Confluence of Term Rewriting Systems

Confluence

Allows us to forget about non-determinism:

Whatever rewriting is done we will converge later.

Back with the simple game

The rules of the game:

$$
\begin{aligned}
\bullet\bullet &\rightarrow \circ \\
\circ\circ &\rightarrow \circ \\
\bullet\circ &\rightarrow \bullet \\
\circ\bullet &\rightarrow \bullet
\end{aligned}
$$

A starting point:

$$\bullet \circ \bullet \circ \bullet \circ \bullet \bullet \bullet \bullet \circ \circ \bullet \circ \circ \bullet \bullet\circ$$

From a given start, is the result determinist?

Analysing the different cases

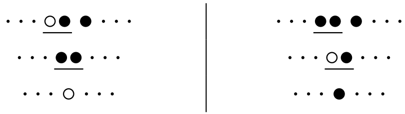
Disjoint redexes:

$$\begin{array}{c}\cdots \underline{\otimes\otimes} \cdots \otimes\ \otimes \cdots \\ \cdots \otimes \cdots \underline{\otimes\otimes} \cdots \\ \cdots \otimes \cdots \otimes \cdots\end{array}$$

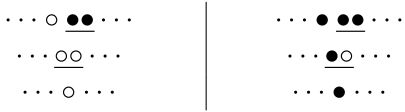
is the same as:

$$\begin{array}{c}\cdots \otimes\ \otimes \cdots \underline{\otimes\otimes} \cdots \\ \cdots \underline{\otimes\otimes} \cdots \otimes \cdots \\ \cdots \otimes \cdots \otimes \cdots\end{array}$$

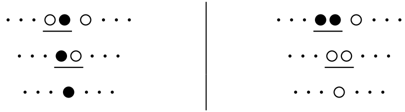
No disjoint redexes (central black):



but



or



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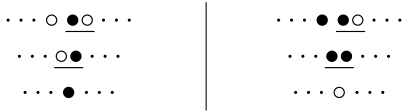

No disjoint redexes (central white):

$\cdots\ \underline{\bigcirc\bigcirc}\ \bullet\ \cdots$	$\cdots\ \underline{\bullet\bigcirc}\ \bullet\ \cdots$
$\cdots\ \underline{\bigcirc\bullet}\ \cdots$	$\cdots\ \underline{\bullet\bullet}\ \cdots$
$\cdots\ \bullet\ \cdots$	$\cdots\ \bigcirc\ \cdots$

but

$\cdots\ \bigcirc\ \underline{\bigcirc\bullet}\ \cdots$	$\cdots\ \bullet\ \underline{\bigcirc\bullet}\ \cdots$
$\cdots\ \underline{\bigcirc\bullet}\ \cdots$	$\cdots\ \underline{\bullet\bullet}\ \cdots$
$\cdots\ \bullet\ \cdots$	$\cdots\ \bigcirc\ \cdots$

or

$\cdots\ \underline{\bigcirc\bigcirc}\ \bigcirc\ \cdots$	$\cdots\ \underline{\bullet\bigcirc}\ \bigcirc\ \cdots$
$\cdots\ \underline{\bigcirc\bigcirc}\ \cdots$	$\cdots\ \underline{\bullet\bigcirc}\ \cdots$
$\cdots\ \bigcirc\ \cdots$	$\cdots\ \bullet\ \cdots$

but

$\cdots\ \bigcirc\ \underline{\bigcirc\bigcirc}\ \cdots$	$\cdots\ \bullet\ \underline{\bigcirc\bigcirc}\ \cdots$
$\cdots\ \underline{\bigcirc\bigcirc}\ \cdots$	$\cdots\ \underline{\bullet\bigcirc}\ \cdots$
$\cdots\ \bigcirc\ \cdots$	$\cdots\ \bullet\ \cdots$

Thus:

$$
\begin{tikzcd}
 & t_0 \arrow[dl] \arrow[dr] & \\
t_1 \arrow[dr, dashed] & & t_2 \arrow[dl, dashed] \\
 & t_3 &
\end{tikzcd}
$$

but what about:

$$
\begin{tikzcd}
 & t_0 \arrow[dl, "*"'] \arrow[dr, "*"] & \\
t_1 \arrow[dr, dashed, "*"] & & t_2 \arrow[dl, dashed, "*"'] \\
 & t_3 &
\end{tikzcd}
$$

Definitions

Consider an ARS ($\mathcal{T}$,$\rightarrow$)

- An element $t \in \mathcal{T}$ is a $\rightarrow$-**normal form** if there exists no $t' \in \mathcal{T}$ such that $t \rightarrow t'$.
- The relation $\rightarrow$ is **terminating** (or **strongly normalizing**, or **noetherian**) if every reduction sequence is finite.
- The relation $\rightarrow$ is **weakly normalizing** (or weakly terminating) if every element $t \in \mathcal{T}$ has a normal form.
- The relation $\rightarrow$ has the **unique normal form property** if for any $t, t' \in \mathcal{T}$, $t \xleftrightarrow{*} t'$ and t, t' are normal forms imply $t = t'$.

