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Nikolai I. Lobachevsky

# Pangeometry

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European Mathematical Society

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# Foreword

Lobachevsky wrote his *Pangeometry* in 1855, the year before his death, at a time when he was completely blind. He dictated two versions of that work, a first one in Russian, and a second one in French. The *Pangeometry* is a résumé of Lobachevsky's work on non-Euclidean geometry and its applications, and can be considered as his clearest account of the subject. It is also the conclusion of his life's work, and his last attempt to acquire recognition.

In this memoir, after a survey of the foundations of hyperbolic geometry, taken from his earlier treatises, Lobachevsky presented the bases of a complete theory of differential and integral calculus in hyperbolic space, and then developed the applications of this theory to the computation of definite integrals. Computing definite integrals using techniques from hyperbolic geometry was one of Lobachevsky's favourite topics, on which he had worked from his initial writings. It was also the main subject of his memoir *Géométrie Imaginaire*, first published in Russian in 1835, and then in French, in *Crelle's Journal*, in 1837. Lobachevsky's integrals represent areas of surfaces and volumes of bodies in hyperbolic 3-space. Computing the area or the volume of the same object in different manners turns out to be an efficient way of finding attractive formulae for some definite integrals. But besides the intrinsic value of the results obtained, there are several reasons why Lobachevsky worked out these computations. First of all, using hyperbolic geometry for the computation of integrals was a way of showing the usefulness of hyperbolic geometry (which at that time was a controversial subject) in another branch of mathematics, namely analysis. At another level, drawing consequences of the new axiom system, like finding values of known integrals using these new methods was a way of checking that the new geometric system was not self-contradictory, which indeed was a major concern for Lobachevsky.

The *Pangeometry* has already been translated several times. Two German translations were published in 1858 and in 1902, an Italian translation appeared in 1867, and an English translation of a small part of this work appeared in 1929. See [102] for the exact references.[1] The present edition provides, to the best of my knowledge, the first complete English translation of the *Pangeometry* to appear in print. It is made from Lobachevsky's French version. It is accompanied by notes and followed by a commentary.

There are several reasons for producing the present edition. The first and obvious one is that Lobachevsky's *Pangeometry* is an exposition of its author's original and most profound ideas, and it deserves to be widely accessible to the mathematical world and to historians of mathematics and science. The second reason is that reading this memoir will also be most instructive for students in geometry. Although the text is more than 150 years old, it will give most readers a fresh point of view on the subject, namely, a model-free point of view which is not to be found in most of the more modern textbooks on hyperbolic geometry. Working with models is fine, and generally very useful, because it is efficient for making computations, namely, by using techniques

[1] The bibliographical references are collected at the end of this volume.

of Euclidean geometry, including those of linear algebra. Furthermore, models of the hyperbolic space are also useful in other branches of mathematics. For instance, Poincaré's upper-half plane and disk models play essential roles in number theory, in complex analysis and in the theory of differential equations. Finally, working in models gives a systematic way of drawing pictures (which generally turn out to be nice pictures). Despite all these advantages, learning hyperbolic geometry through models is, in my opinion, intellectually and aesthetically less satisfying than in the model-free point of view, which requires more imagination and deep thought. I realised this fact only after having taught hyperbolic geometry for several years, using models as is usually done, and after I went through some of the ancient texts on the subject. In any case, the reader will form a personal opinion and choose between the model and the model-free points of view.

Students in geometry can use the *Pangeometry* at different levels. They can read this text at a superficial level, without going into the technical details. This will give them at least a flavour of the subject. But they can also try to go deeply into Lobachevsky's constructions and computations. This will need some investment of time and energy, but it is worth doing. Indeed, although it is relatively easy to follow the main stream of Lobachevsky's ideas, some of the details in the *Pangeometry* are rather obscure, and some of the computations are difficult to follow.

Let me recall Gauss's letter to the mathematician and astronomer C. L. Gerling, written on 8 February 1844, in which he compared Lobachevsky's writings to a "jungle through which it is difficult to find a passage and perspective, without first becoming acquainted with each tree that composes it."[2] Let me also quote Gauss's letter to the astronomer H. C. Schumacher, dated 28 November 1846: "In developing the subject, the author followed a road different from the one I took myself; Lobachevsky carried out the task in a masterly fashion and in a truly geometric spirit. I consider it a duty to call your attention to this book, since I have no doubt that it will give you a tremendous pleasure ..."[3]

Gauss's comments concern Lobachevsky's earlier writings. They show the respect that Gauss had for Lobachevsky's work. Concerning the *Pangeometry* itself, let me quote G. B. Halsted (1835–1922), a mathematician whose life was entirely dedicated to the defence and popularisation of non-Euclidean geometry, and in particular of Lobachevsky's writings, some of which he translated into English. In his biographical article *Lobachevsky* [62], Halsted wrote: "Though Lobachevsky's *Geometric Researches on the Theory of Parallels*, published in 1840, of which my English translation is now in its fourth edition and has been beautifully reproduced in Japan, remains even today the simplest introduction to the subject which has ever appeared; yet in it Lobachevsky has not reached that final breadth of view given first in John Bolyai's

[2] [die mehr einem verworrenen Walde gleichen, durch den es, ohne alle Bäume erst einzeln kennen gelernt zu haben, schwer ist, einen Durchgang und Übersicht zu finden.]

[3] [... materiell für mich Neues habe ich also im Lobatschewskyschen Werke nicht gefunden, aber die Entwickelung ist auf anderm Wege gemacht, als ich selbst eingeschlagen habe, und zwar von Lobatschewsky auf eine meisterhafte Art in ächt geometrischem Geiste. Ich glaube, Sie auf das Buch aufmerksam machen zu müssen, welches Ihnen gewiss ganz exquisiten Genuss gewähren wird.]

*Science of Absolute Space*, but also attained in Lobachevsky's last work *Pangéométrie*, which name is explicitly used as expressive of this final view."

Finally, let me quote J. Hoüel (1823–1886), another famous defender and translator (into French) of non-Euclidean geometry texts, who wrote in his *Notice sur la vie et les travaux de N.-I. Lobatchefsky* [76] that the *Pangeometry* is "one of Lobachevsky's most remarkable works regarding the clarity of exposition".

Reading the major texts of our mathematical ancestors is an expression of the esteem and gratitude that we owe them. This alone is a sufficient reason for publishing this English translation of the *Pangeometry*, and making it easily accessible to the mathematical public.

While I was working on this book, I not only thought about mathematics, but I also tried to enter into Lobachevsky's mysterious intimate world, and to understand his tragic life. This led me to include in this book, besides the mathematical material, historical and biographical elements. Beyond the particular case of Lobachevsky, I think that including historical comments in mathematical texts can be very helpful in making the reader feel the charm of the subject.

I would like to thank Manfred Karbe and Ernest B. Vinberg for their kind interest in this translation and the commentary, and for the invaluable information they provided on various editions of Lobachevsky's works. I am also grateful to Dmitry V. Millionschikov and Grigory M. Polotovsky for help in the references, to A. S. Mishchenko who provided copies of the original versions of the Pangeometry, and to Evgenyi N. Sosov, Daniar H. Mushtari and Yi Zhang who kindly transmitted pictures from Kazan. The pictures on p. 212 and 216, and the upper picture on p. 214 belong to the museum of history of Kazan State University, and were published in the book *N. Lobachevsky*, in the series "Outstanding scientists and graduates of Kazan State University".

I am also grateful to Edwin Beschler and Guillaume Théret for their help in polishing the final manuscript and to Boris R. Frenkin who translated into Russian part of my commentary, see [116]. At the occasion of that translation, I discussed with Frenkin several points in that commentary. I am also grateful to Sergei S. Demidov who read my notes on Lobachevsky's biography, corrected inaccuracies, and sent information which was unknown to me. I am especially grateful to Jeremy Gray for an illuminating long conversation about this book. Gray also read an early version of the manuscript and suggested several valuable improvements.

Part of this work was done at the Max-Planck-Institut für Mathematik (Bonn) to whom I am grateful for its hospitality.

Finally, I would like to take this opportunity to thank Norbert A'Campo from whom I learned a lot of Lobachevsky (model-free) hyperbolic geometry, Marie-Pascale for her care and patience, and Irene Zimmermann for her excellent work in the final editing.

