Poincaré, Intuitionism, Intuition and Convention
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Arend Heyting in the introduction of his famous volume on *Mathematische Grundlagenforschung, Intutionismus, Beweistheorie*, speaks of Poincaré's influence on "contemporary intuitionists" *i.e.,* mathematicians who agree with the opinion that

> (1) Mathematics has not only a formal but also a contentual signification
>
> (2) Mathematical objects are immediately grasped by our minds and are also independent of experience [Heyting 1934, 3].

In addition, according to Heyting, the second characteristic is essentially connected with ontology and admits two different interpretations:

> a) Mathematical entities don't exist independently of our thinking (anti-realism)
>
> b) Mathematical entities exist independently of us but are only knowable through construction.

Heyting attributed the first interpretation to Brouwer and called the second position semi-intuitionistic. The term was motivated by systematic considerations. One of these was the fact that the grasping of the entity might not be immediate. Another was that it might not be compatible with b), taken in a strong sense. Heyting saw the semi-intuitionistic rejection of a construction-transcendent notion of existence as approximating Kronecker's view, and he saw it as even in a wider sense the view of the "French realists or empirists," among which were Borel, Lebesgue, Baire, Kaufmann, Skolem and Richard. I think he is right to have omitted Poincaré from this list. Clearly, for Poincaré, the question of mathematical entities existing independent of our mind is a question of old metaphysics about which he had nothing to say [SH,XXV].

La négation de toute métaphysique, c'est encore une métaphysique, et c'est précisément lÃ ce que j'appelle la métaphysique moderne. [Lettre adressée à Camille Flammarion le]

Since Kant, mathematical intuition has been connected with construction. But constructivism could neither be identified with intuitionism (anti-realism) nor with realism. What it is that is supposed to be constructible, objects or proofs, remains a subject of debate and is linked neither with thesis (a) nor with thesis (b). Brouwer recognized mathematics as an autonomic interior *constructional* mental activity [Brouwer 1955, 551; my emphasis]. Moreover, although constructivism was a key word in Kronecker's attack against Cantor's platonism and, in a wider sense, against traditional (axiomatic) mathematics, the question of construction has a definite sense even for a Platonist. This is illustrated by Gödel's attitude towards constructible sets. Generally speaking, Gödel believed that mathematical objects can be perceived through a special cognitive ability, i.e., "intuition", in something like the same way that empirical objects are grasped through sensory perception. As Hintikka put it, the intuitionist is more interested in what we can *know* whereas the constructivist is more interested in what we can *do* [Hintikka 1996, p. 237]. In any case, constructivism and intuitionism are both neutral with respect to realism and anti-realism. Anti-realism doesn't imply intuitionism, and realism doesn't imply an anti-intuitionistic constructivism. In addition, since historical positions don't necessarily coincide with these normative views, and their opposition is rather contrived (see [Mooij 1966, 136]).

It was Brouwer himself who pressed the difference between (a type of) constructivism and intuitionism by comparing Poincaré's position with his own standpoint. He saw Poincaré as guilty of a "confusion between the act of constructing mathematics and the language of mathematics." [Brouwer 1907, 176]. Although, according to Brouwer, the constructional mental activity of intuitionism may have a useful linguistic expression and can be applied to an exterior world, it has "neither in its origin nor in the essence of its method [. . .] anything to do with language or an exterior world" [Brouwer 1955, 551].

Poincaré was far from taking the nonlinguistic and intuitive construction as the only basis for mathematics. The role of linguistic conventions and language in mathematics is a point that separates him from Brouwer. For Brouwer, "the mistake of logistics lies in the fact that it creates nothing else other than a linguistic structure, which can never be transformed into mathematics proper" [Brouwer 1907, 176], for Poincaré, mathematics proper beyond linguistic structures is not a knowable domain. The evidence of object-constructions has to be linked with their description (By the way, this "pragmatistic" turn of Poincaré's philosophy is the French origin of analytic philosophy). Nevertheless, Brouwer is quite right when he emphasizes that Poincaré "reestablished on the one hand the essential difference in character between logic and mathematics, and on the other hand the autonomy of logic and of a part of mathematics" [Brouwer 1952, 140]. In fact, according to Poincaré, the program of founding number theory on logic involves a *petitio principii* because this foundation presupposes already some number theory. It seems that even his theory of predicative definitions was taken to be independent of mathematics [cf. Goldfarb 1988, 64]. Expressed in other words, even though Poincaré may have agreed with Brouwer that "mathematical language by itself can never create new mathematical systems" [Brouwer 1952, 141], mathematical language is nevertheless a necessary tool for mathematical creation (and not, or at least not merely, in a some psychological way). I will discuss this point later.

