Groups and manifolds : the variety and the unity of Weyl's space problems 1913-1927

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Introduction

An overestimation of WEYL's work around 1920

Hermann WEYL is well known for his contributions in general relativity (1916-1923),

- WEYL 's purely infinitesimal geometry (as a generalization of Riemannian geometry) and his unified field theory,
- formulation and resolution of his space problem (characterization of « pythagorean manifolds »).

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The secondary literature on WEYL is mostly devoted to his work in general relativity and on the foundations of mathematics. Two biases :

- restriction to a very short period (around 1920),
- a wide part of his scientific production is neglected.

Some exceptions

- WEYL's early work at Göttingen (HAWKINS, ECKES),
- A complex connection with the scientific / philosophical « milieu » at the ETH during a longer period (STAMMBACH, FREI, SCHOLZ, SIEROKA),
- his work on Lie groups and Lie algebras, including his lecture courses at the Institute for Advanced Study in 1934-1935, (HAWKINS, ECKES),
- his work in quantum mechanics (SCHOLZ, SCHNEIDER).

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But still very few historical studies on WEYL's productions in analytic number theory. No reference to his famous lectures in algebraic number theory (1938-1939), although WEYL's great friend Erich HECKE is specialized in algebraic number theory.

Some issues

First issue : how to do more than proposing a nth philosophical interpretation of WEYL's space problem, as if it was a self-sufficient theme in his work ?

- A change of historical scale is required.
- We also need to look at possible connections between WEYL's work on his space problem and other parts of his scientific production.
  - Example in another topic : WEYL's early investigations on integral equations (1908-1912) and on Riemann surfaces (1911-1913) play a determining role in his later research on Lie groups (1925-1927).

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Second issue : Is there any common denominator between his productions on Riemann surfaces, on the space problem and on Lie groups ?

A possible answer

All these works are various expressions of a unique project which consists in studying the interconnections between GROUP THEORY and RIEMANN'S GEOMETRIC IDEAS.

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By RIEMANN's *geometric ideas*, WEYL means his contributions in complex analysis (RIEMANN, 1851, 1859) and his *Habilitationsvortrag* (1854, first publication in 1867, new publication by WEYL himself in 1919, 1921 and 1923).

WEYL's 1925 text

WEYL's paper on RIEMANN's geometric ideas was initially written for an edition of Lobachevsky's work by the Russian government (1925) (WEYL's text is published for itself in 1988). It is devided into two main parts :

(i) The continuum, with a clear reference to RIEMANN's and KLEIN's work on Riemann surfaces (roughly speaking uniformization problem and covering space theory),

(ii) the structure, (= metric, conform, affin or projective structure on a manifold), with a clear reference to his group theoretical space problem and to CARTAN.

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Two aspects of WEYL's production are related in this text : his work on Riemann surfaces and his work on the space problem.

WEYL's tribute to KLEIN (1929)

Before replacing HILBERT at the university of Göttingen in 1930, WEYL gives a famous address on the occasion of the inauguration of the Mathematics Institute at Göttingen (3. December 1929). This talk is merely a tribute to Felix KLEIN.

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In this talk, WEYL aims at describing KLEIN's contributions in pure mathematics and his underlying conception of the unity of mathematics.

Weyl's tribute to Klein (1929)

Before replacing Hilbert at the university of Göttingen in 1930, Weyl gives a famous address on the occasion of the inauguration of the Mathematics Institute at Göttingen (3. December 1929). This talk is merely a tribute to Felix Klein.

In this talk, Weyl aims at describing Klein's contributions in pure mathematics and his underlying conception of the unity of mathematics.

According to Weyl,
(1) Klein's work in (pure) mathematics is mainly characterized by the combination of separated « branches », distinct theories and different methods

WEYL's tribute to KLEIN (1929)

(2) KLEIN's way of unifying mathematics is based on two ingredients : (1) intuition and (2) group theory. (1) « Das Hauptorgan von KLEINs mathematischer Methodik war das *intuitive, die Zusammenhänge erschauende Verstehen* ». (cf. also WEYL's late writing on constructive and axiomatic procedures in mathematics (after 1953) : « The chief organ of KLEIN's own productivity was this intuitive perception of interconnections and relations between separate fields ».)

(3) In his mathematical work, KLEIN underlines several interconnections between *group theory* and RIEMANN's *geometric ideas* (particularly in *analysis situs* and in complex analysis). WEYL : « "*Verbindung von Galois und Riemann*" hieß die Parole ».

Several interconnections between groups and manifolds

From a methodoligical point of view, Catherine GOLDSTEIN's paper entitled « Local et global » (in *Dictionnaire d'histoire et de philosophie des sciences*, 1999) is a key reference : C. GOLDSTEIN describes various links between the words « local » and « global » mostly during the XIXth and the XXth century by referring to several mathematical domains (complex analysis, number theory, differential geometry).

