Defining Term Rewriting

First-order terms

Signature and first-order terms

$\mathcal{F}_0$ a set of symbols of arity 0 (the constants)

$\mathcal{F}_i$ a set of symbols of arity i

$\mathcal{F} = \cup_n \mathcal{F}_n$

$\mathcal{X}$ a set of arity 0 symbols called variables.

$\mathcal{T}_{\mathcal{F}}(\mathcal{X})$ is the smallest set such that:

- $\mathcal{X} \subseteq \mathcal{T}_{\mathcal{F}}(\mathcal{X})$,
- $\forall f \in \mathcal{F}, \forall t_1, \ldots, t_n \in \mathcal{T}_{\mathcal{F}}(\mathcal{X}) : \quad f(t_1, \ldots, t_n) \in \mathcal{T}_{\mathcal{F}}(\mathcal{X})$.

$\mathcal{T}_{\mathcal{F}}(\emptyset) = \mathcal{T}_{\mathcal{F}}$ is the set of ground terms.

Terms as mappings: $(\mathcal{N}, .) \rightarrow \mathcal{F}$

$t = f(a + x, h(f(a, b)))$ is represented by:

$$
\begin{array}{lcr}
\textit{position} & \mapsto & \textit{symbol} \\
\top & \mapsto & f \\
1 & \mapsto & + \\
1.1 & \mapsto & a \\
1.2 & \mapsto & x \\
2 & \mapsto & h \\
2.1 & \mapsto & f \\
2.1.1 & \mapsto & a \\
2.1.2 & \mapsto & b
\end{array}
$$

$\mathcal{D}om(t) = \{\top, 1, 1.1, 1.2, 2, 2.1, 2.1.1, 2.1.2\}$

Examples and (some) terminology

With the following signature:

$\mathcal{F} = \{f, a\}$ with $arity(f) = 2$, $arity(a) = 0$, $x, y, z \in \mathcal{X}$:

what are the following terms?

$f(a, a)$

$f(x, f(a, x))$

$f(x, f(y, z))$

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f is ill-formed (since f is of arity 2)

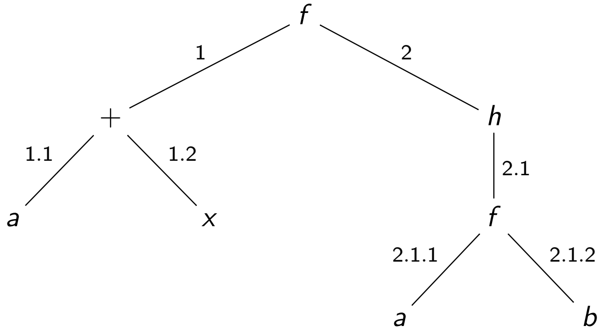
Subterms

$t[s]_\omega$ denotes the term t with s as **subterm** at **position** (or **occurrence**) ω.
$t|_\omega$ denotes the subterm at occurrence ω.

$$f(a + x, h(f(a, b)))|_2 = h(f(a, b))$$

Terms as trees

$t = f(a + x, h(f(a, b)))$ is represented by:



$|t|$ is the size of t i.e. the cardinality of $\mathcal{D}om(t)$.

$$|f(a + x, h(f(a, b)))| = 8$$

$\mathcal{V}ar(t)$ denotes the set of variables in t.

$$\mathcal{V}ar(f(a + x, h(f(a, b)))) = \{x\}$$

Simple questions

What is $f(f(a,b),g(a))|_{1.1}$?

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What is the arity of f just above?

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What is the arity of f just above? — 2

What is the arity of a just above?

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What is the arity of f just above? — 2

What is the arity of a just above? — 0

What are the variables of $f(f(a,b),g(a))|_{1.2}$?

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What is $f(f(a,b),g(a))|_{1.2}$? — b

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What is the arity of a just above? — 0

What are the variables of $f(f(a,b),g(a))|_{1.2}$? — $\emptyset$

What are the variables of $f(f(x,x),g(a))|_{1.2}$?

Simple questions

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Substitutions

Substitution

A substitution σ is a mapping from the set ot variables to the set of terms:

$$\sigma : \mathcal{X} \mapsto \mathcal{T}_{\mathcal{F}}(\mathcal{X})$$

It is extended as a morphism from terms to terms:

$$\sigma : \mathcal{T}_{\mathcal{F}}(\mathcal{X}) \mapsto \mathcal{T}_{\mathcal{F}}(\mathcal{X})$$
$$\sigma(f(t_1, t_2)) = f(\sigma(t_1), \sigma(t_2))$$

If $\sigma = \{x \mapsto a, y \mapsto f(a, g(z)), z \mapsto g(z)\}$, then $\sigma(f(x, f(a, z))) = f(a, f(a, g(z)))$.