Hence there are epistemological features that bring some single *results* of Poincaré's position close to those of Brouwer's; however, Poincaré's thinking cannot be assimilated to Heyting's definition of semi-intuitionism.

Let us return to Heyting's characterization of intuitionists that we began with. Can one say that (1) and (2) were *influenced* by Poincaré? If so, in what sense(s)?

Concerning the first point, regarding the claim that mathematics has not only a formal but also a contentual signification, one can, following Colin McLarty, contrast three kinds of interpretation: *banal, expansive and restrictive* [Mc Larty 1997, 97]. The banal interpretation merely says mathematics requires something beyond formal rigor[1]. The expansive interpretation claims that the content of mathematics goes beyond any formalization. The restrictive interpretation rejects the view that some standard mathematics is inaccessible to intuition.

Brouwer was a *restrictive* intuitionist. The application of his mental construction principle to semantics and ontology lead to his rejecting both the universal validity of the principle of the Excluded Middle (by conceiving a restrictive alternative logic) and the continuum of classical analysis. The question is whether Poincaré was more than a banal "intuitionist". His oft-quoted statement [cf., for example, Largeault 1993, 37 or Resnik 1996, 459] in favor of banal intuitionism is this: "Logic, which alone give certainty, is the instrument of demonstration; intuition is the instrument of invention" [VS, 23 (37), SM, (130)]. This, however, was intended only as a summary of a discussion about the distinction between *sensible intuition* and analytic procedures. Some pages later, he underlined that *pure intuition* gives certainty too and enables us to demonstrate and to invent [VS, 25 (39)][2].

[1] This is expressed by Pierre Boutroux [1920, 170]: 'There is some other thing beyond the logical form [of mathematics]".

[2] Poincaré, in considering pure intuition as a faculty of the intellect, draws on the Neo-kantian heritage.

One may expect from the literature that, as a *philosopher*, Poincaré was both, expansive and restrictive concerning foundational questions. Clearly, Poincaré didn't reject classical logic. His view, formulated in modern terms, was that the various formal-logical theories don't express the proof-theoretical structures that constitute our understanding of mathematics [Cf. SM, 149 (159)]. He rejected the invariance of mathematical reasoning with regards to its contents and promoted, so to speak, a "local" conception of mathematical reasoning according to which "a 'gap' is no longer a logical gap but, rather, a gap in *mathematical understanding*" [Detlefsen 1992, 366, 360] and, as such, a gap that cannot be eliminated through formalization. "Mathematical reasoning is thus no longer to be seen as a primarily logical relationship between *propositions* , but rather as an epistemic relationship between *judgments*" [Detlefsen 1993, 270].

Concerning the restrictive character of Poincaré's philosophical position, Brouwer recalled quite rightly his predicativism and his rejection of the actual infinite that were both consequences of his pragmaticism. Poincaré, he noted, "blames in logistics the *petitio principii* and in cantorism the hypothesis of the actual infinite" [Brouwer 1907, 176].

Let's assume that it makes sense to distinguish Poincaré as philosopher from Poincaré as mathematician. Could one also say that even the *mathematician* Poincaré was more than a banal "intuitionist" in the sense of attributing to mathematics some contentual significance?

In order to discuss this question we need to clarify in what sense the content of mathematics goes "beyond any formalization".

Taking the term "intuitionism" for the view "that mathematics must ultimately be based on the irreducible intuition of counting, of the natural numbers as a potentially infinite sequence", Harold Edwards [1988, 140], thinks to have found a common factor for proponents "including Henri Poincaré, L.E.J. Brouwer, Herrmann Weyl and Errett Bishop" [ibid.]. According to this perspective, Poincaré could perhaps be called semi-intuitionist because he was intuitionist only in arithmetic, or perhaps also in set theory concerning the methods of proofs used therein. I examine only the first possibility, arithmetic. What does it mean that we have an *irreducible* intuition of counting? Here the second point of Heyting's characteristic is important: it requires an irreducible intuition of counting which, at the same time, has to be immediately grasped by our minds independently of our experience. But unfortunately, Poincaré did not think that the sequence of natural numbers is completely independent of all experience:

"We have the faculty of conceiving that a unit may be added to a collection of units. *Thanks to experiment, we have had the occasion of exercising this faculty and are conscious of it* ;" (SH, 24; my emphasis)

In other words, the schema R :

$$N(I)$$

$$\forall x[N(x) \rightarrow N(xI)],$$

is occasioned by experience without having to be empirical.