Athanase Papadopoulos
Strasbourg and Bonn, October 2010

# Contents

# On the present edition

The present volume contains my translation of the *Pangeometry*, followed by some additional material.

The translation itself is accompanied by footnotes. These footnotes mostly concern the first part of the *Pangeometry*, in which Lobachevsky briefly reviewed his previous works. The notes are intended to help the reader who does not know the bases of Lobachevsky geometry to find his way more easily in the subject.

After this translation, we have reproduced the original French and Russian texts of the *Pangeometry*. This will allow the readers of the French or Russian languages to access Lobachevsky's original writings directly.

After the translation and the original texts, I have included a translation of a French biography of Lobachevsky, which was originally written as a preface to the 1886 edition of his *Collected geometric works*, published in Kazan. I wrote an introduction to this 1886 biography, and I accompanied the translation by footnotes. The introduction and the footnotes are intended to update the biography.

After this biography, I have included a commentary on the *Pangeometry*. It is intended to help the interested reader to get a quick overview on the whole memoir, and to put this memoir in a larger perspective. The commentary concerns some mathematical points, as well as historical material and references.

My translation of the *Pangeometry* is not completely literal. The reason is that, as I have already said, I think the *Pangeometry* will be read not only for its historical value, but also by geometers and students in geometry who want to learn more about the subject. Therefore I made a few minor formal changes that will make its reading easier. Apart from the formal changes that are listed below, I tried to be most faithful to the original text.

Here is the list of changes that I made:

• I highlighted certain definitions (angle of parallelism, side of parallelism, limit circle, etc.) by putting the corresponding words in italics.

• I corrected a few misprints, and I included some small corrections that were done in the German 1902 edition by Liebmann.

• I added figures to the text. The original text does not contain figures. The German edition by Liebmann contains figures, and some of my figures are the same as Liebmann's.

• Lobachevsky used the word "line" as a generic word denoting a segment, the length of a segment, and sometimes a distance. It was still common, in nineteenth-century mathematical writing, to identify a segment with its length.[4] To render the text clearer to the modern reader, I used the terms "segment", "length" and "distance" when appropriate.

• Likewise, Lobachevsky used the word "surface" to denote "area", and he sometimes talked about the "value" of a region, to designate its area. I used the word "area" when appropriate.

[4]This is also the case in Euclid's *Elements*.

• Lobachevsky's sentences in French are often very long. For clarity of exposition, and since having long sentences is unusual in English, I divided certain long sentences into several short ones. In particular, I divided long sentences containing several definitions into shorter sentences, each containing an individual definition. For the same reasons, I divided some paragraphs into shorter ones, when the same paragraph deals with several ideas.

• I sometimes replaced the future tense by the present, when the future was not necessary. For instance, instead of "the two quantities will be equal", I wrote "the two quantities are equal".

• I added an index (see page 78).

Except for these non-significant details, the terminology and the notation used in this memoir, which is more than 150 years old, are comparable with those of an extremely well-written modern mathematical text. This is probably not so surprising, due to the fact that Lobachevsky was ahead of his time in every sense of the word.

# I. Pangeometry

The notions upon which elementary geometry is based are not sufficient to imply a proof of a theorem saying that the sum of the three angles of an arbitrary rectilinear triangle is equal to two right angles,[5] a theorem whose validity has been doubted by nobody up to now, since one does not encounter any contradiction in the consequences that are deduced from it, and because the direct measurements of angles of rectilinear triangles, up to the limits of error of the most perfect measurements, agree with that theorem.

The insufficiency of the fundamental notions for the proof of such a theorem has forced geometers explicitly or implicitly to admit auxiliary suppositions, which, although they appear simple, nevertheless appear arbitrary, and consequently, inadmissible.[6] Thus, for instance, we admit that a circle of infinite radius coincides with a straight line, that a sphere of infinite radius coincides with a plane, and that the angles of an arbitrary rectilinear triangle only depend on the ratios of its edges and not on the actual edges, or, finally, as is usually done in elementary geometry,[7] that from a given point in a plane we can draw only one line parallel to another given line, whereas all the other lines drawn from the same point and in the same plane must necessarily, if they are extended far enough, cut the given line.[8] We mean by the expression "a line parallel to a given line" a line which, however far we extend it from both sides, never cuts the line to which it is parallel. This definition is not sufficient in itself, in particular because it does not characterise in a sufficient way a straight line. We can say the same thing about most of the definitions that are ordinarily given in the *Elements of geometry*, because these definitions not only do not indicate the generation of the magnitudes that are being defined, but also, they even do not show that these magnitudes can exist. Thus, a straight line and a plane are defined by one of their properties; we say that the straight lines are the lines that always coincide as soon as they have two common

[5]In *neutral geometry*, that is, in the geometrical system where Euclid's axioms without the parallel postulate are satisfied, the statement that the angle sum in an arbitrary triangle is equal to two right angles is equivalent to the parallel postulate. Let me note right away that in these footnotes and in the commentary that follows, I include, as part of the Euclidean axioms, the so-called Archimedean axiom, which was not explicitly stated by Euclid but which was implicitly assumed. This axiom says (roughly, and using modern terminology) that each line is identified with the real numbers. The axiom was also obviously assumed by Lobachevsky. A discussion of which results in Euclidean geometry (or in neutral geometry) use the Archimedean axiom is beyond our scope here. It is well-known that Hilbert, in his work on the axiomatization of geometry, thoroughly studied this question; we also mention that there is an important work by Max Dehn on the subject, see [38]. Dehn obtained a result which can be stated as follows: there exist non-Archimedean geometries in which the angle sum in each triangle is equal to two right angles and in which the parallel postulate does not hold.

[6]Lobachevsky is probably talking here about the fact that the erroneous proofs that were given of the parallel postulate were usually based on auxiliary assumptions that were considered as self-evident, but which were in fact equivalent to that postulate.

[7]The meaning could also be: "in the Elements of geometry", referring to Euclid.

[8]Each statement in this sentence, under the axioms of neutral geometry (including the Archimedean axiom) is equivalent to the parallel postulate.

points, and that a plane is a surface in which a straight line is included as soon as the straight line has two points in common with it.[9]

Instead of starting geometry with the plane and the straight line, as it is usually done, I preferred to start it with the sphere and the circle, whose definitions are not subject to the reproach of being incomplete, since they contain the generation of the magnitudes that are being defined.

Then, I define the plane as the geometric locus of the intersections of equal spheres[10] described by two fixed points as centres.

Finally, I define the straight line as the geometric locus of the intersections of equal circles situated in the same plane and having two fixed points in that plane as centres.[11]

These definitions of the plane and of the straight line being accepted, all the theory of planes and of perpendicular lines can be presented and demonstrated with much simplicity and brevity.

Given a line and a point in a plane, I call *parallel* to the given line drawn from the given point a line passing through the given point and which is the limit between the lines that are drawn in the same plane, that pass through the same point and that, when extended from one side of the perpendicular dropped from that point on the given line, cut the given line, and those that do not cut it (Figure 1).

I published a complete theory of parallels under the title "*Geometrische Untersuchungen zur Theorie der Parallellinien*. Berlin 1840. In der Finckeschen Buchhandlung".[12] In this work, I first presented all the theorems that can be demonstrated

[9] It seems that Lobachevsky is pointing out here that the straight lines and planes, as primary notions, are only "defined" by their properties, e.g. that two lines coincide as soon as they have two points in common, that a plane contains a straight line as soon as it contains two distinct points of that line, and so on. This contrasts with Euclid's *Elements* point of view, where primary notions, although they are abstractions, are nevertheless defined using sensible adjectives. For instance, in the *Elements*, a *point* is defined as something which "has no part", a line is defined as something which "has length and no breadth", etc., see [72], Vol. I, p. 153. Stylianos Negrepontis pointed out to me that this definition of "line" in Euclid's *Elements* is due to Plato, and that it does not apply only to straight lines, but to all lines, which, in the *Elements*, can be either straight lies or circles. Aristotle, at several places, gave similar definitions, e.g. "that which is wholly indivisible and has position is called a point", cf. the *Metaphysics*, 1016b25 ([8], Vol. 1, p. 235) and the *Posterior Analytics* 87a36 ([9], Book I, p. 155). Despite the fact that Euclid gave such definitions, it is clear that he considered the primary notions of geometry as abstractions, as we do today. His proofs never use the fact that a point "has no part" and so on. Euclid's attempts to "define" the primary notions is certainly aimed to say that these notions correspond to objects that we can intuitively feel. We also note that Euclid's definitions have been criticised by ancient commentators, see for instance Proclus (412–485 A.D.) [122]. Beyond the details of this particular discussion, one notes Lobachevsky's questioning of the nature of the primary elements of geometry.

