. Weyl chose to take up and to generalize two motifs of the classical PoS: to justify the Riemannian nature of the metric (or a generalization of it) in general relativistic spacetimes by group theoretic methods, and to mimic free mobility of rigid matter by allowing what he considered the “greatest liberty” for the distribution of (non-rigid) matter in space and time.

In 1918 Weyl developed his *purely infinitesimal geometry* which generalized Riemannian geometry by introducing a slightly more involved concept of a differential metric, a *gauge metric*. Such a gauge metric can be characterized by equivalence classes $[(g, \varphi)]$ of conformally related pseudo-Riemannian metrics $g = (g_{ij})$ and differential 1-forms $\varphi = (\varphi_i)$, where the equivalence relation was given by (scaling) *gauge transformations* (Weyl 1918).²⁷ That allowed a direct comparison of lengths of vectors only if the vectors were attached to infinitesimally close points. For such a length comparison between different but infinitesimally close points, separated by δx , Weyl introduced a differential form φ which served as an infinitesimal *length connection* ($l \mapsto l' = l + \delta l$ with $\delta l \sim -\varphi(\delta x)$ and $\delta l \sim l$). As a crucial ingredient of the geometry, Weyl found that there is *exactly one affine connection* Γ compatible with the gauge metric. That is, a vector v of length l , infinitesimally parallel translated by Γ along the infinitesimal path u (again a vector in the tangent space), changed its length by the amount $\delta l = -\varphi(u)l$.²⁸ He called this result the *fundamental theorem* of infinitesimal geometry.

That was the basis for Weyl's first attempt of geometrically unifying gravity and electromagnetism. It brought him praise and criticism from the side of Einstein and other physicists.²⁹ Einstein praised Weyl's conceptual and mathematical ingenuity, but criticized the latter's attempt to “improve”

²⁶(Sigurdsson 1991, Eckes 2013/14).

²⁷The shortest possible, although only formal, explanation of the equivalence is: $(\tilde{g}, \tilde{\varphi}) \sim (g, \varphi) \iff \tilde{g} = \Omega^2 g, \quad \tilde{\varphi} = \varphi - d \log \Omega$, with Ω a strictly positive function on the manifold (spacetime). Weyl's underlying idea was that units of measurement are given by “local” (point dependent) specifications, which allow no immediate comparison. Comparison needs a new structure, the length connection. For more detail see (Weyl 1918, Weyl 1923b), (Adler e.a. 1975, chap. 5.2) or (Scholz 2011a), embedded in a more historical perspective (Ryckman 2005, Scholz 2001a).

²⁸ $\delta l = |v + \delta v| - |v|$, where $\delta v = -\Gamma(u)v$ or, in index notation, $\delta v^i = -\Gamma_{jk}^i u^j v^k$ (Weyl 1923a, 12). With the covariant derivative ∇ with regard to Γ , compatibility of Γ and the Weylian metric $[(g, \varphi)]$ can be restated by the condition $\nabla g + 2\varphi \otimes g = 0$. For more details see, among others, (Scholz 1999, Ryckman 2005, Eckes 2011).

²⁹See (Vizgin 1994, Goenner 2004, Goldstein/Ritter 2003, Scholz 2009, Eckes 2013/14).

the field theories involved. Among the various points of differences discussed in their correspondence during the year 1918, Einstein could not see why in Weyl's view only length standards had to be "localized" (present physicists' language), while angle measures were kept immune against the purely infinitesimal relativization. He even ironized that, if one accepted the first, there could just as well come a particularly clever "Weyl II" who might propose localization of angle measurements (Einstein 1998, 551, 31.06.1918).

Although at this time Weyl was probably already convinced that similarity and congruence at each point of the spacetime manifold had to be specified by some group theoretic argument, he did not immediately counter this point of Einstein's criticism. But the challenge may have contributed to his more conceptual analysis of the foundations for infinitesimal geometry, to which he turned in 1921. At that time he started to loose his faith in an immediate success for his unified field theory.³⁰

In the following year, a reedition of Riemann's inuagural lecture gave him the chance to comment on some fundamental aspects of differential geometry (Riemann 1919). Riemann had selected the (square root) of a positive definite differential form $ds^2 = \sum g_{ij} dx_i dx_j$ (modernized symbolism) among all homogeneous expressions $f(\lambda dx_1, \dots, \lambda dx_n) = |\lambda| f(dx_1, \dots, dx_n)$ (f differentiable, positive), later called *Finsler metrics*.³¹ But he had given only a quite pragmatic argument for his choice of specialization.

Weyl conjectured that among the wider class of Finsler metrics the "Pythagorean" ones were singled out by the condition that to any such metric there exists a uniquely determined compatible affine connection, i.e. essentially a parallel displacement of vectors in the manifold.³² It has to be added that Weyl called a (Finsler type) metric of *Pythagorean* nature, if it could be described by a non-degenerate quadratic differential form of any signature (in particular of Lorentz signature). In this way, he transformed the core insight of what he had identified as the fundamental of infinitesimal geometry into a criterion for a "good" differential geometric metric. He did not pursue this conjecture further in this form,³³ but turned towards an even

³⁰(Scholz 2004a)

³¹(Finsler 1918)

³²Two years earlier Levi-Civita had proved that each Riemannian manifold admits exactly one parallel displacement compatible with the metric, i.e. preserving lengths of vectors and angles between them. Weyl himself had distilled from Levi-Civita's construction a symbolic definition of an abstract "affine connection" which allowed to introduce a kind of parallel displacement in any differentiable manifold and a kind of differentiation adapted to this geometrical constellation, called a "covariant derivative" (Reich 1992, Bottazzini 1999, Scholz 1999). For a detailed discussion of Weyl's analysis of the space problem in the light of Finsler metrics see (Coleman/Korté 2001).

³³The conjecture (Weyl's "first space problem") was proven 40 years later by (Laugwitz 1958). Freudenthal later observed that, in a way, it could be considered solved by the answer to Weyl's (second) PoS (Freudenthal 1960). In this discussion, he overlooked the specific difference between Weyl's gauge geometry and Riemannian geometry, but the core of his argument is justified: Weyl's proof also works if one restricts the structure group

deeper conceptual analysis of metrical structures for infinitesimal geometry. The motif of a *uniquely determined affine connection*, however, stabilized and reappeared also in his more fundamental investigations on the PoS.

In the rewritten fourth edition of *Raum, Zeit, Materie* (manuscript finished in November 1920) Weyl added a new section on the *group theoretical outlook on the spatial metric* (Weyl 1921c, §18). Here he sketched his new view at the PoS, introduced general characterizations of infinitesimal congruences at any point p and between two infinitesimally close points p, p' , and formulated two postulates P1, P2 (later postulate of freedom (P1) and of coherence (P2) (Weyl 1923a, lecture 7)). He evaluated the consequences of the postulates for the infinitesimal Lie group (Lie algebra) $\mathfrak{g}$ and conjectured that the only groups satisfying the postulates might be the generalized special orthogonal groups $\mathfrak{g} = so(p, q)$ ($p + q = n$ dimension of the manifold) (Weyl 1921c, 132).³⁴ Optimistically he declared that, if the general proof could be given, “the essence of space (Wesen des Raumes) would have been made comprehensible” by deep mathematical considerations. After an eulogy on the long historical course of events leading to this result (Euclid – Newton – Gauss – Riemann – Einstein) he declared in an often quoted remark

The example of space is most instructive for that question of phenomenology that seems to me particularly decisive: to what extent the delimitation of the essentialities (Wesenheiten) rising up to consciousness express a characteristic structure of the domain of the given itself, and to what extent mere conventions participate in it. (Weyl 1921c, 133f.) ⁽ⁱⁱ⁾

This remark is often taken as proof for Weyl’s close affiliation to Husserl’s phenomenology.³⁵ But one may doubt that this was the driving force of his thought. Declaring to understand the specific relationship between the *structure of the given itself* and the *conventional* as “particularly decisive” sounded rather like a critical commentary to Poincaré’s latest discussion of this question in (Poincaré 1912), even though it was stated in a language which clearly did not hide Weyl’s involvement in the discourse of German idealistic philosophy. His epistemic orientation followed a road passing equidistantly between an empirically oriented positivism and idealism.³⁶ It was governed by a hope expressed by Riemann half a century earlier that scientific knowledge is able, step by step, to go “behind the surface of appearances”.

In the next two years Weyl elaborated the consequence of his analysis of the PoS (Weyl 1921a, Weyl 1922b, Weyl 1921b). In April 1922, in lectures

G^* of his PoS to the congruence group G , see below.

³⁴Weyl stated that he had been able to prove the conjecture for the cases $n = 2, 3$, but not yet in general (Weyl 1921c, 133)

³⁵For example in (Ryckman 2005, 157f.), from which I have taken the translation of the Weyl quote.

