PART V. THEORIES ON THE NATURE OF CHARGED ELEMENTARY PARTICLES

63. The electron and the special theory of relativity

For a long time endeavours have been made to derive all the mechanical properties of the electron from electromagnetic principles. In such a case the equation of motion

$$\frac{d\mathbf{G}}{dt} = \mathbf{K} \tag{463}$$

(where $\mathbf{G}$ = momentum of the electron, $\mathbf{K}$ = *external* force) is interpreted as follows[374]: It is postulated, first, that all forces acting on an electron are electromagnetic in origin, i.e. they are given by the Lorentz expression (212 a); and secondly, that the total force acting on an electron should always vanish,

$$\int \rho \left\{\mathbf{E} + \frac{1}{c}(\mathbf{v} \wedge \mathbf{H})\right\} dV = 0, \tag{464}$$

where the integration is to be carried out over the volume of the electron. This total force can be split into two parts. One is a force due to the external field

$$\int \rho \left\{\mathbf{E}^{\text{ext}} + \frac{1}{c}(\mathbf{v} \wedge \mathbf{H}^{\text{ext}})\right\} dV = \mathbf{K}$$

which is on the right-hand side of Eq. (463). The other is the force exerted by the electron on itself, which can be put equal to

$$\int \rho \left\{\mathbf{E}^{\text{int}} + \frac{1}{c}(\mathbf{v} \wedge \mathbf{H}^{\text{int}})\right\} dV = -\frac{d\mathbf{G}}{dt}.$$

In this expression, $\mathbf{G}$ is the electromagnetic momentum of the self-field of the electron. For accelerations which are not too large (quasi-stationary motion), it can be regarded as the momentum which corresponds to a uniform motion of the electron with the relevant instantaneous velocity. It depends, of course, on the charge distribution inside the electron.

The most natural assumption to make was that of a perfectly rigid electron. The theory for this case was worked out completely by Abraham.[375] In 1904, however, it was shown by H. A. Lorentz[376] that the deductions which can be made from the velocity dependence of the electron momentum will only be in agreement with the principle of relativity when it is assumed that a contraction of the electron takes place in the direction of motion, in the ratio $\sqrt{(1-\beta^2)}:1$. And Einstein[377]

[374] Cf. footnote 4, p. 1, H. A. Lorentz, *ibid.*, § 21.

[375] M. Abrâham, *Ann. Phys., Lpz.*, **10** (1903) 105. See also footnote 4, p. 1, H. A. Lorentz, *ibid.*, § 21.

[376] Cf. footnote 10, p. 2, *Versl. gewone Vergad. Akad. Amst., loc. cit.*

[377] Cf. footnote 12, p. 2, *ibid.*, § 10.

showed afterwards that the velocity dependence of the mass, energy and momentum follow from the principle of relativity alone, without the need for any assumptions about the nature of the electron (cf. § 29). Conversely, therefore, no information on the nature of the electron can be obtained from observations on the changes in mass.

It is however easy to see that the principle of relativity must necessarily lead to the existence of an energy of a non-electromagnetic kind for the electron, at least so long as the Maxwell–Lorentz theory is retained. This was first pointed out by Abraham.[378] Let us first assume that the charge distribution in a stationary electron is spherically symmetrical. We then obtain the energy and momentum of a moving electron from (351), in as far as they are electromagnetic in origin and given by the Maxwell–Lorentz expressions,

$$\mathbf{G} = \mathbf{u}\frac{\frac{4}{3}(E_0/c^2)}{\sqrt{(1-\beta^2)}}, \qquad E = \frac{E_0(1+\frac{1}{3}u^2/c^2)}{\sqrt{(1-\beta^2)}}. \tag{351}$$

If these expressions represented the total energy and momentum as well, we should have, from (317), (318),

$$E = \int\left(\mathbf{u}\cdot\frac{d\mathbf{G}}{dt}\right)dt = \int\left(\mathbf{u}\cdot\frac{d\mathbf{G}}{d\beta}\right)d\beta.$$

This is however not the case. Instead, the integral on the right-hand side has the value

$$\frac{\frac{4}{3}E_0}{\sqrt{(1-\beta^2)}} + \text{const.}$$

If, in contrast to the energy, the *momentum* is assumed to be purely electromagnetic, the total energies $\bar{E}_0$ and $\bar{E}$ of the stationary and moving electron, respectively, and its rest mass, become

$$\bar{E} = \frac{\bar{E}_0}{\sqrt{(1-\beta^2)}}, \qquad \bar{E}_0 = \tfrac{4}{3}E_0, \qquad m_0 = \frac{\bar{E}_0}{c^2} = \frac{4}{3}\frac{E_0}{c^2}. \tag{465}$$

The rest mass m_0 is here defined by

$$\mathbf{G} = \frac{m_0\,\mathbf{u}}{\sqrt{(1-\beta^2)}}.$$

These relations are in agreement with the theorem of the inertia of energy, as required (with the additive constant in $\bar{E}$ already fixed so as to fit in with this theorem). *The total energy of the stationary electron is equal to four-thirds of its electromagnetic energy as given by Lorentz.*

It would seem from the above arguments as if the rigid electron of the "absolute" theory were compatible with a purely electromagnetic world picture (or rather, with the particular electromagnetic world picture which is based on the Maxwell–Lorentz theory), in contrast to the electron

[378] M. Abraham, *Phys. Z.*, 5 (1904) 576: *Theorie der Elektrizität*, Vol. 2, (1st edn., Leipzig 1905), p. 205.

as demanded by the theory of relativity. But this is not correct, and for the following reason: The hypothesis of a rigid electron is a concept which is quite foreign to electrodynamics. If it had not been introduced, we would have had to postulate that not only the *total* force acting on the electron (Eq. (464)) should vanish, but even the force acting at each individual point,

$$\rho\left\{\mathbf{E}+\frac{1}{c}(\mathbf{v}\wedge\mathbf{H})\right\}=0.$$

It is clear that a stationary charge ($\mathbf{v}=0$) is incompatible with this postulate, for it follows that $\rho=0$ (note that div $\mathbf{E}=\rho$). We therefore see that *the Maxwell-Lorentz electrodynamics is quite incompatible with the existence of charges, unless it is supplemented by extraneous theoretical concepts*. From a purely electromagnetic point of view, therefore, the electron of the "absolute" theory is really no better then the electron in relativity. It will, in any case, be necessary to introduce forces which hold the Coulomb repulsive forces of the electron charge on itself in equilibrium, and such forces are not derivable from Maxwell–Lorentz electrodynamics. Poincaré[379] already recognized the need for this and purely formally introduced a scalar cohesive pressure p, on whose nature he could not make any statement. Generally speaking, the problem of the electron has to be formulated as follows: The energy–momentum tensor S_{ik} of Maxwell–Lorentz electrodynamics has to have terms added to it in such a way that the conservation laws

$$\frac{\partial T_i{}^k}{\partial x^k}=0 \tag{341}$$

for the total energy–momentum tensor become compatible with the existence of charges. These additional terms will certainly have to depend on physical quantities which are causally determined by differential equations. (In § 42 we had made the phenomenological "Ansatz" $\mu_0 u_i u_k$ for the energy tensor of an isolated electron.) How far this formulation will have to be modified from the point of view of the *general* theory of relativity, will be discussed in §§ 65 and 66.