Now, as is well known, Poincaré considered the principle of complete induction to be a synthetic *a priori* principle because it is "imposed upon us with such a force that we could not conceive of the contrary proposition" (SH, 48)[3]. But why is it not analytical? Poincaré's concept of an analytical judgment is not the

[3] Poincaré's understanding of Kant on synthetic a priori judgments is debatable; see (Ben-Menahem 2001, 476; especially note 14).

Kantian one but that of Frege and concerns the means of proof: the principle is not analytic because it is "inaccessible by analytical proof" [SH, 12]. One can fix an operation which, if applied to every chosen numeral n, leads to a proof of $N(n)$. But how are we to be sure that *all* members can be generated by an application of R? And how can we secure the validity of arithmetic without applying induction? This would clearly be a vicious circle. And there is even yet no uniform proof-skeleton, because the length of proof that is the number of applications of *modi ponentes* depends on n. It follows, then, that a single proof is not to be read as apprehension of a schema-instance:

> "A construction only becomes interesting when it can be placed side by side with other analogous constructions for forming species of the same genus. [. . .] The analytical process 'by construction' [. . .] leaves us at the same level. We can only ascend by mathematical induction" [SH, 16].

According to Poincaré, mathematical induction cannot be identified with an 'analytical' construction process. It is not on the same level because it expresses, so to say, an infinity of *modi ponentes*, which means that it is synthetic. Technically, we have the fact that the direct clauses of the inductive definition of numerals (that is R) do not analytically imply the principle of induction (or equivalently, the final clause[4]). The principle of induction

> "is only the *affirmation* of the power of the mind which knows it can conceive of the indefinite repetition of the same act, when the act is once possible. *The mind has a direct intuition of this power, and experiment can only be for it an occasion of using it, and thereby of becoming conscious of it.*" [SH, 13, my emphasis].

The awareness of the mind's power of indefinite repetition (R) is also occasioned by experience, *e.g.* of concrete stroke concatenation, and it is the affirmation of this capacity that yields the induction principle. The intuition of the mind's power is *a priori* because the action-schema is the result of our own creation. A pure intuition can be distinguished from simple evidence by the fact that it refers to what can be *done* instead of merely to something that is. The principle of induction is synthetic because it is not generated but only *represented* by the action schema: the principle postulates a *survey* of a potentially reiterated stroke-concatenation, or a *survey* of a potentially reiterated *modus ponens*.

What is important here is that the principle is thought to be the structural component of empirical constructions: the induction principle is occasioned by experience without having to be empirical and constitutes as such the theoretical part of our knowledge. Experience is the *ratio cognoscendi* of the principle of induction. So it seems clear why the negation of the principle is not conceivable: the principle of induction is a *form of understanding*, in the sense that "we can form no conception of an experience which would be best interpreted as a violation of it" (cf. [Folina 1992, 32]). The universal proposition of induction, as used by Poincaré, is neither used qua convention nor qua intuition in the sense of Heyting's criteria. So, Poincaré's influence on this latter, though not very deep, is nonetheless not completely disconnected[5]. In general, experience is for Poincaré by no means a sufficient source of knowledge; it plays, above all, the role of making us aware of the existence of certain structures of the mind to which we have to accommodate our experience, directly as in the case of complete induction or, by the introduction of conventions, as in geometry.

Moreover, Poincaré himself asserted that complete induction is only the simplest of all "other similar principles, offering the same essential characteristics" [SM 149/150 (159/160)]. Such analogous principles are especially the awareness of our capacity to construct a continuum of any dimension, called topological intuition. This concept pre-exists, according to Poincaré, in our mind as a form of reason and the awareness of them is occasioned by experience [Cf. DP, 157, 134sq.; SH, 87sq (107)].

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[4] Cf. [Poincaré 1906, 303]. The final clause says that the only instances are those provided by the clauses of R (cf. Kleene 1980, 21/22).