³⁶The idealistic aspect of Weyl’s view has been emphasized by (Ryckman 2005, Bernard 2013).

at Barcelona (in French) and Madrid (partially in Spanish), he had the chance to present his analysis to a Spanish audience. He edited his lecture notes in German as a booklet, complemented by 12 mathematical appendices (Weyl 1923a). This publication contains the most extended presentation of his thoughts on the PoS and its embedding in the wider mathematical context.

Already in his first presentation of the new PoS Weyl declared that one could perceive as many different metrical determinations as there are “essentially different” (i.e., non conjugate) subgroups of the general linear group $G \subset GL(n, \mathbb{R})$. He posed the question which of these could justifiably be considered as *rotations* in a generalized sense (Weyl 1921c, 125). Such “rotations” had to be considered in the infinitesimal neighbourhoods of points; they operated on the “vector body (Vektorkörper)” attached to each point p separately (today, on the tangent spaces $T_p M$). As first constraint he introduced volume invariance, i.e., $G \subset SL(n, \mathbb{R})$.

If, with regard to a (generalized) rotation group G , conjugation by a linear transformation u does not modify the group, $u^{-1} \cdot G \cdot u = G$, the transformation leaves the “metrical determination” correlated with G unchanged, i.e., it could serve as a (generalized) similarity. Therefore Weyl considered the normalizer G^* of G as the *similarity group* of the rotations G . A *metric connection* between any two infinitesimally close points would be given, so Weyl, by specifying a linear connection Λ expressed, after choice of coordinates and of a basis of the “vector body” (tangent space in our terminology) by a system of real numbers (Λ_{jk}^i) ($1 \leq i, j \leq n$), not necessarily symmetric in the lower indices (in Cartan’s terminology possibly with torsion), and transforming correctly.

This “metric connection” should not be confounded with a parallel displacement (affine connection) which only stood in indirect relation to it. The attribute “metric” only indicated that a transfer of vectors or figures defined by vectors led to vectors or figures which ought to be considered “congruent” in the geometry under consideration. Therefore any composition of a *congruence displacement* with a *rotation* in the target space had to be considered just as well as a metric or congruent displacement. Thus Weyl considered metric displacements to be defined by a linear connection Λ up to additive modifications by a point dependent infinitesimal rotation, i.e., up to addition of differential one form $A = (A_{jk}^i)$ with values in $\mathfrak{g}$.³⁷ In consequence the metric connection was given only up to equivalence

$$\Lambda_{jk}^i \sim \tilde{\Lambda}_{jk}^i \longleftrightarrow \tilde{\Lambda}_{jk}^i = \Lambda_{jk}^i + A_{jk}^i$$

for some differential form $A_{jk}^i dx^k$ with values in $\mathfrak{g}$.

All this, including the explanation of the concept of *parallel displacement* by an affine connection $\Gamma = (\Gamma_{jk}^i)$ symmetric in the lower indices j, k , was

³⁷That means $A = A_{jk}^i dx^k$, $(A_j^i) \in \mathfrak{g}$, using Einstein sum convention.

considered by Weyl as “mere conceptual analysis, an explication of what is contained in the concepts metric, metric connection, and parallel displacement as such”. The specific postulates which ought to be satisfied added something more to the analysis, it belonged to “synthetic part [of the analysis of PoS] in the Kantian sense” (Weyl 1923*a*, 49). Roughly formulated the postulates were:³⁸

P1 (“Principle of freedom”): G allows the “widest conceivable range of possible congruence transfers” in one point. . . .

P2 (“Principle of coherence”): To each congruent transfer Λ_{jk}^i exists *exactly one equivalent affine* connection.

Weyl explained the meaning of “widest conceivable range” by the possibility that the group G had to be large enough that at a given point p the coefficients of the metric connection (Λ_{jk}^i) could take any values in the reals, i.e., the connection had degree of freedom n^3 .

In the Barcelona lectures he added that this postulate assured greatest possible liberty for matter to shape the metric and ought therefore be considered as the relativistic equivalent to free mobility in the classical PoS. This argument was surprisingly close to Helmholtz’s *early* (1845) characterization of the concept of space. There Helmholtz had stipulated that the concept of space ought to allow “all possible changes of matter”. Only twenty later he boiled down this rather general description to free mobility.³⁹

Because homogeneity of space was the result of free mobility, and free mobility the expression of allowing “all possible changes of matter” in the classical PoS, Weyl had good reasons to argue:

The possibility *to subject the metric field to arbitrary virtual changes, keeping the nature of the metric fixed*, has taken the place of homogeneity of the metric field postulated by Helmholtz. (Weyl 1923*a*, 46)⁽ⁱⁱⁱ⁾

The relation of Weyl’s postulate P1 to Helmholtz’s free mobility becomes much clearer in the light of the *Nachlass* notes from 1845; but it has to be added that we do not have any information whether Weyl had read Koenigsberger’s edition of the early Helmholtz text. That may have been the case, but he could just as well have arrived at his generalization by his own conceptual analysis.

The two postulates led to constraints for the infinitesimal Lie group $\mathfrak{g}$, expressed by

³⁸For more details and Weyl’s motivational arguments see, e.g. (Scholz 2004*b*).

³⁹See above fn. (4).

Lemma 1 *If a group $G \subset SL(n, \mathbb{R})$ with infinitesimal rotations (Lie algebra) $\mathfrak{g}$ characterizes the “rotations” in an n -dimensional generalized metric structure, in the sense of the postulates P1, P2, the following holds:*

1. $N := \dim \mathfrak{g} = \frac{n}{2}(n-1)$,
2. $\text{tr } A = 0$ for any $A \in \mathfrak{g}$,
3. *the only system of N matrices $A_k = (A_{jk}^i)$ in $\mathfrak{g}$ ($1 \leq k \leq N$) which is symmetric in the lower two indices, $A_{jk}^i = A_{kj}^i$, is identical 0. (Weyl 1921c, 131f.), (Weyl 1923a, 51).*

In 1920 he conjectured that the only groups satisfying the conditions of Lemma 1 are the special orthogonal groups of arbitrary signature, $G = SO(p, q, \mathbb{R})$. He was able to give a proof at first in the dimensions $n = 2, 3$, then, in a case by case study, for general n (Weyl 1922b), (Weyl 1923a, app. 12). Already his first case by case study brought him closer to the theory of infinitesimal Lie groups, but according to Hawkins he did not yet delve deeply into it (Hawkins 2000).

Cartan, on the other hand, read Weyl’s first presentation of the PoS in 1921 or early 1922. He tried to make sense of it in terms of his newly developed “non-holonomous” spaces and reformulated Weyl’s condition in such a way that his classification of infinitesimal Lie groups became applicable. By this move he could solve a (modified) main theorem of the space problem without much additional technical work (*Cartan’s space problem*, see below). In October 1922 he wrote Weyl about this result. Weyl was impressed and was attracted to Cartan’s theory, but did not yet try to assimilate it (Hawkins 2000, 438ff). He continued to rework the proof of the extended Barcelona lectures in his own approach.

In a systematic study of the (complex) rotation group Weyl brought it into a form which he found finally satisfying (Weyl 1923c). The result can be stated as

Theorem 1 *The only complex infinitesimal Lie groups $\mathfrak{g}$ satisfying conditions (1) to (3) of Lemma 1 are the Lie algebras of special orthogonal groups, $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ ($n \geq 2$).*

Over the complex numbers any nondegenerate quadratic form can be brought into normal form with positive coefficients, even normalized to 1, while for real subspaces positive definiteness may be lost. The result was exactly what Weyl had conjectured from the outset. He drew the resumé:

Just as the old conception of a metrical structure inbuilt into space by a priori reasons independently of matter leads to Helmholtz-Lie’s characterization of the rotation group, the modern view due to Einstein’s relativity theory, according to which the metric

structure is variable and is causally dependent on matter, leads to acknowledging a different property of the rotation group as decisive (...). (Weyl 1923*c*, 348)^(iv)

For Weyl, the conditions (1) to (3) of the lemma took over a role exactly analogous to flag transitivity in the old one.⁴⁰ By complexification they led to the inclusion of non-definite rotation groups in the real case.

During the long path toward a satisfying proof of his main theorem Weyl found sufficient occasion to rethink his position with regard to the foundations of mathematics. In 1921 he was an ardent defender of what he considered Brouwerian intuitionism, among others he considered the principle of excluded middle as illegitimate for infinite collections.⁴¹ Already in his first systematic case by case proof Weyl realized how difficult proofs could become if one was to avoid the true/false dichotomy of classical logic. As a result he chose to weaken the intuitionistic proof criteria for the continuum in order to arrive at clean case distinctions.⁴² But he still promised to “come back to this point, once a new analysis in more definite form has been elaborated than is the case at the moment” (Weyl 1922*b*, 295).