[10] "Equal spheres" (and, below, "equal circles"), means spheres (respectively circles) of equal radii.

[11] In other words, a plane is defined here as the equidistant locus from two points in 3-space, and a line is defined as the equidistant locus from two points in a plane, but without using the words "distance" or "equidistance". The idea of such a definition is attributed by Heath (with some care) to Leibniz who, in a letter to Giordano, defined a plane *as that surface which divides space into two congruent parts*, see [72], Vol. I, p. 174.

[12] Lobachevsky refers here to his *Geometrische Untersuchungen zur Theorie der Parallellinien* (Geometric investigations on the theory of parallels) [100]. At several places of the *Pangeometry*, Lobachevsky refers to his *Geometrische Untersuchungen*, even for results that are contained in his earlier memoirs. The reason is probably that the previous memoirs were written in Russian, and published in Russia, and Lobachevsky, who did not get any recognition for his work in Russia, addresses in the *Pangeometry* the Western-European

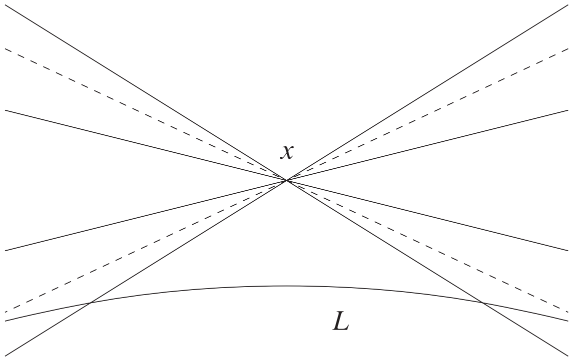


Figure 1. The two dashed lines represent the two parallels through $x$ to the line $L$. Each of these parallel lines is a limit line, in the pencil of lines containing $x$, separating the family of lines that cut $L$ from the family of lines that do not cut $L$.

without the help of the theory of parallels.[13] Among these theorems, the theorem that gives the ratio of the area of any spherical triangle over the area of the entire sphere on which it is drawn is particularly remarkable (*Geometrische Untersuchungen* §27). If $A$, $B$, $C$ denote the angles of a spherical triangle and if $\pi$ denotes two right angles, then the ratio of the area of that triangle to the area of the sphere to which it belongs is equal to the ratio of

$$\frac{1}{2}(A + B + C - \pi)$$

to four right angles.

Then, I proved that the sum of the three angles of any rectilinear triangle can never exceed two right angles (*Geometrische Untersuchungen* §19) and that if this sum is equal to two right angles in some arbitrary rectilinear triangle, then it is so in all of them (*Geometrische Untersuchungen* §20).[14] Thus, there are only two possible assumptions: either the sum of the three angles of any rectilinear triangle is equal to two right angles,

reader, for whom the paper written in German and published in Berlin might have been more easily accessible.

[13] The meaning here is: without the use of the parallel postulate.

[14] The result concerns results in neutral geometry and it is usually attributed to Giovanni Girolamo Saccheri (1667–1733). The result was not explicitly stated by Saccheri, but it easily follows from his work *Euclides ab omni naevo vindicatus* published in 1733 [134], [139]; see Pont [121], pp. 352–368, Gray [55], p. 81, and Bonola [26], §13 and §15, for a discussion of Saccheri's work. Saccheri stated that result in terms of the geometry of quadrilaterals with two opposite edges of equal length meeting a third common edge at right angles (which Bonola calls *two right-angled isosceles quadrilaterals*). Besides its two right angles, such a quadrilateral has two other angles that have a common value. Saccheri studied separately the three hypotheses: either this common value is equal to $\pi/2$ (a case which occurs in Euclidean geometry), or this value if $> \pi/2$ (which occurs in spherical geometry), or this value is $< \pi/2$, a hypothesis which is valid in hyperbolic geometry. Proposition VII of Book I of Saccheri's *Euclides ab omni naevo vindicatus* says (in Halsted's translation) that "if even in a single case the hypothesis of acute angle is true, always in every case it alone is true". Saccheri, with a faulty reasoning, thought that this cannot occur. The quadrilaterals involved in Saccheri's work are usually called *Saccheri quadrilaterals*, sometimes *birectangles*, and sometimes also *Khayyam–Saccheri quadrilaterals* (see e.g. Rosenfeld [131] and Pont [121]). The reason for the latter terminology is that these quadrilaterals were, long before Sacccheri, analysed by the medieval mathematician

and this assumption leads to the known geometry – or in any rectilinear triangle this sum is less than two right angles and this assumption serves as a basis for another geometry, to which I gave the name *Imaginary geometry*, but which it may be more convenient to call *Pangeometry*, because this name designates a general geometric theory which includes ordinary geometry as a particular case.[15]

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Omar Khayyam (1048–1131 ca.). Omar Khayyam, in Book I of his *Commentaries on the Difficulties encountered in certain postulates in the book of Euclid* made (under the axioms of neutral geometry) an analysis of quadrilaterals with two congruent opposite edges making right angles with a common third edge. He showed that the two remaining angles are congruent, and he studied the case where this angle is right, acute, or obtuse. Preceding Saccheri in his analysis, Khayyam also preceded Saccheri in his mistake, since he (erroneously) refuted the second and third possibility, and he used this to deduce a proof of the parallel postulate; see [123], p. 324 ff., and [132], p. 467. Besides being a mathematician, Omar Khayyam was a philosopher and an astronomer. A valuable edition of Khayyam's work is due to R. Rashed and B. Vahabzadeh [123]. According to Rashed, it cannot be said with certainty that Khayyam the mathematician and Khayyam the poet, author of the celebrated *Rubayiat* (quatrains), are the same person, see [123], p. 5. Khayyam worked in Samarkand, Bukhara, Ispahan and Merva. See [78] and [123] for the mathematical texts of Khayyam. The result quoted here by Lobachevsky was rediscovered, after Saccheri, by Johann Heinrich Lambert (1728–1777) and by Adrien-Marie Legendre (1752–1833). The result was (and is sometimes still) attributed to Legendre, the first part being called the First Legendre Theorem and the second part called the Second Legendre Theorem. Some authors attribute the first part to Saccheri and the second part to Legendre. Bonola (see [26], p. 56) considers that it is a mistake to attribute any of the two parts to Legendre, since both of them are contained in Saccheri's work. The reason of that confusion is that Saccheri's work was forgotten soon after its publication, and was rediscovered and rehabilitated only in 1889, the year Beltrami published a paper entitled *Un precursore italiano di Legendre e di Lobatschewski* [19]. See also the discussion in Pont [121], who noted on p. 251 that Lobachevsky wrote a proof of both parts of this theorem in his 1816–1817 notes, that is, about 15 years before Legendre gave his proof [94]. Lobachevsky, in his *New elements* (see [96], p. 8 of the French translation) wrote that Legendre proved the second part of the statement in 1833, whereas he himself had already written a proof of that part in his *Exposition succinte* in 1826. Lambert's proof of the theorem of Saccheri was given in his *Theorie der Parallelinien* [139] written in 1766 and published posthumously. This proof involves the analysis of quadrilaterals with three right angles, for which Lambert made and analysed a trichotomy that is analogous to the one Saccheri made for his quadrilaterals. Lobachevsky's proof of the second part of the statement that is given in [100], §20, is more direct than the ones of Saccheri and Lambert. It is also worth noting here that Dehn showed in [38] that the first part of the statement cannot be proved without some Archimedean axiom, and he gave a proof of the second part of the statement that does not use such an axiom. Finally, we note that the first part of the statement of the theorem of Saccheri is contained in the work of Nasir al-Din at-Tusi (1201–1274), see [72], Vol. I, p. 209. Nasir al-Din at-Tusi, in two treatises, called *Treatise that cures doubts in parallel lines* and *Exposition of Euclid*, also studied the geometry of quadrilaterals with two congruent opposite edges making right angles with a common third edge, and he made an analysis of the three possibilities for the remaining two (equal) angles (see [132], p. 468). Nasir al-Din at-Tusi was a Persian philosopher, astronomer, theologian and doctor, and he was one of the most valuable medieval mathematicians. He translated and commented Euclid, and this led him to study Khayyam-Saccheri quadrilaterals, continuing the work done by Omar Khayyam; see his mathematical texts in [78].

