Chapter 2
Euclid: Doing and Showing

Reading older mathematical texts always involves a hermeneutical dilemma: in order to make sense of the mathematical content of a given old text one wants to interpret it in modern terms; in order to see the difference between the modern mathematical thinking and older ways of mathematical thinking one wants to avoid anachronisms and understand the old text in its own terms (Unguru 1975). Any scholar studying older mathematics needs to find a way between the Scylla of "antiquarianism" that seeks the scholar's conversion into a person living during a different historical epoch, and the Charybdis of radical "presentism" that finds in older texts nothing but a minor part of today's standard mathematical curricula and wholly ignores the historical change of basic patterns of mathematical thinking.[1] My way through the channel is the following. I read Euclid's text verbatim (relying on Heiberg's edition of the original Greek (Euclides 1883–1886) and using Fitzpatrick's new English translation Euclid 2011), consider its most important modern interpretations (including overtly anachronistic ones), criticize some of these interpretations on the basis of textual evidences, and finally suggest some alternative interpretations. In order to prevent the risk of losing the main argument behind the following historical details I formulate now my general conclusion. Contrary to popular opinion Euclid's geometry is not a system of propositions some of which have a special status of axioms while some other are derived from the axioms according to certain rules of logical inference. It can be rather described after Friedman as "a form of rational argument" (Friedman 1992, p. 94),[2] where some non-propositional content (including non-propositional first principles) is indispensable. Precipitating what follows (see particularly Sect. 3.6) let me mention

[1]Being between Scylla and Charybdis is an idiom deriving from Greek mythology. Scylla and Charybdis were mythical sea monsters noted by Homer. Scylla was rationalized as a rock shoal (described as a six-headed sea monster) on the Italian side of the strait and Charybdis was a whirlpool off the coast of Sicily. They were regarded as a sea hazard located close enough to each other that they posed an inescapable threat to passing sailors; avoiding Charybdis meant passing too close to Scylla and vice versa. (after WikipediA)

[2]See the full quote from Friedman in the end of Sect. 3.5.

A. Rodin, *Axiomatic Method and Category Theory*, Synthese Library 364,
DOI 10.1007/978-3-319-00404-4_2,

that certain non-propositional principles also make part of modern formal theories in the form of *syntactic rules*. As we shall now see in Euclid the non-propositional aspect of mathematical reasoning plays a more prominent role.

2.1 Demonstration and "Monstration"

All Propositions of Euclid's *Elements* (with few easily understandable exceptions) fit into the scheme described by Proclus in his *Commentary* (Proclus 1970) as follows:

> Every Problem and every Theorem that is furnished with all its parts should contain the following elements: an *enunciation*, an *exposition*, a *specification*, a *construction*, a *proof*, and a *conclusion*. Of these *enunciation* states what is given and what is being sought from it, a perfect *enunciation* consists of both these parts. The *exposition* takes separately what is given and prepares it in advance for use in the investigation. The *specification* takes separately the thing that is sought and makes clear precisely what it is. The *construction* adds what is lacking in the given for finding what is sought. The *proof* draws the proposed inference by reasoning scientifically from the propositions that have been admitted. The *conclusion* reverts to the *enunciation*, confirming what has been proved. (Proclus 1970, p. 203, italic is mine)

It is appropriate to notice here that the term "proposition", which is traditionally used in translations as a common name of Euclid's problems and theorems, is not found in the original text of the *Elements*: Euclid numerates these things throughout each Book without naming them by any common name. (The reader will shortly see why this detail is important.) The difference between problems and theorems is explained in Sect. 2.4 below. Let me now show how Proclus' scheme applies to Proposition 5 of the First Book (Theorem 1.5), which is a well-known theorem about angles of the isosceles triangle. References in square brackets are added by the translator; some of them will be discussed later on. Words in round brackets are added by the translator for stylistic reason. Words in bold are borrowed from the above Proclus' quote. Throughout this Chap. 2 I write these words in italic when I use them in Proclus' specific sense (Fig. 2.1).

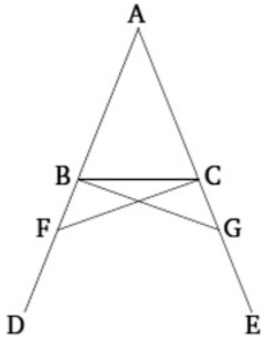
enunciation:

> For isosceles triangles, the angles at the base are equal to one another, and if the equal straight lines are produced then the angles under the base will be equal to one another.

exposition:

> Let *ABC* be an isosceles triangle having the side *AB* equal to the side *AC*; and let the straight lines *BD* and *CE* have been produced further in a straight line with *AB* and *AC* (respectively). [Postulate 2].

Fig. 2.1 Isosceles triangle theorem (Theorem 1.5 of Euclid's *Elements*)



specification:

I say that the angle *ABC* is equal to *ACB*, and (angle) *CBD* to *BCE*.

construction:

For let a point *F* be taken somewhere on *BD*, and let *AG* have been cut off from the greater *AE*, equal to the lesser *AF* [Proposition 1.3]. Also, let the straight lines *FC*, *GB* have been joined. [Postulate 1]

proof:

In fact, since *AF* is equal to *AG*, and *AB* to *AC*, the two (straight lines) *FA*, *AC* are equal to the two (straight lines) *GA*, *AB*, respectively. They also encompass a common angle *FAG*. Thus, the base *FC* is equal to the base *GB*, and the triangle *AFC* will be equal to the triangle *AGB*, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Proposition 1.4]. (That is) *ACF* to *ABG*, and *AFC* to *AGB*. And since the whole of *AF* is equal to the whole of *AG*, within which *AB* is equal to *AC*, the remainder *BF* is thus equal to the remainder *CG* [Axiom 3]. But *FC* was also shown (to be) equal to *GB*. So the two (straight lines) *BF*, *FC* are equal to the two (straight lines) *CG*, *GB* respectively, and the angle *BFC* (is) equal to the angle *CGB*, while the base *BC* is common to them. Thus the triangle *BFC* will be equal to the triangle *CGB*, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Proposition 1.4]. Thus *FBC* is equal to *GCB*, and *BCF* to *CBG*. Therefore, since the whole angle *ABG* was shown (to be) equal to the whole angle *ACF*, within which *CBG* is equal to *BCF*, the remainder *ABC* is thus equal to the remainder *ACB* [Axiom 3]. And they are at the base of triangle *ABC*. And *FBC* was also shown (to be) equal to *GCB*. And they are under the base.

conclusion:

> Thus, for isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another. (Which is) the very thing it was required to show.