During the work process in the following year he seems to have changed his mind. He was obviously satisfied with his final proof in (Weyl 1923*c*) which still needed dichotomic case distinctions. The only remnant of his formerly declared goal of intuitistic proofs was hidden under a weakly self-ironic remark at the end of the article:

Our game on the chess board of matrix schemata has been played to the end ... (ibid., 372)

Readers who knew his commentaries on the foundations of mathematics could easily recognize the parallel to earlier critical remarks of Weyl on Hilbert’s formalism as an epistemically empty game (chess). But in 1922/23 Weyl was sure that the game had an epistemical surplus value provided by the context of the problem of space.

For him, the context went far beyond the algebraic characterization of the rotation groups. With the geometrical characterization of infinitesimal congruence structures by the postulates P1 and P2 it seemed clear that over the reals the congruences in the infinitesimal were characterized by a rotation group $G = SO(p, q, \mathbb{R})$ of fixed signature (by reasons of continuity). After the choice of a basis in the “vector bodies” (tangent spaces), they could

⁴⁰Already early in his proof attempts Weyl realized that the trace condition (2) could be omitted for $n > 2$ (Weyl 1923*a*, 112), (Weyl 1923*c*, 354)

⁴¹In fact, Weyl’s perception of intuitionism differed from Brouwer’s in certain respects (Hesseling 2003, van Dalen 2000).

⁴²He declared his own proof as “... nur zwingend, wenn wir in einem Zahlbereich bewegen, wie z.B. den rationalen Zahlen, in welchem eine Zahl entweder = 0 oder $\neq 0$ ist. Die Fallunterscheidungen der Elementarteilerttheorie sind von diesem Standpunkt aus ein besonders bedenklicher Ausgangspunkt.” (Weyl 1922*b*, 295)

be specified as those of a pseudo-Riemannian metric $g = (g_{ij})$. Pointwise change of the congruence groups was mediated by conjugation in the similarity group $G^* = \mathbb{R}^+ \times SO(p, q, \mathbb{R})$ (in modernized terminology the structure group). Thus displacement of vectors or a vector constellation (a figure) by the “metrical connection” from a point p to an infinitesimally close one p' would lead to a *similar* constellation (figure). The deviation of the parallel displacement (unique among the all equivalent metrical connections) from conserving metrical values relative to the chosen $g = (g_{ij})$ was then expressible by a Weylian length connection $\varphi = (\varphi_i)$ depending on g . Thus Weyl arrived at

Result 1 *The structure of a Weylian metric $[(g, \varphi)]$ can be reconstructed from infinitesimal congruences in the sense of the new PoS (postulates P1 and P2), just as the Riemannian metric had been reconstructed by Helmholtz, Lie, and Killing in the classical PoS (Weyl 1923a, 51f.).*

In this sense, Einstein’s “Weyl II” objection was shown to be immaterial, if one accepted Weyl conceptual analysis and his “synthetic” postulates (at least P2).⁴³ The structure of Weylian metric had been given a conceptually outstanding status, although Einstein probably would have been unimpressed by such a combination of philosophical and mathematical arguments in Weyl’s analysis, in case he came to an understanding of the complicated argument.⁴⁴ At the time of Weyl’s work on the PoS Einstein was already on his path toward metric-affine theories (Sauer 2004), although in some moments he still tried to make sense of Weyl’s field unification.⁴⁵

For Weyl, on the other hand, his way of treating the PoS may have contributed to pave the way for his later work in generalizing gauge structures from the similarities to phase factors in Dirac theory (group extension by

⁴³In the later language of fibre bundles, Einstein’s objection can be stated as the question, why one would not expect a general affine connection, not reducible to an orthogonal group, if one was prepared to give up Riemannian geometry anyhow. (Physicists speak of a *metric affine* theory of gravity.) Eddington’s affine field theory went in that direction, even in a more radical form; no wonder that it played a crucial role in Einstein’s early turn toward unified field theories.

⁴⁴A side remark in Einstein’s letter from 06. 06. 1922 to Weyl shows that he was aware of Weyl’s analysis of the space problem: “Ich bemühe mich gerade um das Verständnis Ihrer Arbeit über die mathematische Vorzugsstellung der quadratischen Form. Physikalisch komme ich nicht weiter. Ich glaube nicht an den Zusammenhang zwischen elektrischem Feld und Streckenkrümmung. . . .” (Einstein 1987ff., **13**, doc 219). The second phrase shows that Einstein still struggled with Weyl’s unified field theory. — Thanks to D. Lehmkuhl whom I owe the information about Einstein’s (at least marginal) interest in Weyl’s PoS.

⁴⁵In his travel diary Oct. 1922 – March 1923 (Japan, Palestine, Spain) he noted (09. 10. 1922): “Gestern habe ich die elektromagnetischen Vakuum-Gleichungen (. . .) im Sinne der Weylschen Geometrie umgerechnet, in der Hoffnung, einen Ausdruck für die Stromdichte zu finden. Es kommt aber unbrauchbares Resultat $\varphi^{\mu\alpha}\varphi_\alpha$ heraus.” (Einstein 1987ff., **13**, doc 379).

$U(1)$ rather than by $\mathbb{R}^+$ (Weyl 1929).⁴⁶ Of course this shift from similarities to phase was part of a much wider turn from classical unified field theories to the question of how to make the views of general relativity coherent with the new wave functions (Schrödinger and/or Dirac) in quantum theory. For Weyl, the turn went in hand with a rising awareness that a priori type arguments were only of limited help for physics, while the empirical input had to be considered as the finally decisive factor.⁴⁷ It may have been this shift which induced Weyl to ponder about the difference between purely *geometrical* (in the sense of mathematical geometry) and *physical automorphisms* in physical theories.

In the English translation of his *Philosophy of Mathematics and Natural Sciences* and in the unpublished notes of a talk, given apparently about the time of preparing (Weyl 1949),⁴⁸ Weyl clearly distinguished between different types of symmetries in physical theories: between those that are of mathematical, in particular geometrical, importance (*mathematical* or *geometrical automorphisms*) and those that relate to the “physical world” (*physical automorphisms*).

Discussing classical geometry and classical physics he explained the group of Euclidean isometries, introduced as proper or improper “motions”, and denoted it by Δ . He then went on:

A far deeper aspect of the group Δ than that of describing the mobility of rigid bodies is revealed by its role as the group of *automorphisms of the physical world*. In physics we have to consider not only points but also various other types of physical quantities, velocity, force, electromagnetic field strength, etc. ... All the laws of nature are invariant under the transformation thus induced by the group Δ . (Weyl 1949, 82f., emphasis in original)

On the other hand, he introduced the (classical) similarity group Γ as the normalizer of Δ (in the diffeomorphism group of space) and explained its importance for characterizing classical metrical (Euclidean) geometry. Different to what he had thought at the turn to the 1920s, he now drew upon the empirical insights from atomic physics:

The atomic constants of charge and mass of the electron and Planck’s quantum of action h fix an absolute standard of length, that through the wave lengths of spectral lines is also made available for practical measurements. ... The orthogonal transformations of signature – must be included in Δ . For there is no

⁴⁶(Vizgin 1994, O’Raifeartaigh/Straumann 2000, Scholz 2004b)

⁴⁷(Eckes 2013/14, Scholz 2005a, Scholz 2005b).

⁴⁸H. Weyl “Similarity and congruence: a chapter in the epistemology of science”, Bibliothek ETH Zürich, Hs 9a:31 (23 pp.), formally undated typescript, written “about 30 years” after “an attempt was made by the speaker to build up a unified field theory of gravitation and electromagnetism ...” (ibid, p. 20).

indication in the laws of nature of an intrinsic difference between left and right. (ibid.)