[15]One should not understand this sentence as saying that pangometry, in Lobachevsky's sense, designates neutral geometry, that is, a geometry based on an axiom system that allows both hyperbolic and Euclidean geometry. The correct meaning of the word "pangeometry" can be understood from the lines that follow that sentence, where properties that Lobachevsky attributes to pangeometry are properties that are valid in hyperbolic geometry, and not in Euclidean geometry; cf. for instance the properties of perpendicular lines that are described a few lines after this statement. The statement that is made here, that pangeometry includes ordinary geometry as a particular case, should be interpreted in the light of the fact, observed by Lobachevsky, that hyperbolic 3-space contains a model of Euclidean space, namely, the horosphere. It is important to stress the fact that Lobachevsky, here and elsewhere in his work, realised the importance of getting models of one geometry in another geometry, and that one of his aims in emphasising these facts

It follows from the principles adopted in Pangeometry that a perpendicular $p$ dropped from a point on a given line to one of its parallels makes with the first line two angles, one of which is acute. I call this angle the *angle of parallelism*, and the side of the first line to which this angle belongs (and this side is the same for all points on that line) the *side of parallelism*. I denote this angle by $\Pi(p)$, since it depends on the length of the perpendicular (see Figure 2). In ordinary geometry, $\Pi(p)$ is always equal to a

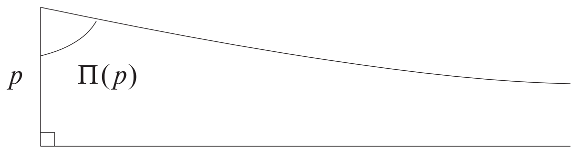



Figure 2. The parallelism angle $\Pi(p)$ corresponding to the segment $p$.

right angle, independently of the value of $p$. In Pangeometry, the angle $\Pi(p)$ passes through all the values, from zero, corresponding to $p = \infty$, to $\Pi(p)$ = a right angle, corresponding to $p = 0$ (*Geometrische Untersuchungen* §23). To give the function $\Pi(p)$ a more general analytical value, I consider that the value of this function for negative $p$ (a case to which the original definition does not apply) is determined by the following equation[16]:

$$\Pi(p) + \Pi(-p) = \pi.$$

Thus, for any angle $A > 0$ and $< \pi$, we can find a segment $p$ such that $\Pi(p) = A$, where the segment $p$ is positive if $A < \frac{\pi}{2}$. Conversely, there exists for any segment $p$ an angle $A$ satisfying $A = \Pi(p)$.

I call *limit circle*[17] a circle whose radius is infinite.[18] A limit circle can be drawn

was his attempt to prove the relative consistency of the various geometries with respect to each other.

[16]This extension of the angle of parallelism function is very simple, but it turned out to be extremely useful. It was used by Lobachevsky since his published memoirs (see the *Elements of geometry* (1829) [95], §8, p. 11 of the German translation).

[17]The notion of limit circle is already contained in Lobachevsky's first published memoirs, see the *Elements of geometry* (1829) [95], §19, p. 28 of the German translation, and the *New elements of geometry* (1835) [96], §112, p. 185 of the German translation. In his *Geometrische Untersuchungen* §31, Lobachevsky used the word "oricycle", composed of the two Greek words "hori" (boundary) and "cyclos" (circle). The word "horizon" has the same root. The word "horicycle" was later on transformed in the English and French literature into "horocycle" (for a reason which is unknown to the translator).

[18]The description given here of a limit circle is rather rough. In §31 of his *Geometrische Untersuchungen*, Lobachevsky defined a limit circle as a curve in the plane for which all perpendiculars erected at midpoints of chords are parallel to each other, see also §112 of his *New elements of geometry* [96]. Then he established the existence of such curves, by an explicit construction (§32), and showed in the same section how a family of circles whose radii tend to infinity converges to a limit circle. We note that Beltrami, in his famous paper *Saggio di interpretazione della geometria non-Euclidea*, gave a description of limit circles on his pseudo-sphere model. He wrote in [16] (p. 26 of Stillwell's translation): "The geodesic circles with centre at infinity obviously correspond to the *horocycles* in Lobachevsky's geometry (*Theory of parallels*, nos. 31 and 32). We can therefore say that a system of concentric horocycles is transformed, by a suitable flexion of the surface, into a system of parallels on the surface of revolution generated by a curve with constant tangent." [D'altra parte le circonferenze geodetiche col centro all'infinito corrispondono manifestamente agli *oricicli* di Lobatschewsky (l. c. n. 31 e 32). Conservando questa denominazione noi possiamo dunque

by approximation, by constructing as many points of this limit circle as one wishes, in the following manner.

Let us take a point on an infinite line, let us call this point the *vertex*, and let us call this line the *axis* of the limit circle. Let us construct an angle $A$ whose value is $> 0$ and $< \frac{\pi}{2}$, whose vertex[19] coincides with the vertex of the limit circle, and of which the axis of the limit circle is one of the sides. Finally, let $a$ be the segment that satisfies $\Pi(a) = A$ and let us construct on the other side of the angle, starting from the vertex, a segment of length $2a$. The terminal point of this segment will be on the limit circle (see Figure 3). To continue tracing the limit circle from the other side of the axis, one has to repeat this construction from that other side. It follows that all the lines that are parallel to the axis of the limit circle can be taken as axes.[20]

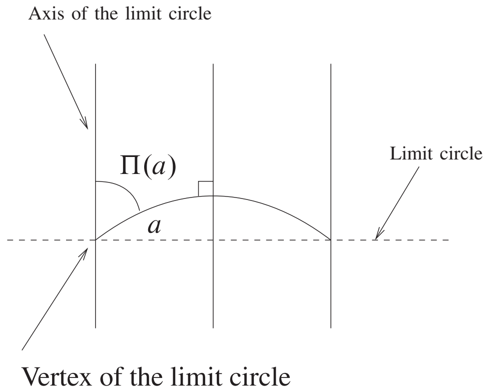



Figure 3. Construction of the limit circle, using the angle of parallelism: Starting from a line (which is an axis of the limit circle), and from a point on that line (called vertex of the limit circle), we take an angle between 0 and $\pi$ (strictly), as shown in the figure. This angle is equal to $\Pi(a)$ for some positive $a$. The point on the side of that angle that is not the axis we started with, situated at distance $2a$ from the vertex, is a point on the limit circle. Note that the construction provides two different limit circles passing through the given point and having the given axis.

The revolution of the limit circle around one of its axes[21] produces a surface which I call a *limit sphere*,[22] a surface which consequently is the limit towards which the sphere approaches as its radius increases to infinity. We shall call the axis of revolution, and, consequently, any line that is parallel to this axis of revolution, an *axis* of the limit

dire che un sistema di oricicli si trasforma, mediante una flessione opportuna della superficie nel sistema dei paralleli della superficie di rotazione generata dalla linea delle tangenti costanti.]

[19]An *angle* is defined as two rays having a common origin. This origin is called the *vertex* of the angle, and the two rays are called the *sides* of the angle.

[20]Each line in the pencil of lines that are parallel (relatively to the right specified direction) to an axis of a limit circle (respectively of a limit sphere, defined in the paragraph that follows) is also an axis of that limit circle (respectively of the limit sphere).

[21]Here, the hyperbolic plane is considered as being embedded in hyperbolic 3-space.

[22]In his *Geometrische Untersuchungen* §34, Lobachevsky used the word "orisphere", which was later on transformed into "horosphere".

sphere.[23] We shall call a *diametral plane*[24] any plane that contains one or more of the axes of the limit sphere. The intersections of the limit sphere with its diametral planes are limit circles. A portion of the surface of the limit sphere that is bounded by three limit circle arcs is called a *limit sphere triangle*. The limit circle arcs are called the *edges*, and the dihedral angles between the planes of these arcs are called the *angles* of the limit sphere triangle.[25]

Two straight lines that are parallel to a third one are parallel (*Geometrische Untersuchungen* §25).[26] It follows that all the axes of the limit circle and of the limit sphere are mutually parallel.