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Accordingly, we would like to describe various links between group theory and Riemann's geometric ideas in WEYL's work, without limiting our investigations to his productions in differential geometry. We also aim at studying WEYL's underlying conception of the unity of mathematics.

Several interconnections between Groups and manifolds

First part : (discrete) groups and manifolds (1913-1916)
(a) in his book *Die Idee der Riemannschen Fläche* and in a 1916 article, WEYL classifies the different (unramified) covering surfaces of a Riemann surface (analytic MANIFOLD of one complex dimension) by referring to their associated DECK TRANSFORMATION GROUPS.

- this first connection between groups and manifolds gives rise to a complex analogy between Galois theory, class field theory and the theory of (unramified) covering surfaces of a Riemann surface.

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- this first connection between groups and manifolds gives rise to a complex analogy between Galois theory, class field theory and the theory of (unramified) covering surfaces of a Riemann surface.

(b) At the end of his famous 1916 article in analytic number theory, WEYL aims at solving a part of the so-called CLIFFORD-KLEIN space problem (determination of all COMPACT RIEMANNIAN MANIFOLDS OF CONSTANT CURVATURE). We will see that WEYL refers to CRYSTALLOGRAPHIC GROUPS as a TOOL in the construction of compact euclidean manifolds.

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WEYL's space problem consists in characterizing « pythagorean manifolds » in a broader sense. WEYL doesn't limit his investigations to Riemannian manifolds. In particular, he extends the expression « pythagorean manifolds » to the class of manifolds he introduced in 1918 as a geometric framework for his unified field theory.

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WEYL solves this problem by REWRITING IT in a group theoretical language (theory of linear Lie groups and their corresponding Lie algebras).

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A great part of his reasoning is based on the consideration of compact Lie groups and their associated universal covering group (topological and integral viewpoint).

In other words, WEYL extends to (compact) Lie groups a series of theoretical developments he had initially applied to Riemann surfaces.

- Here, RIEMANN's geometric ideas PLAY A GUIDING ROLE in the study of Lie groups.

(discrete) groups and manifolds (1913-1916)

A first contact with RIEMANN's work

Die Idee der Riemannschen Fläche (1st ed. 1913) is the published version of an *advanced lecture course* given by WEYL during the winter-semester 1911-1912 at Göttingen.

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Concrete Riemann surfaces (consisting of several sheets glued together over the complex plane $\mathbb{C}$ or over the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$) were originary introduced by RIEMANN in his *Dissertation* (1851) in order to study algebraic functions of one complex variable, i.e. implicit functions obtained by solving an algebraic equation of the form $P(x, y) = 0$, where $P \in \mathbb{C}[X, Y]$ is an irreducible polynomial.

- Algebraic equations such as $y^2 - x = 0$ define multivalued functions. For instance, there exist two determinations of the algebraic function $\sqrt{z}$ over $\mathbb{C} \setminus [0, +\infty[$: $f_1(z) = \sqrt{r}e^{i\theta/2}$ and $f_2(z) = -\sqrt{r}e^{i\theta/2}$, where $z = re^{i\theta}$, with $r > 0$ and $\theta \in]0, 2\pi[$.

A first contact with RIEMANN's work

To avoid this ambiguity, we can replace $\mathbb{C} \setminus [0, +\infty[$ by a surface Σ which consists of two copies S_1 and S_2 of $\mathbb{C} \setminus [0, +\infty[$, glued together along their cut (the positive real axis). These procedures lead to a well defined function (i.e. a one-valued function) over Σ.

- The Riemann surface $\widehat{\Sigma}$ of $\sqrt{z}$ is obtained by replacing $\mathbb{C}$ by $\widehat{\mathbb{C}}$ in the preceding construction.
- The Riemann surface of an algebraic function is a compact analytic manifold.

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(Compact) Riemann surfaces can also be considered for themselves (abstract viewpoint). From this perspective, compact Riemann surfaces lie at the FOUNDATION of the theory of algebraic functions (of one complex variable).

A first contact with Riemann's work

This abstract (and geometric) viewpoint is fully developed by Felix Klein in his 1881-1882 lectures entitled *Riemanns Theorie der algebraischen Funktionen und ihrer Integrale*. Weyl is fully in accordance with this line of thought. Precisely, his 1913 book is dedicated to Klein. Nevertheless, Weyl avoids Klein's physical approach, based on the consideration of conductive surfaces. In the first part of his book (concept of a Riemann surface),

(i) Weyl develops an « arithmetized » construction of a Riemann surface, based on Weierstrass' analytic continuation process (local point of view) [§§ 1-3].

(ii) He gives an axiomatized definition of a two-dimensional topological manifold and of a Riemann surface, in a hilbertian vein [§§ 4-7].