We are now in a position to answer the question raised by Ehrenfest[380], whether a uniform translational motion in all directions, in the absence of forces, is possible for an electron which is not spherically symmetric, even when at rest. For in such a case the electromagnetic momentum of the moving electron will not always be in the direction of its velocity, so that the electromagnetic forces will exert a rotational couple on the electron. It was stressed by Laue[381] that the situation is quite analogous to that in Trouton and Noble's experiment. In this experiment, the electromagnetic couple is compensated by a couple generated by the elastic

[379] Cf. footnote 11, p. 2, *R.C. Circ. mat. Palermo.*, *loc. cit.*
[380] P. Ehrenfest, *Ann. Phys., Lpz.*, 23 (1907) 204; remarks on this by A. Einstein, *Ann. Phys., Lpz.*, 23 (1907) 206.
[381] Cf. footnote 240, p. 129, *ibid.*

energy current. In our case, the compensatory effect is similarly due to an energy current which is represented by the above-mentioned additional terms in the energy–momentum tensor. The introduction of such additional terms becomes necessary not just for the case of a moving electron, but even for that of a stationary one. Ehrenfest's question must therefore be answered in the affirmative.

We still have to discuss what can be said about the *dimensions* of the electron, both from the above theoretical and from the experimental points of view. *Experimentally* it is known with a high degree of probability that, in the last analysis, all matter consists of hydrogen nuclei and electrons. *Naturally, all our previous statements about the electron hold equally for the hydrogen nucleus.* Experiment has only told us about the dimensions of these particles that they are certainly not larger than 10^{-13} cm. In other words, two such particles will behave practically like point charges over this distance, as far as the forces which they exert on each other are concerned. The possibility that the dimensions of the particles could be very much smaller than 10^{-13} cm is not excluded by the experiments hitherto carried out. *Theoretically*, definite statements can only be made from Lorentz's point of view, as follows: A sphere of radius a and continuous surface charge distribution has an energy

$$E = \frac{e^2}{8\pi a},$$

where e is the total charge, measured in Heaviside units. It follows from (465) that

$$m_0 = \frac{e^2}{6\pi a c^2}, \qquad a = \frac{e^2}{6\pi m_0 c^2}. \tag{466}$$

If a different charge distribution were assumed, this would only modify the numerical coefficient, but not the order of magnitude of a. The value for a is obtained from the known rest masses of the electron and hydrogen nucleus. For the former it is of order 10^{-13} cm, for the latter about 1800 times smaller, corresponding to its greater mass. But it must be remarked that these considerations rest on very weak theoretical foundations. For we have seen that they are based on the following hypotheses:

(i) The charge distribution of a stationary electron (hydrogen nucleus) is spherically symmetric.

(ii) The total momentum of a moving electron (hydrogen nucleus) is given by the expression

$$\mathbf{G} = \frac{1}{c}\int \mathbf{E}\wedge\mathbf{H}\, dV$$

of the Maxwell–Lorentz theory; this is therefore assumed to be valid also for extreme concentrations of the charges and fields.

The second hypothesis, in particular, appears doubtful. From the experimental data available so far, no experimental support whatever can be found for the dimensions calculated in this way, in particular not for the theoretical requirement that the radius of the hydrogen nucleus should be very much smaller than that of the electron.[381a]

64. Mie's theory

The first attempt to set up a theory which could account for the existence of electrically charged elementary particles, was made by Mie[382]. He set himself the task to generalize the field equations and the energy–momentum tensor in the Maxwell–Lorentz theory in such a way that the Coulomb repulsive forces in the interior of the electrical elementary particles are held in equilibrium by other, *equally electrical*, forces, whereas the deviations from ordinary electrodynamics remain undetectable in regions outside the particles.

Mie retains the first set of Maxwell's equations

$$\frac{\partial F_{ik}}{\partial x^l}+\frac{\partial F_{li}}{\partial x^k}+\frac{\partial F_{kl}}{\partial x^i}=0, \tag{203}$$

from which the existence of a four-vector potential follows,

$$F_{ik}=\frac{\partial \phi_k}{\partial x^i}-\frac{\partial \phi_i}{\partial x^k} \tag{206}$$

(cf. § 28). Furthermore, the four-current density will certainly have to satisfy the continuity equation

$$\frac{\partial s^k}{\partial x^k}=0. \tag{197}$$

From this follows the existence of an antisymmetrical tensor of rank 2, $H^{ik}=-H^{ki}$, which satisfies the equation

$$s^i=\frac{\partial H^{ik}}{\partial x^k} \tag{467}$$

Here, H_{ik} combines the two vectors **D** and **H**, just as F_{ik} combines the vectors **E** and **B**. It is thus seen that, for $H^{ik}=F^{ik}$, the equations reduce to those of ordinary electrodynamics and that they agree formally with those of the phenomenological electrodynamics of ponderable bodies.

Now however these field equations acquire a new physical content by virtue of the following, decisive, assumption: *The vectors H^{ik} and s^k are to be universal functions of F_{ik} and ϕ_i,*

$$H^{ik}=u_{ik}(\mathsf{F},\varphi), \qquad s^k=v_k(\mathsf{F},\varphi). \tag{467a}$$

[381a] On this point, we cannot agree with the view presented in M. Born's book, *Die Relativitätstheorie Einsteins* (Berlin 1920), p. 192.

[382] G. Mie, 'Grundlagen einer Theorie der Materie', *Ann. Phys., Lpz.*, **37** (1912) 511; **39** (1912) 1; **40** (1913) 1. Cf. also the version by M. Born, *Nachr. Ges. Wiss. Göttingen*, (1914) 23, which brings out the analogy between the derivation of the energy–momentum law from the variational principle of Mie's theory, and the derivation of the energy conservation law from Hamilton's principle, in ordinary mechanics. See also H. Weyl, *Raum-Zeit-Materie* (1st edn., 1918) § 25, p. 165; (3rd edn., 1920) § 25, p. 175.

The first six relations differ inherently from those of phenomenological electrodynamics, in that H^{ik} also depends on ϕ_i explicitly. In Mie's theory, not only the potential difference, but also the absolute value of the potential acquires a real meaning. The equations do not remain unchanged if one replaces φ by $\varphi+const$. We shall see later that this feature of Mie's theory leads to a serious difficulty. The last four equations (467 a) are essential for the existence and for the equations of motion of the material particles (electron and hydrogen nucleus). Mie calls, more or less arbitrarily, ϕ_i and F_{ik} *"quantities of intensity"*, s^k and H^{ik} *"quantities of magnitude"*.

By means of (467a), no less than ten universal functions are introduced into the theory. As Mie found, however, the energy principle brings about a great simplification, reducing the ten unknown universal functions to a single one. For it is seen that an equation of the form

$$\frac{\partial W}{\partial t}+\operatorname{div}\mathbf{S}=0$$

$$(W=\text{energy density},\quad \mathbf{S}=\text{energy current})$$

can only be derived from (206) and (457), when an invariant $L(\mathsf{F},\varphi)$ (in the first place, relative to the Lorentz group) exists, from which H^{ik} and s^i can be obtained by differentiation,

$$H^{ik}=\frac{\partial L}{\partial F_{ik}},\qquad s^i=-\frac{1}{2}\frac{\partial L}{\partial \phi_i}, \tag{468}$$

so that

$$\delta L=H^{ik}\,\delta F_{ik}-2s^i\,\delta\phi_i. \tag{468a}$$

A simple calculation then shows that equations (467) follow from the action principle

$$\delta\int L\,d\Sigma=0, \tag{469}$$

provided the variation fulfils the condition that the relations (206) also remain valid for the field which is to be varied.

One can make a number of general statements about the invariant L, which is often called the world function. First of all, the only independent invariants which can be formed from the antisymmetric tensor F_{ik} and the vector ϕ_i, are the following:

(1) The square of the tensor F_{ik} : $\frac{1}{2}F_{ik}F^{ik}$.

(2) The square of the vector ϕ_i : $\phi_i\phi^i$.