In this sense the group Δ characterized “physical automorphisms”. For the geometrical (mathematical) automorphisms Weyl concluded now:

The group of geometrical automorphisms, by virtue of the very meaning of this term, is the normalizer Γ of Δ . It is larger than Δ as it includes the dilatations This divergence between Δ and Γ proves conclusively that *physics can never be reduced to geometry* as Descartes had hoped [and Weyl himself had believed between 1918 and ca. 1923, E.S.]. (Weyl 1949, 83)

In other words, it was the *distinction between the automorphism groups* of mathematics/geometry and physics which led Weyl to the conclusion that physics could not be “reduced” to mathematics.⁴⁹

A little later in the chapter Weyl discussed the new situation after the advent of general relativity. He emphasized that the metric is here no longer part of the spatial structure as such (“the extensive medium of the world”), but rather part of its changing material content. By the argument that “the free mobility of bodies without changes in measure is regained, since a body in motion will ‘take along’ the metric field that is generated or deformed by it” (ibid, 87), Weyl argued for accepting diffeomorphism with dragging of physical fields as part of the physical automorphisms of GR. Moreover, he presented the method of moving (orthogonal) frames, denoted by $e_{i\beta}$, ($1 \leq i, \beta \leq n$), used by Cartan and by himself in his second (phase) gauge theory of the general relativistic Dirac field (Weyl 1929). Then he went on:

What has happened in the transition from special to general theory is obviously this. The physical automorphisms forming the group Δ as described in the previous section⁵⁰ have been split into their translatory and rotatory parts. The group of translations has been replaced by that of all possible transformations of the coordinates, whereas the rotations have remained Euclidean rotations but are now tied to a center P and must be allowed to vary freely while the center P moves over the manifold. Space, the extensive medium of the material world, is clearly the seat

⁴⁹ Although this insight can be defended, and probably would better be defended, by different arguments and even may seem obvious to unbiased reflection, this conclusion was important for Weyl, because in his radical years 1918–1920 he had tended to believe in such reducibility. Weyl’s clearcut distinction between physical and mathematical automorphisms became blurred again after the rise of modern gauge theories in late 20th century physics (see below).

⁵⁰ Weyl skipped the discussion of special relativity as an intermediate step and referred the reader to another chapter of his book, if he or she wanted to see “how time is included as a fourth coordinate in the above scheme” (Weyl 1949, 83).

of the group Ω of coordinate transformations; but the group Δ_0 [the point dependent representation of the rotation group, E.S.] seems to have its origin in the ultimate elementary particles of matter. The quantities $e_{i\beta}$ thus mediate between matter and space. (Weyl 1949, 89)

About 1928/1929, Weyl's characterization of the problem of space had changed from his earlier strongly aprioristic philosophical view to an empirically imbued one.⁵¹ The old gauge idea (scale geometry) now appeared to him as part of the "geometrical automorphisms", not of the physical ones.⁵² The physical automorphisms he had now in sight included, slightly rephrased, the diffeomorphism auf the spacetime manifold ("all possible coordinate transformation"), point dependent Lorentz rotations, extended by (also point dependent) phase transformations, not unlike the similarity extension of 1918 and the early 1920s. Clearly, the extension group, $U(1)$, was no longer part of the geometry (it did not operate on the "external medium of the world"), but operated on the dynamical space of Dirac spinors or the wave fields. This did not hinder Weyl to consider physical automorphisms with point dependent group operations (i.e., gauge groups of physical theories) as carrying the "character of general relativity", (Weyl 1929, 246).⁵³

The tendency of intertwining geometry with matter in Einstein's theory became strengthened with the new, although still embryonic, insights in the physics of elementary particles at the end of the 1940s. At the time of rewriting his *Philosophie der Mathematik und Naturwissenschaften* Weyl even dared to speculate that the (Lorentz) rotations and their extension "have their origin in the ultimate elementary particles of matter". Mathematically, the localized group "could be considered as an abstract group of which various representations by linear transformations are characteristic for various physical quantities" (Weyl 1949, 89). Groups and their representations, gauge structures, geometry and quantum physics intermingled in Weyl's view at the turn to the 1950s. The problem of space could no longer be separated from the problem of matter.

⁵¹In the decade 1920-1929 Weyl thus moved from a heavy leaning toward the a priori side of a "relativized a priori" in the sense of (Friedman 2001) to the historically relativized one, taking account of new empirical constraints (atomic and quantum physics) and taking the empirical input of knowledge more serious than before.

⁵²The reappearance of scale in- and covariance in field physics in the second half of the 20th century brought new aspects into this question. These cannot be discussed here.

⁵³About a decade later, Weyl reconsidered the Lagrangian of the general relativistic Dirac field and found a "slight discordance between affine connection and metric", i.e., a (con)torsion effect resulting from the spin current of the Dirac field (Weyl 1950). He thus realized that there may be physical reasons to relax the condition of *affine* connection in general relativity, which in 1923 he had considered a "synthetical" postulate "a priori" (see section 4). But there is no indication that he started to rethink the question of "physical automorphisms" in full-fledged Cartan geometric terms.

5. Cartan's view of the space problem

When Cartan got to know Weyl's PoS (in 1921 or in early 1922) he could already build upon a huge expertise in the theory of infinitesimal Lie groups which he had collected over a period of roughly thirty years.⁵⁴ Among others he had classified the simple complex Lie groups in his doctoral thesis 1894, twenty years later (1913/14) the real ones (Hawkins 2000, part III). Moreover he had brought to perfection the usage of *differential forms* ("Pfaffian forms") in differential geometry (Katz 1985). In 1910 he had started to describe the differential geometry of classical motions by generalizing Darboux' method of "trièdres mobiles" (moving frames) (Cartan 1910). During the year 1921 he started to develop, and in 1922 he published, a series of articles in which he presented his new theory of generalized *non-holonomous* spaces (later *Cartan spaces*), a kind of "infinitesimalized" Kleinian geometries which allowed a supplementary translational curvature called *torsion*.⁵⁵

The idea for such an additional aspect of curvature seems to have been a result of Cartan's attempts to geometrize certain aspects of E. and F. Cosserat's theory of generalized elastic media. The brothers had investigated a variational principle for elastic media with an action density dependent on an infinitesimal "trièdre trirectangulaire" (orthonormal frame) and invariant under infinitesimal Euclidean motions.⁵⁶ Under such assumptions the ensuing stress tensor was no longer necessarily symmetric. Moreover, an "infinitesimal" surface element on the boundary of an elastic body under external influences could be the carrier of an elastic torque in addition to an elastic force.⁵⁷

At the end of his *Comptes Rendus* note presented on February 27, Cartan (1922a) remarked that this generalization was close to the Cosserat's investigation in (Cosserat 1909a) and to Weyl's generalized concept of space in the PoS.⁵⁸ In the light of Weyl's recent studies of spaces with infinitesimal con-

⁵⁴In his first note on Weyl's PoS Cartan (1922d) referred to the French translation of Weyl's *Raum - Zeit - Materie* (Weyl 1922a). It may very well be that he knew from Weyl's PoS already from the German fourth edition, from which the translation was made; at least he cited the *German* title of the book.

⁵⁵See (Nabonnand 2009) and P. Nabonnand's contribution on Cartan in this volume. For the mathematical background see, e.g., (Sharpe 1997).

⁵⁶Compare fn. (14) and, for more details, (Brocato/Chatzis 2009). For a more systematic physical discussion see, among others, (Hehl/Obukhov 2007).

⁵⁷"... nous appellerons *effort extérieur et moment de déformation extérieur au point M de la surface S*, qui limite le milieu déformé, *rapporté à l'unité d'aire de surface S_o* les segments issus de ce point *M* et dont les projections sur les axes *Mx'*, *My'*, *Mz'* sont respectivement $-F'_o$, $-G'_o$, $-H'_o$ et $-I'_o$, $-J'_o$, $-K'_o$ " (Cosserat 1909b, 598). The quantities F'_o ... and I'_o arose as coefficients of infinitesimal deformation vector, respectively infinitesimal rotation vector, in the surface action density (after application of Stokes theorem) (Cosserat 1909b, 597).

⁵⁸"J'ajouterais que les considérations précédentes qui, du point de vue mécanique, s'apparentent aux beaux travaux de MM. E. et F. Cosserat sur l'action euclidienne, s'apparentent également à la théorie des espaces généralisés de H. Weyl et peuvent elles-

gruence structures the conceptual problems of general relativity presented a challenging context for developing Cartan's ideas on connections with a translational component. Why should not, perhaps, the relativistic "ether" (to use Einstein's and Weyl's expression for the structure field of general relativity) behave as sophisticated as the Cosserats' solid state media could (at least hypothetically)?

As a consequence, Cartan's generalization went far beyond the one envisaged by Weyl. He developed it in a multitude of publications extending over years, and even decades, to come. As a side activity to his main work he considered it worthwhile to give his own answer to the question posed by Weyl the new PoS (Cartan 1922*d*, Cartan 1923*b*). But his approach differed in three respects from Weyl's:

- He reformulated Weyl's characterization of the PoS in terms of his representation of connections by differential forms.
- As a result of his reformulation he unknowingly suppressed the specific Weylian aspect of the metric and reduced the geometrical analysis of the PoS to the (pseudo-) Riemannian special case.
- The conditions arising from the PoS for the infinitesimal groups then acquired such a form that Cartan's general structure theory of simple and semi-simple Lie groups could be applied, leading to a much shorter proof of the main theorem.