Weyl makes an explicit connection between Riemann's *Disseration* (1851) and Riemann's *Habilitationsvotrag* (1854).

A first contact with RIEMANN's work

WEYL (1913 / 1955) [§ 4] : « the concept "two-dimensional manifold" or "surface" will not be associated with the points in three-dimensional space ; rather it will be a much more general abstract idea. If any set of objects (which will play the role of points) is given and a continuous coherence between them, similar to that in the plane, is defined, then we shall speak of a two-dimensional manifold. Since all ideas of continuity may be reduced to the concept of neighborhood, two things are necessary to specify a two dimensional manifold :

- to state what entities are the "points" of the manifold ;
- to define the concept of "neighborhood ».

In other words, an (abstract) two dimensional topological manifold is locally homeomorphic to $\mathbb{R}^2$ (the HAUSDORFF condition will be added later).

A first contact with Riemann's work

Weyl (1913 / 1955) [§ 7 / § 6] : « One can talk of a continuous function on a surface for which only analysis-situs properties are considered ; but one cannot talk of "continuously differentiable," "analytic" (or even "entire rational"), and such like functions. In order to pursue analytic function theory on a surface $\mathfrak{F}$, in a fashion analogous to function theory in the plane, an explanation (besides the definition of the surface) must be given to specify the meaning of the term "analytic function on the surface" ; this specification must be such that all theorems which are valid "in the small" for analytic functions in the plane carry over to this more general situation ».

- A Riemann surface is an analytic manifold of one complex variable (conformal structure).

A first contact with RIEMANN's work

In his 1913 book, WEYL axiomatizes the concept of a (topological / analytic) manifold, clarifying the meaning of the expression « continuous manifold » as it was used by RIEMANN in his *Habilitationsvortrag*.

- In particular, WEYL makes a clear distinction between a topoligical level and an analytic level in his definitions.

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Later (in 1918-1919), WEYL reads RIEMANN's *Habilitationsvortrag* from a different perspective (critical analysis of RIEMANN's presuppositions concerning the metrical determinations on a continuous manifold).

A connection with Galois theory

In §§ 8-9 of his book, Weyl classifies the (unrafimied) covering surfaces of a Riemann surface. This classification plays a central role in the proof of the Koebe-Poincaré uniformization theorem according to which each simply connected Riemann surface is conformally equivalent to the unit disc, the complex plane or the Riemann sphere.

- A covering of a Riemann surface $\mathfrak{F}$ is defined by the data of a (Riemann) surface $\overline{\mathfrak{F}}$ together with a continuous application $p : \overline{\mathfrak{F}} \to \mathfrak{F}$ satisfying some local trivialization property.

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WEYL introduces the group of automorphisms $\mathrm{Aut}(\overline{\mathfrak{F}})$ of $\overline{\mathfrak{F}}$ *over* $\mathfrak{F}$. The covering $p : \overline{\mathfrak{F}} \to \mathfrak{F}$ is said to be *regular* if $\mathrm{Aut}(\overline{\mathfrak{F}})$ acts transitively on each fiber $p^{-1}(x)$, $x \in \mathfrak{F}$.

A connection with Galois theory

Under this assumption, the group $\mathrm{Aut}(\overline{\mathfrak{F}})$ plays exactly the same role as the galois group $\mathrm{Gal}(K|k)$, where $k \subset \mathbb{C}$ is a number field, K a normal extension of k and $\mathrm{Gal}(K|k)$ the group of k-automorphisms of K.

- Weyl shows in 1916 that the theory of covering surfaces (of Riemann surfaces) and Galois theory have exactly the same structure. This connection is based on the concept of automorphism group.

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Weyl, « Strenge Begründung der Charakteristikentheorie auf zweiseitigen Flächen » : « Eine *reguläre* (unverzweigte, unbegrenzte) Überlagerungsfläche $\overline{\mathfrak{F}}$ über $\mathfrak{F}$ kann man gemäß der in der Zahlkörpertheorie üblichen Terminologie als eine Galoissche Überlagerungsfläche (relativ zu $\mathfrak{F}$) bezeichnen und die Gruppe der Decktransformationen als die Galoissche Gruppe dieser Fläche (relativ zu $\mathfrak{F}$) ».

And with class field theory

Then, WEYL considers the abelian coverings of a compact Riemann surface (i.e. for which $\mathrm{Aut}(\overline{\mathfrak{F}})$ is abelian).

(i) He proves an analogue of HILBERT's conjecture known as the *Hauptidealsatz* (first proof by FURTWÄNGLER in 1929).

- principal ideal theorem : let K be a number field, L its class field, i.e. the maximal unramified — at the finite and at the infinite places of K — abelian extension of K, for any ideal $\mathfrak{i}$ of $\mathcal{O}_K$, the ideal $\mathfrak{i}\mathcal{O}_L$ is principal.