[5] Beth's judgment seems quite striking: 'Poincaré's ideas stimulated Brouwer', not more but not less [Beth 1956, 233].

We will see in the following, that the elaboration of *conventions* in geometry is analogous to the "genesis" of the induction principle although the *a priori* status of complete induction clearly contrasts with that of conventions. Thereby the edge of Poincaré's creative virtue in arithmetic, *i.e.* repetition, is also necessary in the psychophysiological genesis of geometry:

> "Repetition [. . .] has given to space its essential characteristics; now, repetition supposes time; this is enough to tell that time is logically anterior to space" [VS, 72]

Surely, there is a 'familiar resemblance' with Brouwer's time progression but only if one substitutes 'ontologically' for 'logically' [Février 1981,

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II. — Poincaré is conscious of the fact, that sensible reproduction can only be arranged in sensible space, and that we do not represent geometrical figures, but only reason about them (SH, 56). Under these circumstances, how do we link the sensible space to geometric space?[1]

The solution turns on a very sophisticated psycho-physiological genesis:[2] geometrical space is obtained by choosing the language of groups to serve as the tool of reasoning about representations of muscular sensations. As the main result of this genesis, Poincaré finds that certain actions, when accompanied by muscular sensations, define equivalence classes, called *displacements,* and that each set of displacement classes forms a group in the mathematical sense. Thereby the edge of Poincaré's creative virtue in arithmetic, *i.e.* repetition, is also necessary in the psychophysiological genesis of geometry:

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[1] The essential properties of geometric space are: "1st, it is continuous; 2nd, it is infinite; 3rd, it is of three dimensions; 4th, it is homogeneous—that is to say, all its

points are identical one with another, 5th, it is isotropic. Compare this now with the framework of our representations and sensations, which I may call representative space" (SH, 52). In fact, "it is neither homogeneous nor isotropic; we cannot even say that it is three dimensional" (SH, 56).

After arguing the idea that the genesis of space depends on the genesis of a group of transformations in his first papers about that question, Poincaré seems, in 1903 to concede to the idea of the existence of an amorphous continuum as a kind of primary frame for the genesis of geometry and geometrical space. Poincaré, for some reasons of coherence, is compelled to revise parts of his original theory and to renounce his claim that there is no need to presuppose a "3-dimensional *Zahlenmannigfaltigkeit*" as the matter on which groups of transformations act. He revised his theory in two papers first published in the *Revue de métaphysique et de morale* and later in *La Valeur de la science.* In these two papers, Poincaré placed a new emphasis on topology which he presented as qualitative geometry as a *prior* to quantitative geometries.

In his paper about the nature of geometry[6], after claiming the impossibility of the arguing convincingly that Euclidean geometry is *a priori* – mathematically or physically, Brouwer nevertheless conceded the existence of an *a priori* form of the perception of the world of experience that he called "the intuition of time or intuition of two-in-one"[7] which he defined as the mathematical intuition of continuous time, extraneous to the idea of measure as conceived by a singular subject. Because of this claim, it can be argued that Brouwer was already defending a kind of intuitionist point of view in geometry, at least in the weak understanding in the sense of Edwards or Heyting.

For Brouwer, geometry is the science of "properties of spaces of one or several dimensions". Geometers study and classify sets, geometrical transformations and groups of transformations in these spaces. Brouwer confer a particular status to the groups of topology, that is, the groups of continuous transformations because of their vastnes and because "a figure only unfolds its internal properties while for its definition only invariants of the largest possible group are used" [Brouwer 1909, 118].

The question is: Does Poincaré's revision of his claim and his putting *Analysis situs* at the basis of geometry make him a defender of a point of view analogous to Brouwer's? Such a claim may be reinforced by the strong link between Poincaré's definition of topological dimension and that of Brouwer (inspite of the latter's criticism of the former's argumentation). Our claim is that this analogy is totally superficial and that the geometrical conceptions of Poincaré are radically different from those of Brouwer for the same reason as in arithmetic.

The geometrical conventionalism of Poincaré is well-known but the conventional aspects of geometry are not restricted only to the choice of the most "commode" group of transformations. In his paper, published in the American journal *The Monist* in 1898 which at that time was the more

[6] [Brouwer 1909].
[7] [Brouwer 1909], p. 116.

profound of his contributions to this question, Poincaré did not elude the problems of compensations and approximation nor the more general problem of the relationships between experiences and laws: we impose on Nature through some active operations of the mind these compensations or these laws but this imposition permanently needs the use of conventions.

