Calcul tensoriel : exemples

- Démontrer que $\alpha Tr\left(\underline{\underline{T}}\right)=Tr\left(\alpha\underline{\underline{T}}\right)$

$$\left.\begin{aligned}
&\alpha Tr\left(\underline{\underline{T}}\right)=\alpha\underline{\underline{T}}:\underline{\underline{I}}=\alpha\left(T_{ij}\underline{e_i}\otimes\underline{e_j}\right):\left(\delta_{kl}\underline{e_k}\otimes\underline{e_l}\right)=\alpha\left(T_{ij}\underline{e_i}\otimes\underline{e_j}\right):\left(\underline{e_k}\otimes\underline{e_k}\right)=\alpha T_{kk}\\
&Tr\left(\alpha\underline{\underline{T}}\right)=\left(\alpha\underline{\underline{T}}\right):\underline{\underline{I}}=\left(\alpha T_{ij}\underline{e_i}\otimes\underline{e_j}\right):\left(\delta_{kl}\underline{e_k}\otimes\underline{e_l}\right)=\left(\alpha T_{ij}\underline{e_i}\otimes\underline{e_j}\right):\left(\underline{e_k}\otimes\underline{e_k}\right)=\alpha T_{kk}
\end{aligned}\right\}CQFD$$

- Démontrer la relation suivante : $Tr\left(\underline{\underline{T}}\cdot\underline{\underline{Grad}}\left(\underline{v}\right)\right)=T_{ij}\dfrac{\partial v_j}{\partial x_i}$

$$Tr\left(\underline{\underline{T}}\cdot\underline{\underline{Grad}}\left(\underline{v}\right)\right)=Tr\left(\left(T_{ij}\underline{e_i}\otimes\underline{e_j}\right)\cdot\left(\frac{\partial v_k}{\partial x_l}\underline{e_k}\otimes\underline{e_l}\right)\right)=Tr\left(T_{ij}\frac{\partial v_j}{\partial x_l}\underline{e_i}\otimes\underline{e_l}\right)=T_{ij}\frac{\partial v_j}{\partial x_i}$$

- Donner l'expression de $\underline{\underline{T}}$ dans sa base principale $\left(\underline{e_I},\underline{e_{II}},\underline{e_{III}}\right)$

$$\underline{\underline{T}}=T_I\underline{e_I}\otimes\underline{e_I}+T_{II}\underline{e_{II}}\otimes\underline{e_{II}}+T_{III}\underline{e_{III}}\otimes\underline{e_{III}}$$

- Démontrer que $\underline{x}\cdot\underline{\underline{T}}^T\cdot\underline{y}=\underline{y}\cdot\underline{\underline{T}}\cdot\underline{x}$

$$\left.\begin{aligned}
\underline{x}\cdot\underline{\underline{T}}^T\cdot\underline{y}&=\left(x_i\underline{e_i}\right)\cdot\left(T_{kj}\underline{e_j}\otimes\underline{e_k}\right)\cdot\left(y_l\underline{e_l}\right)\\
&=\left(x_iT_{ki}\underline{e_k}\right)\cdot\left(y_l\underline{e_l}\right)\\
&=x_iT_{ki}y_k=y_kT_{ki}x_i\\
\underline{y}\cdot\underline{\underline{T}}\cdot\underline{x}&=\left(y_i\underline{e_i}\right)\cdot\left(T_{jk}\underline{e_j}\otimes\underline{e_k}\right)\cdot\left(x_l\underline{e_l}\right)\\
&=\left(y_iT_{ik}\underline{e_k}\right)\cdot\left(x_l\underline{e_l}\right)\\
&=y_iT_{ik}x_k
\end{aligned}\right\}CQFD$$

- Démontrer que si $\underline{\underline{T}}=\underline{\underline{A}}\cdot\underline{\underline{B}}$ alors $\underline{\underline{T}}^T=\underline{\underline{B}}^T\cdot\underline{\underline{A}}^T$

$$\left.\begin{aligned}
&\underline{\underline{T}}=\underline{\underline{A}}\cdot\underline{\underline{B}}=A_{ij}B_{jk}\underline{e_i}\otimes\underline{e_k}\quad\Rightarrow\quad\underline{\underline{T}}^T=A_{kj}B_{ji}\underline{e_i}\otimes\underline{e_k}\\
&\underline{\underline{B}}^T\cdot\underline{\underline{A}}^T=\left(B_{ji}\underline{e_i}\otimes\underline{e_j}\right)\cdot\left(A_{lk}\underline{e_k}\otimes\underline{e_l}\right)=B_{ji}A_{lj}\underline{e_i}\otimes\underline{e_l}=A_{lj}B_{ji}\underline{e_i}\otimes\underline{e_l}
\end{aligned}\right\}CQFD$$