(iii) At the end of his paper, WEYL uses this analogy in a reverse sense : the theory of covering surfaces may be a source of inspiration in the development of a class-field theory in the non-abelian case.

Galois and Riemann

To sum up,

- In his book on Riemann surfaces, WEYL gives formalized definitions of a topological manifold and of an analytic manifold. Consequently, he makes a clear connection between RIEMANN's *Dissertation* and RIEMANN's *Habilitationsvortrag*.

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To sum up,

- In his book on Riemann surfaces, WEYL gives formalized definitions of a topological manifold and of an analytic manifold. Consequently, he makes a clear connection between RIEMANN's *Dissertation* and RIEMANN's *Habilitationsvortrag*.
- Moreover, WEYL shows the existence of a very strong link between the theory of (unramified) coverings of a Riemann surface, Galois theory and Class field theory. This « Verbindung von Galois und Riemann » (KLEIN's motto) is based on the concept of AUTOMORPHISM GROUP.

WEYL's 1916 article in analytic number theory

WEYL publishes in 1916 a very famous article in analytic number theory entitled « Über die Gleichverteilung von Zahlen modulo Eins ». WEYL finds an analytic criterion of equidistribution modulo 1 (on $\mathbb{R}/\mathbb{Z}$, i.e. the circle with circumference one) for a sequence of real numbers. « Modulo one », means that any two real numbers with the same fractional part may be identified.

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A sequence $(u_n)_{n\in\mathbb{N}}$ is said to be uniformly distributed on $[0,1]$ if for any subinterval $[a,b]\subset[0,1]$ we have

$$\lim_{N\to\infty}\frac{\#\{u_n\in[a,b] \mid n\leq N\}}{N}=b-a.$$

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WEYL's criterion : a sequence $(u_n)_{n\in\mathbb{N}}$ of real numbers is equi-distributed modulo 1, if, and only if, for each non-zero integer p

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}e^{2i\pi p u_k}=0.$$

WEYL's 1916 article in analytic number theory

He then generalizes his reasoning to the *n*-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

- The set $\mathbb{Z}^n$ is a lattice in $\mathbb{R}^n$, i.e. a discrete SUBGROUP of $\mathbb{R}^n$ (for addition), which spans $\mathbb{R}^n$.

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He uses this opportunity to pay attention to COMPACT sub-manifolds of $\mathbb{R}^n$. As underlined by R. CHORLAY, the appendix of this article is devoted to compact connected euclidean submanifolds of $\mathbb{R}^n$, i.e. compact connected submanifolds of $\mathbb{R}^n$ with constant (sectional) curvature zero.

WEYL's 1916 article in analytic number theory

In fact, WEYL solves a specific aspect of the « Clifford-Klein space form problem » (classification of complete connected compact Riemannian manifolds with constant sectional curvature), first studied by CLIFFORD (1873), KLEIN (1890) and KILLING (1891).

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Inspired by BIEBERBACH's last achievements on crystallographic groups (1911, 1912), WEYL sketches a proof of the following statement : every compact euclidean Clifford-Klein form is the quotient of $\mathbb{R}^n$ by a (torsion-free) crystallographic group Γ (i.e. a Bieberbach group).

- A crystallographic group $\Gamma \subset \mathrm{Iso}(\mathbb{R}^n)$ is a discret, co-compact group (i.e. $\mathbb{R}^n/\Gamma$ is compact).
- A Bieberbach group is a *torsion-free* crystallographic group, i.e. $\mathrm{id} \in \Gamma$ is the only element of finite order.

WEYL's 1916 article in analytic number theory

This example shows how WEYL combines different mathematical domains : analytic number theory, Riemannian geometry, study of crystallographic groups, theory of covering spaces, etc. This way of reasoning reminds us of KLEIN's holistic view on mathematical knowledge. Moreover, the appendix shows that WEYL is interested in the CLIFFORD-KLEIN space problem.

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To sum up,

- WEYL solves part of a very famous space problem *before* his involvement in the mathematization of general relativity,
- He shows anew his attachment to KLEIN's work,
- He underlines a strong connection between a class of Riemannian MANIFOLDS and crystallographic GROUPS (as a TOOL in the CONSTRUCTION and in the CLASSIFICATION of these manifolds).

Lie groups and « pythagorean manifolds » (1921-1923)

Which periodization for the problem of space ?

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WEYL's texts on this topic are mainly published within a very short period (1921-1923) :

(a) *Raum, Zeit, Materie* (4th ed.) formulation and partial resolution of his *Raumproblem*,
(b) « Das Raumproblem » (1921, Jahresbericht), trans. into French R. CHORLAY,
(c) « Die Einzigartigkeit der Pythagoreischen Maßbestimmung » (*Mathematische Zeitschrift*, 1922), complete solution,
(d) « Zur Charakterisierung der Drehungsgruppe » (characterization of $\mathrm{SO}(p,q,\mathbb{R})$ with $n = p + q$ and its Lie algebra,
(e) *Die mathematische Analyse des Raumproblems* (1923) (trans. into French R. ANDUREAU and J. BERNARD).