(3) The square of the vector $F_{ik}\phi^k$: $F_{ir}\phi_s F^{is}\phi^r$.

(4) The square of the vector $F^*_{ik}\phi^k$ or, which amounts to the same, the square of the tensor $F_{ik}\phi_l+F_{li}\phi_k+F_{kl}\phi_i$.

L must therefore be a function of these four invariants.† If L becomes equal to the first of the above invariants, the field equations of Mie's theory

† See suppl. note 20.

degenerate into the ordinary equations of electron theory for charge-free space. *Thus, L can only be markedly different from $\frac{1}{2}F_{ik}F^{ik}$ in the interior of the material particles.* Nothing further can be said about the world function L. It is impossible to narrow down the number of alternatives to such an extent that one would be led unambiguously to a quite definite world function. Rather, there remains an infinite manifold of possibilities.

We next have to determine the energy–momentum tensor T_{ik} as a function of the field quantities. Hilbert[383] and Weyl[384] showed that the corresponding calculations can be simplified considerably by writing the equations in the notation of the general theory of relativity and then using the method for varying the g_{ik} which was introduced in § 55. Only then do the formal connections become clearly apparent. We prepared the ground for this here by our notation for the above formulae and we differ in this respect from Mie, who of course based his papers of 1912 and 1913 on the special theory of relativity. First of all, just as for the case of ordinary electrodynamics dealt with in § 54, the set of equations (203), (208) remains valid also for an arbitrary G-field. On the other hand, (197) and (467) have to be replaced by

$$\frac{\partial \mathfrak{s}^i}{\partial x^i} = 0 \tag{197a}$$

$$\mathfrak{s}^i = \frac{\partial \mathfrak{H}^{ik}}{\partial x^k}. \tag{467b}$$

As was noticed by Weyl, the "quantities of magnitude" appear here as tensor *densities* (i.e. multiplied by $\sqrt{(-g)}$), whereas the "quantities of intensity" remain ordinary tensors. Equally, the relations (468), (468a) and Hamilton's principle (469) remain valid, and the latter can of course also be written in the form

$$\delta \int \mathfrak{L}\, dx = 0. \tag{469a}$$

In order to find the energy tensor T_{ik} we need only determine the variation of the action function which is obtained by varying the G-field. Since, in this case, L is independent of the derivatives of the g_{ik}, the following relation holds if the electromagnetic field is kept constant,

$$\delta \mathfrak{L} = \mathfrak{T}_{ik}\, \delta g^{ik},$$

so that

$$T_{ik} = \frac{\partial L}{\partial g^{ik}} - \tfrac{1}{2} L g_{ik}, \qquad T_i{}^k = \frac{\partial L}{\partial g^{ir}} g^{rk} - \tfrac{1}{2} L \delta_i{}^k. \tag{470}$$

If, on the other hand, we substitute into the general expression

$$\delta L = \frac{\partial L}{\partial g^{ik}} \delta g^{ik} + H^{ik} \delta F_{ik} - 2 s^i \delta \phi_i,$$

[383] Cf. footnote 99, p. 63, *ibid.*, I.

[384] H. Weyl, *Raum-Zeit-Materie* (1st edn., 1918), p. 184 *et seq.*; (3rd edn., 1920), p. 199.

resulting from (468 a), a variation of the field quantities produced by an infinitesimal coordinate transformation (as given by (163), (164)), δL must vanish identically. In other words,

$$2\frac{\partial \xi^i}{\partial x^k}\left(\frac{\partial L}{\partial g^{ir}}g^{rk} - H^{kr}F_{ir} + s^k\phi_i\right) \equiv 0,$$

which is only possible if the bracket itself vanishes identically. Finally, we can take the value of $(\partial L/\partial g_{ir})g^{rk}$ from there and substitute it in (470). We then obtain

$$T_i^k = H^{kr}F_{ir} - s^k\phi_i - \tfrac{1}{2}L\,\delta_i^k. \tag{470a}$$

It is seen from their derivation that the corresponding covariant components T_{ik} are symmetrical. On the basis of the results of § 55 we can see, moreover, that the energy–momentum law is a consequence of the field equations. In the absence of gravitational fields it is of the form

$$\frac{\partial T_i^k}{\partial x^k} = 0, \tag{341}$$

and in the presence of gravitational fields,

$$\frac{\partial \mathfrak{T}_i^k}{\partial x^k} - \tfrac{1}{2}\mathfrak{T}^{rs}\frac{\partial g_{rs}}{\partial x^i} = 0. \tag{341a}$$

The expression (470 a) for the energy–momentum tensor is identical with that obtained by Mie by direct calculation.

Let us now consider once more the problem of the equation of motion, and whether it is possible for material particles to exist. In ordinary electrodynamics the electric field strength is defined as the force acting on a (stationary) charge. This simple meaning of the field strength no longer applies to the interior of a material particle in Mie's theory. Indeed, the ponderomotive force is everywhere equal to zero. Nevertheless, the practical meaning of the total charge of a particle still remains. For, let us consider a charged particle in an external field. In this case, it follows from (341) that

$$\frac{d}{dx^4}\int T_i^4\,dx^1\,dx^2\,dx^3 = -\int T_i^k n_k\,d\sigma. \qquad (i, k = 1, 2, 3)$$

(n_k = unit normal to the surface)

The second integral is taken over a surface sufficiently far removed from the material particle. Since ordinary electrodynamics is valid on this surface, the surface integral has the same value as there, i.e. it represents the Lorentz force. We have thus given an electrodynamical proof of the equation of motion (210) of an electron, within the framework of Mie's theory. At the same time, it is seen that the rest mass of the particle is determined by

$$m_0 = \frac{E_0}{c^2} = -\int T_4^4\,dx^1\,dx^2\,dx^3, \tag{471}$$

according to the theorem of the inertia of energy. For T_4^4 we have to substitute the expression obtained from (470).

Mie assumes the field of the stationary electron to be static and spherically symmetric. As was explained in the previous section, the latter assumption is admittedly not justified by our experimental knowledge alone, but recommends itself for its simplicity. We will then have to look for those solutions of the field equations which are regular everywhere—for $r = 0$ as well as for $r = \infty$. A world function which is to correspond to reality will have to lead to one, and one only, solution for every kind of electricity. *It has not been possible, so far, to find a world function which satisfies this condition.* On the contrary, the trial expressions for L which have so far been discussed, lead to the conclusion, in contradiction with experiment, that elementary particles with arbitrary values of their total charge can exist. This in itself is not sufficient reason for abandoning Mie's electrodynamics, since it has not been proved that a world function *cannot* exist which is compatible with the existence of *certain* elementary particles.

A much more serious difficulty, it seems to us, is caused by a fact already noticed by Mie. Once we have found a solution for the electrostatic potential φ of a material particle of the required kind, $\varphi + const.$ will not be another solution, because the field equations of Mie's theory contain the absolute value of the potential. *A material particle will therefore not be able to exist in a constant external potential field.* This, to us, seems to constitute a very weighty argument against Mie's theory. In the theories which we are going to discuss in the following sections, this kind of difficulty does not arise.

An attempt by Weyl should be mentioned here, in which he tries to make the asymmetry between the two kinds of electricity understandable from the point of view of Mie's theory.† If the world function L is not a rational function of $\sqrt{(-\phi_i\phi^i)}$, we can put

$$L = \tfrac{1}{2}F_{ik}F^{ik} + w(+\sqrt{(-\phi_i\phi^i)}) \qquad \text{for } \phi_i\phi^i < 0, \phi_4 > 0$$

and

$$L = \tfrac{1}{2}F_{ik}F^{ik} + w(-\sqrt{(-\phi_i\phi^i)}) \qquad \text{for } \phi_i\phi^i < 0, \phi_4 < 0$$

where w denotes any function which is not even. For the statical case the field equations will not remain invariant for an interchange of φ with $-\varphi$ (positive and negative electricity). Quite generally, if L is a many-valued function of the four above-mentioned fundamental invariants, it is possible to choose one of its branches as world function for positive electricity, and another for negative electricity. We shall come back to this possibility in § 67.