> When experience teaches us that a certain phenomenon does not correspond *at all* to these law**s**, we strike it from the list of displacements. When it teaches us that a certain change obeys them *only approximatively*, we consider the change, *by an artificial convention*, as the resultant of two other component changes. The first component is regarded as a displacement rigorously satisfying the laws of which I have just spoken, while the second component, which is small, is regarded as a qualitative alteration. [Poincaré 1898, 11]

Poincaré stressed several times that one of the consequences of his theory is that there is no need **for** a primary notion of space; the notion of space follows from the consideration of properties of displacements (which are changes of sensations). In 1898, there was no problem for Poincaré: Geometry precedes Space.

From this point of view, Poincaré's conceptions are original compared to predecessors such as Riemann, Helmholtz, Lie or Klein, who presupposed the existence of an amorphous *continuum* in the development off their geometrical theories.

> […] we must distinguish in a group the form and the matter [material]. For Helmholtz and Lie the matter of the group existed previously to the form, and in geometry the matter is a *Zahlenmannigfaltigkeit* of three dimensions.[8] […] It is only by the introduction of the group, that they made of it a measurable magnitude, that is to say a veritable space. Again, the origin af this non-measurable continuum of three dimensions remains imperfectly explained. [Poincaré 1898, 40]

This "3-dimensional non measurable *continuum*" is the space of *analysis situs* which will be considered later by Poincaré as the qualitative geometry. In 1898, Poincaré claimed that he didn't need the prerequisite of a three dimensional continuum. Very often, the hypothesis of a general space on which the geometry is constructed comes with the idea of a general geometry or a qualitative geometry which more or less embraces all metrical geometries. We understand why at that time, Poincaré, for his philosophical thinking could not agree with the general geometry of Calinon which presupposes the existence of an amorphous space. Moreover, it is untenable to use Calinon's theory to defend a psycho-physiologic point of view of the genesis of geometry since for Calinon, geometry has no experimental foundation. In the same manner, Poincaré rejected theories (like Russell's)[9] which put projective geometry at the beginning of the genesis of metrical geometry. All these theories reduce the importance of experience to the choice of the metrical geometry in which we describe our common or physical experience, all admitting a general *a priori* geometrical framework.

In his 1898 paper, Poincaré put forward a (mathematical) explanation of the three dimensions of space. He observed that the Euclidean group, selected after many conventions, can be seen as acting

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[8] Poincaré adds that for him, « on the contrary, the form exists before the matter ».

[9] See [Nabonnand 2000a].

on a space of three, four or five dimensions.[10] The choice of a three dimensional space is justified by considerations of commodity. Unfortunately, Poincaré's argument is vicious because the choice of the Euclidean group was grounded on Lie's classification of groups of transformations acting on $\mathbb{R}^3$ and admitting a fundamental invariant. In 1903, Poincaré (after his works about analysis situs and his first debate with Russell about the status of axioms of geometry) returnes to the question of dimension of space in a paper titled "L'espace et ses trois dimensions"[11].

This paper deals with the question of the status of the space in Analysis situs which Poincaré presents as the common background for the Euclidean and non-Euclidean spaces. After claiming once more that Euclidean geometry could not be a "form imposed on our sensibility", he asks if the amorphous continuum of the Analysis situs is a "form imposed to our sensibility":

> And then comes a question: Is not this amorphous that our analysis has allowed to survive a form imposed upon our sensibility? If so, we should have to enlarge the prison in which this sensibility is confined, but it would still remain a prison. [Poincaré 1905, 40]
>
> The same questions concerning the truths of Euclidean geometry come up anew about the theorems of *analysis situs*. Are they obtainable by deductive reasoning? Are they the disguised conventions? Are they experimental truths? Are they the characteristics of a form imposed either upon our sensibility or upon our understanding? [Poincaré 1905, 40-41][12]

Poincaré answers to these questions with the analysis of the genesis of the geometry. The general properties of the amorphous continuum are obtained with the same process as the metrical properties. In 1898, Poincaré argued that the notion of displacement was obtained as a change of the sensible (or representative) space. It is as if the displacements set up a continuous group of transformations without the groupt acting on the representative space:

> The complexus of our sensations has without doubt furnished us with a sort of matter, but there is a striking contrast between the grossness of this matter and the subtle precision of the form of our group. [Poincaré 1898, 41]

At the same time that the experiment of the composition of displacements makes us to do as if the displacements form a group, we consider the repetition of displacements. When the displacement D is very small, we experiment what Poincaré calls physical continuum. We can sum up this notion by the formula:

$$9D = 10\ D,\ 10D = 11D,\ 9D < 11D$$

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[10] For more details, see [Nabonnand 2000b].