In a letter to Weyl, written in October 1922, Cartan politely announced his second publication on the PoS and sent him a preprint ("un exemplaire d'un mémoire que je viens de faire paraître dans le *Journal de Mathématiques*") for information. Cartan modestly played down the achievements of his general structure theory of infinitesimal Lie groups, but tried to interest Weyl for his methods to describe connections by systems of differential forms which we today prefer to read as differential forms with values in a Lie algebra.⁵⁹

Weyl's characterization of the PoS, with its basic ingredient of a generalized "group of rotations" G operating on the vectors attached to a point ("vecteur issu de P " in Cartan's language) was easy to accept for Cartan. But Weyl's way of dealing with similarity operations G^* for the displacements of

mêmes se généraliser." (Cartan 1922*a*, 595).

⁵⁹"... une foi le théorème reconnu vrai, le plus ou moins de simplicité de sa démonstration n'est rien auprès de sa portée philosophique ... Les procédés de calcul qu'il y emploie ne sont pas ceux du calcul différentiel absolu; ... ils sont tout aussi bien adaptés je crois à l'étude des multiplicités, non seulement à connexion affine (pour employer votre terminologie), mais encore à connexion projective, ou conforme, etc. ..." (Cartan 1922ff, Cartan to Weyl, 09. 10. 1922). I thank P. Nabonnand for having made transcripts of this letter available to me.

vectors between infinitesimally close points p and p' remained incomprehensible to him, because Weyl used a form of group extension ($G \subset G^*$) specific for his conception of gauge geometries.⁶⁰ Moreover, the characterization of the postulate of freedom (P1) with the specification that the “metric connection” Λ_{jk}^i should have “greatest liberty” for matter influence and could acquire any of n^3 values (at a point), could not easily be rephrased (if at all) in Cartan’s terminology of moving frames and differential forms.⁶¹ So it is not surprising that after rephrasing Weyl’s first postulate, Cartan continued with an interpretation of his own.

Cartan used a base choice of the vector spaces ($T_q M$, $q \in U$) in a (finite) neighbourhood U of p , such that the operation of the “rotation group”, or its infinitesimal version, was given by constant coefficients. In other words, his way of making precise Weyl’s speech of “different orientations” of the group G at different points consisted in assuming a choice of frames (vector bases) in each $T_q M$, chosen such that the operation of G has the same matrix representation for all $q \in U$.⁶² The next step contained Cartan’s simplification of Weyl’s approach:

This given, the meaning of Mr. H. Weyl’s first axiom is the following: We may choose at each point P of the space the *orientation* of the group arbitrarily, i.e., the coefficients $a_{ij} \dots$ [which describe the operation of the group, ES]. After this choice, the metrical connection of the space is determined: one of the congruence transfers ... is the one in which two corresponding vectors have *the same* components ... (Cartan 1923b, 172)^(vi)

Thus Cartan assumed that the frames chosen for the representation of the G -operation could be considered as *congruent* without further specification. Put in the language applied by him and Einstein in the late 1920s, he considered a (Cartan-) connection corresponding to a *distant parallelism* defined by the frame choice as one of the possible “metrical connections”.⁶³ Note that this was *no faithful representation* of Weyl’s description of “metrical”

⁶⁰Cartan’s own approach to non-holonomous spaces also used a group and a subgroup ($L \supset G$), but for the completely different purpose of constructing homogeneous spaces L/G which were to describe infinitesimal tangent structures. See below or, in a systematic perspective, (Sharpe 1997).

⁶¹If one wanted to express Weyl’s P1 in modernized terminology, one had to exploit the interplay of a connection with values in the general linear group, which can acquire any value at a point, although it can be reduced to G^* , respectively its Lie algebra $\mathfrak{g}^*$. Moreover one has to observe that the equivalence classes of postulate P2 are constructed by differential forms with values in $\mathfrak{g} = \text{Lie } G$.

⁶²In modernized terminology one could see in Cartan’s description a well chosen trivialization of the tangent bundle, adapted to the operation of G on the tangent spaces.

⁶³So far, the specification of the translational part of the Cartan connection could be left open. It entered the investigation only when Cartan characterized the condition of vanishing torsion (below PA).

infinitesimal geomery with its *specific difference* between infinitesimal *similarities* $\mathfrak{g}^*$ (for metrical displacements) and *congruences* $\mathfrak{g}$ (for “rotations” at a point). In his mode of rephrasing Weyl’s idea, Cartan suppressed this specific difference.⁶⁴

In the next step, Cartan considered, like Weyl, the whole collection of “metrical connections” derivable from the distant parallel displacement by adding a differential form with values in $\mathfrak{g}$ to the one given by the frame choice. He then translated Weyl’s stipulation of symmetry for an affine connection into his own terms:⁶⁵

PA Whatever the choice of the translational part, among all Cartan connections (values in $\mathbb{R}^n + \mathfrak{g}$) with rotational part formed in this way there is at least one with vanishing torsion (ibid. 173).

As “second axiome de M. H. Weyl” he stated the uniqueness postulate:

PB For a metrical connection in the sense above, satisfying postulat PA, the torsion free connection is uniquely determined.

Of course, Cartan shifted again the content of Weyl’s principles: He skipped completely the content of the “postulate of freedom” P1 and separated the “postulate of coherence” P2 into two parts, existence (PA) and uniqueness (PB) of a compatible torsion free connection. From a mathematical point of view, both modifications made sense (but Cartan wrote as if he did not notice the shifts). Although Weyl considered P1 as philosophically important and although it led to an easy accessible counting argument for the dimension of G , it turned out, in the end, as mathematically redundant. Not only did it become superfluous in Cartan’s modified space problem; E. Scheibe later showed that also for the Weyl’s original PoS it was unnecessary (Scheibe 1957, Scheibe 1988).⁶⁶

The second modification (separation of P2 in two parts, PA and PB) was helpful for reformulating the existence condition of vanishing torsion in Cartan’s own terms and for structuring the proof of the main theorem of the PoS in the framework of his theory of infinitesimal Lie groups. Already the existence assumption of a compatible torsion free connection (PA) allowed Cartan to select a certain number of simple groups as possible candidates for G from his general list (Cartan 1923b, 181ff.).⁶⁷ Adding the uniqueness

⁶⁴In modern parlance, Weyl’s construction is not described by a principal fibre bundle with structure group G , but by a bundle $E \rightarrow M$ with fibre G , and the tangent bundle TM associated to it. The structure group of E is G^* which operates on G by conjugation.

⁶⁵“Par suite, l’axiome I de M. H. Weyl peut s’énoncer de la manière suivante: ...” (Cartan 1923b, 173)

⁶⁶The dimension condition follows already from the existence postulate of a compatible affine connection.

⁶⁷The list for the homogeneous part G was: $SL(n, \mathbb{R})$, subgroup of $SL(n, \mathbb{R})$ keeping a line fixed, $SO(p, q; \mathbb{R})$, $Sympl(2n, \mathbb{R})$.

demand (PB) then reduced the allowable groups to the special orthogonal ones:

The linear group of a non-degenerate quadratic real form is therefore the only one which satisfies the conditions of the axioms I and II of Mr. H. Weyl [here abbreviated as PA and PB, ES]. In this way we have proven, on the basis of these axioms, that the Pythagorean form of the metric in the universe is necessary.^(vii)
(Cartan 1923b, 192, emphasis in original)

Geometrically, Cartan’s adaptation of the modern PoS led to the

Result 2 *Infinitesimal congruences in the sense of Cartan’s PoS, satisfying postulates PA and PB allow to reconstruct the structure of a Riemannian metric, comparable to the reconstruction of the Riemannian metric by Helmholtz, Lie, and Killing in the classical PoS.*

Cartan seemed to be satisfied, philosophically, to have found a characterization of Riemannian geometry among his much more general class of infinitesimal geometries, just like Weyl had been about his slightly different argument for his gauge generalization of Riemannian manifolds.

Of course this excursion into the PoS, triggered by Weyl’s example, contained only a selected aspect of Cartan’s own thoughts about the question of how to modify the concept of space in the light of relativity. He pursued a much vaster program for revising the “classical” concept(s) of space, oriented at the goal of achieving for Klein’s view of geometries a similar infinitesimalization (present physicists’ language “localization”) which Riemann had proposed for Euclidean geometry. In his talk at the Toronto ICM Cartan explained this view in general terms

...while a Riemannian space does not possess absolute homogeneity, it possesses a kind of infinitesimal homogeneity; in the immediate neighbourhood of a point it can be assimilated to a Euclidean space. (Cartan 1924a, 85)^(viii)

For the investigation of such infinitesimal laws of homogeneity, Cartan extended the method of *moving frames* (“*répères mobiles*”) and *Pfaffian (differential) forms* taken over from his teacher Darboux, but generalized both in a highly sophisticated way. Infinitesimal neighbourhoods (modernized T_pM) of a Riemannian manifold could be “assimilated” (Cartan in Toronto) to an Euclidean space, which again could be constructed as a homogeneous space $\mathbb{E}^n \cong L/G$, for example from the Euclidean isometries $L \cong \mathbb{R}^n \rtimes SO(n, \mathbb{R})$ and the rotations $G = SO(n, \mathbb{R})$.