An obvious difference between Proclus' analysis of the above theorem and its usual modern analysis is the following. For a modern reader the proof of this theorem begins with Proclus' *exposition* and includes Proclus' *specification*, *construction* and *proof*. Thus for Proclus the *proof* is only a part of what we call today the proof of this theorem. Also notice that Euclid's theorems conclude with the words "which ... was required to *show*" (as correctly translates Fitzpatrick) but not with the words "what it was required to *prove*" (as inaccurately translates Heath 1926). The standard Latin translation of this Euclid's formula as *quod erat demonstrandum* is also inaccurate. These inaccurate translations conflate two different Greek verbs: "apodeiknumi" (English "to prove", Latin "demonstrare") and "deiknumi" (English "to show", Latin "monstrare"). The difference between the two verbs can be clearly seen in the two Aristotle's *Analytics*: Aristotle uses the verb "apodeiknumi" and the derived noun "apodeixis" (proof) as technical terms in his syllogistic logic, and he uses the verb "deiknumi" in a broader and more informal sense when he discusses epistemological issues (mostly in the *Second Analytics*). Without trying to trace here the history of Greek logical and mathematical terminology and speculate about possible influences of some Greek writers on some other writers, I would like to stress the remarkable fact that Aristotle's use of verbs "deiknumi" and "apodeiknumi" agrees with Euclid's and Proclus'. In my view this fact alone is sufficient for taking seriously the difference between the two verbs and distinguishing between *proof* and "showing" (or otherwise between *demonstration* and *monstration*).[3]

One may think that the difference between the current meaning of the word "proof" in today's mathematics and logic and the meaning of Proclus' *proof* (Greek "apodeixis") is a merely terminological issue, which is due to difficulties of translation from Greek to English. I shall try now to show that this terminological difference points on a deeper problem, which is not merely linguistic. In today's logic the word "proof" stands for a logical inference of certain conclusion from some given premises. In fact this is what by and large was meant by proof also by Aristotle and Proclus. Indeed, looking at the *proof* (in Proclus' sense) of Euclids Theorem 1.5 we see that it also qualifies as a proof in the modern sense: we have here a number of premises (which I make explicit in the next Section) and certain

[3] As far as mutual influences are concerned two things are certain: (i) Proclus read Aristotle and (ii) Aristotle had at least a basic knowledge of the mathematical tradition, on which Euclid later elaborated in his *Elements* (as Aristotle's mathematical examples clearly show Heath 1949). It remains unclear whether Aristotle's work could influence Euclid. In my view this is unlikely. However Aristotle's logic certainly played an important role in later interpretations and revisions of Euclid's *Elements*. I leave this interesting issue outside of the scope of this book.

conclusions derived from those premises. It is irrelevant now whether or not this particular inference is valid according to today's logical standards; what I want to stress here is only the general setting that involves some premises, an inference (probably invalid) and some conclusions. This core meaning of the word "proof" (Greek "apodeixis") hardly changed since Proclus' times.

So we get a problem, which is clearly not only terminological: Is it indeed justified to describe the *exposition*, the *specification* and the *construction* as elements of the proof or one should rather follow Proclus and consider these things as independent constituents of a mathematical theorem?

The question of *logical significance* of the *exposition*, the *specification* and the *construction* in Euclid's geometry has been discussed in the literature; in what follows I shall briefly describe some tentative answers to it. However before doing this I would like to stress that this question may be ill-posed to begin with. As far as one assumes, first, that the theory of Euclid's *Elements* is (by and large) sound and, second, that any sound mathematical theory is an axiomatic theory in the modern sense, then, in order to make these two assumptions mutually compatible, one has to describe the *exposition*, the *specification* and the *construction* of each Euclid's theorem as parts of the proof of this theorem and specify their logical role and their logical status. I shall not challenge the usual assumption according to which Euclid's mathematics is by and large sound. (I say "by and large" in order to leave some room for possible revisions and corrections of Euclid's arguments and thus avoid controversies about the question whether a given interpretation of Euclid is authentic or not. Although I pay more attention to textual details than it is usually done in modern logical reconstructions of Euclid's reasoning, I am not going to criticize these reconstructions by pointing to their anachronistic character.) However I shall challenge the other assumption according to which any sound mathematical theory is an axiomatic theory in the modern sense. Since I do not take this latter assumption for granted I do not assume from the outset that the problematic elements of Euclid's reasoning (the *exposition*, the *specification* and the *construction*) play some *logical* role, which only needs to be made explicit and appropriately understood. In what follows I try to describe how these elements work without making about them any additional assumptions and only then decide whether the role of these elements qualifies as logical or not.

2.2 Are Euclid's Proofs Logical?

Let's look at Euclid's Theorem 1.5 more attentively. I begin its analysis with its *proof*. Among the premisses of this *proof*, one may easily identify Axiom (Common Notion) 3 according to which

(Axiom 3): If equal things are subtracted from equal things then the remainders are equal and the preceding Theorem 1.4 according to which

> (Proposition 1.4): If two triangles have two corresponding sides equal, and have the angles enclosed by the equal sides equal, then they will also have equal bases, and the two triangles will be equal, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles.

I shall not comment on the role Theorem 1.4 in this *proof* (which seems to be clear) but say few things about the role of the Axiom 3.

Here is how exactly the Axiom (Common Notion) 3 is used in the above Euclid's *proof*. First, *by construction* we have

Con1: $BF \equiv AF - AB$ and **Con2**: $CG \equiv AG - AC$

which is tantamount to saying that point B lays between points A, F and point C lays between points A, G. Second, *by hypothesis* we have

Hyp: $AB = AC$

and once again *by construction*

Con3: $AF = AG$

Now we see that we have got the situation described in Axiom 3: equal things are subtracted from equal things. Using this Axiom we conclude that $BF = CG$.

Notice that Axiom 3 applies to all "things" (mathematical objects), for which the relation of *equality* and the operation of *subtraction* make sense. In Euclid's mathematics this relation and this operation apply not only to straight segments and numbers but also to geometrical objects of various sorts including *figures*, angles and solids. Since Euclid's equality is not interchangeable with identity I use for the two relations two different symbols: namely I use the usual symbol for Euclid's equality (even if this equality is not quite usual), and use symbol $\equiv$ for identity. My use of symbols $+$ and $-$ is self-explanatory.[4]

The other four Euclid's Axioms (not to be confused with Postulates!) have the same character. This makes Euclid's Axioms in general, and Axiom 3 in particular, very unlike premises like **Con1–3** and **Hyp**, so one may wonder whether the very idea of treating these things on equal footing (as different *premises* of the same inference) makes sense. More precisely we have here the following choice. One option is to interpret Axiom 3 as the following implication:

$$\{(a \equiv b - c)\&(d \equiv e - f)\&(b = e)\&(c = f)\} \rightarrow (a = d)$$

[4]The *difference* $A - B$ of two figures A, B is a figure obtained through "cutting" B out of A; the *sum* $A + B$ is the result of *concatenation* of A and B. These operations are not defined up to *congruence* of figures (for there are, generally speaking, many possible ways, in which one may cut out one figure from another) but, according to Euclid's Axioms, these operations are defined up to Euclid's *equality*. This shows that Euclid's *equality* is weaker than *congruence*: according to Axiom 4 congruent objects are equal but, generally, the converse does not hold. In the case of (plane) figures Euclid's equality is equivalent to the equality (in the modern sense) of their air.

and then use it along with **Con1–3** and **Hyp** for getting the desired conclusion through *modus ponens* and other appropriate rules. This standard analysis involves a fundamental distinction between premises and conclusion, on the one hand, and rules of inference, on the other hand. It assumes that in spite of the fact that Euclid (as most of other mathematicians of all times) remains silent about logic, his reasoning nevertheless follows some implicit logical rules. The purpose of logical analysis in this case is to make this "underlying logic" (as some philosophers like to call it) explicit.