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In fact, WEYL's *Raumproblem* is related to

- his pure infinitesimal geometry (1918). WEYL's *Raumproblem* can be viewed as a mathematical justification of his generalization of Riemannian geometry,
- the three editions of RIEMANN's *Habilitationsvortrag* (1919-1923) (formulation of a first version of WEYL's *Raumproblem*).

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- the three editions of RIEMANN's *Habilitationsvortrag* (1919-1923) (formulation of a first version of WEYL's *Raumproblem*).

Two consequences : (a) WEYL's *Raumproblem* can't be treated independently from his initial project of a *unified field theory* (he admits in 1921 the physical inconsequences of this theory). (b) there exist two versions of WEYL's *Raumproblem* :

(i) characterization of Riemannian manifolds within Finsler manifolds,

(ii) characterization of « pythagorean manifolds ».

Three roots of Weyl's *Raumproblem*

(i) The Riemann-Helmholtz-Lie space problem (1854 / 1867, 1868, 1893) which is based on the condition of free mobility of rigid bodies. This problem leads to a group-theoretical characterization of Euclidean, hyperbolic and elliptic geometry.
   - The Clifford-Klein space problem consists in classifying the different topological space forms in each case (Euclidean, hyperbolic, elliptic).
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(iii) WEYL's pure infinitesimal geometry and its mathematical justification (answer to an objection by EINSTEIN).

Weyl's pure infinitesimal geometry (1918)

Riemannian geometry is not a pure infinitesimal geometry for the following reason : « in the Riemannian geometry (...) there is contained a residual element of rigid geometry — with no goof reason, as far as I can see ; it is due only to the accidental development of Riemannian geometry from Euclidean geometry. The metric (...) allows the magnitudes of two vectors to be compared, not only at the same point, but at any two arbitrarily separated points ».

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A Riemannian manifold M is a smooth manifold endowed with a Riemannian metric g that is, roughly speaking, a family of positive definite symmetric bilinear forms (inner product) on each tangent space T_pM. Let $p \in M$ and $v_p \in T_pM$. The MAGNITUDE of v_p is not modified by a parallel transport along a loop with basepoint p. However, in a Riemannian manifold, the DIRECTION of a vector changes under parallel transport.

Weyl's pure infinitesimal geometry

WEYL defines a general class of manifolds (now called weylian manifolds) which generalizes Riemannian manifolds.

First step : He considers a smooth four-dimensional manifold M endowed with a conformal structure ; i.e. the metric in M is given up to *conformal* equivalence.

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More precisely, two Riemannian or Lorentzian metrics (g_{ij}) and (g'_{ij}) are conformally equivalent if there exists a strictly positive real valued function $\lambda(x)$ such that

$$g'_{ij}(x) = \lambda(x) \cdot g_{ij}(x), \quad \lambda(x) > 0,$$

$\lambda(x)$ is called a *calibration* or a *gauge function*. A conformal class of Riemannian or Lorentzian metrics will be denoted by $[g]$. WEYL defines this conformal structure « because *no natural unit* should be assumed in geometry a priori » (E. SCHOLZ).

Weyl's pure infinitesimal geometry

Second step : The manifold M must be endowed with a scale connection which allows parallel transport on M and comparison of quantities at different points on M. This metric connection is defined by a differential form φ which depend on the choice of the gauge (i.e. the choice of the metric in $[g]$).

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To sum up, a Weylian metric on a smooth manifold M is given by a pair (g, φ), modulo the equivalence relation

$$g'_{ij} = \lambda(x) \cdot g_{ij} \quad \varphi' = \varphi - \frac{\mathrm{d}\lambda}{\lambda} = \varphi - \mathrm{d}\log\varphi$$

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Weyl, RZM, (1919) : « The metrical character of a manifold is characterized relatively to a system of reference (co-ordinate system + calibration) by two fundamental forms, namely a quadratic differential form $Q = \sum_{ik} g_{ik}\mathrm{d}x_i\mathrm{d}x_k$ and a linear one $d\varphi = \sum_i \varphi_i \mathrm{d}x_i$ ».

Connection with WEYL's space problem

Many scientific and epistemological objections will be opposed to WEYL's related unified field theory (cf. EINSTEIN, REICHENBACH, HILBERT, PAULI, SOMMERFELD, etc.). WEYL will progressively abandon this project :

- In the late summer of 1920, WEYL recognizes that the structure of matter can't be described within a classical field theory,
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But from a mathematical standpoint, he remains convinced that his geometry has « important conceptual advantages in comparison with more special (Riemannian) or other (Finsler) infinitesimal geometries » (E. SCHOLZ, 2004). With his *Raumproblem*, WEYL answers EINSTEIN's objection according to which his pure infinitesimal geometry is an *arbitrary* generalization of Riemannian geometry.