Definitions

Localy confluent (LC)

$$\begin{tikzcd} & t_0 \arrow[dl] \arrow[dr] & \\ t_1 \arrow[dr, dotted, "*"'] & & t_2 \arrow[dl, dotted, "*"] \\ & t_3 & \end{tikzcd}$$

Diamond property (DP)

$$\begin{tikzcd} & t_0 \arrow[dl] \arrow[dr] & \\ t_1 \arrow[dr, dotted] & & t_2 \arrow[dl, dotted] \\ & t_3 & \end{tikzcd}$$

Church Rosser (CR)

$$\begin{tikzcd} t \arrow[rr, leftrightarrow, "+"] \arrow[dr, dotted, "*"'] & & s \arrow[dl, dotted, "*"] \\ & u & \end{tikzcd}$$

Confluent (C)

$$\begin{tikzcd} & t_0 \arrow[dl, "*"'] \arrow[dr, "*"] & \\ t_1 \arrow[dr, dotted, "*"'] & & t_2 \arrow[dl, dotted, "*"] \\ & t_3 & \end{tikzcd}$$

Local versus global confluence

1. $C \Rightarrow LC$
2. $LC \Rightarrow C$?

Local versus global confluence

1. $C \Rightarrow LC$
2. $LC \Rightarrow C$?
   - Consider four distinct elements a, b, c, d of $\mathcal{T}$ and the relation:
     $a \to b$
     $b \to a$
     $a \to c$
     $b \to d$

$$\begin{tikzcd} a \arrow[r, bend left] \arrow[d] & b \arrow[l, bend left] \arrow[d] \\ c & d \end{tikzcd}$$

Newman's lemma

Lemma[Newman 1942]

Provided the relation $\rightarrow$ is terminating

then

$\rightarrow$ is confluent iff it is locally confluent

Proof:

- locally confluent if confluent
  obvious
- confluent if locally confluent
  ?

Noetherian induction: a fondamental tool

Let $(\mathcal{T}, >)$ be an ordered set s.t. $>$ is well-founded.

Let $\mathcal{P}$ be a proposition:

1. $\forall t \in \mathcal{T},\ [\forall t' \in \{t' \mid t > t'\},\ \mathcal{P}(t')] \Rightarrow \mathcal{P}(t)$
2. $\mathcal{P}(t)$ is provable for all minimal element t,

then $\forall t \in \mathcal{T}, \mathcal{P}(t)$.

Noetherian induction: a fondamental tool

Consider $(\mathcal{T}, \rightarrow)$

$$
\begin{tikzcd}
 & & t \arrow[dl] \arrow[dr] & & \\
 & t_1' \arrow[dl, "m-1"'] \arrow[dr, "*"] & & t_2' \arrow[dl, "*"'] \arrow[dr, "n-1"] & \\
t_1 \arrow[dr, "*"] & & t' \arrow[dl, "*"'] & & t_2 \arrow[ddll, "*"'] \\
 & t_1'' \arrow[dr, "*"] & & & \\
 & & u & &
\end{tikzcd}
$$

Proving local confluence

Local confluence is equivalent to the confluence of the critical pairs of the relation.

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A non-variable term t' and a term t *overlap* if there exists a position ω in t such that $t_{|\omega}$ and t' are unifiable (with $t_{|\omega}$ not a variable).

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Where does $(x + y) + z$ and $(x' + y') + z'$ overlap?

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Where does $(x + y) + z$ and $(x' + y') + z'$ overlap?

Where does $0 + x \to x$ and $s(x') + y \to s(x' + y)$ overlap?

How about $H \vdash \overline{\overline{P}} \to H \vdash P$ and $H' \wedge P' \vdash P' \to T$?

Paranthesis

Unification problems

Solve an equation

Does it exist x, y, z such that

$$x + (z * y) = y + (x * z)$$

An infinity of solutions,
but a most general one

$$x = y = z$$

Unification problem.