The other option that I have in mind is to interpret Axiom 3 itself as a rule rather than as a premiss. Following this rule, which can be pictures as follows:

$$\frac{(a \equiv b - c), (d \equiv e - f), (b = e), (c = f)}{(\mathrm{a} = \mathrm{d})} \text{ (Axiom 3)}$$

one derives from **Con1–3** and **Hyp** the desired conclusion. So interpreted Axiom 3 hardly qualifies as a *logical* rule because it applies only to propositions of a particular sort (namely, of the form $X = Y$ where X, Y are some *mathematical* objects of appropriate types). This Axiom cannot help one to prove that Socrates is mortal. Nevertheless the domain of application of this rule is quite vast and covers the whole of Euclid's mathematics. An important advantage of this analysis is that it doesn't require one to make any assumption about hidden features of Euclid's thinking: unlike the distinction between logical rules and instances of applications of these rules the distinction between axioms and premises like **Con1–3** and **Hyp** is explicit in Euclid's *Elements*.

There is also a historical reason to prefer the latter reading of Euclid's Common Notions. Aristotle uses the word "axiom" interchangeably with the expressions "common notions", "common opinions" or simply "commons" for what we call today logical laws or logical principles but not for what we call today axioms. Moreover in this context he systematically draws an analogy between mathematical common notions and his proposed logical principles (laws of logic). This among other things provides an important historical justification for calling Euclid's Common Notions by the name of Axioms. It is obvious that mathematics in general and mathematical common notions (axioms) in particular serve for Aristotle as an important source for developing the very idea of logic. Roughly speaking Aristotle's thinking, as I understand it, is this: behind the basic principles of mathematical reasoning spelled out through mathematical common notions (axioms) there are other yet more general principles relevant to reasoning about all sorts of beings and not only about mathematical objects. The fact that Euclid, according to the established chronology, is younger than Aristotle by some 25 years (Euclid's dates unlike Aristotle's are only approximate) shouldn't confuse one. While there is no strong evidence of the influence of Aristotle's work on Euclid, the influence on Aristotle of the same mathematical tradition, on which Euclid elaborated, is clearly documented in Aristotle's writings. In particular, Aristotle quotes Euclid's Axiom 3 (which, of course, Aristotle could know from another source) almost verbatim.[5]

However important Aristotle's argument in the history of Western thought may be, there is no reason to take it for granted every time when we try today to interpret Euclid's *Elements* or any other old mathematical text. Whatever is one's philosophical stance concerning the place of logical principles in human reasoning one can see what kind of harm can be made if Aristotle's assumption about the primacy of logical and ontological principles is taken straightforwardly and uncritically: one treats Euclid's Axioms on equal footing with premisses like **Con1–3** and **Hyp** and so misses the law-like character of the Axioms. Missing this feature doesn't allow one to see the relationships between Greek logic and Greek mathematics, which I just sketched.

Having said that I would like to repeat that Euclid's *proof* (apodeixis) is the part of Euclid's theorems, which more resembles what we today call proof (in logic) than other parts Euclid's theorems. For this reason in what follows I shall call inferences in Euclid's *proofs*, which are based on Axioms, *protological* inferences

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[5]Here are some quotes:

> By first principles of proof [as distinguished from first principles in general] I mean the common opinions on which all men base their demonstrations, e.g. that one of two contradictories must be true, that it is impossible for the same thing both be and not to be, and all other propositions of this kind. (Met. 996b27-32, Heath's translation, corrected)

Here Aristotle refers to a logical principle as "common opinion". In the next quote he compares mathematical and logical axioms:

> We have now to say whether it is up to the same science or to different sciences to inquire into what in mathematics is called axioms and into [the general issue of] essence. Clearly the inquiry into these things is up to the same science, namely, to the science of the philosopher. For axioms hold of everything that [there] is but not of some particular genus apart from others. Everyone makes use of them because they concern being qua being, and each genus is. But men use them just so far as is sufficient for their purpose, that is, within the limits of the genus relevant to their proofs. Since axioms clearly hold for all things qua being (for being is what all things share in common) one who studies being qua being also inquires into the axioms. This is why one who observes things partly [=who inquires into a special domain] like a geometer or a arithmetician never tries to say whether the axioms are true or false. (Met. 1005a19-28, my translation)

Here is the last quote where Aristotle refers to Axiom 3 explicitly:

> Since the mathematician too uses common [axioms] only on the case-by-case basis, it must be the business of the first philosophy to investigate their fundamentals. For that, when equals are subtracted from equals, the remainders are equal is common to all quantities, but mathematics singles out and investigates some portion of its proper matter, as e.g. lines or angles or numbers, or some other sort of quantity, not however qua being, but as [...] continuous. (Met. 1061b, my translation)

The "science of philosopher" otherwise called the "first philosophy" is Aristotle's logic, which in his understanding is closely related to (if not indistinguishable from) what we call today ontology. After Alexandrian librarians we call today the relevant collection of Aristotle's texts by the name of *metaphysics* and also use this name for a relevant philosophical discipline.

and distinguish them from inferences of another type that I shall call *geometrical* inferences. This analysis is not incompatible with the idea (going back to Aristotle) that behind Euclid's protological and geometrical inferences there are inferences of a more fundamental sort, that can be called *logical* in the proper sense of the word. However I claim that Euclid's text as it stands provides us with no evidence in favor of this strong assumption. One can learn Euclid's mathematics and fully appreciate its rigor without knowing anything about logic just like Moliere's M. Jourdain could well express himself long before he learned anything about prose!

Whether or not the science of logic really helps one to improve on mathematical rigor – or this is rather the mathematical rigor that helps one to do logic rigorously – is a controversial question that I shall discuss throughout this book and suggest an answer only in the last Chapter. The purpose of my present reading of Euclid is at the same time more modest and more ambitious than the purpose of logical analysis. It is more modest because this reading doesn't purport to assess Euclid's reasoning from the viewpoint of today's mathematics and logic but aims at reconstructing this reasoning in its authentic archaic form. It is more ambitious because it doesn't take the today's viewpoint for granted but aims at reconsidering this viewpoint by bringing it into a historical perspective.