Connection with WEYL's space problem

EINSTEIN's objection [EINSTEIN to WEYL, 31. 6. 1918], sumed up by E. SCHOLZ (2005) : « If Weyl proposed to weaken the possibility of direct comparison of lengths to vectors at the same point of the manifold, allowing for changes in transfer from one point to another, some "Weyl II" might come along some day and propose another weakening of classical Riemannian geometry with the same justification as Weyl I : One ought to accept only angle comparisons at the same point, while the parallel transfer of vectors might change the angles ».

- In other words, how to justify WEYL's generalization of Riemannian geometry ? Why does WEYL allow direct comparison of angles in his *pure* infinitesimal geometry ?

the problem(s) of space

WEYL's problems of space (first and second version) are more general than the RIEMANN-HELMHOLTZ problem : WEYL aims at characterizing *entire classes of metrical manifolds which are not a priori supposed to be homogenous*.

- "Reformulation" of RIEMANN-HELMHOLTZ problem in the framework of general relativity.

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- "Reformulation" of RIEMANN-HELMHOLTZ problem in the framework of general relativity.

WEYL's space problem (first version, 1919) $=$ "[mathematical] ground of RIEMANN's assumption that the metric of a manifold is given by a quadratic differential form, as opposed to the other possibilities afforded by the wider class of Finsler metrics ».

- A Finsler metric is given by the data of a function f of the coordinates differentials $\mathrm{d}x_i$ satisfying

$$f(\lambda \mathrm{d}x_1, \ldots, \mathrm{d}x_n) = |\lambda| f(\mathrm{d}x_1, \ldots, \mathrm{d}x_n), \quad f \geqslant 0.$$

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WEYL's conjecture : « [Riemannian metrics are] singled out, among all possible classes of Finsler's metrics, by the condition that to any such metric there exists a uniquely determined "affine connection" ».

- cf. LEVI-CIVITA's theorem, 1917 : « each Riemannian manifold admits exactly one parallel displacement compatible with the metric, i.e. preserving lengths of vectors and angles between them ».

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WEYL's space problem (second version, 1921) = characterization of "pythagorean manifolds" in a generalized sense

- metric structure defined by a family of non-degenerate quadratic forms (not necessarily positive definite),
- extension to Weylian manifolds.

WEYL's conjecture : there exists a unique affine connection compatible with their metric structure.

History of this distinction

(i) Erhard Scheibe, doctoral thesis (defended in 1955) : *Über das Weylsche Raumproblem* (Journal für die reine und angewandte Mathematik).

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(iv) Careful historical analysis of these two problems by E. SCHOLZ in many publications.

the problem of space

WEYL interprets and his solves the second problem in the framework of (linear) Lie groups and Lie algebras. Let M be a smooth n-dimensional manifold. Roughly speaking

(i) He formulates a series of postulates for the metrical structure on M — in particular, the existence of a uniquely determined affine connection compatible with its metrical structure.

(ii) he interprets them as conditions on the Lie algebras of the groups of congruences G_p acting on T_pM (these Lie algebras are isomorphic, so they can be denoted by the single letter $\mathfrak{g}$),

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(ii) he interprets them as conditions on the Lie algebras of the groups of congruences G_p acting on T_pM (these Lie algebras are isomorphic, so they can be denoted by the single letter $\mathfrak{g}$),

   (a) the trace of each matrix is $=0$, i.e. $\mathfrak{g} \subset \mathfrak{sl}(n,\mathbb{R})$,

   (b) each non-zero matrix must satisfy some asymmetry condition,

   (c) the dimension of $\mathfrak{g}$ is $N = \frac{n(n-1)}{2}$.

- he proves that the only Lie algebras corresponding to these conditions are the Lie algebras of the special orthogonal groups of arbitrary signature $\mathrm{SO}(p,q,\mathbb{R})$ with $(p+q=n)$.

the problem of space

In fact, he only proves this result for $n = 3$ in § 18 of RZM (4th ed.). A general proof can be found in « Die Einzigartigkeit der Pythagoreischen Maßbestimmung » (1922).

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Thanks to this mathematical result, he shows the following philosophical implication : the "nature" of the metric of the universe does not depend on experience, i.e. we can prove *a priori* that infinitesimally pythagorean manifolds admit a unique affine connection compatible with their metric structure.

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« es ist nicht richtig zu sagen, daß der Raum oder die Welt an sich, vor aller materiellen Erfüllung, lediglig eine formlose stetige Mannigfaltigkeit im Sinne der Analysis situs ist ; die *Natur* der Metrik ist ihm an sich eigentümlich, nur die gegenseitige Orientierung der Metriken in den verschiedenen Punkten ist zufällig, a posteriori und abhängig von der materiellen Erfüllung ».