Matching

Matching

Finding a substitution σ such that

$$\sigma(l) = t$$

is called *the matching problem* from l to t.

This is denoted $l \ll^{?} t$

It is decidable in linear time in the size of t.

It induces a relation on terms called *subsumption*

Matching: A rule based description

$$
\begin{array}{llll}
\textit{Delete} & & t \ll^{?} t \wedge P & \\
& \rightarrow & P & \\
\\
\textit{Decomposition} & & f(t_1, \ldots, t_n) \ll^{?} f(t'_1, \ldots, t'_n) \wedge P & \\
& \rightarrow & \bigwedge_{i=1,\ldots,n} t_i \ll^{?} t'_i \wedge P & \\
\\
\textit{SymbolClash} & & f(t_1, \ldots, t_n) \ll^{?} g(t'_1, \ldots, t'_m) \wedge P & \\
& \rightarrow & \textit{Fail} & \text{if } f \neq g \\
\\
\textit{SymbolVariableClash} & & f(t_1, \ldots, t_n) \ll^{?} x \wedge P & \\
& \rightarrow & \textit{Fail} & \text{if } x \in \mathcal{X}
\end{array}
$$

Matching: A rule based description

$$\begin{array}{llll}
\textit{Delete} & & t \ll^{?} t \wedge P & \\
 & \rightarrow & P & \\[2ex]
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 & \rightarrow & \mathit{Fail} & \text{if } x \in \mathcal{X} \\[2ex]
\textit{MergingClash} & & x \ll^{?} t \wedge x \ll^{?} t' \wedge P & \\
 & \rightarrow & \mathit{Fail} & \text{if } t \neq t'
\end{array}$$

Find a match

$$x + (y * 3) \ll^? 1 + (4 * 3)$$

$$\Rightarrow_{\text{Decomposition}}\ x \ll^? 1 \ \wedge\ y * 3 \ll^? 4 * 3$$

$$\Rightarrow_{\text{Decomposition}}\ x \ll^? 1 \ \wedge\ y \ll^? 4 \ \wedge\ 3 \ll^? 3$$

$$\Rightarrow_{\text{Delete}}\ x \ll^? 1 \ \wedge\ y \ll^? 4$$

Find a match

$$x+(y*3) \ll^? 1+(4*3)$$

$$\Rightarrow_{\text{Decomposition}}\ x \ll^? 1 \ \wedge\ y*3 \ll^? 4*3$$

$$\Rightarrow_{\text{Decomposition}}\ x \ll^? 1 \ \wedge\ y \ll^? 4 \ \wedge\ 3 \ll^? 3$$

$$\Rightarrow_{\text{Delete}}\ x \ll^? 1 \ \wedge\ y \ll^? 4$$

$$x+(y*y) \ll^? 1+(4*3)$$

$$\Rightarrow_{\text{Decomposition}}\ x \ll^? 1 \ \wedge\ y*y \ll^? 4*3$$

$$\Rightarrow_{\text{Decomposition}}\ x \ll^? 1 \ \wedge\ y \ll^? 4 \ \wedge\ y \ll^? 3$$

$$\Rightarrow_{\text{MergingClash}}\ \mathit{Fail}$$

Matching rules

Does it terminate?
Do we always get the same result?

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Do we always get the same result?

Theorem

The normal form by the rules in Match, of any matching problem $t \ll^? t'$ such that $\mathcal{V}ar(t) \cap \mathcal{V}ar(t') = \emptyset$, exists and is unique.

1. *If it is Fail, then there is no match from t to t'.*
2. *If it is of the form $\bigwedge_{i \in I} x_i \ll^? t_i$ with $I \neq \emptyset$, the substitution $\sigma = \{x_i \mapsto t_i\}_{i \in I}$ is the unique match from t to t'.*
3. *If it is empty then t and t' are identical: $t = t'$.*

Matching is used everywhere

ML (OCaml, SML, Scala, ...)
TOM
XQUERY
"pattern matching" in general

Term subsumption

$$s \ll t \Longrightarrow \sigma(s) = t$$

Vocabulary:
t is called an instance of s
s is said more general than t or
s subsumes t
σ is a match from s to t.
$\ll$ is a quasi-ordering on terms called subsumption.

$$f(x, y) \ll f(f(a, b), h(y))$$

Theorem: [Huet78]
Up to renaming, the subsumption ordering on terms is well-founded.