2.3 Instantiation, Objecthood and Objectivity

Let us now see where the premises **Hyp** and **Con1–3** come from. As I have already mentioned they actually come from two different sources: **Hyp** is assumed *by hypotheis* while **Con1–3** are assumed *by construction*. I shall consider here these two cases one after the other.

The notion of hypothetic reasoning is an important extension of the core notion of axiomatic theory outlined above; it is well-treated in the literature and I shall not cover it here in full. I shall consider only one particular aspect of hypothetical reasoning as it is present in Euclid. The hypothesis that validates **Hyp**, informally speaking, amounts to the fact that Theorem 1.5 tells us something about isosceles triangles (rather than about objects of another sort). The corresponding definition (Definition 1.20) tells us that two sides of the isosceles triangle are equal. However to get from here to **Hyp** one needs yet another step. The *enunciation* of Theorem 1.5 refers to isosceles triangles *in general*. But **Hyp** that is involved into the *proof* of this Theorem concerns only *particular* triangle ABC. Notice also that the *proof* concludes with the propositions $ABC = ACB$ and $FBC = GCB$ (where ABC, ACB, FBC and GCB are angles), which also concern only *particular* triangle ABC. This conclusion differs from the following *conclusion* (of the whole Theorem), which almost verbatim repeats the *enunciation* and once again refers to isosceles triangles and their angles in general terms.

The wanted step that allows Euclid to proceed from the *enunciation* to **Hyp** is made in the *exposition* of this Theorem, which introduces triangle ABC as an "arbitrary representative" of isosceles triangles (in general). In terms of modern logic this step can be described as the *universal instantiation*:

$$\forall x P(x) \Longrightarrow P(a/x)$$

where $P(a/x)$ is the result of the substitution of individual constant a at the place of all free occurrences of variable x in $P(x)$. The same notion of universal instantiation allows for interpreting Euclid's *specification* in the obvious way. The reciprocal backward step that allows Euclid to obtain the *conclusion* of the Theorem from the conclusion of the *proof* can be similarly described as the *universal generalization*:

$$P(a) \Longrightarrow \forall x P(x)$$

(which is a valid rule only under certain conditions that I skip here).

As long as the *exposition* and the *specification* are interpreted in terms of the universal instantiation these operations are understood as logical inferences and, accordingly, as element of proof in the modern sense of the word. A somewhat different – albeit not wholly incompatible – interpretation of Euclid's *exposition* and *specification* can be straightforwardly given in terms of Kant's *transcendental aesthetics* and *transcendental logic* developed in his *Critique of Pure Reason* (Kant 1999). Kant thinks of the traditional geometrical *exposition* not as a logical inference of one proposition from another but as a "general procedure of the imagination for providing a concept with its image"; a representation of such a general procedure Kant calls a *schema* of the given concept (A140). Thus for Kant any individual mathematical object (like triangle *ABC*) always comes with a specific *rule* that one follows constructing this object in one's imagination and that provides a link between this object and its corresponding concept (the concept of isosceles triangle in our example). According to Kant the representation of general concepts by imaginary individual objects (which Kant for short also describes as "construction of concepts") is the principal distinctive feature of mathematical thinking, which distinguishes it from a philosophical speculation.

> Philosophical cognition is rational cognition from concepts, mathematical cognition is that from the construction of concepts." But to construct a concept means to exhibit a priori the intuition corresponding to it. For the construction of a concept, therefore, a non-empirical intuition is required, which consequently, as intuition, is an individual object, but that must nevertheless, as the construction of a concept (of a general representation), express in the representation universal validity for all possible intuitions that belong under the same concept, either through mere imagination, in pure intuition, or on paper, in empirical intuition.... The individual drawn figure is empirical, and nevertheless serves to express the concept without damage to its universality, for in the case of this empirical intuition we have taken account only of the action of constructing the concept, to which many determinations, e.g., those of the magnitude of the sides and the angles, are entirely indifferent, and thus we have abstracted from these differences, which do not alter the concept of the triangle.
>
> Philosophical cognition thus considers the particular only in the universal, but mathematical cognition considers the universal in the particular, indeed even in the individual...(A713-4/B741-2).

Kant's account can be understood as a further explanation of what the instantiation of mathematical concepts amounts to; then one may claim that the Kantian interpretation of Euclid's *exposition* and *specification* is compatible with its interpretation as the universal instantiation in the modern sense. However the Kantian interpretation doesn't suggest by itself to interpret the instantiation as a logical procedure, i.e., as an inference of a proposition from another proposition. As the above quote makes it clear Kant describes the instantiation as a cognitive procedure of a different sort.

Now coming back to Euclid we must first of all admit that the *exposition* and the *specification* of Theorem 1.5 as they stand are too concise for preferring one philosophical interpretation rather than another. Euclid introduces an isosceles triangle through Definition 1.20 providing no rule for constructing such a thing. (This example may serve as an evidence against the often-repeated claim that every geometrical object considered by Euclid is supposed to be constructed on the basis of Postulates beforehand.) Nevertheless given the important role of constructions in Euclid's geometry, which I explain in the next Section, the idea that every geometrical object in Euclid has an associated construction rule, appears very plausible. There is also another interesting textual feature of Euclid's *specification* that in my view makes the Kantian interpretation more plausible.

Notice the use of the first person in the *specification* of Theorem 1.5: "I say that . . . .". In *Elements* Euclid uses this expression systematically in the *specification* of every theorem. Interpreting the *specification* in terms of universal instantiation one should, of course, disregard this feature as merely rhetorical. However it may be taken into account through the following consideration. While the *enunciation* of a theorem is a general proposition that can be best understood á la Frege in the abstraction from any human or inhuman thinker, i.e., independently of any thinking *subject*, who might believe this proposition, assert it, refute it, or do anything else about it, the core of Euclid's theorems (beginning with their *exposition*) involves an individual thinker (individual subject) that cannot and should not be wholly abstracted away in this context. When Euclid *enunciates* a theorem this *enunciation* does not involve – or at least is not supposed to involve – any particularities of Euclid's individual thinking; the less this *enunciation* is affected by Euclid's (or anyone else's) individual writing and speaking style the better. However the *exposition* and the*specification* of the given theorem essentially involve an *arbitrary* choice of notation ("Let *ABC* be an isosceles triangle...”), which is an individual choice made by an individual mathematician (namely, made by Euclid on the occasion of writing his *Elements*). This individual choice of notation goes on par with what we have earlier described as *instantiation*, i.e. the choice of one individual triangle (triangle *ABC*) of the given type, which serves Euclid for proving the general theorem about *all* triangles of this type. The *exposition* can be also naturally accompanied by drawing a diagram, which in its turn involves the choice of a particular shape (provided this shape is of the appropriate type), to leave alone the choices of its further features like color, etc.

Thus when in the *specification* of Theorem 1.5 we read "I say that the angle *ABC* is equal to *ACB*" we indeed do have good reason to take Euclid's wording seriously.