Lie groups as manifolds (1925-1927)

WEYL's involvement in the theory of Lie groups

During the year 1924, WEYL works mainly in pure mathematics and he studies for itself the (representation) theory of Lie groups, Lie algebras. He is already well aware of some applications of this theory

- in the resolution of the space problem,
- in tensor calculus.

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WEYL has a direct contact with his two main sources of inspiration, namely Elie CARTAN and Issai SCHUR. WEYL's main goal consists in proving the complete reducibility theorem for semi-simple Lie algebras. He publishes a first (non-algebraic) proof of this theorem in his 1925-1926 article on Lie groups and Lie algebras (tr. into French and commentary by ECKES and THUILLIER).

Some mathematical definitions

A Lie algebra $\mathfrak{g}$ (at that time *infinitesimal group*) is a vector space over $\mathbb{R}$ or $\mathbb{C}$ together with a *bracket*, i.e. a bilinear map

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

which is *skew-symmetric* :

$$[X, Y] = -[Y, X]$$

and which satisfies the *Jacobi Identity*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

A Lie algebra is *semisimple* if it has no nonzero solvable ideals. A semisimple Lie algebra can be written as the direct sum of simple Lie algebras.

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Complete reducibility theorem : every finite dimensional representation of a (complex) semi-simple Lie algebra satisfies the complete reducibility property (it can be decomposed into a direct sum of irreducible subrepresentations).

Plan of WEYL's article

- FIRST PART : study of $\mathrm{SL}(n,\mathbb{C})$; proof of the complete reducibility theorem in this case ; theory of characters ; connection between the irreducible representations of $\mathrm{SL}(n,\mathbb{C})$ and the irreducible representations of the symmetric group.
- SECOND PART : study of $\mathrm{SO}(n)$, $\mathrm{O}(n)$ and $\mathrm{Sp}(2n,\mathbb{C})$.
- THIRD PART : the general theory of semi-simple Lie algebras.
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Goal of my commentary : to show that « Riemann's geometric ideas » play a central role in WEYL's proof of this theorem. WEYL extends to compact Lie groups the theory of covering spaces he initially studied in the framework of Riemann surfaces.

Polyphonic harmony between algebra and topology

WEYL studies the representations of (semi-simple) Lie algebras from two different perspectives : an algebraic one, represented by CARTAN and based on the study of "infinitesimal groups" (i.e. Lie algebras) ; an integral one, represented by HURWITZ (1897) and SCHUR (1924), which involves topological properties of some Lie groups (as « continuous » MANIFOLDS).

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WEYL doesn't give a purely algebraic proof of his complete reducibility theorem (such proofs will be found later by CASIMIR, VAN DER WAERDEN and BRAUER). His strategy consists in integrating over some compact Lie groups (WEYL also considers their universal covering group).

- WEYL's 1925-1926 article is a good example of his holism in pure mathematics, as inherited from KLEIN.

A « paradigmatic example » : $\mathrm{SL}(n,\mathbb{C})$

In order to show the role played by Riemann's geometric ideas in WEYL's reasoning, we can limit our commentary to the case of the special linear group $\mathrm{SL}(n,\mathbb{C})$.

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A « paradigmatic example » : $\mathrm{SL}(n,\mathbb{C})$

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The special linear group can be viewed as a *paradigmatic example* in this text : this *particular* case gives access to the *general theory* of (semi-simple) Lie algebras.

Again some definitions

(i) The *special linear group* $\mathrm{SL}(n,\mathbb{C})$ is the group of $n \times n$ invertible matrices with complex entries having determinant one.

(ii) Its *Lie algebra* $\mathfrak{sl}(n,\mathbb{C})$ is the space of all $n \times n$ complex matrices with trace zero.

(iii) The *special unitary group* $\mathrm{SU}(n)$ is the group of unitary matrices with determinant one.

- A unitary matrix A is a $n \times n$ complex matrix which satisfies the condition $A^* A = AA^* = I_n$, where I_n is the identity matrix and A^* is the conjugate transpose of A, i.e. $(A^*_{i,j}) = (\overline{A_{j,i}})$.

(iv) Its *Lie algebra* $\mathfrak{su}(n)$ is the space of all $n \times n$ skew-hermitian matrices with trace zero.

- A matrix A is skew-hermitian, if $(A^*_{i,j}) = -(\overline{A_{j,i}})$.

WEYL's unitarian trick

WEYL proves that the special linear group satisfies the complete reducibility property in I, § 5. *First issue* : $\mathrm{SL}(n,\mathbb{C})$ is not *compact*. WEYL restricts his investigations to $\mathrm{SU}(n) \subset \mathrm{SL}(n,\mathbb{C})$ which is compact. Consequently, it satisfies the complete reducibility property.