Taking a consequently infinitesimal perspective, Cartan preferred to pass over to the corresponding infinitesimal Lie groups, in later terminology the

Lie algebras $\mathfrak{l} = \text{Lie } L, \mathfrak{g} = \text{Lie } G$; then the infinitesimal neighbourhood could be “assimilated to” a homogeneous space of infinitesimal Lie groups⁶⁸

$$T_p M \cong \mathfrak{l}/\mathfrak{g} \cong \mathfrak{h}. \quad (1)$$

where $\mathfrak{h}$ was here the trivial subalgebra $\mathbb{R}^n$, invariant under the adjoint operation of $\mathfrak{g}$ (today *reductivity*), and $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{g}$. Similarly for other pairs of groups L, G for which this condition ((1) and reductivity) was satisfied. Among them Cartan studied for L the affine, the conformal and the projective groups, while G was the respective isotropy group of a point. Cartan described such an assimilation by introducing increasingly general (and tricky) (*Cartan*) *frames* (“répères”). For completing the idea of “homogénéité infinitésimale” Cartan explained, how the group operation of L , respectively its infinitesimal version $\mathfrak{l}$ binds the infinitesimal neighbourhoods together. To do this he introduced a collection of “Pfaffian” (i.e., differential) forms ω^i ($1 \leq i \leq n$) and ω_k^j ($1 \leq j, k \leq \dim \mathfrak{g}$).⁶⁹ They parametrized the infinitesimal operations of $\mathfrak{l}/\mathfrak{g} \cong \mathfrak{h}$ and $\mathfrak{g}$ with regard to the chosen Cartan frame. They prescribed infinitesimal group operations in $\mathfrak{l}/\mathfrak{g} \cong \mathbb{R}^n$, respectively $\mathfrak{g}$ for any infinitesimal variation $\delta x = (\delta x_1, \dots, \delta x_n)$ of coordinates of a given point. In slightly modernized terminology

$$\begin{aligned} (\omega^i) & \quad \text{is a differential 1-form with values in } \mathfrak{l}/\mathfrak{g} \cong \mathbb{R}^n \\ (\omega_k^j) & \quad \text{is a differential 1-form with values in } \mathfrak{g} \end{aligned}$$

Together they form what later would be called a *Cartan connection*. The (ω^i) represent the generalized translational and the (ω_k^j) the isotropic (generalized rotational) contribution of the connection.⁷⁰

If one follows (integrates) such motions along a path the result is, in general, path dependent. Infinitesimal closed contours lead, like in the case of the Levi-Civita connection of Riemannian geometry, to curvatures (Ω^i) (translational part) and (Ω_k^j) (isotropic part). Cartan was able to express these curvatures by differential 2-forms with values in the respective infinitesimal groups and depending on the “sides” of an infinitesimal parallelogram

$$\begin{aligned} \Omega^i &= d\omega^i - \omega^k \wedge \omega_k^i, \\ \Omega_k^j &= d\omega_k^j - \omega_i^k \wedge \omega_k^j. \end{aligned}$$

For the translational part of the curvature, Ω^i , Cartan coined the term *torsion*, the spaces he called *non-holonomous*.

During the 1920s Cartan studied non-holonomous spaces of increasingly complex type:

⁶⁸Cf. fn. (10).

⁶⁹Different from Cartan’s notation, an upper and lower index notation is used here in analogy to the usage in co-contravariant tensor calculus.

⁷⁰Compare the more detailed presentation in Nabonnand, this volume.

- Poincaré group in papers on the geometrical foundation of general relativity (Cartan 1922*b*, Cartan 1923*a*, Cartan 1924*b*). For torsion $\Omega^i = 0$ such a Cartan space reduced to a Lorentz manifold and could be used for treating Einstein's theory in Cartan geometric terms,
- inhomogeneous similarity group (for torsion = 0, this case reduced to Weylian manifolds),
- conformal group (Cartan 1922*c*),
- projective group (Cartan 1924*c*).

In this way, Cartan developed an impressive conceptual frame for studying different types of differential geometries, Riemannian, Lorentzian, Weylian, affine, conformal, projective. All were enriched by the possibility to allow for the new phenomenon of torsion, and all arose from Cartan's unified method of adapting the Kleinian viewpoint to infinitesimal geometry.

For him, the look at PoS in the sense of Weyl was only a small subject in a much wider field. Nevertheless Cartan mentioned it prominently at the end of his Toronto talk:

Before I come to the end, I have to indicate the remarkable researches in which Mr. H. Weyl has taken up the old philosophical problem of the problem of space, which was treated formerly by Helmholtz and Lie; he has it now adapted to the new viewpoints introduced by relativity theory. The concept of group is here again at the base even of the formulation of the problem posed by Mr. H. Weyl. I cannot dream, however, to enter into even a comprehensive discussion of this important question which would demand a special presentation of its own. (Cartan 1924*a*, 93f.) ^(ix)

This could be read as an invitation to mathematicians for taking the modern PoS seriously as a research problem. Although Cartan himself had subsumed it under his more general perspective of non-holonomous spaces and (Cartan-) connections, he did not consider the PoS as closed by his short articles at the beginning of the 1920s.

6. Outlook: mathematical and physical aspects of the PoS in the second half of the 20th century

Most *mathematicians* of the next (post-Cartan/Weyl) generation learned the modernized PoS from Cartan, and thus separated from Weyl's gauge geometric perspective (Tits, Freudenthal, Nomizu, Klingenberg, e.a.). With the rise of the fibre bundle language, even Cartan's geometry tended to be reduced to the homogeneous (Riemannian) standpoint, and its generalization

of G -structures (on the tangent bundle of M). Only at Göttingen two young mathematicians (the second one later turned toward philosophy of science) did research on the specific Weylian versions (first and second) of the PoS keeping close to Weyl's specific geometric contextualization of his first (1919) and second (1921–23) discussions of the space problem (Laugwitz 1958, Scheibe 1957).

The other authors of the next generation approached the problem from views closer to main developments in the field. J. Tits and H. Freudenthal analyzed general topological characterizations of the group operation. They thus continued and expanded Hilbert's characterization of the foundations of geometry by topological group principles in the notes of 1902/1903 (Tits 1952, Tits 1953, Freudenthal 1956). Moreover they simplified the group theoretic part of the investigation and updated it from the point of view of the structure classification of Lie algebras, which became famous in their generation. The topological group aspect of the PoS had been investigated in the decades before by a series of researchers, among them Brouwer (1909/10), R.L. Moore (1919), Cairns (1923), Süss (1925–27), Lubben (1928), Kolmogorov (1930), Montgomery/Zippin (1940), Kérékarto (1950).⁷¹ In their topological characterization, Tits and Freudenthal built along the line proposed by Kolmogorov. For the underlying space S they assumed no longer the manifold property, but only metrizability, local compactness and local connectedness. The transformation group G had to satisfy some kind of “rigidity”, completeness, and a separation property for orbits. Already these quite general conditions led to specifications for the group that could be evaluated in the light of the classification theory of simple Lie algebras.⁷²

The differential geometers of the next generation were strongly influenced by the Cartan tradition. They developed a modernized language for infinitesimal connections, (Ehresmann 1951, Chern 1951, Nomizu 1956) which fitted well to the new theory of fibre bundles established and studied by topologically oriented geometers (Steenrod 1951, Hirzebruch 1956). Cartan's analysis of the PoS was translated into this newly established setting. In this framework Cartan's PoS was generalized to any type of G -bundle of frames $E_G \rightarrow M$ in the tangent bundle over a (differentiable) manifold M , for any closed subgroup $G \subset GL(n, \mathbb{R})$. Klingenberg then posed the generalized PoS in the form:

We now ask for which closed subgroups G of $GL(n, \mathbb{R})$ there always exists a canonical linear connection in E_G , i.e., a connection in some way specified by the structure of G . . . (Klingenberg 1956, 300)(x)

⁷¹List according to (Freudenthal 1960, 16); see there for bibliographical references.

⁷²(Tits 1953, Freudenthal 1960). Freudenthal commented that Hilbert's 3-closedness condition could be considered as a topological kind of “rigidity”.

He showed that, e.g., for $2n$ -dimensional almost complex manifolds M a result analogous to Cartan's holds: If for a closed subgroup $G \subset GL(n, \mathbb{C})$ any G -frame bundle E_G over M allows exactly one connection for which the mixed terms of the torsion vanish, then G is a real form of $GL(n, \mathbb{C})$ and vice versa.⁷³ Thus Cartan's PoS developed into a wider research field later called the study of G -structures on M (Kobayashi/Nomizu 1963, 288ff.).