If three planes intersect pairwise in three parallel lines and if we consider the region that is situated between these parallels, the sum of the three dihedral angles that these planes form is equal to two right angles (*Geometrische Untersuchungen* §28). It follows from this theorem that the angle sum of any limit sphere triangle is equal to two right angles and, consequently, everything that we prove in ordinary geometry concerning the proportionality of edges of rectilinear triangles can be proved in the same manner in the Pangeometry of limit sphere triangles,[27] by only replacing the lines that are parallel

[23]Lobachevsky uses here the fact that the intersection of the limit sphere with any plane containing a line parallel to the axis of revolution is a limit circle (a fact which is stated a few lines later), and that the limit sphere is invariant under rotations around such a line.

[24]In his *Geometrische Untersuchungen* §34, Lobachevsky calls such a plane a *principal plane*.

[25]This terminology of limit sphere triangle, limit circle edge and limit sphere angle that Lobachevsky introduces here is of paramount importance, because as he explains a few lines after this, these objects are the elements of a geometry on the limit sphere which will turn out to be Euclidean.

[26]In the reference given, Lobachevsky proved this statement in full generality for lines in hyperbolic 3-space and not only for coplanar lines.

[27]Lobachevsky deduces that the geometry of the limit sphere is Euclidean from the fact that in that geometry, the axioms of neutral geometry are satisfied and the angle sum of triangles is equal to two right angles. The last fact was known to be equivalent to Euclid's parallel postulate. The other Euclidean axioms are considered to be clearly satisfied in this geometry. This is an interesting instance, first noticed by Lobachevsky, of a plane embedded in hyperbolic 3-space and which is equipped with a natural geometry which is Euclidean. We note that Beltrami, in his *Teoria fondamentale degli spazii di curvatura costante* [17], obtained a concrete description of limit spheres. As a matter of fact, Beltrami obtained a differential-geometric description of limit cycles and of limit spheres, Using this description of limit spheres, he gave a different point of view on the fact that their geometry is Euclidean. Using today's language, Beltrami's point of view says that the *length structure* induced by the metric of 3-dimensional hyperbolic space on a limit sphere is Euclidean, whereas Lobachevsky's point of view (concerning this point) is closer to the axiomatic level. In a letter to Hoüel, dated 1st of April 1869, Beltrami wrote: "From the formula $ds = \text{const} \cdot \sqrt{d\eta_1^2 + d\eta_2^2}$ that I established on page 21 of my last Memoir [Beltrami refers here to his *Teoria fondamentale degli spazii di curvatura costante* [17]), we can deduce (or rather, we can check, because this is already contained in Lobachevsky) that the geometry of the limit sphere is nothing else than that of the Euclidean plane. This is exactly what I meant when I said that the curvature of this surface is zero. In other words, I meant that all the metric properties of this surface are the same as those of the ordinary plane, because the linear elements in both surfaces are identical. [...] For me, I would say that the limit sphere is one of the forms under which the Euclidean plane exists in the non-Euclidean space, *considering the Euclidean plane as being defined by the property of having zero curvature*. [De la formule $ds = \text{const} \cdot \sqrt{d\eta_1^2 + d\eta_2^2}$, que j'ai établie à la page 21 de mon dernier Mémoire on tire (ou plutôt on vérifie, car cela se trouve dans Lobatcheffsky) que la géométrie de la sphère-limite n'est pas autre chose que celle du plan euclidien. En disant que la courbure de cette surface est nulle je n'ai pas voulu dire autre chose. En d'autres termes j'ai voulu dire que toutes les propriétés métriques de cette surface sont les mêmes que celles du plan ordinaire, à cause de l'identité des

to one of the edges of the rectilinear triangle by limit circle arcs drawn from the points on one of the edges of the limit sphere triangle, and all of them making the same angle with that edge.[28]

Thus, for instance, if $p$, $q$, $r$ are the edges of a right limit sphere triangle and if $P$, $Q$, $\frac{\pi}{2}$ are the angles opposite to these edges, we must adopt, in the same way as for the right rectilinear triangles of ordinary geometry, the following equations:

$$p = r \sin P = r \cos Q,$$
$$q = r \cos P = r \sin Q,$$

$$P + Q = \frac{\pi}{2}.$$

In ordinary geometry, one proves that the distance between two parallel lines is constant.

On the contrary, in Pangeometry,[29] one proves that the distance $p$ from a point on a line to a parallel line decreases on the side of parallelism, that is, on the side towards which the parallelism angle $\Pi(p)$ is directed.

Now let $s, s', s'', \ldots$ be a sequence of limit circle arcs bounded by two parallel lines, serving as axes for all these limit circles, and let us assume that the pieces of parallel lines bounded by two consecutive such arcs are all equal to a quantity we denote by $x$.[30] Let us call $E$ the ratio of $s$ to $s'$,

$$\frac{s}{s'} = E$$

where $E$ is some number greater than 1.

Let us first assume that $E = \frac{n}{m}$, with $m$ and $n$ being integers, and let us subdivide the arc $s$ into $m$ equal parts. From the division points, let us draw lines that are parallel to the axis of the limit circles. These parallels divide each of the arcs $s'$, $s''$, ... into $m$ equal parts. Let $AB$ be the first part of $s$, $A'B'$ the first part of $s'$, $A''B''$ the first part of $s''$ and so on, where $A$, $A'$, $A''$, ... are the points situated on one of the given parallels (see Figure 4). Let us transport the segment $A'B'$ and superpose it on the segment $AB$ in such a way that $A$ and $A'$ coincide and $A'B'$ falls on $AB$.[31] Let us repeat this superposition $n$ times. Since by assumption $\frac{s}{s'} = \frac{n}{m}$, it necessarily follows that $nA'B' = mAB$ and, consequently, the second endpoint of $A'B'$ coincides, after

élements linéaires chez l'une et chez l'autre. [...] Pour mon compte je dirais que la sphère-limite est une des formes sous lesquelles le plan euclidien existe dans l'espace non-euclidien, *en considérant le plan euclidien comme défini par la propriété d'avoir sa courbure nulle*.] cf. [23], pp. 87–88. A similar explanation on limit spheres is contained in Beltrami's letter to Hoüel, dated 12 October 1869, [23], p. 87–88.

[28] Lobachevsky explains here how one can construct similar (i.e. homothetic) triangles on the limit sphere, in a way analogous to the construction of similar triangles in the Euclidean plane. We recall that in neutral geometry, making the assumption that there exist non-congruent similar triangles is equivalent to adding Euclid's parallel postulate.

[29] This is also an indication that the word "pangeometry", in the way Lobachevsky used it, does not mean neutral geometry, since the property stated here does not hold in Euclidean geometry. Thus, pangeometry is not a geometry that includes, from the axiomatic point of view, both Euclidean and hyperbolic geometry.

[30] Thus, $x$ is the distance between two consecutive limit circles.

[31] Such an argument, using the notion of "transport", would be described today as an "isometry argument".

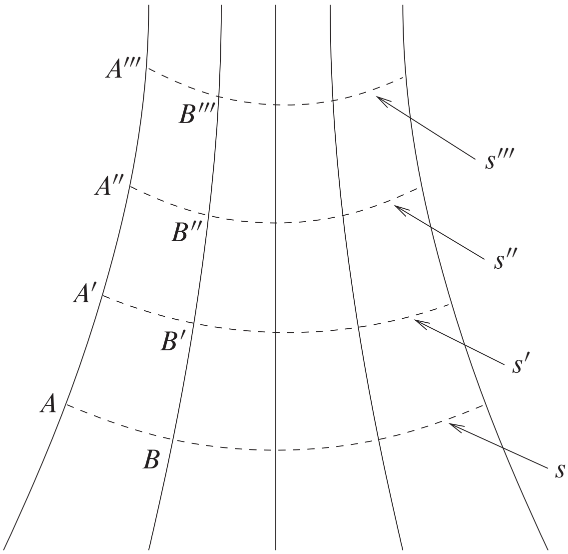



Figure 4. Equidistant limit circle arcs (in dashed lines) bounded by two axes of these limit circles. The letters $s, s', s'', \ldots$ denote the lengths of the arcs. We have $\frac{s}{s'} = \frac{s'}{s''} = \frac{s''}{s'''} = \cdots$ .

the $n$-th superposition, with the second endpoint of $s$, which is thus divided into $n$ equal parts. Each of $s'$, $s''$, ... is also divided into $m$ equal parts by lines parallel to the two given parallel lines. But if we imagine that when we perform the above superposition, $A'B'$ covers the part of the plane bounded by this arc and by the two parallels drawn from the endpoints, it is clear that, at the same time, $n$ times $A'B'$ covers the arc $s$ entirely, $nA''B''$ covers the arc $s'$ entirely, and so forth, since in this case the parallels must coincide in all their extent. In this way, we get

$$nA''B'' = mA'B'$$

or, equivalently,

$$\frac{s'}{s''} = \frac{n}{m} = E; \quad \frac{s'}{s''} = E, \ldots$$

which is the desired result.