65. Weyl's theory‡

In a series of papers[385] Weyl has developed an extremely profound theory, which is based on a generalization of Riemannian geometry and

† See suppl. note 21.

‡ See suppl. note 22.

[385] H. Weyl, *S.B. preuss. Akad. Wiss.* (1918) 465; *Math. Z.* **2** (1918) 384; *Ann. Phys., Lpz.*, **59** (1919) 101; *Raum-Zeit-Materie*, (3rd edn., 1920), Chaps. II and IV, §§ 34 and 35, p. 242 *et seq.*; *Phys. Z.*, **21** (1920) 649.

which claims to interpret all physical events in terms of gravitation and electromagnetism, and these, in turn, in terms of the world metric. The theory's basic structure and results obtained to date will be discussed at *this* juncture, because the theory also makes certain statements on the nature of material particles.

(α) *Pure infinitesimal geometry. Gauge invariance.* The transition from Euclidean to Riemannian geometry was seen in Part II to be effected by no longer assuming that the transference of the ***direction*** of a vector from a point P to a point P' is independent of the path taken. Weyl takes this a step further by permitting a corresponding path dependence for the transference of length. It will then only be possible to compare lengths measured *at one and the same* world point, but not those at different world points. In other words, only the relations between the g_{ik} can be determined by measurements, but not these quantities themselves. Let us first assume some arbitrary (continuous) absolute values for the g_{ik} and define

$$ds^2 = g_{ik}\,dx^i\,dx^k$$

to be the square of the length of a measuring rod, where dx^i are the coordinate differences of its end points. (For brevity, we speak here, and in what follows, of the length of a measuring rod, but the same arguments apply of course to the period of a clock in the case of a time-like line element.) If now the measuring rod is displaced along a definite cu $x^i = x^i(t)$ from the point $P(t)$ to the point $P'(t+dt)$, the square of the length $ds^2 = l$, will then be changed. We shall assume axiomatically that it will always be changed by a definite fraction of l,

$$\frac{dl}{dt} = -l\frac{d\varphi}{dt} \tag{472}$$

where φ is a certain function of t and is independent of l. We introduce as a second axiom that $d\varphi/dt$ only depends on the first derivatives of the coordinates, dx^i/dt. Since, moreover, equation (472) has to hold for an arbitrary choice of the parameter t, *$d\varphi/dt$ must be a homogeneous function of first degree of the dx^i/dt.* This function can be specified still more by making use of the concept of a parallel displacement, which had been discussed in § 14. This concept had been defined there by two conditions: The first stated that the components of a vector would remain unchanged for an *infinitesimal* parallel displacement *in a suitably chosen coordinate system*, whereas the second expressed the fact that the length of a vector would remain unaltered during a parallel displacement. The first assumption can be retained unchanged; it leads to the expression (64) for the change in the vector components,

$$\frac{d\xi^i}{dt} = -\Gamma^i_{rs}\frac{dx^s}{dt}\xi^r \tag{64}$$

with

$$\Gamma^i_{rs} = \Gamma^i_{sr}. \tag{65}$$

The second assumption, however, loses its meaning here, because the lengths of two vectors at different points can no longer be compared. Instead, the assumption has to be replaced by the condition that the length should change according to (472),

$$\frac{d}{dt}(g_{ik}\xi^i\xi^k) = \frac{d}{dt}(\xi_i\xi^i) = -g_{ik}\xi^i\xi^k\frac{d\varphi}{dt}, \tag{473}$$

for a parallel displacement. Substituting (64), it first of all follows that $d\varphi/dt$ must be a linear form of the dx^i/dt,

$$d\varphi = \phi_i\,dx^i: \tag{474}$$

Only in this case, therefore, is a parallel displacement possible. Furthermore, we have from

$$\Gamma_{i,rs} = g_{ik}\Gamma^k{}_{rs}, \qquad \Gamma^i{}_{rs} = g^{ik}\Gamma_{k,rs}, \tag{66}$$

that

$$\frac{\partial g_{ir}}{\partial x^s} + g_{ir}\phi_s = \Gamma_{i,rs} + \Gamma_{r,is}. \tag{475}$$

The geodesic components of Weyl's geometry are thus different from those of Riemannian geometry. We shall always denote by an asterisk expressions in Riemannian geometry. They reduce to those of Weyl's geometry for the case $\phi_i = 0$. If therefore the quantities (69) are denoted by $\Gamma^*{}_{i,rs}$, then

$$\Gamma_{i,rs} = \Gamma^*{}_{i,rs} + \tfrac{1}{2}(g_{ir}\phi_s + g_{is}\phi_r - g_{rs}\phi_i). \tag{476}$$

We had fixed the absolute values of the g_{ik} quite arbitrarily. Instead of the set of values g_{ik} we could equally well have used the set λg_{ik}, where λ is an arbitrary function of position. All elements of length would in that case have to be multiplied by λ and, because of (472), we would have found the set of values

$$\phi_i - \frac{\partial \log\lambda}{\partial x^i} = \phi_i - \frac{1}{\lambda}\frac{\partial\lambda}{\partial x^i},$$

instead of ϕ_i. Fixing the factor λ, the gauge, in Weyl's geometry is thus on a par with the choice of coordinates in Riemannian geometry. *Just as there we had postulated the invariance of all geometrical relations and physical laws under arbitrary coordinate transformations, so we have now to postulate, in addition, their invariance with respect to the substitutions*

$$\bar{g}_{ik} = \lambda_{ik}, \qquad \bar{\phi}_i = \phi_i - \frac{1}{\lambda}\frac{\partial\lambda}{\partial x^i}, \tag{477}$$

i.e. with respect to changes in the gauge (gauge invariance).

(β) *Electromagnetic field and world metric.* From (472) it follows by integration that

$$\log l \Big|_P^{P'} = -\int_P^{P'} \phi_i \, dx^i,$$

(478)

$$l_{P'} = l_P \exp\left\{-\int_P^{P'} \phi_i \, dx^i\right\}.$$

When the linear form $\phi_i \, dx^i$ is a total differential, the length of a vector becomes independent of the path along which it is transferred, and we revert to the Riemannian case. The necessary and sufficient condition for this is the vanishing of the expressions

$$F_{ik} = \frac{\partial \phi_k}{\partial x^i} - \frac{\partial \phi_i}{\partial x^k}. \tag{479}$$

We see from (477) that, in this case, the vector ϕ_i can always be made to vanish by a suitable choice of the gauge. In the general case, however, the quantities F_{ik} will be different from zero. They then form the covariant components of an antisymmetric tensor of rank 2, which moreover remain unchanged for changes in the gauge, because of (477). In addition, they satisfy the equations

$$\frac{\partial F_{ik}}{\partial x^l} + \frac{\partial F_{li}}{\partial x^k} + \frac{\partial F_{kl}}{\partial x^i} = 0, \tag{480}$$

which follow from (479). We see that relations (479) and (480) are of exactly the same form as equations (206) and (203) of electron theory. But the analogy can be carried still further. If we take the view (contrary to the assumptions of Mie's theory) that electromagnetic phenomena are primarily determined by the space and time changes of the field strengths alone, and that the potentials, on the other hand, are only looked upon as mathematical auxiliary quantities, then all potential values ϕ_i which lead to the same field strengths are physically completely equivalent; thus a gradient $\partial\psi/\partial x^i$ remains undetermined in the ϕ_i. But as we have seen, exactly the same applies to the metric vector ϕ_i. Following Weyl, we are thus led to identify both sets of quantities ϕ_i, F_{ik}: *The metric vector ϕ_i which determines the behaviour of lengths according to* (478), *is to be identical with the electromagnetic four-vector potential* (*apart from a numerical factor*). Just as in Einstein's theory the gravitational effects are closely linked with the behaviour of measuring rods and clocks, such that they follow from it unambiguously, so the same holds in Weyl's theory for electromagnetic effects. In this sense, both gravitation and electricity appear as a consequence of the world metric in this theory.