Parallel to this mathematical research on the PoS, following the lead of Cartan and Weyl, *physicists* developed their own understanding of infinitesimal, local and global structures of spacetime and the dynamical fields upon it. Here we can, of course, only scratch upon the surface of this complex and many-sided history, which covers large parts of fundamental physics of the second half in the 20th century. We can only roughly indicate two developments in which physicists started to see the role of "localized" symmetries in spacetime (as they used to call point dependent infinitesimal operations of Lie groups) or acting in dynamical spaces above it (*internal spaces*). Both are most closely related to Weyl's and Cartan's treatment of the PoS.

These developments come from the two branches of physical theory which, about the middle of the last century, started to be built about gauge structures and turned the latter into a central paradigm in fundamental physics of the outgoing 20th century. The most spectacular development in this direction arose from the growth of gauge field theory in elementary particle physics. It was opened up by the work of Mills/Yang (1954) and others, less well known but earlier and more general, Utiyama who sketched an integrated gauge panorama for both, gravity and nuclear fields.⁷⁴ The other one consisted in the deepening usage of Cartan geometric structures in gravity theory.⁷⁵

At the end of section 3, we saw a widening gap in general relativity between mathematical infinitesimal symmetries and their physical interpretation, as far as energy conservation was concerned. That stood in clear contrast to special relativity, where the generalized "motions" (automorphisms of Minkowski space) not only could be used to describe transformations of rigid measuring devices between different inertial systems. On a much deeper level, these automorphisms turned out to be the structural origin of conservation laws, in particular of energy-momentum. In GR the question of how to treat energy conservation was more involved and remained a challenge for a long time. In general relativity, infinitesimal translations could

⁷³Klingenberg generalized a result by Chern: There is exactly one connection with vanishing mixed torsion terms for the unitary group $U(n)$ and an almost complex manifold (Chern 1946). In this way Weyl's demand of torsion free connection was weakened to the vanishing mixed torsion terms.

⁷⁴See (Pickering 1988, Hoddgeson e.a. 1997, O'Raifeartaigh 1997, O'Raifeartaigh/Straumann 2000).

⁷⁵See (Blagojević/Hehl 2012).

be modelled in two ways: by infinitesimal diffeomorphisms of the manifold or through infinitesimal translations in the infinitesimal homogeneous spaces of Cartan geometries. Both ways, the variation of a dynamical Lagrangian led to energy momentum currents, the *canonical energy momentum* $\mathfrak{t}_{ij}$ for variation with regard to infinitesimal translations, while the *dynamical energy momentum*, T_{ij} , was derived from varying the gravitational potential (in Einstein gravity g_{ij} , in the Cartan geometric generalizations the respective connections). At first it was a problem how to make these two expressions consistent. First answers were given in the 1940s by Belinfante, Rosenfeld (see end of section 3).

D. Sciama and T. Kibble, both working in London, at King’s College and Imperial College respectively, investigated different but methodologically closely related approaches for a consequent “covariantization” of a special relativistic field theory, i.e., a generalization and import into general relativity. Their idea was to “localize” the symmetries of Minkowski space, i.e., the Poincaré group (Sciama 1962, Kibble 1961). In this enterprise they made use of the basic structures of Cartan geometry, although with quantities expressed in classical tensor calculus, without explicit recourse to Cartan’s calculus of differential forms.⁷⁶

It did not take long until Cartan’s geometric methods, sometimes combined with tensor calculus, were introduced into this research program. A. Trautman and F. Hehl, with their Warsaw respectively Cologne collaborators, seem to have been among the earliest to do so (Trautman 1973, Hehl e.a. 1976).⁷⁷ Their work showed that Cartan geometry offered a tailor-made geometric framework for infinitesimalizing (“localizing” in the language of physicists) the currents known from Minkowski space and special relativity.⁷⁸

In the Sciama-Kibble approach, Cartan geometry was allowed to have both, rotational curvature and torsion, and was additionally endowed with a Riemannian metric (*Riemann-Cartan* geometry). The gravitational Lagrangian was modelled as closely as possible to the Hilbert action of Einstein

⁷⁶Both quoted (Weyl 1929) which used certain aspects of Cartan’s work and anticipated a gauge theoretic formulation of general relativity. Sciama referred to Weyl as his source for the use of “vierbeins”. He did not mention Cartan nor his differential forms. Notice the paradoxical historical analogy with the transmission of Weyl’s gauge theory of 1929 to Yang and Mills: Yang and Mills did not know of Weyl’s work but absorbed its basic idea in a simplified form through Pauli’s article on wave mechanics and generalized it.

⁷⁷More detailed historical research on this question is necessary. At the moment the best information on this complex can be found in the reader (Blagojević/Hehl 2012) which contains extremely helpful commentaries. For ex-post surveys see (Trautman 2006, Hehl 2014).

⁷⁸That had not been done in the earlier generation, maybe because Cartan himself, as well as Einstein, Weitzenböck and other physicists of the first generation of relativists, used Cartan geometry to rewrite Einstein gravity in terms of *distant parallelism*, see (Sauer 2006, Goenner 2004). In this setting the deviation of flat space was encoded in the torsion part of curvature, while rotational curvature was set to zero.

gravity, but the geometrical structure was fully Cartanian (with Poincaré group). As the infinitesimal neighbourhoods of the (spacetime) manifold were identified with homogeneous spaces constructed from the Poincaré group, i.e., affine rather than vector spaces, point dependent infinitesimal translations could be handled in analogy to gauge transformations in elementary particle physics.⁷⁹ In the dynamical equations of this approach (a modified Einstein equation and another one regulating spin-torsion coupling) the canonical energy-momentum appears on the right hand side of the modified Einstein equation (similarly the torsion equation has the spin current as source). It is conserved, due to the underlying Cartan geometric symmetry structure (Sciama 1962, Hehl e.a. 1976, Trautman 2006, Hehl 2014). This was a striking *structural insight*, independent of the empirical relevance of the spin-torsion extension. Moreover, but far beyond the scope of this contribution, the program was extended to studying more general Lagrangians (e.g., quadratic in curvature and torsion) and more general Cartan geometries (in particular metric affine Cartan geometry).⁸⁰

In this way, the central guiding idea of Cartan in the development of his “non-holonomic” (Cartan) spaces was realized not only in geometrical but also in dynamical terms. In his Toronto talk Cartan had put his view like this:

...it is nothing but the development of relativity theory, bound by the paradoxical obligation to interpret, in a non-homogeneous universe, the numerous experiences made by observers which believe in the homogeneity of this universe, which allows to fill up, at least partially, the trench which separates Riemannian spaces from Euclidean space. (Cartan 1924*a*, 86)^(xi)

The Sciama-Kibble-Hehl-Trautman e.a. research program underpinned the dynamical nature of the infinitesimal translations component of what Weyl had called the physical automorphisms of general relativity (see end of section 4) and gave it a clearer mathematical expression in the “localized” Poincaré group.

Also in the 1970s, Weyl’s original gauge geometry with infinitesimalized scale invariance was taken up again by Omote, Utiyama e.a. in Japan and independently by Dirac, Ehlers, Pirani and Schild and others in Europe.⁸¹ This retake was no longer motivated by Weyl’s original hope for a geometrically unified field theory of electromagnetism and gravity, it rather arose

⁷⁹Transformations of these geometric *Cartan gauges*, in the terminology of (Sharpe 1997), consist of two components, one translational transforming as a differential form, the other rotational transforming as one expects from gauge connections.

⁸⁰For these developments see (Blagojević/Hehl 2012).