For the sentence "angle *ABC* is equal to *ACB*" unlike the sentence "for isosceles triangles, the angles at the base are equal to one another" has a feature that is relevant only to one particular presentation (and to one particular diagram if any), namely the use of letters A,B,C rather than some other letters.[6] The words "I say that ..." in the given context stress this situational character of the following sentence "angle *ABC* is equal to *ACB*". What matters in these words is, of course, not Euclid's personality but the reference to a particular act of speech and cognition of an individual mathematician. Proving the same theorem on a different occasion Euclid or anybody else could use other letters and another diagram of the appropriate type.

A competent reader of Euclid is supposed to know that the choice of letters in Euclid's notation is arbitrary and that Euclid's reasoning does not depend of this choice. The arbitrary character of this notation should be distinguished from the general arbitrariness of linguistic symbols in natural languages. What is specific for the case of *exposition* and *specification* is the fact that here the arbitrary elements of reasoning (like notation) are sharply distinguished from its invariant elements. To use Kant's term we can say that behind the notion according to which the choice of Euclid's notation is arbitrary (at least at the degree that letters used in this notation are permutable) and according to which the same reasoning may work equally well with different diagrams (provided all of them belong to the same appropriate type) there is a certain invariant *schema* that sharply limits such possible choices. This schema not simply *allows* for making some arbitrary choices but *requires* every possible choice in the given reasoning to be wholly arbitrary. This requirement is tantamount to saying that subjective reasons behind choices made by an individual mathematician for presenting a given mathematical argument are strictly irrelevant to the "argument itself" (in spite of the fact that the argument cannot be formulated without making such choices). In general talks in natural languages there is no similar sharp distinction between arbitrary and invariant elements. When I write this paper I can certainly change some wordings without changing the sense of my argument but I am not in a position to describe precisely the scope of such possible changes and identify the intended "sense" of my argument with a mathematical rigor. This is because my present study is philosophical and historical but not purely mathematical.

Thus Euclid's *exposition* serves for the formulation of a given universal proposition in terms, which are suitable for a particular act of mathematical cognition made by an individual mathematician. This aspect of the *exposition* is not accounted for by the modern notion of universal instantiation. It may be argued that this aspect of the *exposition* needs not be addressed in a *logical* analysis of Euclid's mathematics that aims at explication of the *objective meaning* of Euclid's reasoning and may well leave aside cognitive aspects of this reasoning. I agree that this latter issue lies out of the scope of logical analysis in the usual sense of the term but I disagree that the objective meaning of Euclid's reasoning can be properly understood without addressing this issue. Euclid's mathematical reasoning is *objective* due

[6] Although the choice of letters in Euclid's notation is arbitrary the *system* of this notation is not. This traditional geometrical notation has a relatively stable and rather sophisticated syntax, which I briefly describe in what follows.

to a mechanism that allows one to make universally valid inferences through one's individual thinking. Whatever the "objective meaning" might consist of this mechanism must be taken into account.

2.4 Proto-Logical Deduction and Geometrical Production

Remind that the *proof* of Euclid's Theorem 1.5 uses not only premiss **Hyp** assumed "by hypothesis" but also premisses **Con 1–3** (as well as a number of other premisses of the same type) assumed "by construction". I turn now to the question about the role of Euclid's *constructions* (which, by the way, are ubiquitous not only in geometrical but also in arithmetical Books of the *Elements*) and more specifically consider the question how these *constructions* support certain premisses that are used in following *proofs*.

As it is well-known Euclid's geometrical constructions are supposed to be realized "by ruler and compass". In the *Elements* this condition is expressed in the *Elements* through the following three

Postulates:

1. Let it have been postulated to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any center and radius.

(I leave out of my present discussion two further Euclid's Postulates including the controversial Fifth Postulate.)

Before I consider popular interpretations of these Postulates and suggest my own interpretation let me briefly discuss the very term "postulate", which is traditionally used in English translations of Euclid's *Elements*. Fitzpatrick translates Euclid's verb "aitein" by English verb "to postulate" following the long tradition of Latin translations, where this Greek verb is translated by Latin verb "postulare". However according to today's standard dictionaries the modern English verb "to postulate" does not translate the Greek verb "aitein" and the Latin verb "postulare" in general contexts: the modern dictionaries translate these verbs into "to demand" or "to ask for". This clearly shows that the meaning of the English verb "to postulate" that derives from Latin "postulare" changed during its lifetime.[7]

Aristotle describes a postulate (aitema) as what "is assumed when the learner either has no opinion on the subject or is of a contrary opinion" (*An. Post.* 76b); further he draws a contrast between postulates and *hypotheses* saying that the latter

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[7] I reproduce here Fitzpatrick's footnote about Euclid's expression "let it be postulated":

> The Greek present perfect tense indicates a past action with present significance. Hence, the 3rd-person present perfect imperative *Hitesthw* could be translated as "let it be postulated", in the sense "let it stand as postulated", but not "let the postulate be now brought forward". The literal translation "let it have been postulated" sounds awkward in English, but more accurately captures the meaning of the Greek.

appear more plausible to the learner than the former (ibid.). It is unnecessary for my present purpose to go any further into this semantical analysis trying to reconstruct an epistemic attitude that Euclid might have in mind "demanding" the reader to take his Postulates for granted. The purpose of the above philological remark is rather to warn the reader that the modern meaning of the English word "postulate" can easily mislead when one tries to interpret Euclid's Postulates adequately. So I suggest to read Euclid's Postulates as they stand and try to reconstruct their meaning from their context forgetting for a while what one has learned about the meaning of the term "postulate" from modern sources.

Euclid's Postulates are usually interpreted as propositions of a certain type and on this basis are qualified as axioms in the modern sense of the term. There are at least two different ways of rendering Postulates in a propositional form. I shall demonstrate them at the example of Postulate 1. This Postulate can be interpreted either as the following *modal* proposition:

(PM1): Given two different points it is always possible to draw a (segment of) straight-line between these points
or as the following *existential* proposition:

(PE1): For any two different points there exists a (segment of) straight-line lying between these points.

Propositional interpretations of Euclid's Postulates allow one to present Euclid's geometry as an axiomatic theory in the modern sense of the word and, more specifically, to present Euclid's geometrical constructions as parts of proofs of his theorems. Even before the modern notion of axiomatic theory was strictly defined in formal terms many translators and commentators of Euclid's *Elements* tended to think about his theory in this way and interpreted Euclid's Postulates in the modal sense. Later a number of authors (Hintikka and Remes 1974 and Dean et al. 2009) proposed different formal reconstructions of Euclid's geometry based on the existential reading of Postulates. According to Hintikka and Remes

> [R]eliance on auxiliary construction does not take us outside the axiomatic framework of geometry. Auxiliary constructions are in fact little more than ancient counterparts to applications of modern instantiation rules. (Hintikka and Remes 1976, p. 270)

The instantiation rules work in this context as follows. First, through the *universal instantiation* (which under this reading correspond to Euclid's *exposition* and *specification*) one gets some propositions like **Hyp** about certain particular objects (individuals) like AB and AC. Then one uses Postulates 1–3 reading them as existential axioms according to which the existence of certain geometrical objects implies the existence of certain further geometrical objects, and so proves the (hypothetical) existence of such further objects of interest. Finally one uses another instantiation rule called the rule of *existential instantiation*:

$$\exists x P(x) \Longrightarrow P(a)$$

and thus "gets" these new objects. Under this interpretation Euclid's *constructions* turn into logical inferences of sort. As Hintikka and Remes emphasize in their paper the principal distinctive feature of Euclid's *constructions* (under their interpretation) is that these constructions introduce some *new* individuals; they call such individuals "new" in the sense that these individuals are not (and cannot be) introduced through the universal instantiation of hypotheses making part the *enunciation* of the given theorem.