This point of view had to modified subsequently by Weyl. For, the basic assumptions of the theory in its original form lead to deductions which seem to be in contradiction with experiment. This was stressed by Einstein.[386] Let us think of an electrostatic field connected with a static

[386] A. Einstein, *S.B. preuss. Akad. Wiss.* (1918) 478, including Weyl's reply.

G-field. The spatial components ϕ_i ($i = 1, 2, 3$) then vanish, and the time component $\phi_4 = \varphi$ as well as the g_{ik} are time independent. The gauge is thus fixed to within a *constant* factor. If we apply relation (478) to the period of a stationary clock, it follows directly that

$$\tau = \tau_0 \, e^{\alpha \varphi t}, \tag{481}$$

where α is a factor of proportionality. The meaning of this equation is the following: Let two identical clocks C_1, C_2, going at the same rate, be placed at first at the point P_1, at an electrostatic potential φ_1. Let clock C_2 then be taken to a point P_2, at potential φ_2, for t secs. and finally returned to P_1. The result will be that the rate of clock C_2, compared with that of clock C_1, will be increased or decreased, respectively, by a factor $\exp[-\alpha(\varphi_2-\varphi_1)t]$ (depending on the sign of α and of $\varphi_2-\varphi_1$). In particular, this effect should be noticeable in the spectral lines of a given substance, and spectral lines of definite frequencies could not exist at all. For, however small α is chosen, the differences would increase indefinitely in the course of time, according to (481). Weyl's present attitude to this problem is the following: *The ideal process of the congruent transference of world lengths, as determined by* (472), *has nothing to do with the real behaviour of measuring rods and clocks; the metric field must not be defined by means of information taken from these measuring instruments.* In this case the quantities g_{ik} and ϕ_i are, by definition, no longer observable, in contrast to the line element ds^2 of Einstein's theory. This relinquishment seems to have very serious consequences. While there now no longer exists a direct contradiction with experiment, the theory appears nevertheless to have been robbed of its inherent convincing power, from a physical point of view.[387] For instance, the connexion between electromagnetism and world metric is not now essentially physical, but purely formal. For there is no longer an immediate connection between the electromagnetic phenomena and the behaviour of measuring rods and clocks. There is only an interrelation between the former and the ideal process which is mathematically defined as congruent transference of vectors. Besides, there exists only formal, and not physical, evidence for a connection between world metric and electricity. This is quite in contrast to the connexion between world metric and gravitation, for which strong empirical support can be found in the equality of gravitational and inertial mass, and which is a rigorous consequence of the principle of equivalence and of special relativity.

(γ) *The tensor calculus in Weyl's geometry.* Before writing down the field equations, we still have to describe briefly the formal rules for the setting up of gauge-invariant equations. It is clear that in Weyl's theory the concept of a tensor has to be modified in such a way that a system of equations which expresses the vanishing of all the components of a tensor must not only remain invariant for an arbitrary change in the coordinates, but also for an arbitrary change in the gauge (477). In fact, it is found convenient to denote only those quantities as tensors, which are merely

[387] A. Einstein believes that, even in this version, the theory will not stand up to a comparison with reality (*Phys. Z.*, **21** (1920) 651; see also footnote 18, p. 4, *ibid.*).

multiplied by a power of λ, λ^e, for a transformation (477); e is called the weight of the tensor. Thus, g_{ik} is of weight $+1$, g^{ik} of weight -1, $\sqrt{(-g)}$ in a four-dimensional world of weight 2, $\Gamma^r{}_{ik}$ is, according to (64) or (476), *absolutely* gauge-invariant, i.e. of weight 0.

All those operations which are solely based on the concept of a parallel displacement can naturally be directly carried over into Weyl's geometry; only, we have to use expressions (66), (476) in place of (66), (69) for $\Gamma^r{}_{ik}$. Thus, the geodesic lines can be defined here, too, by the condition that their tangents should always remain parallel to themselves; they again satisfy equations (80). Eqs. (77 a) ($u_i u^i =$ const.), however, have to be replaced by

$$\frac{d}{d\tau}(u_i u^i) = -(u_i u^i)(\phi_k u^k)$$

because of (472) and (474). If, in particular, $u_i u^i = 0$ at a given point on the geodesic line, this relation will always remain valid. This makes it possible to determine geodesic null lines. The property of geodesic lines, that they are also shortest lines, is dropped in Weyl's geometry, because the concept of a curve length becomes meaningless here. As in § 16 we arrive at the curvature tensor by means of a parallel displacement of a vector along a closed curve,

$$R^h{}_{ijk} = \frac{\partial \Gamma^h{}_{ij}}{\partial x^k} - \frac{\partial \Gamma^h{}_{ik}}{\partial x^j} + \Gamma^h{}_{k\alpha}\Gamma^\alpha{}_{ij} - \Gamma^h{}_{j\alpha}\Gamma^\alpha{}_{ik}. \qquad (86)$$

The components in this equation are of weight 0, and the components of R_{hijk} therefore of weight 1. The symmetry relations for this curvature tensor are however different from those for the Riemannian tensor, which are determined by (92). Weyl has discussed this in more detail and has also calculated explicitly the expression (86) for the curvature tensor by substituting (476). As in § 17, we obtain the contracted curvature tensor R_{ik}, (94), whose covariant components are of weight 0, and the invariant R, (95), of weight -1. Finally, all operations of §§ 19 and 20 also remain valid in Weyl's theory, provided, first, the differentiated components of tensors or tensor densities are of weight 0, and secondly, expressions (66) and (476) are used for the quantities $\Gamma^r{}_{ik}$. It will be noticed that it is quite sufficient for the proofs of most of the theorems to realize that the concept of a parallel displacement, according to (64), is determined in an invariant way with the help of the $\Gamma^r{}_{ik}$, without having to know the connection between the metric quantities g_{ik}, ϕ_i. In the latest versions of his theory, Weyl stressed this point very strongly, by developing his geometry in three stages. In the first, he derived those theorems which are valid in arbitrary manifolds; in the second, the relations which are based on the concept of parallel displacement (called "affine connexion" by Weyl); and finally, the conclusions resulting from the existence of the two fundamental metric forms: the quadratic $g_{ik}\, dx^i\, dx^k$ (gravitation) and the linear $\phi_i\, dx^i$ (electricity). The linking of these two ranges of phenomena, which had been separate in the earlier theories, finds expression in the formal fact that the g_{ik} and ϕ_i occur simultaneously in the geodesic components Γ^r_{ik}, and thus also in most of the other gauge-invariant equations.