Unification Problems

$\mathcal{F}$ a set of function symbols,
$\mathcal{X}$ a set of variables,
$\mathcal{A}$ an $\mathcal{F}$-algebra.
A $<\mathcal{F}, \mathcal{X}, \mathcal{A}>$-unification problem
is a disjunction of existentially quantified formulas

$$P_j = \exists \vec{z} \bigwedge_{i \in I_j} s_i =^?_{\mathcal{A}} t_i$$

sometimes abbreviated

$$P_j = \exists \vec{z} \ \{s_i =^?_{\mathcal{A}} t_i\}_{i \in I_j}.$$

A unifier to such a problem is a *substitution* σ such that
$\exists j, \forall i \in I_j, \quad \mathcal{A} \models \exists \vec{z} \ \sigma_{|\mathcal{X}-\vec{z}}(s_i) = \sigma_{|\mathcal{X}-\vec{z}}(t_i)$.

Rules for syntactic unification

$$
\begin{array}{lll}
\textit{Delete} & & P \wedge s =^? s \\
& \rightarrow & P
\end{array}
$$

Rules for syntactic unification

$$
\begin{array}{lrl}
\textit{Delete} & & P \wedge s =^{?} s \\
 & \to & P \\
 & & \\
\textit{Decompose} & & P \wedge f(s_1, \ldots, s_n) =^{?} f(t_1, \ldots, t_n) \\
 & \to & P \wedge s_1 =^{?} t_1 \wedge \ldots \wedge s_n =^{?} t_n
\end{array}
$$

Rules for syntactic unification

$$\begin{array}{lcll}
\textit{Delete} & & P \wedge s =^? s & \\
 & \to & P & \\
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 & \to & P \wedge s_1 =^? t_1 \wedge \ldots \wedge s_n =^? t_n & \\
 & & & \\
\textit{Conflict} & & P \wedge f(s_1, \ldots, s_n) =^? g(t_1, \ldots, t_p) & \\
 & \to & \textit{Fail} & \text{if } f \neq g
\end{array}$$

Rules for syntactic unification

$$
\begin{array}{llll}
\textit{Delete} & & P \wedge s =^{?} s & \\
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& & & \\
\textit{Decompose} & & P \wedge f(s_1, \ldots, s_n) =^{?} f(t_1, \ldots, t_n) & \\
& \rightarrow & P \wedge s_1 =^{?} t_1 \wedge \ldots \wedge s_n =^{?} t_n & \\
& & & \\
\textit{Conflict} & & P \wedge f(s_1, \ldots, s_n) =^{?} g(t_1, \ldots, t_p) & \\
& \rightarrow & \textit{Fail} & \text{if } f \neq g \\
& & & \\
\textit{Coalesce} & & P \wedge x =^{?} y & \\
& \rightarrow & \{x \mapsto y\}P \wedge x =^{?} y & \text{if } x, y \in \mathcal{V}ar(P) \wedge x \neq y
\end{array}
$$

Rules for syntactic unification

$$\begin{array}{rll} \textit{Eliminate} & P \wedge x =^? s & \\ \to & \{x \mapsto s\}P \wedge x =^? s & \quad \text{if } x \notin \mathcal{V}ar(s), s \notin \mathcal{X}, x \in \mathcal{V}ar(P) \end{array}$$

Rules for syntactic unification

$$
\begin{array}{lll}
\textit{Eliminate} & P \wedge x =^? s & \\
\rightarrow & \{x \mapsto s\}P \wedge x =^? s & \text{if } x \notin \mathcal{V}ar(s), s \notin \mathcal{X}, x \in \mathcal{V}ar(P) \\
 & & \\
\textit{Merge} & P \wedge x =^? s \wedge x =^? t & \\
\rightarrow & P \wedge x =^? s \wedge s =^? t & \text{if } 0 < |s| \leq |t|
\end{array}
$$

Rules for syntactic unification

$$
\begin{array}{llll}
\textit{Eliminate} & P \wedge x =^? s & & \\
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& & & \\
\textit{Merge} & P \wedge x =^? s \wedge x =^? t & & \\
\rightarrow & P \wedge x =^? s \wedge s =^? t & & \text{if } 0 < |s| \leq |t| \\
& & & \\
\textit{Check} & P \wedge x =^? s & & \\
\rightarrow & \textit{Fail} & & \text{if } x \in \mathcal{V}ar(s) \textit{ and } s \notin \mathcal{X}
\end{array}
$$