The propositional interpretations of Euclid's Postulates are illuminating because they allow for analyzing traditional geometrical constructions in modern logical terms. However they require a paraphrasing of Euclid's wording, which from a logical point of view is far from being innocent. In order to see this let us leave aside the epistemic attitude expressed by the verb "postulate" and focus on the question of *what* Euclid postulates in his Postulates 1–3. Literally, he postulates the following:

P1: to draw a straight-line from any point to any point.
P2: to produce a finite straight-line continuously in a straight-line.
P3: to draw a circle with any center and radius.

As they stand expressions P1–3 don't qualify as propositions; they rather describe certain *operations*! And making up a proposition from something which is not a proposition is not a innocent step. My following analysis is based on the idea that Postulates are *not* primitive truths from which one may derive some further truths but are primitive operations that can be combined with each other and so produce into some further operations. In order to make my reading clear I paraphrase P1–3 as follows:

(OP1): drawing a (segment of) straight-line between its given endpoints
(OP2): continuing a segment of given straight-line indefinitely ("in a straight-line)"
(OP3): drawing a circle by given radius (a segment of straight-line) and center (which is supposed to be one of the two endpoints of the given radius).

Noticeably none of OP1–3 allows for producing geometrical constructions out of nothing; each of these fundamental operation produces a geometrical object out of some other objects, which are supposed to be *given* in advance. The table below specifies inputs (operands) and outputs (results) of OP1–3:

Operation	Input	Output
OP1	Two (different) points	Straight segment
OP2	Straight segment	(Bigger) Straight segment
OP3	Straight segment and one of its endpoints	Circle

PE1 as it stands does not imply that there exists at least one point or at least one line in Euclid's geometrical universe. If there are no points then there are no lines either. Similar remarks can be made about the existential interpretation of other Euclid's Postulates. Thus the existential interpretation of Postulates by itself does not turn these Postulates into existential axioms that guarantee that Euclid's universe is non-empty and contains all geometrical objects constructible by ruler and compass. To meet this purpose one also needs to postulate the existence of at least two different points – and then argue that the absence of any counterpart of such an axiom in Euclid is due to Euclid's logical incompetence. My proposed interpretation of Postulates 1–3 as operations doesn't require such ad hoc stipulations and thus is more faithful not only to Euclid's text but also to a deeper structure of his reasoning.[8]

Hintikka and Remes describe Euclid's geometrical constructions as *auxiliary*. Such a description may be adequate to the role of geometrical constructions in today's practice of teaching the elementary geometry but not to the role of constructions in Euclid's *Elements*. Remind that Euclid's so-called Propositions are of two types: some of them are Theorems while some other are Problems (see again the above quotation from Proclus' *Commentary*). In the *Elements* Problems are at least as important as Theorems and arguably even more important: in fact the *Elements* begin and end with a Problem but not with a Theorem. As we shall now see when a given *construction* makes part of a problem rather than a theorem it cannot be described as auxiliary in any appropriate sense. We shall also see the modern title "proposition" is not really appropriate when we talk about Euclid's Problems: while *enunciations* of Theorems do qualify as propositions in the modern logical sense of the term *enunciations* of Problems do not.

I shall demonstrate these features at the well known example of Problem 1.1 that opens Euclid's *Elements*; my notational conventions remain the same as in the example of Theorem 1.5 (Fig. 2.2).

enunciation:

To construct an equilateral triangle on a given finite straight-line.

exposition:

Let AB be the given finite straight-line.

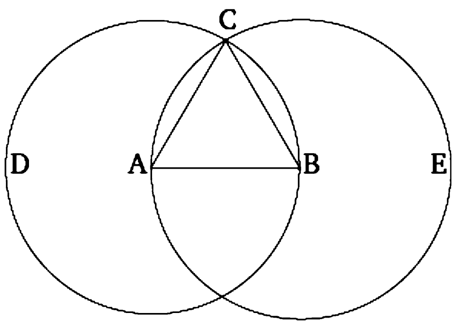

Fig. 2.2 Construction of regular triangle (Problem 1.1 of Euclid's *Elements*)

[8]Remind that the concepts of infinite straight line and infinite half-line (ray) are absent from Euclid's geometry; thus the result of OP2 is always a finite straight segment. However the result of OP2 is obviously not fully determined by its single operand. This shows that OP2 doesn't really fit the today's usual notion of algebraic operation.

specification:

So it is required to construct an equilateral triangle on the straight-line *AB*.

construction:

Let the circle *BCD* with center *A* and radius *AB* have been drawn [Postulate 3], and again let the circle *ACE* with center *B* and radius *BA* have been drawn [Postulate 3]. And let the straight-lines *CA* and *CB* have been joined from the point *C*, where the circles cut one another, to the points *A* and *B* [Postulate 1].

proof:

And since the point *A* is the center of the circle *CDB*, *AC* is equal to *AB* [Definition 1.15]. Again, since the point *B* is the center of the circle *CAE*, *BC* is equal to *BA* [Definition 1.15]. But *CA* was also shown (to be) equal to *AB*. Thus, *CA* and *CB* are each equal to *AB*. But things equal to the same thing are also equal to one another [Axiom 1]. Thus, CA is also equal to CB. Thus, the three (straight-lines) *CA*, *AB*, and *BC* are equal to one another.

conclusion:

Thus, the triangle *ABC* is equilateral, and has been constructed on the given finite straight-line *AB*. (Which is) the very thing it was required to do.

As one can see at this example *enunciations* of Problems are expressed in the same grammatical form as Postulates 1–3, namely in the form of infinitive verbal expressions. I read these expressions in the same straightforward way, in which I read Postulates: as descriptions of certain geometrical *operations*. The obvious difference between (*enunciations* of) Problems and Postulates is this: while Postulates introduce basic operations taken for granted (drawing by ruler and compass) Problems describe complex operations, which in the last analysis reduce to these basic operations. Such reduction is made through a *construction* of a given Problem: it performs the complex operation described in the *enunciation* of the problem through combining basic operations OP1–3 (and possibly some earlier performed complex operations). The procedure that allows for performing complex operations by combining a small number of repeatable basic operations I shall call a *geometrical production*. In Problem 1.1 the construction of regular triangle is (geometrically) *produced* from drawing the straight-line between two given points (Postulate 1) and drawing a circle by given center and radius (Postulate 3). This is, of course, just another way of saying that the regular triangle is constructed by ruler and compass; the unusual terminology helps me to describe Euclid's geometrical constructions more precisely.