Of particular importance for the physical applications are the modifications and extentions, in Weyl's theory, of the statements on infinitesimal coordinate transformations and invariant integrals which we made in § 23. First of all, the infinitesimal changes in the gauge appear on an equal footing with the infinitesimal coordinate transformations. From (477) we have, with $\lambda = 1 + \epsilon\pi(x)$,

$$\delta g_{ik} = \epsilon\pi g_{ik}. \qquad (\delta g^{ik} = -\epsilon\pi g^{ik}), \qquad \delta\phi_i = -\epsilon\frac{\partial\pi}{\partial x^i}. \tag{482}$$

Furthermore, it is evident that, in Weyl's theory, only scalar densities $\mathfrak{W}$ of weight 0 lead to invariant integrals $\int \mathfrak{W}\,dx$. The corresponding scalars are then of weight -2, because of the factor $\sqrt{(-g)}$ in a four-dimensional world. Scalars of this kind will therefore play an important rôle in what follows. Among them, there are four which are rational combinations of the components of the curvature tensor,[388]

$$\tfrac{1}{2}F_{ik}F^{ik}, \qquad R_{hijk}R^{hijk}, \qquad R_{ik}R^{ik}, \qquad R^2. \tag{483}$$

The invariant R which appears in the action principle of Einstein's theory is however of weight -1. It was pointed out by Weyl that, because the scalar densities belonging to (483) have the weight 0, a four-dimensional world has a distinct advantage over a metrical manifold with a different number of dimensions. In fact, in such a manifold no scalar densities can be constructed which are of weight 0 and would have such a simple structure.

(δ) *Field equations and action principle. Physical deductions.* We now have to look for gauge-invariant physical laws. According to Weyl, all processes must be reducible to electromagnetic and gravitational effects. There are thus the fourteen independent quantities of state ϕ_i, g_{ik} available. But since gauge invariance has now been added to the invariance under coordinate transformations, the general solution of the field equations must contain five, instead of four, arbitrary functions; there must therefore also be five identities between the fourteen field equations. We shall see that four of the identities express the energy–momentum law, analogously to those in Einstein's theory, and that the fifth identity expresses the conservation of charge.

We shall first try to retain Maxwell's equations and to identify the material energy tensor with that of Maxwell, and shall only replace the curvature tensor of Riemannian geometry in Einstein's equations by that of Weyl's theory. It will however be seen that only the former "prescription" is possible, and not the latter. Let us first of all investigate the Maxwell theory. The first set of Maxwell's equations are automatically satisfied, as has already been mentioned. But since the field strengths F_{ik} are of weight 0, the same will be true, in a four-dimensional world,

[388] It was shown by R. Weitzenböck, *Wiener Ber., math.-nat. Kl.*, IIa, 129 (1920) 683 and 697, that the invariants given here are the only ones of this kind.

of the contravariant components $\mathfrak{F}^{ik}$ of the corresponding tensor density. The equations

$$\frac{\partial \mathfrak{F}^{ik}}{\partial x^k} = \mathfrak{s}^i$$

are therefore gauge-invariant: *Maxwell's equations remain invariant when g_{ik} is replaced by λg_{ik}.* Bateman's theorem, that Maxwell's equations are invariant under conformal transformations (§ 28), is contained in this statement as a special case. Such transformations do indeed change the normal values δ_i^k, which the g_{ik} have in special relativity, into $\lambda\delta_i^k$. The gauge invariance of Maxwell's equations is connected with the fact that the action integral

$$J = \int \tfrac{1}{4} F_{ik} \mathfrak{F}^{ik}\, dx$$

from which they are produced is itself gauge-invariant. We wish to remark at this point that the reason for the, seemingly accidental, vanishing of the scalar of the Maxwell energy tensor (§ 30, Eq. (223)) is also to be found in the gauge invariance of this action integral. For, a variation of the action integral, keeping the F_{ik} constant, leads to

$$\delta J = \int \mathfrak{S}_{ik}\, \delta g^{ik}\, dx,$$

according to § 55. If we now look for the condition that J should remain unchanged for the infinitesimal change in gauge, $\lambda = 1+\epsilon\pi(x)$, it follows directly from (482) that $\mathfrak{S}_i^i = 0$, as required.[389]

The situation is quite different when we come to deal with Einstein's theory, as opposed to Maxwell's. To start with, the law that the world lines of point-masses and light rays are geodesics is no longer generally valid in Weyl's theory. A point-mass only moves along a geodesic world line in the absence of electromagnetic fields, and for a light ray the equation of the geodesic line loses its meaning for the following reason. Even in the absence of gravitational fields, the terms which contain the four vector potential ϕ_i will introduce into the equation of the geodesic line oscillatory functions of the light period. Only the gauge-invariant equation

$$g_{ik}\, dx^i\, dx^k = 0$$

of the null cone remains valid for the world lines of the light rays. The

[389] The ϕ_i need not be varied here, since J contains them only in the form of the gauge-invariant F_{ik}. This connexion admits of an interesting application to Nordström's gravitational theory. As was mentioned in § 56, the line element in this theory is of the form

$$ds^2 = \Phi \sum_i (dx^i)^2.$$

It follows first of all from the gauge invariance of Maxwell's equations that they also remain valid in the presence of gravitational fields in Nordström's theory, so that gravitational fields have no influence on electromagnetic processes (e.g. no bending of light rays). Conversely, because of the vanishing of Maxwell's energy scalar, the electromagnetic energy does not generate gravitational fields in Nordström's theory, since the gravitational field equations only contain the energy scalar. It is seen from the above that this circumstance, too, has as its formal basis the gauge invariance of Maxwell's equations.

attempt to make use of the field equations of Einstein's theory in Weyl's theory (by replacing the Riemannian curvature quantities by the more general ones of Weyl) is finally wrecked by the fact that in the equation

$$G_{ik} = -\kappa T_{ik}$$

the left-hand side would be of weight 0, the right-hand side of weight -1; the latter is easily seen by taking Maxwell's energy tensor as an example. This failure is due to the fact that the action integral $\int \mathfrak{R}\, dx$ from which Einstein's field equations are produced is not gauge-invariant, since the integral is of weight 1 instead of 0. If, therefore, we wish to retain the principle of gauge invariance, we have to abandon the Einstein field equations. This last remark, however, already points the way to a method of obtaining gauge-invariant field equations. An action principle

$$\delta \int \mathfrak{W}\, dx = 0 \tag{484}$$

has to be set up, in which the integral is invariant with respect to changes in the gauge, too. If the variations vanish at the boundary, and if in general for varying the ϕ_i and g_{ik}

$$\delta \int \mathfrak{W}\, dx = \int (\mathfrak{w}^i \delta\phi_i + \mathfrak{W}^{ik} \delta g_{ik})\, dx, \tag{485}$$

then the physical laws are expressed by

$$\mathfrak{w}_i = 0, \qquad \mathfrak{W}^{ik} = 0. \tag{486}$$

By looking for conditions that the integral $\int \mathfrak{W}\, dx$ should be invariant with respect to infinitesimal coordinate transformations and infinitesimal changes in the gauge, we obtain five identities between these fourteen equations, as had been postulated above on the grounds of causality. These are

$$\frac{\partial \mathfrak{w}^i}{\partial x^i} + \mathfrak{W}_i{}^i \equiv 0 \tag{487}$$

and

$$\frac{\partial \mathfrak{W}_i{}^k}{\partial x^k} - \Gamma^r{}_{si} \mathfrak{W}_r{}^s + \tfrac{1}{2} F_{ik} \mathfrak{w}^k \equiv 0. \tag{488}$$

We can, next, consider variations of the action integral which do not vanish at the boundary. This will enable us to construct unambiguously from the action invariant a vector density $\mathfrak{s}^i$ and the density of an *affine* tensor $\mathfrak{S}_i{}^k$, which satisfy the relations

$$\frac{\partial \mathfrak{s}^i}{\partial x^i} \equiv \frac{\partial \mathfrak{w}^i}{\partial x^i} \quad \text{and} \quad \frac{\partial \mathfrak{S}_i{}^k}{\partial x^k} \equiv \frac{\partial \mathfrak{W}_i{}^k}{\partial x^k} \tag{489}$$

identically, without vanishing themselves by virtue of the physical laws. Weyl therefore calls $\mathfrak{s}^i$ the four-current, and $\mathfrak{S}_i{}^k$ the energy components. We thus see: *In Weyl's theory the charge conservation law is formally on exactly the same footing as the energy conservation law. Both are doubly derived* *from the physical laws, thus producing the necessary five identities between them.* The components of the total energy, which only form an affine tensor even in Einstein's theory (i.e. which are only covariant under *linear* transformations), can now no longer be split into a part due to gravitation and one due to matter itself; thus, an energy–momentum tensor $T_i{}^k$ of matter does not exist at all in this theory. It has to be admitted that the action principle enables us to recognize these relationships in a particularly simple and transparent manner. But it should be added that it is not at all self-evident, from a physical point of view, that the physical laws should be derivable from an action principle. It would, on the contrary, seem far more natural to derive the physical laws from purely physical requirements, as was done for Einstein's theory in § 56.