To prove the same thing in the case where $E$ is an incommensurable number, we can employ one of the methods used for similar cases in ordinary geometry. For brevity, I omit the details. Thus,

$$\frac{s}{s'} = \frac{s'}{s''} = \frac{s''}{s'''} = \cdots = E.$$

With this, it is not difficult to conclude[32] that

$$s' = sE^{-x}.$$

Here, $E$ is the value of $\frac{s}{s'}$ when $x$, the distance between the arcs $s$, $s'$, is equal to one.[33]

One must note that this ratio $E$ does not depend on the length of the arc $s$, and that it is unchanged if the two given lines are farther away from each other or closer to each other. The number $E$, which is necessarily greater than 1, depends only on the unit of length, which is the distance between two consecutive arcs and which can be chosen completely arbitrarily.[34] The property that we just proved concerning the arcs $s$, $s'$, $s''$, ... remains true for the areas $P$, $P'$, $P''$, ... bounded by consecutive arcs and by the two parallel lines.[35] Therefore we have

$$P' = PE^{-x}.$$

If we put $n$ similar areas $P$, $P'$, $P''$, ..., $P^{(n-1)}$ side by side, the total area is then

$$P \cdot \frac{1 - E^{-nx}}{1 - E^{-x}}.$$

For $n = \infty$ this expression gives the area of the plane region contained between two parallel lines, bounded on one side by the arc $s$, and unbounded in the direction of parallelism. The value of this area is

$$\frac{P}{1 - E^{-x}}.$$

If we choose as unit area the area $P$ that corresponds to an arc $s$ which is also of unit length and to $x = 1$, then this area becomes in the general case, for an arbitrary arc $s$,

$$\frac{Es}{E - 1}.$$

In ordinary geometry, the ratio that we denoted by $E$ is constant and equal to 1. It follows that in ordinary geometry, two parallel lines are everywhere equidistant and that the area of the region in the plane contained between two parallel lines and bounded from one side by a common perpendicular to these lines is infinite.[36]

---

[32]The argument can be as follows: Denoting by $f$ the function that associates to each $x$ the ratio $\frac{s}{s'}$ of two limit circle arcs that are at distance $x$ from each other, with $s'$ corresponding to the arc that is in the direction of parallelism of the axes (which, as we recall, are parallel lines) with respect to $s$, this function satisfies $f(x)f(y) = f(x + y)$. This is the functional equation satisfied by the exponential.

[33]The contraction property of the distance between two parallel lines is already proved in Lobachevsky's first written memoir that reached us, the *Elements of geometry* ([95], §22, p. 32 of the German translation).

[34]Today, we would say that the constant $E$ depends on the curvature of the space. If this curvature is equal to $-1$, then $E$ is equal to $e$, the basis of natural logarithms. (We recall that Lobachevsky did not have the notion of curvature of a surface).

[35]Lobachevsky now considers the case where the *areas* of the pieces bounded by the two parallel lines and the limit circle arcs have a common value equal to $x$.

[36]We recall that in neutral geometry, adding the assumption that there exist two equidistant lines is equivalent to assuming that Euclid's parallel postulate is satisfied.

Now consider a right rectilinear triangle with edges $a$, $b$, $c$ and let $A$, $B$, $\frac{\pi}{2}$ be the angles opposite to these edges. The angles $A$, $B$ can be taken to be angles of parallelism $\Pi(\alpha)$, $\Pi(\beta)$ of segments of positive lengths $\alpha$, $\beta$ respectively. Let us also make the convention that from now on, we denote by a letter with an accent a segment whose length corresponds to an angle of parallelism which is the complement to a right angle of the angle of parallelism corresponding to the segment whose length is denoted by the same letter without accent.[37] Thus,

$$\Pi(\alpha) + \Pi(\alpha') = \frac{\pi}{2},$$
$$\Pi(b) + \Pi(b') = \frac{\pi}{2}.$$

Let us denote by $f(a)$ the portion of a parallel to the axis of a limit circle that is bounded by the perpendicular to that axis drawn from the vertex of the limit circle and the limit circle itself, if this parallel passes through a point on the perpendicular whose distance to the vertex is $a$. Finally let $L(a)$ be the length of the arc from the vertex to that parallel (see Figure 5).

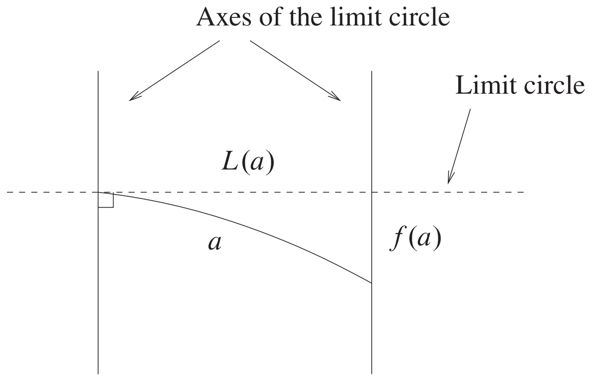



Figure 5. The functions $f(a)$ and $L(a)$ are obtained by taking a rectilinear segment of length $a$ tangent to a limit circle at one endpoint and then projecting the other endpoint of that segment on the limit circle. Then, $f(a)$ is the length of the perpendicular, and $L(a)$ is the length of the arc joining the initial point to the projection.

In ordinary geometry, we have, for any segment $a$,

$$f(a) = 0$$

and

$$L(a) = a.$$

[37]An equivalent form of this convention was already used in Lobachevsky's first published memoir (cf. the *Elements of geometry* (1829) [95], §11, p. 17 of the German translation). This convention turns out to be very useful in terms of providing economical notation in the trigonometric formulae that follow.

Let us draw a perpendicular[38] $AA'$ to the plane of the right rectilinear triangle whose edges have been denoted by $a$, $b$, $c$, this perpendicular passing through the vertex $A$ of angle $\Pi(\alpha)$. Consider two planes containing this perpendicular, one, which we call the first plane, containing the edge $b$, and the other, which we call the second plane, containing the edge $c$. Let us construct in the second plane the line $BB'$ parallel to $AA'$ and passing through the vertex $B$ of angle $\Pi(\beta)$ and let us take a third plane passing through $BB'$ and the edge $a$ of the triangle. This third plane cuts the first one in a line $CC'$ parallel to $AA'$.[39] Let us now consider a sphere having $B$ as centre and having an arbitrary radius, but smaller than $a$. This sphere will consequently cut the two edges $a, c$ of the triangle and the segment $BB'$ in three points. We call $n$ the first point, $m$ the second point, and $k$ the third point. (See Figure 6.)