[11] [Poincaré 1903]

[12] Et alors une question se pose : ce continuum amorphe, que notre analyse a laissé subsister, n'est-il pas une forme imposée à notre sensibilité ? Nous aurions élargi la prison dans laquelle cette sensibilité est enfermée, mais ce serait toujours une prison.

Les mêmes questions qui se posaient à propos des vérités de la géométrie euclidienne, se posent de nouveau à propos des théorèmes de l'Analysis Situs. Peuvent-ils être obtenus par un raisonnement déductif ? Sont-ce des conventions déguisées ? Sont-ce des vérités expérimentales ? Sont-ils les caractères d'une forme imposée soit à notre sensibilité, soit à notre entendement ?

As Poincaré claimed, this formula is intolerable for our reason and this empirical data impose on us the need to create the mathematical *continua* so as ensure the continuity of the group of displacements. Of course, this solution imposes itself because it is in agreement with other properties of the group. If these properties were incompatible with the hypothesis of continuity, we would be forced to choose another hypothesis. For Poincaré, the group of displacements resulting from the analysis of changes of sensations and the geometrical groups of transformations entertain the same kind of relationship as do physical and mathematical continua.

> We first study its form agreeably to the formula of the physical continuum, and since there is something repugnant to our reason in this formula we reject it and substitute for it that of the continuous group which, potentially, pre-exists in us, but which we originally know only by its form. The gross matter which is furnished us by our sensations was but a crutch for our infirmity, and served only to force us to fix our attention upon the pure idea which we bore about in ourselves previously. [Poincaré 1898, 41]

The choice of the mathematical continuity results from convention inferences; the geometrical space is continuous because the group of geometrical transformations is. The aim of *analysis situs* is to study properties of mathematical space which are linked with the property of continuity; the choice of the 3-dimensionality of space is the result of inference commodity concerning the internal action of the group of transformations on the set of its subgroups.[13]

In 1905, Poincaré advocated for a significantly different explanation of the choice of the hypothesis of the continuity of space. Poincaré began by asking what " is meant when we say that a mathematical continuum or that a physical continuum has two or three dimensions "[14]. He introduced the notion of cut in the context of physical continuum. If any two (physically) discernible elements of a physical continuum *C* can be linked by a path of indiscernible elements (a physically continuous path), the physical continuum *C* is called "all in one piece (*d'un seul tenant*)". A cut is an arbitrary set of elements of *C*. According **to** the choosen cut, the physical continuum *C* reamins "d'un seul tenant" (i.e. any two discernable element can be linked by a path of indiscernable elements without leaving *C* and without crossing the cut) or on the contrary, it will be divided by the cut. Poincaré pointed out that these definitions are all suggested by experience:

> It will be noticed that all these definitions are constructed in setting out solely from this very simple fact, that two manifolds of impressions sometimes can be discriminated, sometimes can not be. [Poincaré 1905, 43]

We say that a physical continuum is 1-dimensional if we cut it by one or several discernable elements. A physical continuum is said 2-dimensional if we can cut it by one or several 1-dimensional cut (and implied it cannot be cut by one or several discernable elements). And so on.[15]

[13] For more details, see [Nabonnand 2000].
[14] [Poincaré 1905], 42.
[15] Poincaré specified that this definition is the same as the one of geometers who defined surfaces as boundaries of volumes, and lines as boundaries of surfaces and points as boundaries of lines. He added that this form of definition is "applicable not to the mathematical continuum, but to the physical continuum, which alone is susceptible of representation" [Poincaré 1905, 44].