Notice that

$s \ll t \quad \not\Rightarrow \quad f(u, s) \ll f(u, t)$
since
$x \ll a$ but $f(x, x) \not\ll f(x, a)$

$s \ll t \quad \not\Rightarrow \quad \sigma(s) \ll \sigma(t)$
since
$x \ll a$ but $(x \mapsto b)x \not\ll (x \mapsto b)a$

Rewriting

Definition of rewriting

It relies on 5 notions:

- The objects: terms and rewrite rules
- The actions
  - matching
  - substitutions
  - replacement

and, given a rule and a term, it consists in:

- finding a subterm of the term
- that matches the left hand side of the rule
- and replacing that subterm by the right hand side of the rule instanciated by the match

Formally

t rewrites to t' using the rule $l \to r$ if

$$t_{|p} = \sigma(l) \quad \text{and} \quad t' = t[\sigma(r)]_p$$

This is denoted

$$t \longrightarrow_p^{l \to r} t'$$

Rewrite relation

A term rewrite system R (a set of rewrite rules) determines a relation on terms denoted $\longrightarrow_R$:

$$u \to_R v$$

iff
there exist t, $l \to r \in R$, an occurrence ω in t, such that
$u = t[\sigma(l)]_\omega$
and
$v = t[\sigma(r)]_\omega$

$$t[\sigma(l)]_\omega \to_R t[\sigma(r)]_\omega$$

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USUALLY, when defining the rewriting relation, one requires the all rewrite rules satisfy $Var(r) \subseteq Var(l)$.

Simple examples

Consider the rewrite system R:

$$\begin{array}{lcl} x + x & \to & x \\ (a + x) + y & \to & y + x \end{array}$$

How many redexes are in $(a + a) + (a + a)$?

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Consider the rewrite system R:

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How many redexes are in $(a+a)+(a+a)$? — 4

Draw the rewrite derivation tree issued from $(a+a)+(a+a)$.

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— no

Why?

Expressiveness of rewriting

[Max Dauchet 1989]
A Turing machine can be simulated by a single rewrite rule
This unique rewrite rule can further be left linear and regular!

... Termination of a rewrite relation

On the use of term rewriting

- for programming (ASF, ELAN, MAUDE, ML, OBJ, Stratego, ...)
- for proving (Completion procedures, proof systems, ...)
- for solving (Constraint manipulations, ...)
- for verifying (Exhaustive (and may be intelligent) search)

What are the typical problems of the field?

Confluence
Termination
Control of rewriting : strategies
Conditional rewriting
Theorem proving and rewriting
Rewriting and higher-order features : ρ-calculus
Types and rewriting

Extended notions of rewriting

Conditional rules

$$l \rightarrow r \;\; \textbf{if} \;\; c$$

- $l, r \in \mathcal{T}_{\mathcal{F}}(\mathcal{X})$,
- c a boolean term
- $\mathcal{V}ar(r) \cup \mathcal{V}ar(c) \subseteq \mathcal{V}ar(l)$

The rule applies on a term t provided the matching substitution σ allows $c\sigma$ to reduce to *true*.

Applying a conditional rewrite rule

$$\begin{array}{rcll} \mathit{even}(0) & \to & \mathit{true} & \\ \mathit{even}(s(x)) & \to & \mathit{odd}(x) & \\ \mathit{odd}(x) & \to & \mathit{true} & \textbf{if} \quad \mathit{not}(\mathit{even}(x)) \\ \mathit{odd}(x) & \to & \mathit{false} & \textbf{if} \quad \mathit{even}(x) \end{array}$$

$$\mathit{even}(s(0)) \longrightarrow \mathit{odd}(0) \longrightarrow \mathit{false}$$

Generalized rules

$$l \to r \textbf{ where } p_1 := c_1 \ldots \textbf{ where } p_n := c_n$$

- $l, r, p_1, \ldots, p_n, c_1, \ldots, c_n \in \mathcal{T}_{\mathcal{F}}(\mathcal{X})$,
- $\mathcal{V}ar(p_i) \cap (\mathcal{V}ar(l) \cup \mathcal{V}ar(p_1) \cup \cdots \cup \mathcal{V}ar(p_{i-1})) = \emptyset$,
- $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l) \cup \mathcal{V}ar(p_1) \cup \cdots \cup \mathcal{V}ar(p_n)$
- $\mathcal{V}ar(c_i) \subseteq \mathcal{V}ar(l) \cup \mathcal{V}ar(p_1) \cup \cdots \cup \mathcal{V}ar(p_{i-1})$.

where *true* $:= c$ is equivalently written **if** c.
p_i is oftern reduced to a variable x.

Generalized rule application

$$l \rightarrow r \textbf{ where } p_1 := c_1 \ldots \textbf{ where } p_n := c_n$$

To apply this rewrite rule on t, the matching substitution σ from l to t (i.e. such that $l\sigma = t$) is successively composed with each match μ_i from p_i to $c_i\sigma\mu_1 \ldots \mu_{i-1}$, for all $i = 1, \ldots, n$.

$$\mathit{move}(S) \rightarrow C(x, y) \;\; \textbf{where} \;\; < x, y > := \mathit{position}(S) \;\; \textbf{if } x > y$$

Simple exercises

Peano integers

A Peano integer n is represented by the term $\mathbf{s(s(\ldots(s(}0\mathbf{))))}$ with n occurrences of s (successor). The sum

$$x + \text{zero} = x$$

$$x + s(y) = s(x + y)$$

is computed by the following rewriting system

```
x + zero -> x
x + suc(y) -> suc(x + y)
```

Compute $2 + 2$ and $1 + 1 + 1 + 1$.