The arcs of great circles that are intersections of this sphere with three planes passing through $B$ and connecting pairwise the points $n$, $m$, $k$, form a spherical triangle with a right angle at $m$, whose edges are $mn = \Pi(\beta)$, $km = \Pi(c)$, $kn = \Pi(a)$. The spherical angle $knm$ is equal to $\Pi(b)$ and the angle $kmn$ is a right angle. The three lines being mutually parallel, the sum of the three dihedral angles formed by the portions of the planes $AA'BB'$, $AA'CC'$, $BB'CC'$ situated between the lines $AA'$, $BB'$, $CC'$, are equal to two right angles. It follows that the third angle of the spherical triangle is equal to $mkn = \Pi(\alpha')$. Thus, we see that to each right rectilinear triangle with edges $a$, $b$, $c$ and with opposite angles $\Pi(\alpha)$, $\Pi(\beta)$, $\frac{\pi}{2}$ corresponds a right spherical triangle with edges $\Pi(\beta)$, $\Pi(c)$, $\Pi(a)$ and with opposite angles $\Pi(\alpha')$, $\Pi(b)$, $\frac{\pi}{2}$.[40]

Let us construct another right rectilinear triangle whose perpendicular edges are $\alpha'$, $a$, whose hypotenuse is $g$, whose angle opposite to the edge $a$ is $\Pi(\lambda)$, and whose angle opposite to the edge $\alpha'$ is $\Pi(\mu)$. Let us pass from this triangle to the corresponding spherical triangle in the same way as the spherical triangle $kmn$ corresponds to the triangle $ABC$. Consequently, the edges of this new spherical triangle are

$$\Pi(\mu),\quad \Pi(g),\quad \Pi(a)$$

[38]Lobachevsky uses here the notion of perpendicularity in 3-space. In non-Euclidean geometry, perpendicularity of a line $L$ to a plane $P$ is defined as in Euclidean geometry: the line $L$ intersects $P$ in a point $p$, and it is perpendicular (in the sense of plane geometry) to any line in $P$ passing through $p$.

[39]The fact that $CC'$ is parallel to $AA'$ (and to $(BB')$) is a general result proven by Lobachevsky in his *Geometrische Untersuchungen*, §25). Using the present notation, this result says that if two coplanar lines $BB'$ and $AA'$ are parallel and if a plane containing $AA'$ intersects a plane containing $BB'$ in a line $CC'$, then $CC'$ is parallel to $AA'$ and to $BB'$. We note that the three parallel lines $AA'$, $BB'$ and $CC'$ define, together with the triangle $ABC$ a tetrahedron with one vertex at infinity. The perpendicularity properties between edges make this tetrahedron a *birectangular tetrahedron*, or an *orthoscheme* in the sense of Schläfli, a terminology also used by Coxeter (see [35], p. 156). Birectangular tetrahedra turned out to be very important objects, and extensive research on the volumes of these solids was revived during the last two decades of the twentieth century, see e.g. Kellerhals [81] and Vinberg [151].

[40]This construction of a right spherical triangle whose angles and edge lengths are explicitly given in terms of those of a right hyperbolic triangle is already contained in Lobachevsky's first published memoir *On the elements of geometry* (1829), cf. [95], §11, p. 18 of the German translation. The construction was repeated in Lobachevsky's *New elements of geometry, with a complete theory of parallels* (1835–1838), cf. [96], §136, p. 210 of the German translation, and in his *Geometrische Untersuchungen zur Theorie der Parallellinien* (Geometrical researches on the theory of parallels) cf. [100], §35, p. 36 of the English translation.

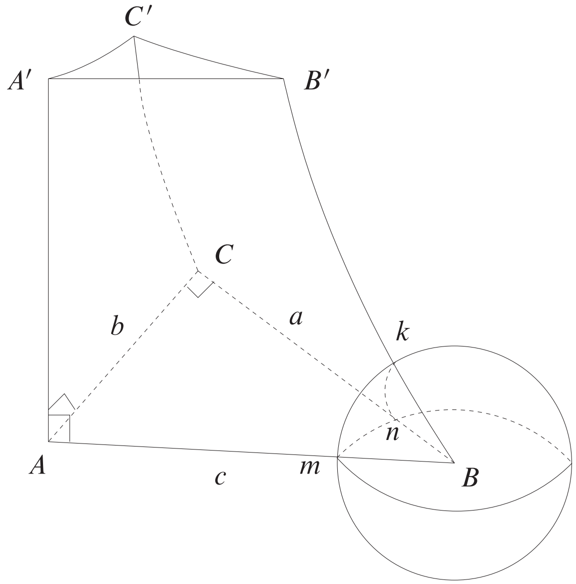



Figure 6. The figure represents the right spherical triangle $mnk$ associated by Lobachevsky to the right hyperbolic triangle $ABC$. If the edges of the right hyperbolic triangle are $a$, $b$, $c$ and the corresponding opposite angles $\Pi(\alpha)$, $\Pi(\beta)$, $\frac{\pi}{2}$, then the edges of the spherical triangle are $\Pi(\beta)$, $\Pi(c)$, $\Pi(a)$, and its opposite angles $\Pi(\alpha')$, $\Pi(b)$, $\frac{\pi}{2}$. (In this figure, the right angle is at $m$, $\Pi(b)$ is at $n$ and $\Pi(\alpha')$ is at $k$.)

and its opposite angles

$$\Pi(\lambda'),\ \ \Pi(\alpha'),\ \ \frac{\pi}{2}.$$

Furthermore, this triangle has all its elements equal to the corresponding elements of the spherical triangle $kmn$, since the edges of the latter were

$$\Pi(c),\ \ \Pi(\beta),\ \ \Pi(a)$$

and its opposite angles

$$\Pi(b),\ \ \Pi(\alpha'),\ \ \frac{\pi}{2},$$

which shows that these spherical triangles have equal hypotenuses, and an equal adjacent angle.

It follows that

$$\mu = c, \quad g = \beta, \quad b = \lambda.'$$

Thus, the existence of a right rectilinear triangle with edges

$$a,\ b,\ c$$

and opposite angles

$$\Pi(\alpha),\ \ \Pi(\beta),\ \ \frac{\pi}{2}$$

implies the existence of another right rectilinear triangle with edges

$$a,\ \ \alpha',\ \ \beta$$

and opposite angles

$$\Pi(b'), \Pi(c), \frac{\pi}{2}.$$

We express the same thing by saying that if

$$a,\ \ b,\ \ c,\ \ \alpha,\ \ \beta$$

are the elements of a right rectilinear triangle, then

$$a,\ \ \alpha',\ \ \beta,\ \ b',\ \ c$$

are the corresponding elements of another right rectilinear triangle.

If we construct a limit sphere with axis the perpendicular $AA'$ to the plane of the given right rectilinear triangle and with vertex the point $A$, we get a triangle situated on the limit sphere, obtained as the intersection of this limit sphere with the three planes directed by the three edges of the given triangle. Let us denote the three edges of this limit sphere triangle by $p$, $q$, $r$ in such a way that $p$ is the intersection of the limit sphere with the plane containing $a$, that $q$ is the intersection of the limit sphere with the plane passing by $b$, and that $r$ is the intersection of the limit sphere with the plane containing $c$ (see Figure 7). The angles opposite to these edges are: $\Pi(\alpha)$ opposite to $p$, $\Pi(\alpha')$ opposite to $q$, and a right angle opposite to $r$.[41] From the conventions adopted above, we have $q = L(b)$ and $r = L(c)$. The limit sphere cuts the line $CC'$ in a point whose distance to $C$ is, according to these same conventions, $f(b)$. In the same way, $f(c)$ is the distance from the intersection point of the limit sphere with the line $BB'$ to the point $B$.

It is easy to see that we have

$$f(b) + f(a) = f(c).$$

In the triangle whose edges are the limit circle arcs $p$, $q$, $r$, we have

$$p = r \sin \Pi(\alpha);\ \ q = r \cos \Pi(\alpha).$$

Multiplying both sides of the first of these two equations by $E^{f(b)}$, we obtain

$$pE^{f(b)} = r \sin \Pi(\alpha)\, E^{f(b)}.$$

But we have

$$pE^{f(b)} = L(a),$$

[41] The construction of a right Euclidean triangle whose angles and edge lengths are explicitly given in terms of those of a right hyperbolic triangle is already contained in Lobachevsky's first published memoir *On the elements of geometry* (1829), cf. [95], §17, p. 26 of the German translation. The construction was repeated in Lobachevsky's *New elements of geometry, with a complete theory of parallels* (1835–1838), cf. [96], §136, p. 211 of the German translation.