Let us see in more detail how works Euclid's geometrical production. Basic operations OP1–3 like other (complex) operations need to be *performed*: in order to produce an output they have to be fed by some input. This input is provided through the *exposition* of the given Problem (the straight line *AB* in the above example).

OP1–3 are composed in the usual way well-known from today's algebra: outputs of earlier performed operations are used as inputs for further operations.[9]

Just like Postulates 1–3 *enunciations* of Problems can be read as modal or existential propositions (in the modern logical sense of the term). Then the (modified) *enunciation* of Problem 1.1 reads:

(1.1.M) it is possible to construct a regular triangle on a given finite straight-line:

or

(1.1.E) for all finite straight-line there exists a regular triangle on this line.

As soon as the *enunciations* of Euclid's Problems are rendered into the propositional form the Problems turn into theorems of a special sort. In the case of existential interpretation Problems turn into *existential* theorems that state (under certain hypotheses) that there exist certain objects having certain desired properties. However this is not what we find in Euclid's text as it stands. Every Euclid's Problem ends with the formula "the very thing it was required to do", not "to show" or "to prove". I can see no evidence in the *Elements* that justifies the idea that in Euclid's mathematics *doing* is less significant than *showing* and that the former is in some sense reducible to the latter. In the Second Part of this paper I shall argue that *doing* remains as much important in today's mathematics as it was in Greek mathematics, and that the idea of reducing mathematics to *showing* or *proving* (in the precise sense of modern logic) is a unfortunate philosophical misconception.

According to another popular reading Euclid's Problems are tasks or questions of sort. This version of modal propositional interpretation of Euclid's Problems involves a deontic modality rather than a possibility modality:

(1.1.D) it is required to construct a regular triangle on a given finite straight-line:

Indeed geometrical problems similar to Euclid's Problems can be found in today's Elementary Geometry textbooks as exercises. However the analogy between Euclid's Problems and school problems on construction by ruler and compass is not quite precise. *Enunciations* of Euclid's Problems just like the *enunciations* of Euclid's Theorems prima facie express no modality whatsoever. A deontic expression appears only in the *exposition* of the given Problem ("it is required to construct an equilateral triangle on the straight-line *AB*"). I don't think that this fact justifies the deontic reading of the *enunciation* because, as I have already explained

[9]Problem 1.1 involves a difficulty that has been widely discussed in the literature: Euclid does not provide any principle that may allow him to construct a point of intersection of the two circles involved into the *construction* of this Problem. This flaw is usually described as a *logical* flow. In my view it is more appropriate to describe this flow as properly*geometrical* and fill the gap in the reasoning by the following additional postulate (rather than an additional axiom):

Let it have been postulated to produce a point of intersection of two circles with a common radius.

Even if this additional postulates is introduced here purely ad hoc, the way in which it is introduced gives an idea of how Euclid's Postulates could emerge in the real history.

above, the *exposition* describes reasoning of an individual mathematician rather than presents this reasoning in an objective form. That every complex construction must be performed through Postulates and earlier performed constructions is an epistemic requirement, which is on par with the requirement according to which every theorem must be proved rather than simply stated. Remind that the *expositions* of Euclid's Theorems have the form "I say that…". This indeed makes an apparent contrast with the *expositions* of Problems that have the form "it is required to ….". However this contrast doesn't seem me to be really sharp. Euclid's expression "I say that…" in the given context is interchangeable with the expression "it is required to show that…", which matches the closing formula of Theorems "(this is) the very thing it was required to show". Euclid's expression "it is required to…" that he uses in the *expositions* of Problems similarly matches the closing formula of Problems "(this is) the very thing it was required to do". The requirement according to which every Theorem must be "shown" or "monstrated" doesn't imply, of course, that the *enunciation* (statement) of this Theorem has a deontic meaning. The requirement according to which every Problem must be "done" doesn't imply either that the *enunciation* of this Problem has something to do with deontic modalities.

The analogy between axioms and theorems, on the one hand, and postulates and problems, on the other hand, may suggest that Euclid's geometry splits into two independent parts one of which is ruled by (proto)logical deduction while the other is ruled by geometrical production. However this doesn't happen and in fact problems and theorems turn to be mutually dependent elements of the same theory. The above example of Problem 1.1 and Theorem 1.5 show how the intertwining of problems and theorems works. Theorems, generally, involve *constructions* (called in this case auxiliary), which may depend (in the order of geometrical production) on earlier treated problems (as the *construction* of Theorem 1.5 depends on Problem 1.3.) Problems in their turn always involve appropriate *proofs* that prove that the *construction* of the given theorem indeed performs the operation described in the *enunciation* of this theorem (rather than performs some other operation). Such *proofs*, generally, depend (in the order of the protological deduction) on certain earlier treated theorems (just like in the case of *proofs* of theorems). Although this mechanism linking problems with theorems may look unproblematic it gives rise to the following puzzle. Geometrical production produces geometrical objects from some other objects. Protological deduction deduces certain propositions from some other propositions. How it then may happen that the geometrical production has an impact on the protological deduction? In particular, how the geometrical production may justify premises assumed "by construction", so these premises are used in following *proofs*?

In order to answer this question let's come back to the premise **Con3** ($AF = AG$) from Theorem 1.5 and see what if anything makes it true. $AF = AG$ because Euclid or anybody else following Euclid's instructions constructs this pair of straight segments in this way. How do we know that by following these instructions one indeed gets the desired result? This is because the *construction* of Problem 1.3 that contains the appropriate instruction is followed by a *proof* that proves that this *construction* does exactly what it is required to do. *Construction* 1.3 in its turn uses *construction* 1.2 while *construction* 1.2 uses *construction* 1.1 quoted above. In other words *construction* 1.1 (geometrically) produces *construction* 1.2 and *construction* 1.2 in its turn produces *construction* 1.3. This geometrical production produces the relevant part of *construction* 1.5 (the construction of equal straight segments AF and AG) from first principles, namely from Postulates 1–3. In order to get the corresponding protological deduction of premise **Con3** from first principles we should now look at *proofs* 1.1, 1.2 and 1.3 and then combine these three proofs into a single chain. For economizing space I leave now details to reader and only report what we get in the end. The result is somewhat surprising from the point of view of the modern logical analysis. The chain of *constructions* leading to *construction* 1.5 involves a number of circles (through Postulate 3). Radii of a given circle are equal by definition (Definition 1.15). Thus by constructing a circle and its two radii, say, X and Y one gets a primitive (not supposed to be proved) premise $X = Y$. Having at hand a number of premises of this form and using Axioms as inference rules (but not as premises!) one gets the desired deduction of **Con3**. The fact that first principles of the protological deduction of **Con3** appear to be partly provided by a definition helps to explain why Euclid places his definitions among other first principles such as postulates and axioms.