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```
s(s(zero))+s(s(zero))
-> s(s(s(zero))+s(zero))
-> s(s(s(s(zero))+zero))
-> s(s(s(s(zero))))
```

Fibonacci

Define the rewriting system that computes Fibonacci suites:

$$\mathsf{fib}(0) = 1$$

$$\mathsf{fib}(1) = 1$$

$$\mathsf{fib}(n) = \mathsf{fib}(n\text{-}1) + \mathsf{fib}(n\text{-}2)$$

first assuming "regular" numbers, and then Peano numbers

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first assuming "regular" numbers, and then Peano numbers

We need the condition here! Why?

```
fib(0) -> 1
fib(1) -> 1
fib(n) -> fib(n-1)+fib(n-2)      if n>1
```

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$$\text{fib}(1) = 1$$

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fib(0) -> 1
fib(1) -> 1
fib(n) -> fib(n-1)+fib(n-2)      if n>1
```

```
fib(0) -> s(0)
fib(s(0)) -> s(0)
fib(s(s(n))) -> fib(s(n))+fib(n)
```

Greater than, max

Define a rewriting system which computes $>$, assuming the number are Peano integers

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```
s(n) > s(m) -> n > m
0 > n -> false
s(n) > 0 -> true
```

Greater than, max

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```
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```

Define a rewriting system which computes *max*, assuming the number are Peano integers

Greater than, max

Define a rewriting system which computes $>$, assuming the number are Peano integers

```
s(n) > s(m) -> n > m
0 > n -> false
s(n) > 0 -> true
```

Define a rewriting system which computes *max*, assuming the number are Peano integers

```
max(s(n), s(m)) -> max(n, m)
max(0, n) -> n
max(n, 0) -> n
```

Polynomials

We consider polynomials with one variable and natural coefficients (eg $2X^3 + 4X^2 + 5X + 1$). We want to compute the derivatives w.r.t. the variable (X).

$$deriv(n, X) \to 0$$
$$deriv(mX^n, X) \to mnX^{n-1}$$
$$deriv(P1 + P2, X) \to deriv(P1, X) + deriv(P2, X)$$

Try it on $3 * X^2 + X + 7$ and $5 * X^2 + 3 * X + 4$ and on their sum.

Simplifications

The result we obtained is not canonical: one monomial for each power with the monomials sorted by decreasing order.

Define an appropriate rewriting system and apply it to the previous result.

Simplifications

$$m_1 X^n + m_2 X^n \rightarrow_{simpl} (m_1 + m_2) X^n$$

$$m_1 X^{n_1} + m_2 X^{n_2} \rightarrow_{simpl} m_2 X^{n_2} + m_1 X^{n_1} \qquad \text{if } n_1 < n_2$$

$$m_1 X^n + (m_2 X^n + P) \rightarrow_{simpl} (m_1 + m_2) X^n + P$$

$$m_1 X^{n_1} + (m_2 X^{n_2} + P) \rightarrow_{simpl} m_2 X^{n_2} + (m_1 X^{n_1} + P) \qquad \text{if } n_1 < n_2$$

$$(P_1 + P_2) + P_3 \rightarrow_{simpl} P_1 + (P_2 + P_3)$$

Simplifications

$$
\begin{aligned}
&m_1 X^n + m_2 X^n \to_{simpl} (m_1 + m_2) X^n \\
&m_1 X^{n_1} + m_2 X^{n_2} \to_{simpl} m_2 X^{n_2} + m_1 X^{n_1} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{if } n_1 < n_2 \\
&m_1 X^n + (m_2 X^n + P) \to_{simpl} (m_1 + m_2) X^n + P \\
&m_1 X^{n_1} + (m_2 X^{n_2} + P) \to_{simpl} m_2 X^{n_2} + (m_1 X^{n_1} + P) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{if } n_1 < n_2 \\
&(P_1 + P_2) + P_3 \to_{simpl} P_1 + (P_2 + P_3)
\end{aligned}
$$

No need to use the rules

$$
\begin{aligned}
&m_1 X^n + (P + m_2 X^n) \to (m_1 + m_2) X^n + P \\
&m_1 X^{n_1} + (P + m_2 X^{n_2}) \to m_2 X^{n_2} + (m_1 X^{n_1} + P) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{if } n_1 < n_2
\end{aligned}
$$