Rules for syntactic unification

$$
\begin{array}{lll}
\textit{Eliminate} & P \wedge x =^? s & \\
\rightarrow & \{x \mapsto s\}P \wedge x =^? s & \text{if } x \notin \mathcal{V}ar(s), s \notin \mathcal{X}, x \in \mathcal{V}ar(P) \\
 & & \\
\textit{Merge} & P \wedge x =^? s \wedge x =^? t & \\
\rightarrow & P \wedge x =^? s \wedge s =^? t & \text{if } 0 < |s| \leq |t| \\
 & & \\
\textit{Check} & P \wedge x =^? s & \\
\rightarrow & \textit{Fail} & \text{if } x \in \mathcal{V}ar(s) \textit{ and } s \notin \mathcal{X} \\
 & & \\
\textit{Check}* & P \wedge \;\; x_1 =^? s_1[x_2] \wedge \ldots & \\
 & \qquad \ldots \wedge x_n =^? s_n[x_1] & \\
\rightarrow & \textit{Fail} & \text{if } s_i \notin \mathcal{X} \textit{ for some } i \in [1..n]
\end{array}
$$

Examples

$$x+(z*y) =^? y+(x*z)$$

$$\Rightarrow_{\text{decompose}} x =^? y \;\wedge\; z*y =^? x*z$$

$$\Rightarrow_{\text{decompose}} x =^? y \;\wedge\; z =^? x \;\wedge\; y =^? z$$

$$\Rightarrow_{\text{coalesce}} y =^? z \;\wedge\; x =^? z \;\wedge\; z =^? x$$

$$\Rightarrow_{\text{coalesce}} z =^? x \;\wedge\; y =^? x \;\wedge\; x =^? x$$

$$\Rightarrow_{\text{delete}} z =^? x \;\wedge\; y =^? x$$

Examples

$$f(a, f(x, x)) =^? f(x, f(a, y))$$

Examples

$$f(a, f(x, x)) =^? f(x, f(a, y))$$
$$f(x, f(y, z)) =^? f(f(a, b), f(x, b)$$

Examples

$$
\begin{aligned}
&f(a, f(x, x)) =^? f(x, f(a, y)) \\
&f(x, f(y, z)) =^? f(f(a, b), f(x, b) \\
&f(x, f(f(x, b), a)) =^? f(f(a, y), f(y, a))
\end{aligned}
$$

A *tree solved form* for P is any conjonction of equations

$$x_1 =^? t_1 \wedge \cdots \wedge x_n =^? t_n$$

equivalent to P such that $\forall i, x_i \in \mathcal{X}$ and:

$$\begin{array}{ll} (i) & \forall 1 \leq i \leq n, x_i \in \mathcal{V}ar(P), \\ (ii) & \forall 1 \leq i, j \leq n, i \neq j \Rightarrow x_i \neq x_j, \\ (iii) & \forall 1 \leq i, j \leq n, x_i \notin \mathcal{V}ar(t_j). \end{array}$$

Example: $x =^? f(f(y)) \wedge z =^? g(a)$.

Theorem: Starting with a unification problem P and using the above rules repeatedly until none is applicable
— results in *Fail* iff P has no solution, or else it
— results in a tree solved form $x_1 =^? t_1 \wedge \cdots \wedge x_n =^? t_n$ with the same set of solutions than P.
Moreover

$$\sigma = \{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$$

is a most general unifier of P.

Critical pair

A non-variable term t' and a term t overlap if there exists a position ω in t such that $t_{|\omega}$ and t' are unifiable (with $t_{|\omega}$ not a variable).

Where does $(x + y) + z$ and $(x' + y') + z'$ overlap?

Where does $0 + x \to x$ and $s(x') + y \to s(x' + y)$ overlap?

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Critical Pairs

Critical pair

$$\mathbf{l_1} \to \mathbf{r_1} \qquad\qquad \mathbf{l_2}[\mathbf{u}] \to \mathbf{r_2}$$

$$(\mathbf{l_2}[\mathbf{r_1}]\sigma, \mathbf{r}_2\sigma)$$

u is a non-variable sub-term of l_2
σ is the $mgu(u, l_1)$

Critical Pair Lemma

R is locally confluent iff all critical pair satisfies:

$$\mathbf{l}_2[r_1]\sigma \xrightarrow{*}_R \quad \otimes \quad {}_R\xleftarrow{*} r_2\sigma$$

Orthogonal systems

A rewrite system that is both linear (the left-hand side of each rule is a linear term) and non-overlapping is called orthogonal.

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Theorem

If a rewrite system is orthogonal, then it is confluent.