In order to be able to make further deductions from the theory, we must now make specific assumptions about the shape of the action function. The number of possibilities is not as great here as in Mie's theory. There, a new invariant could be derived from an *arbitrary* function $f(J_1, J_2, \ldots)$ of arbitrary invariants $J_1, J_2, \ldots$. This is no longer the case here, because the invariants have to be of weight -2, in order that the corresponding scalar densities should be of weight 0. A new, admissible, action function can therefore only be produced by a function which is at most *homogeneous of first degree* in these invariants. Even so, the manifold of admissible action functions still remains fairly considerable. The most natural assumption is that the action invariant should be a rational function of the curvature components. According to what was said in (γ), the action function will then have to be a linear combination of the invariants (483).[390] The calculation first of all leads to the *validity of Maxwell's equations*

$$\frac{\partial \mathfrak{F}^{ik}}{\partial x^k} = \mathfrak{s}^i \tag{208a}$$

and also to the expression

$$s_i = k\left(\frac{\partial R}{\partial x^i} + R\phi_i\right) \tag{490}$$

for the four-current (R is the curvature invariant in Weyl's geometry, k a constant). For the statical case we obtain from this that

$$R = \text{const.} \tag{491}$$

If charges are present at all, this constant cannot vanish. If, in addition, it is assumed to be positive, *it follows automatically that the curvature of space is positive and that the universe is finite,* so that it is unnecessary to add a special λ-term to the gravitational equations. This is a particular merit of Weyl's theory. As for the gravitational equations themselves, finally, these are not identical with Einstein's equations, even in the absence of an electromagnetic field ($\phi_i = 0$), as might have been expected from earlier arguments, and they are of higher order than the second.

[390] H. Weyl (Cf. footnote 385, p. 192, *Ann. Phys., Lpz., loc. cit.*, and *Raum-Zeit-Materie, loc. cit.*) thinks it probable that in particular the assumption $W = \frac{1}{2} F_{ik} F^{ik} + cR_{hijk} R^{hijk}$ corresponds to reality.

But it can be shown that for the case of a static, spherically symmetric, field in the space outside a "material particle", the gravitational field (421) of Einstein's theory is at the same time a solution of the gravitational equations of Weyl's theory. This case is, in practice, the only important one and is decisive for the perihelion precession of Mercury and the bending of light rays. *Weyl's theory is thus capable of explaining the perihelion precession of Mercury and the bending of light rays just as well as Einstein's theory.*[391]

We still have to discuss what deductions can be made relating to the problem of the structure of matter. Once more, the problem consists in determining those static, spherically symmetric, solutions of the field equations which are nowhere singular. It has again to be demanded of an action function which is to correspond to reality, that it should admit only of one solution each for the two kinds of electricity. An aspect which is essentially new, compared with Mie's theory, appears in the condition for regularity, not at infinity, but on the "equator" of the universe, because of the finiteness of the universe. We are thus led to conjecture that a connection exists between the size of the universe and that of the electron, which might seem somewhat fantastic. The forces which keep the electron together are, in this theory, only partly electrical in nature, and are partly gravitational forces. Even with the special assumptions for the action function, which were discussed above in detail, the differential equations become so complicated that it has not so far been possible to integrate them. Apart from this, the differential equations are the same for positive and negative electricity (cf. § 67), so that the completely asymmetric conditions which obtain in reality are certainly not represented correctly.† *Summarizing, we can say that Weyl's theory has not succeeded in getting any nearer to solving the problem of the structure of matter.* As will be argued in more detail in § 67, there is, on the contrary, something to be said for the view that a solution of this problem cannot at all be found in this way.

66. Einstein's theory

Einstein[392] tried to approach the inquiry into the structure of material particles from a completely different angle. The field equations (401) and (452), respectively, were based on the assumption of a material energy-momentum tensor $\mathfrak{T}_i^k$ which satisfies the equation

$$\frac{\partial \mathfrak{T}_i^k}{\partial x^k} - \tfrac{1}{2}\mathfrak{T}^{rs}\frac{\partial g_{rs}}{\partial x^i} = 0. \tag{341a}$$

This assumption will be retained here. Since Maxwell's energy tensor

$$\mathfrak{S}_i^k = F_{ir}\mathfrak{F}^{kr} - \tfrac{1}{4}F_{rs}\mathfrak{F}^{rs}\delta_i^k \tag{222a}$$

(cf. § 54) only satisfies this condition in charge-free space, some further

[391] Apart from the papers by Weyl, quoted in footnote 385, p. 192, see also W. Pauli, jr., *Verh. dtsch. phys. Ges.*, 21 (1919) 742, where, specifically, the action principle mentioned in footnote 390, p. 201, is used as a basis.

† See suppl. note 21.

[392] A. Einstein, *S.B. preuss. Akad. Wiss.* (1919) 349; also in the collection Lorentz–Einstein–Minkowski, *Das Relativitätsprinzip*, (5th edn., Berlin 1920).

terms have to be added to $\mathfrak{S}_i^k$. Mie had assumed that these terms are of an electrical nature, i.e. that they are functions of the electrical quantities F_{ik}, ϕ_i. Einstein, on the contrary, assumes that *the material particles are held together solely by gravitational forces*, so that these additional terms have to depend on the g_{ik} and their derivatives. Although the Maxwell tensor $\mathfrak{S}_i^k$ cannot now be called the total energy tensor of matter and does not satisfy equation (341 a), Einstein starts quite analogously as in § 56 with the hypothesis that *this energy tensor* $\mathfrak{S}_i^k$ *of Maxwell should be proportional to a differential expression of second order which is formed from the* g_{ik} *alone*. This simple assumption is decisive for Einstein's theory. We conclude from this and from the general covariance requirement, as in § 56, that the field equations must be of the form

$$R_{ik} + \bar{c}Rg_{ik} = -\kappa S_{ik}.$$

The addition of another term proportional to g_{ik} will be seen to be superfluous here. But since equation (341 a) for $\mathfrak{S}_{ik}$ is not valid, we are no longer entitled to put, as previously, $\bar{c} = -\frac{1}{2}$. Instead, $\bar{c}$ is determined by another condition. According to (223), the scalar S_i^i vanishes; we must put $\bar{c} = -\frac{1}{4}$, so that the scalar of the left-hand side of the field equations also vanishes identically. For this case, the field equations read

$$R_{ik} - \tfrac{1}{4}g_{ik}R = -\kappa S_{ik}. \tag{492}$$

In addition, the equations of electron theory,

$$\frac{\partial F_{ik}}{\partial x^l} + \frac{\partial F_{li}}{\partial x^k} + \frac{\partial F_{kl}}{\partial x^i} = 0 \tag{203}$$