This definition of the definition of physical continuum allowed Poincaré to break the vicious circle of his former theory of the 3-dimensionality of space. In a first time, Poincaré abandoned part of his early theory. Certainly, the physical continuum formed by the displacements is 6-dimensional and this result comes from experience. Poincaré claimed he could maintain his previous theory and keep infering the 3-dimensionality of the geometrical space from the internal action of the group of its subgroup. But it would be difficult because he would still have to justify the utilization of Lie's classification of groups of transformations acting on $\mathbb{R}^3$. Poincaré didn't allude to this problem. He simply argued that such a theory could not be completely satisfying because it does not respect the hierarchy of our spatial sensations:

> […]; and when we shall have shown how the notion of this continuum [the group of displacements] can be formed and how that of space may be deduced from it, it might always be asked why space of three dimensions is much more familiar to us than this continuum of six dimensions, and consequently doubted wether it was this detour that the notion of space was formed in the human mind. [Poincaré 1905, 57-58][16]

So Poincaré added to his theory a new stage explaining why we are guided (by experience) towards a 3-dimensional physical continuum. Here, the most essentiel space is that which is associated to the sense of touch; Poincaré shows that a necessary condition for the setting up of the tactile space is that tactile sensations are not falsified by muscular ones which are associated with our moves. Each space which is associated with our fingers is a 3-dimensional physical continuum whereof Poincaré proved on the basis of experiences that we are guided to consider them all as similar; he argued that we are also guided to assimilate the visual space to these tactile spaces.

So if Poincaré in 1905, at the beginning af his genesis of the geometry, included 3-dimensional physical continuum, he didn't do so for epistemological reason, nor for reasons linked to intuition, nor for mathematical reason. For reasons of coherence, Poincaré needed to admit a rough notion of 3-dimensional space to justify his utilization of Lie's classification of 6-parameters groups of transformations acting on $\mathbb{R}^3$ and admiting a significant invariant; in addition, he wanted to explain why the 3-dimensionality of geometrical space is so immediate. But he pointed out that as in his previous theory, experience play an essential role at each stage of the genesis of geometry. Nevertheless, Poincaré did not lapse into fruitless and beatific empiricism. Indeed, as he emphasized, "experience does not prove to us that space has three dimensions; it only proves to us that it is convenient to attribute three to it, because thus the number of fillips is reduced to a minimum"[17].

The choice of conventions is justified according to the cases by considerations concerning accomodation to our experience, or by reasons of practical or mathematical commodity. The choice of a convention opens a space of possibilities of actions, among which, giving meaning and defining new objects. As Poincaré recalled, a convention is never imposed by experience. It is only the result of a capacity of our mind and experience **provides us** the occasion to use this capacity. According to Poincaré, we have innate capacities to set up groups or continuum. Understand the genesis of geometry is to understand how experiences provide the innate capacities of our mind the occasion

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[16] This kind of cognitive and genetic argument is typical of Poincaré. He refused to consider projective geometry as a type of general geometry because "this evidently is not the manner in which our geometrical notions were formed" [Poincaré 1898, 35].

[17] [Poincaré 1905, 69]

to manifest themselves. So conventions are not intuitions themselves, nor the result of an intuition; they are the result of the activation by experiences of one or several innate capacities of our mind. It is the same with the principles of reasoning, for example the recurrence principle. Principle of recurrence is “only the affirmation of the power of the mind which knows it can conceive of the indefinite repetition of the same act, when the act is once possible”.

> The mind has a direct intuition of this power, and experiment can be for it an opportunity of using it, and thereby of becoming conscious of it. [Poincaré 1902, 13]

So experience provides our mind the occasion to use his capacities; when we do not have choice, the role of experience stops here and the result is a principle of reasoning. When we have choice, experience helps us to choose and the result is a convention. By no means, the second characteristic of Heyting can be applied to Poincaré. For Poincaré, mathematical theories and methods are dependent on experience since it provides our mind the occasion to apply its innate capacities.

CONCLUSION (en discussion avec Gerhard)

Il pourrait sembler que par certains côtés les conceptions mathématiques de Poincaré et Brouwer soient proches. On pense d’abord aux critiques du logicisme Poincaré et Brouwer, mais on notera aussi l’importance cruciale accordée à l’idée de répétition indéfinie pour justifier l’induction complète que pour introduire les notions de continu (mathématique) ou celle de groupe de transformations. Tous deux privilégient la présentation de la géométrie à partir de la théorie des groupes de transformations et refusent d’accorder un caractère a priori à la géométrie euclidienne.