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Theorem

If a rewrite system is orthogonal, then it is confluent.

Linearity is needed for non-terminating rewriting system:

$$\begin{array}{lcl} d(x,x) & \to & t \\ d(x,c(x)) & \to & f \\ a & \to & c(a) \end{array}$$

Orthogonal systems

Exemple:

$$\begin{array}{ccc} \bullet\bullet & \longrightarrow & \circ \\ \circ\circ & \longrightarrow & \circ \\ \bullet\circ & \longrightarrow & \bullet \\ \circ\bullet & \longrightarrow & \bullet \end{array}$$

Other systems

Theorem Consider a reduction relation $\rightarrow_R$ and let $\rightarrow_D$ s.t.

- $\rightarrow_R \subseteq \rightarrow_D \subseteq \xrightarrow{*}_R$
- $\rightarrow_D$ has the diamond property

Then, $\rightarrow_R$ is confluent.

Exercices

Show the confluence of the two rewriting systems:

$$x + zero \to x$$
$$zero + x \to x$$
$$suc(y) + x \to suc(x + y)$$
$$x + suc(y) \to suc(x + y)$$

Show the confluence of the two rewriting systems:

$$x + zero \to x$$
$$zero + x \to x$$
$$suc(y) + x \to suc(x + y)$$
$$x + suc(y) \to suc(x + y)$$

CP:

$x + zero = zero + x \Longrightarrow zero + zero$

$x + zero = suc(y) + x \Longrightarrow suc(y) + zero$

$\ldots$

Show the confluence of the two rewriting systems:

$$x + zero \to x$$

$$x * zero \to zero$$

$$suc(y) + x \to suc(x + y)$$

$$x * suc(y) \to x + (x * y)$$

Show the confluence of the two rewriting systems:

$$x + zero \to x$$
$$x * zero \to zero$$
$$suc(y) + x \to suc(x + y)$$
$$x * suc(y) \to x + (x * y)$$

CP:

$x * zero = zero * x \Longrightarrow zero * zero$

$\ldots$

Syntactic filtering

$$
\begin{array}{lll}
 & t \ll^? t \wedge P & \\
\to & P & \\
 & f(t_1, \ldots, t_n) \ll^? f(t'_1, \ldots, t'_n) \wedge P & \\
\to & \bigwedge_{i=1,\ldots,n} t_i \ll^? t'_i \wedge P & \\
 & f(t_1, \ldots, t_n) \ll^? g(t'_1, \ldots, t'_m) \wedge P & \\
\to & \mathcal{F} & \text{if } f \neq g \\
 & x \ll^? t \wedge x \ll^? t' \wedge P & \\
\to & \mathcal{F} & \text{if } t \neq t' \\
 & f(t_1, \ldots, t_n) \ll^? x \wedge P & \\
\to & \mathcal{F} & \text{if } x \in \mathcal{X}
\end{array}
$$

Confluence?

Groups

$$
\begin{aligned}
x + e &\to x \\
e + x &\to x \\
x + (y + z) &\to (x + y) + z \\
x + i(x) &\to e \\
i(x) + x &\to e
\end{aligned}
\qquad\qquad
\begin{aligned}
i(e) &\to e \\
(y + i(x)) + x &\to y \\
(y + x) + i(x) &\to y \\
i(i(x)) &\to x \\
i(x + y) &\to i(y) + i(x)
\end{aligned}
$$

Groups

$$
\begin{array}{lcl}
x + e & \to & x \\
e + x & \to & x \\
x + (y + z) & \to & (x + y) + z \\
x + i(x) & \to & e \\
i(x) + x & \to & e
\end{array}
\qquad
\begin{array}{lcl}
i(e) & \to & e \\
(y + i(x)) + x & \to & y \\
(y + x) + i(x) & \to & y \\
i(i(x)) & \to & x \\
i(x + y) & \to & i(y) + i(x)
\end{array}
$$

Confluence:

Groups

$$
\begin{aligned}
x + e &\to x & \qquad i(e) &\to e \\
e + x &\to x & (y + i(x)) + x &\to y \\
x + (y + z) &\to (x + y) + z & (y + x) + i(x) &\to y \\
x + i(x) &\to e & i(i(x)) &\to x \\
i(x) + x &\to e & i(x + y) &\to i(y) + i(x)
\end{aligned}
$$

Confluence:

$x + e = e + x$

$x + e = i(x) + x$

$x + e = (y + i(x)) + x$

$x + e =_{x+y} i(x + y)$

$e + x =_{y+i(x)} (y + i(x)) + x$