and

$$\frac{\partial \mathfrak{F}^{ik}}{\partial x^k} = \mathfrak{s}^i; \tag{208}$$

are to remain valid. A simple enumeration shows that (203) and (492) contain just four independent equations less than there are unknowns, as required by a general relativistic theory. It should also be mentioned that the field equations do not appear to be derivable from an action principle, in this case. It follows from § 54 (203) and (208), that the divergence of S_{ik} has the value

$$-F_{ik}\mathfrak{s}^k$$

of the (negative) force vector of Lorentz. Therefore the divergence of the field equations (492) leads to the relation

$$F_{ik}s^k - \frac{1}{4\kappa}\frac{\partial R}{\partial x^i} = 0, \tag{493}$$

since the divergence of $R_{ik} - \frac{1}{2}g_{ik}R$ vanishes. *This relation shows that for these field equations the Coulomb repulsive forces are indeed held in equilibrium by a gravitational pressure.* If we put $s^k = \rho_0 u^k$, it also follows that

$$\frac{\partial R}{\partial x^i}u^i = \frac{dR}{d\tau} = 0, \tag{494}$$

i.e. R remains constant on the world line of one and the same element of matter. In charge-free space

$$\frac{\partial R}{\partial x^i} = 0,$$

from (493), so that

$$R = \text{const.} = R_0. \tag{495}$$

In the interior of a material particle, R decreases continuously from the value R_0 to smaller and smaller values, up to the centre of the particle. (493) shows that $(1/4\kappa)R$ represents directly the potential energy of the gravitational forces which hold the particle together.

We now have to look for the material energy–momentum tensor T_{ik}. *For this, the Eq.* (452), *which contains the* λ*-term, is to remain valid.* From (453), we have $R = -4\lambda$ for matter-free space. Comparison with (495) shows that we have to put

$$R_0 = -4\lambda, \qquad \lambda = -\frac{R_0}{4}. \tag{496}$$

It constitutes one of the main advantages of this new formulation that the constant λ is not characteristic of the fundamental law itself, but has the meaning of an *integration constant.* Eq. (452) can now be written

$$G_{ik} + \tfrac{1}{4}R_0 g_{ik} = -\kappa T_{ik},$$

whereas (492) leads to

$$G_{ik} + \tfrac{1}{4}R g_{ik} = -\kappa S_{ik}.$$

Comparison then shows that

$$T_{ik} = S_{ik} + \frac{1}{4\kappa}(R - R_0)g_{ik}. \tag{497}$$

This tensor therefore satisfies automatically the earlier equation (452), because of (492), and thus also Eq. (341 a). In addition it vanishes in matter-free space. We are thus quite justified, from a physical point of view too, to call it the energy tensor of matter. The material energy density $-\mathfrak{T}_4{}^4$ is composed of two parts, one originating from the electromagnetic and one from the gravitational field, both of which are positive. It is easy to see that the spatially finite universe with constant, stationary, mass density ($T_1{}^1 = T_2{}^2 = T_3{}^3 = 0, T_4{}^4 = -\mu_0 c^2$) is a solution of the new field equations. All the relations of § 62(β) remain unaltered. The electromagnetic tensor $S_i{}^k$ is generally calculated from (497) to be

$$S_i{}^k = T_i{}^k - \tfrac{1}{4}T\delta_i{}^k \tag{498}$$

so that, for our case,

$$S_1{}^1 = S_2{}^2 = S_3{}^3 = \tfrac{1}{4}\mu_0 c^2, \qquad S_4{}^4 = -\tfrac{3}{4}\mu_0 c^2. \tag{499}$$

The energy of a spatially finite universe is three-quarters electromagnetic and one-quarter gravitational in origin. This contribution of the electromagnetic to the total energy is exactly the same as that derived in § 63 on the basis of specific (not necessarily correct) assumptions about the electron.

If we now try to determine the field of a material particle from the differential equations (203) or (206), (208) and (492), respectively, we find that we are one equation short for the determination of the unknowns in the static spherically symmetrical case. *According to the theory of Einstein as developed here, every static spherically symmetrical distribution of electricity is in equilibrium.* However satisfactory the foundations of this theory may be, it, too, is not capable of providing an answer to the problem of the structure of matter.

67. General remarks on the present state of the problem of matter†

Each of the theories which we discussed has its particular advantages and drawbacks. Their joint failure prompts us, however, to summarize specifically those shortcomings and difficulties which are common to them all.

It is the aim of all continuum theories to derive the atomic nature of electricity from the property that the differential equations expressing the physical laws have only a discrete number of solutions which are everywhere regular, static, and spherically symmetric. In particular, one such solution should exist for each of the positive and negative kinds of electricity. It is clear that differential equations which have this property must be of a particularly complicated structure. It seems to us that this complexity of the physical laws in itself already speaks against the continuum theories. For it should be required, from a physical point of view, that the existence of atomicity, in itself so simple and basic, should also be interpreted in a simple and elementary manner by theory and should not, so to speak, appear as a trick in analysis.

Furthermore, we have seen that the continuum theories are forced to introduce special forces which keep the Coulomb repulsive forces in the interior of the electrical elementary particles in equilibrium. If we assume that these forces are *electrical in nature*, we have to assign an absolute meaning to the four-vector potential, which leads to the difficulties discussed in § 64. The other alternative, that the electrical elementary particles are held together by *gravitational forces*, is however countered by a very weighty, empirical, argument. For one would expect, in such a case, that a simple numerical relation would exist between the gravitational mass of the electron and its charge. Actually, the relevant dimensionless number $e/(m\sqrt{k})$ ($k =$ ordinary gravitational constant) is of the order of 10^{20} (see also § 59.)!

It must also be required of the field equations that they should account for the asymmetry (difference in mass) in the two kinds of electricity.‡ It is however easy to see that this is in contradiction with their general covariance,[393] from a formal point of view. For the statical case, the field

† See suppl. note 23.
‡ See suppl. note 21.
[393] W. Pauli, jr., *Phys. Z.*, **20** (1919) 457.

equations only contain the electrostatic potential φ as a variable, apart from the g_{ik} ($i, k = 1, 2, 3$ or $i = k = 4$). As a special case of general covariance, the differential equations must, in particular, be covariant under time reversal $x'^4 = -x^4$. But then φ goes over into $-\varphi$, whereas the g_{ik} remain unchanged (in our case, $g_{ik} = 0$ for $i = 1, 2, 3$). If therefore φ, $g_{ik}(g_{i4} = 0)$ is a solution of the field equations, then $-\varphi$, $g_{ik}(g_{i4} = 0)$ is also a solution, in contradiction to the asymmetry of the two kinds of electricity. One could try to avoid such a conclusion by introducing non-rational action functions, as was indicated at the end of § 64. But, first of all, the field equations would then become even more complicated, and, secondly, the selection of the unambiguous branch of the action function is not carried out in a generally covariant manner: covariance no longer exists, e.g. for the time reversal $x'^4 = -x^4$.

Finally, a conceptual doubt should be mentioned.[394] The continuum theories make direct use of the ordinary concept of electric field strength, even for the fields in the interior of the electron. This field strength is however defined as the force acting on a test particle, and since there are no test particles smaller than an electron or a hydrogen nucleus, the field strength at a given point in the interior of such a particle would seem to be unobservable, by definition, and thus be fictitious and without physical meaning.

Whatever may be one's attitude in detail towards these arguments, this much seems fairly certain: new elements which are foreign to the continuum concept of the field will have to be added to the basic structure of the theories developed so far, before one can arrive at a satisfactory solution of the problem of matter.

[394] Cf. footnote 391, p. 202, W. Pauli, jr., *ibid.*, and the 'Nauheimer Diskussion', *Phys Z.*, 21 (1920) 650.