The above analysis allows for disentangling the protological deduction of **Con3** from the geometrical production of straight segments AF, AG and so the aforementioned puzzle remains even after we have looked at Euclid's reasoning under a microscope. Even if we can describe in detail the impact of Problems to Theorems and vice versa it remains unclear how the two kind of things can possibly work together. Here is my tentative answer to this question. Every Euclid's *proof* involves only concrete premises like **Con3** and **Hyp**, which are related to certain individual objects. It is assumed that such a premise is valid only if the corresponding object is effectively constructed. (At least this concerns all premises "by construction"; as we have seen at the example of Theorem 1.5 hypothetic premises sometimes don't respect this rule.) This fundamental principle links Euclid's geometrical production and protological deduction together.

One may argue that my proposed analysis after all is not significantly different from the standard logical analysis of Euclid's geometrical reasoning according to which Euclid first proves that certain geometrical objects exist and only then prove some further propositions concerning properties of these objects. Is there indeed any significant difference between proving that such-and-such object exist and producing this object through what I call the geometrical production? There is of course a difference of a metaphysical sort between these two notions: to produce an object is not quite the same thing as to prove that certain object exists. But arguably this difference has no objective significance and so one may simply ignore it. There is however a further difference between the geometrical production and the mathematical existence, which seems me more important. Euclid's *Elements* contain two sets of rules, namely axioms and postulates, supposed to be applied to operations of two different sorts: axioms tell us how to derive equalities from other equalities while postulates tell us how to produce geometrical objects from other geometrical objects. A logical analysis of Euclid's geometry that involves a propositional (in particular existential) reading of postulates aims at replacing these two sets of rules by a single set of rules called *logical*. I would like to stress again that the results of my proposed analysis do not exclude the possibility of logical analysis. Such a replacement may be or be not a good idea but in any event logical rules are not made in the Euclid's text explicit and I do not see much point in saying that he uses rules of this sort implicitly. The fact that *we* can use today modern logic for interpreting Euclid is a completely different issue. An interpretation of Euclid's geometry by means of logical analysis can be indeed illuminating but one should not confuse oneself by believing that Euclid already had a grasp of modern logic even if could not formulate principles of this logic explicitly.

For further references I shall call the 6-part structure of Euclid's problems and theorems *Euclidean structure*. As the above analysis makes it clear the Euclidean structure does not fit into Hilbert's notion of axiomatic theory even when this latter notion is formulated in very general terms as in the above quotes. While Hilbert and his modern followers assume that a mathematical theory is a set of truths, some of which are assumed as axioms and some other are *logically inferred* from axioms, Euclid builds his theory through a combination of two different procedures, which I call protological deduction and geometrical production. Precipitating what follows I would like to mention here that Hilbert's view on mathematical theory (which is presented more accurately in the next Chapter) is not unique in the twentieth century. An influential alternative view has been put forward by Luitzen Egbertus Jan Brouwer; a relevant part of this alternative view is formulated by Brouwer's student Heyting as follows:

> One of Brouwer's main theses was that mathematics is not based on logic, but that logic is based on mathematics. [..] If mathematics consists of mental constructions, then every mathematical theorem is the expression of a result of a successful construction. The proof of the theorem consists in this construction itself, and the steps of the proof are the same as the steps of the mathematical construction. These are intuitively clear mental acts, and not applications of logical laws. (quoted by Scott 1970, p. 237)

This general description prima facie better fits Euclid's procedures than the modern axiomatic approach. The problem is that this description does not by itself provide us with an alternative general method of building mathematical theories. I postpone the discussion on this matter until Sect. 3.2 and conclude the present Chapter with a observation concerning the relevance of Euclid's way of theory-building to today's mathematical practice.

2.5 Euclid and Modern Mathematics

What has been said above may give one an impression that in Euclid's *Elements* we deal with an archaic pattern of mathematical thinking that has noting to do with today's mathematics. However this impression is wrong. In fact the Euclidean structure is apparently present in today's mathematics, perhaps in a slightly modified form. Consider the following example taken from a standard mathematical textbook (Kolmogorov and Fomin 1976, p. 100, my translation into English):

Theorem 2.3. *Any closed subset of a compact space is compact*

Proof. Let F be a closed subset of compact space T and $\{F_\alpha\}$ be an arbitrary centered system of closed subsets of subspace $F \subset T$. Then every F_α is also closed in T, and hence $\{F_\alpha\}$ is a centered system of closed sets in T. Therefore $\cap F_\alpha \neq \emptyset$. By Theorem 1 it follows that F is compact.

Although the above theorem is presented in the usual for today's mathematics form "proposition-proof", its Euclidean structure can be made explicit without reinterpretations and paraphrasing:

enunciation:

Any closed subset of a compact space is compact

exposition:

Let F be a closed subset of compact space T

specification: **absent**

construction:

[Let] $\{F_\alpha\}$ [be] an arbitrary centered system of closed subsets of subspace $F \subset T$.

proof:

[E]very F_α is also closed in T, and hence $\{F_\alpha\}$ is a centered system of closed sets in T. Therefore $\cap F_\alpha \neq \emptyset$. By Theorem 1 it follows that F is compact.

conclusion: **absent**

The absent *specification* can be formulated as follows:
I say that F is a compact space

while the absent *conclusion* is supposed to be a literal repetition of the *enunciation* of this theorem. Clearly these latter elements can be dropped for parsimony reason. In order to better separate the *construction* and the *proof* of the above theorem the authors could first construct set $\cap F_\alpha$ and only then prove that it is non-empty. However this variation of the classical Euclidean scheme also seems me negligible. I propose the reader to check it at other modern examples that the Euclidean structure remains today at work.

Does this mean that the modern notion of axiomatic theory is inadequate to today's mathematical practice just like it is inadequate to Euclid's mathematics? Such a conclusion would be too hasty. Arguably, in spite of the fact that today's mathematics preserves some traditional outlook it is essentially different. So the

"Euclidean appearance" of today's mathematics cannot be a sufficient evidence for the claim the Euclidean structure remains significant in it. In order to justify this claim a different argument is needed.

Before I try to provide such an argument I would like to point to the fact that the modern notion of axiomatic theory is used in today's mathematics in two rather different ways. First, it is used as a broad methodological idea that determines the general architecture of a theory but has no impact on details. Such an application of the modern axiomatic method is usually called *informal*. Second, the notion of axiomatic theory is used for building *formal* theories that contain a list of axioms and a set of theorems derived from these axioms according to explicitly specified rules of logical inference. In the next Chapter I shall describe the notion of formal axiomatic theory more precisely and try to explain the precise sense in which it is called formal.