Mais cette proximité de conceptions n’est en fait qu’apparente et les points de vue de Poincaré et de Brouwer sont conceptuellement radicalement différents. En effet, la possibilité de la répétition indéfinie est chez Brouwer une sorte d’instantiation de l’intuition pure du temps alors que Poincaré n’y voit que le résultat de l’excitation par l’expérience d’une capacité innée de l’esprit à « concevoir la répétition indéfinie d’un même acte dès que cet acte est une fois possible » [SH41]. De même, les notions de continus mathématiques ou de groupes de transformations sont obtenues grâce à la capacité innée de notre esprit à former des continus ou des groupes mais Poincaré prend bien soin de préciser que l’expérience fournit l’occasion à ces capacités de se manifester. De plus, l’utilisation de ces notions pour rendre compte de l’expérience est une convention.

Poincaré (à partir de 1903) et Brouwer se retrouvent pour accorder à l’Analysis situs un caractère fondamental. Mais chez Brouwer, l’Analysis situs occupe (suivant le principe de subordination de Klein) une place fondamentale car le groupe des transformations continues est le groupe le plus général opérant sur un continuum. Chez Poincaré, l’Analysis situs est la partie (certes communes aux trois géométries euclidienne et non-euclidiennes) qui étudie spécifiquement les propriétés liées à la continuité du groupe de transformations par rapport auquel on rapporte nos considérations spatiales. Enfin, Brouwer admet en prélable l’existence d’un continuum sur lequel opère les transformations continues. La démarche de Poincaré (en 1903) est beaucoup plus génétique ; Poincaré montre que nous avons l’expérience (sensible par le sens du toucher) d’un continuum physique de dimension trois ; l’expérience de ce continuum physique permet de justifier l’utilisation de la classification de Lie des groupes de transformations opérant sur $\mathbb{R}^3$ lorsque Poincaré décrit le processus de choix du groupe euclidien. L'espace géométrique euclidien et ses éléments restent construits à partir de l'étude de la structure de ce groupe.

En adoptant la présentation des géométries par l'étude des groupes de transformation, Poincaré signalait ainsi son insatisfaction par rapport à l'approche purement formelle des mathématiques. Tout en reconnaissant l'équivalence mathématique des présentations axiomatiques ou de celle utilisant la théorie des groupes, Poincaré, dans sa recension des *Grundlagen der Geometrie*, souligne que le choix de l'approche axiomatique par Hilbert montre que "le point de vue logique paraît seul l'intéresser" et qu'il néglige de considérer le "fondement" des axiomes et leur "origine psychologique".[18] Si la présentation axiomatique satisfait aux exigences de rigueur, Poincaré lui préfère pour des raisons cognitives celle par les groupes de transformations.[19] En effet, la notion de transformations géométriques est beaucoup plus adaptée pour montrer comment l'expérience participe à la conceptualisation des notions mathématiques et à leur mise en forme. L'originalité philosophique de Poincaré par rapport aux tenants de l'empirisme, de l'intuitionisme ou du formalisme réside dans la manière avec laquelle il rend compte du rôle de l'expérience dans la genèse des concepts essentiels de la géométrie : ni simple abstraction, ni indépendance, ni réduction à une simple manipulation de signes. Sa théorie permet en particulier de mesurer l'importance du niveau linguistique ; aux contraire des intuitionistes, il ne nie pas que le niveau langagier dans la pratique des mathématiques est fondamental sans pour autant réduire celle-ci à la seule manipulation de symbole abstrait. Le langage est certes essentiel dans la mise en forme mathématique mais il l'est aussi définitivement dans les moments de conceptualisation car le langage est partie intégrante du cadre expérimental dans lequel s'exercent les capacités innées de l'esprit. En ce sens, se résout ce que Detlefsen désignait comme le dilemme de Poincaré:

I. All the inferences used in mathematical proof are of a purely logical character (rigor)

II. Conclusions of mathematical proofs can, and often do, constitue extensions of the mathematical knowledge represented by the premises.

[18] [Poincaré 1902b], 45.
[19] Poincaré précise que les géométries (non-archimédiennes, non-pascaliennes, …) considérées par Hilbert peuvent toutes être présentées comme des géométries associées à un groupe :
[…] les groupes de transformations au sens de Lie ne semblent plus jouer qu'un rôle secondaire. C'est du moins ce qu'il semble quand on lit le texte même de M. Hilbert. Mais, si l'on y regardait de plus près, on verrait que chacune de ces géométries est encore l'étude d'un groupe. » [Poincaré 1902b, 45].

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