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Then, WEYL must prove the following implications

(i) If $\mathrm{SU}(n)$ has the complete reducibility property, then so does $\mathfrak{su}(n)$,

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(ii) If $\mathfrak{su}(n)$ has the complete reducibility property, then so does $\mathfrak{sl}(n,\mathbb{C})$.

He first proves the second implication = "WEYL's unitarian trick". He then realizes that the first implication is not self-evident : although a Lie group has the complete reducibility property, its Lie algebra need not have it.

WEYL's unitarian trick

Second issue. HAWKINS : "the irreducible representations of a Lie algebra $\mathfrak{g}$ associated to a given Lie group G do not necessarily correspond to representations of G but rather to its (...) universal covering group $\widetilde{G}$". WEYL introduces the concept of a universal covering group of a Lie group.

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Second issue. HAWKINS : "the irreducible representations of a Lie algebra $\mathfrak{g}$ associated to a given Lie group G do not necessarily correspond to representations of G but rather to its (...) universal covering group $\widetilde{G}$". WEYL introduces the concept of a universal covering group of a Lie group.

Third issue. The implication "if a Lie group has the compactness property, then so does its universal covering group" is false. Counter-example given by WEYL : the unit circle in the complex plane is compact, its covering group is an "infinite spiral" [WEYL's terminology] which is not compact.

WEYL's unitarian trick

The second and the third problems are crucial in the proof of the complete reducibility theorem for every semi-simple Lie algebra. But in the case of $\mathrm{SL}(n,\mathbb{C})$ (and its restriction to $\mathrm{SU}(n)$), they can be easily avoided because $\mathrm{SU}(n)$ is simply connected (for $n \geqslant 2$).

- However, WEYL doesn't begin with this result,
- He reports the proof of the simple connectivity of $\mathrm{SU}(n)$ (for $n \geqslant 2$) at the very end of his reasoning,
- Hence, there is a *dramatization* of WEYL's proof of the complete reducibility theorem in the case of $\mathrm{SL}(n,\mathbb{C})$: until the end of his proof, WEYL seems to ignore (deliberately) the simple connectivity of $\mathrm{SU}(n)$,
- With this dramatization, WEYL clarifies the main issues that will occur in the *general* case (treated in ch. III and IV).

To sum up

WEYL's initial problem consists in decomposing every (finite-dimensional) representation of a (semi-simple) Lie algebra into irreducible representations.

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To this end, he considers some suitable compact Lie group (as a manifold) and he uses a kind of reasoning which he had first developed in his 1913 book on Riemann surfaces.

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WEYL's initial problem consists in decomposing every (finite-dimensional) representation of a (semi-simple) Lie algebra into irreducible representations.

To this end, he considers some suitable compact Lie group (as a manifold) and he uses a kind of reasoning which he had first developed in his 1913 book on Riemann surfaces.

With this example,

- we can underline the interdependence between several parts of WEYL's mathematical production,
- we can show that Riemann's geometric ideas play a guiding role in WEYL's study of Lie groups and Lie algebras.

conclusion

The variety of « Raumprobleme »

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(iii) WEYL's first space problem (characterization of Riemannian manifolds among Finsler manifolds),

(iv) WEYL's second space problem (characterization of pythagorean manifolds and answer to EINSTEIN's mathematical objection against WEYL's pure infinitesimal geometry).

Which unity ?

COLEMAN-KORTÉ's answer : there is « one »
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My answer is twofold : (i) all these problems are mathematically and historically different but they involve mathematicians who consider RIEMANN's HABILITATIONSVOTRAG as a central source of inspiration. They try to clarify, to justify or to generalize different aspects of this text.

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My answer is twofold : (i) all these problems are mathematically and historically different but they involve mathematicians who consider RIEMANN's HABILITATIONSVOTRAG as a central source of inspiration. They try to clarify, to justify or to generalize different aspects of this text.

(ii) The resolution of these problems (or part of these problems) implies various connections between group theory and manifolds.

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More generally, these connections represent a leitmotiv in WEYL's scientific work and they reveal its unity.

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More generally, these connections represent a leitmotiv in WEYL's scientific work and they reveal its unity.

(a) the concept of automorphism group as a COMMON DENOMINATOR between several THEORIES (Riemann surfaces, Galois theory),

(b) crystallographic groups as an essential tool in the CONSTRUCTION of compact euclidean manifolds (part of the CLIFFORD-KLEIN space forms problem),

(c) INTERPRETATION and RESOLUTION of WEYL's second space problem in the framework of linear Lie groups and Lie algebras,

(d) Compact Lie groups as manifolds to which RIEMANN's geometric ideas can be EXTENDED (proof of the complete reducibility theorem for semi-simple Lie algebras).