⁸¹Initiating papers were (Omote 1971, Ehlers/Pirani/Schild 1972, Dirac 1973, Omote 1974, Utiyama 1975*a*, Utiyama 1975*b*). A draft for a first historical outline can be found in (Scholz 2011*b*).

from an attempt to establish a bridge between a scalar field extended gravity theory of Jordan-Brans-Dicke type and cosmology (Dirac), or to the nuclear forces the theory of which still lay in the dark (Utiyama).⁸²

Weyl's scale gauge geometry offered a chance to build a conceptual bridge between gravity and the conformal structures of fundamental forces, in as much as they were considered under abstraction from mass (with "massless" fermionic and bosonic fields). This chance did not materialize in the short run. One reason, perhaps the crucial one, was the unclear role of the scale covariant scalar field of Weyl-Dirac gravity, which remained a riddle and open for speculations in different directions. So the approach did not lead to results as definite as the torsion-spin coupling and the clarification of the source currents in Einstein-Cartan gravity. In fact, scalar fields remained hypothetical until recently (2012) also in elementary particle physics,⁸³ although a specific one, the *Higgs field*, had become a central gadget already with the rise of the new fundamental theory of matter which developed in the 1970/80s.⁸⁴

In the meantime, gauge ideas also gained ground in electromagnetism and elementary particle physics. "Localized" group extensions of the Lorentz group were explored, in the hope to find mathematical symbolizations of the strong, later also the weak, interaction, or even a combined (perhaps even unified) electro-weak dynamics.⁸⁵ In these approaches the gauge idea was stripped from the explicit relationship to gravity, even though by its very nature gauge fields shared "the character of general relativity", as Weyl had expressed it. Such a link was not obvious, but it was around and served as a heuristic idea to many of the actors of the post-Weyl/Cartan generation. Originally, these attempts were speculative, tentative and laden with difficulties. Before the mid-seventies only few physicists would have expected the striking success which this theory form acquired with the rise of the standard model of elementary particles. The main task was to bring order into the plethora of new particles or resonances, to find principles to structure them, and to look for overarching theoretical explanations of the dynamics. Groups, weight diagrams of group representations, and infinitesimalized operations (gauge structures) played an increasing role in the 1960s and contributed piece by piece to forming the model class which agglomerated and became known, from 1974 onward, as the standard model of electroweak ($SU(2) \times U(1)$) and chromodynamic/strong ($SU(3)$) interactions. Leptons

⁸²Dirac, surprisingly, stuck to the obsolete interpretation of Weyl's scale connection as the potential of the electromagnetic field, the other authors followed different lines. For a recent critical acclaim of the foundational oriented paper of Ehlers/Pirani/Schild see (Trautman 2012).

⁸³(Collaboration ATLAS 2012a, Collaboration CMS 2012b)

⁸⁴For attempts to make use of Weyl geometry in relation to fundamental forces see among others (Cheng 1988, Drechsler 1999, Nishino 2004) or, more conceptually, (Scholz 2012b).

⁸⁵(O'Raifeartaigh 1997, O'Raifeartaigh/Straumann 2000, Karaca 2013).

and quarks became now the fundamental fermionic constituents of matter.⁸⁶

A crucial contribution to the final success of the standard model was the realization of the renormalizability of gauge theories. That was an important precondition for drawing quantitative consequences of the theories by perturbative calculations of the electroweak interaction or, for QCD, by lattice approximations.⁸⁷ For a while, renormalizability was considered as a most important selection criterion for theory approaches. Crucial relations for renormalizability (Ward-Takahashi relation for electromagnetism and the Slavnov-Taylor identities for the non-abelian generalizations) turned out, in due time, as quantum field consequences of infinitesimal gauge symmetries (Itzykson 1980, chaps. 11.2, 12.4), although this was not at all clear to the inventors of the first renormalizability proofs (t’Hooft 1997).⁸⁸

In the last third of the 20th century “localized” symmetries started to play a crucial role in both branches of the relativistic field theories, partly with the intention to trace conserved quantities (gravity theory), partly because of their structural importance for renormalizability (elementary particle theory). In both cases the infinitesimal symmetry structures conceived by Weyl and Cartan in the 1920s spread from the analysis of spacetime to the theory of matter and started to give them a common conceptual framework. In this mutual assimilation, features carrying “the character of general relativity”, to put it in Weyl’s terms (end of section 4), have been imported into the basic structures of elementary particle physics; on the other hand the gauge structures of gravity have become manifest. Weyl’s modified $U(1)$ gauge structure, mimicking characteristic features of infinitesimal symmetries explored in his analysis of the PoS was an inspiring background for this development (indirect for Yang and Mills, direct for Sciama and Kibble), while Cartan’s generalized geometries delivered the appropriate geometric framework for a broader “gauging of gravity”.⁸⁹

Seen from this vantage point, the fundamental theories of the structure of matter and of gravity are now permeated by a *coherent conceptual approach*, even though a unification in the strong sense, hoped for by many physicists and by some philosophers, is still lacking. There might be deep structural reasons for this state of affairs.⁹⁰

⁸⁶(Pickering 1988, Hoddgeson e.a. 1997).

⁸⁷(Kragh 1999, 341f.)

⁸⁸I thank A. Borrelli for pointing this out to me.

⁸⁹(Blagojević/Hehl 2012), here in particular p. 179.

⁹⁰The question, whether a quantization of the geometrical degrees of freedom is advisable, is still wide open. A closer inspection of inter-theory relations is, of course, needed and under way. For a lucid survey of the present situation in this regard, see (Fredenhagen e.a. 2006).

Endnotes (citations in original language)

- (i) Der zehngliedrigen Gruppe der “Bewegungen” im (x, y, z, t) -Raum mit der Maßbestimmung:

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2$$

entsprechend, gelten für den ganzen Körper 10 dem Schwerpunkt-, Flächen- und Energieprinzip der gewöhnlichen Mechanik analoge Sätze. (Herglotz 1911, 511)

- (ii) Das Beispiel des Raumes ist zugleich sehr lehrreich für diejenige Frage der Phänomenologie, die mir die eigentlich entscheidende zu sein scheint: inwieweit die Abgrenzung der dem Bewußtsein aufgehenden Wesenheiten eine dem Reich des Gegebenen selbst eigentümliche Struktur zum Ausdruck bringt und inwieweit an ihr bloße Konvention beteiligt ist. (Weyl 1921*c*, 133f.)
- (iii) An die Stelle der von Helmholtz geforderten Homogenität des metrischen Feldes ist die Möglichkeit getreten, *im Rahmen der feststehenden Natur der Metrik das metrische Feld beliebigen virtuellen Veränderungen zu unterwerfen*. (Weyl 1923*a*, 46, Hervorh. im Original)
- (iv) Ebenso zwingend wie die alte Auffassung einer a priori dem Raume innewohnenden und von der Materie unabhängigen metrischen Struktur zur Helmholtz-Lieschen Kennzeichnung der Drehungsgruppe führt, läßt die moderne, von der Einsteinschen Relativitätstheorie ausgebildeten Auffassung, nach welcher die Maßstruktur veränderungsfähig ist, und in kausaler Abhängigkeit von der Materie steht, eine andere Eigenschaft der Drehungsgruppe als die entscheidende erkennen ... (Weyl 1923*c*, 348)
- (v) Unsere Partie auf dem Schachbrett des Matrizenschemas ist zu Ende gespielt. (ibid., 372)
- (vi) Cela posé, la signification du premier axiome de M. H. Weyl est la suivante. Choisissons arbitrairement en chaque point P de l'espace l'*orientation* du groupe G des rotations, c'est à dire les coefficients a_{ij} ... [which describe the operation of the group, ES]. Un tel choix étant fait, la connexion métrique de l'espace est déterminée: l'une des correspondances par congruences ... est celle pour laquelle les deux vecteurs correspondants ont *mêmes* composantes ... (Cartan 1923*b*, 172)
- (vii) *Il n'y a donc que le groupe linéaire d'une forme quadratique réelle non dégénérée qui satisfasse aux conditions posées par les axiom I et II*

de *M. H. Weyl* [here abbreviated as PA and PB, ES]. Et c'est ainsi qu'est démontrée, en partant de ces axiomes, la nécessité de la forme pythagorienne de la métrique d'Univers. (Cartan 1923*b*, 192, emphasis in original)

- (viii) ... si un espace de Riemann ne possède pas une homogénéité absolue, il possède cependant une sorte d'homogénéité infinitésimale; au voisinage immédiat d'un point donné il est assimilable à une espace euclidien. (Cartan 1924*a*, 85)

- (ix) Je ne puis enfin terminer sans signaler les remarquables recherches dans lesquelles M.H. Weyl a repris l'ancien problème philosophique de l'espace, traité autrefois par Helmholtz et Lie, pour adapter aux points de vue nouveaux introduits par la théorie de relativité; la notion de groupe est, là encore, à la base de l'énoncé même du problème posé par M.H. Weyl. Mais je ne puis songer à entrer dans l'exposition, même sommaire, de cette importante question, qui exigerait à elle seule une conférence spéciale. (Cartan 1924*a*, 93f.)

- (x) Wir fragen nun, für welche abgeschlossenen Untergruppen G von $GL(n, \mathbb{R})$ es in einem G -Bündel E_G stets einen kanonischen, d.h. durch die Struktur von E_G irgendwie ausgezeichneten, linearen Zusammenhang gibt; ... (Klingenberg 1956, 300).

- (xi) Or, c'est le développement même de la thorie de la relativité, lié par l'obligation paradoxale d'interpréter dans et par un Univers non homogène les résultats de nombreuses expériences faites par des observateurs qui croyaient à l'homogénéité de cet Univers, qui permet de combler en partie le fossé qui séparait les espaces de Riemann de l'espace euclidien. (Cartan 1924*a*, 86)

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⁹¹Wrongly dated "22. Mai 1866", correct May 22, 1868, see (Volkert 1993)

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