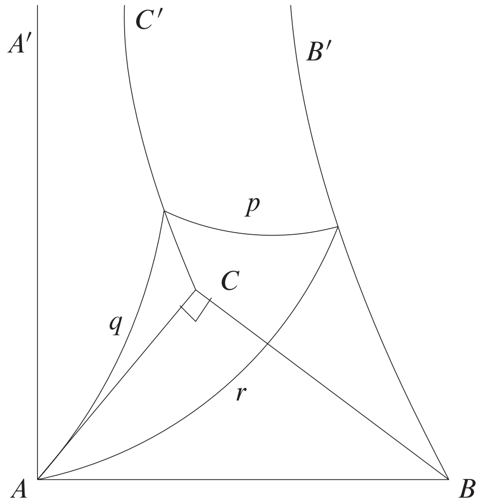



Figure 7. To the right rectilinear triangle $ABC$ whose sides are $a$, $b$, $c$ with opposite angles $\Pi(\alpha)$, $\Pi(\beta)$, $\frac{\pi}{2}$, is associated a Euclidean triangle, constructed on the limit sphere tangent at $A$ to the plane of the triangle $ABC$, with sides $p = r \sin \Pi(\alpha), q = r \cos \Pi(\alpha), r = L(c)$ and opposite angles $\Pi(\alpha)$, $\Pi(\alpha')$, $\frac{\pi}{2}$. The three lines $AA'$, $BB'$ and $CC'$ are parallel to each other, and they define, together with the triangle $ABC$, a tetrahedron with one vertex at infinity, which (in a terminology that Lobachevsky did not use) is called an *orthoscheme*, or a *birectangular tetrahedron*, with one vertex at infinity.

and consequently

$$L(a) = r \sin \Pi(\alpha)\, E^{f(b)}.$$

In the same way, we have

$$L(b) = r \sin \Pi(\beta)\, E^{f(a)}.$$

At the same time, we have

$$q = r \cos \Pi(\alpha),$$

or, equivalently,

$$L(b) = r \cos \Pi(\alpha).$$

Comparing the two values of $L(b)$, we obtain the equation

$$\cos \Pi(\alpha) = \sin \Pi(\beta)\, E^{f(a)}. \tag{1}$$

Substituting $b'$ for $\alpha$ and $c$ for $\beta$ without changing $a$, which is allowed, according to what has already been proved, we obtain

$$\cos \Pi(b') = \sin \Pi(c)\, E^{f(a)},$$

or, equivalently, since

$$\Pi(b) + \Pi(b') = \frac{\pi}{2},$$

$$\sin \Pi(b) = \sin \Pi(c)\, E^{f(a)}.$$

In the same way, we have

$$\sin \Pi(a) = \sin \Pi(c)\, E^{f(b)}.$$

Let us multiply both sides of the last equation by $E^{f(a)}$ and let us substitute $f(b) + f(a)$ by $f(c)$. This gives

$$\sin \Pi(a)\, E^{f(a)} = \sin \Pi(c)\, E^{f(c)}.$$

But since in a right rectilinear triangle the perpendicular edges can vary in such a way that the hypotenuse remains constant, we can set in this equation $a = 0$ without changing $c$. This will give, noting that $f(0) = 0$ and $\Pi(0) = \frac{\pi}{2}$,

$$1 = \sin \Pi(c)\, E^{f(c)}$$

or, equivalently,

$$E^{f(c)} = \frac{1}{\sin \Pi(c)}$$

for any segment $c$.

Now let us take Equation (1),

$$\cos \Pi(\alpha) = \sin \Pi(\beta)\, E^{f(a)},$$

and let us replace $E^{f(a)}$ with $\dfrac{1}{\sin \Pi(a)}$. Then the equation takes the form

$$\cos \Pi(\alpha) \sin \Pi(a) = \sin \Pi(\beta). \tag{2}$$

Replacing $\alpha$, $\beta$ in this equation with $b'$, $c$ without changing $a$, we obtain

$$\sin \Pi(b) \sin \Pi(a) = \sin \Pi(c).$$

Equation (2) gives, after replacing some letters with others,

$$\cos \Pi(\beta) \sin \Pi(b) = \sin \Pi(\alpha).$$

If we replace $\beta$, $b$, $\alpha$ in this equation with $c$, $\alpha'$, $b'$, we get

$$\cos \Pi(c) \cos \Pi(\alpha) = \cos \Pi(b). \tag{3}$$

In the same way, we have

$$\cos \Pi(c) \cos \Pi(\beta) = \cos \Pi(a). \tag{4}$$

Equations (2), (3) and (4) concern a spherical triangle of which, in the sequel, we shall denote the edges by $a$, $b$, $c$ and the angles $A$, $B$ opposite to the edges $a$, $b$, and $\frac{\pi}{2}$ opposite to $c$. In these equations, we can put $a$ instead of $\Pi(\beta)$, $b$ instead of $\Pi(c)$, $c$ instead of $\Pi(a)$, $\frac{\pi}{2} - A$ instead of $\Pi(a)$, and $B$ instead of $\Pi(b)$. In this way, the aforementioned equations become

$$\begin{aligned} \sin A \sin c &= \sin a, \\ \cos b \sin A &= \cos B, \\ \cos a \cos b &= \cos c. \end{aligned} \tag{5}$$

Equations (5) concern a right spherical triangle that can be deduced from a right rectilinear triangle, and whose edges therefore cannot exceed $\frac{\pi}{2}$. Let us add that if we draw an arc of a great circle from the vertex of the angle $A$ perpendicularly to the edge $b$, this arc will intersect the arc $a$ or its extension in such a way that each of the arcs, from the intersection point until $b$, is equal to $\frac{\pi}{2}$, and the angle made by these arcs is $b$. After that, it is not difficult to conclude that in the right spherical triangle,

$$\text{if } c < \frac{\pi}{2} \text{ then } a < \frac{\pi}{2} \text{ and } A < \frac{\pi}{2},$$

$$\text{if } c = \frac{\pi}{2} \text{ then } a = \frac{\pi}{2} \text{ and } A = \frac{\pi}{2},$$

and, finally,

$$\text{if } c > \frac{\pi}{2} \text{ then } a > \frac{\pi}{2} \text{ and } A > \frac{\pi}{2}.$$

It follows that if we assume $a > \frac{\pi}{2}$, we must at the same time assume $c > \frac{\pi}{2}$ and $A > \frac{\pi}{2}$. If in that case we extend the edges $a, c$ beyond the side $b$ until they intersect, we obtain another right spherical triangle whose edges are $\pi - a$, $b$, $\pi - c$ and whose opposite angles are $\pi - A$, $B$, $\frac{\pi}{2}$, that is, a triangle to which Equations (5) apply. But Equations (5) keep the same form if in them we replace $a$ with $(\pi - a)$, $c$ with $(\pi - c)$, and $A$ with $(\pi - A)$. This proves that Equations (5) apply to any right spherical triangle.

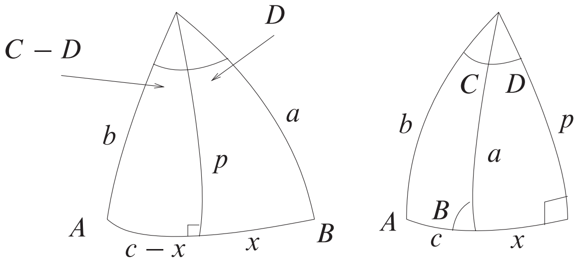
Now let us pass to an arbitrary spherical triangle, whose edges are $a, b, c$ and whose opposite angles are $A$, $B$, $C$, without assuming that one of the angles is a right angle, because so far Equations (5) were proved only in that case.

Let us drop from the vertex of the angle $C$ an arc of great circle $p$ perpendicular to the edge $c$. The following cases may occur:

– either the perpendicular $p$ falls in the interior of the triangle, dividing the angle $C$ into two parts, $D$ and $C - D$, and the edge $c$ into two parts, $x$, opposite to $D$, and $c - x$, opposite to $C - D$ (Figure 8 (a));

– or this perpendicular falls outside the triangle, and it adds an angle $D$ to the angle $C$ and an arc $x$ to the edge $c$ (Figure 8 (b)).

In the first case, the given spherical triangle is the union of two right spherical triangles. The edges of one of these triangles are $p$, $x$, $a$, and its opposite angles are $B$, $D$, $\frac{\pi}{2}$. In the other triangle, the edges are $p$, $c - x$, $b$, and the opposite angles are




(a) (b)

Figure 8

$A$, $C - D$, $\frac{\pi}{2}$. Applying Equations (5) to the first triangle gives

$$\begin{aligned}
&\sin p = \sin a \sin B, \\
&\sin x = \sin a \sin D, \\
&\cos p \sin D = \cos B, \\
&\cos x \sin B = \cos D, \\
&\cos a = \cos p \cos x.
\end{aligned} \tag{A}$$

The second triangle gives in the same manner

$$\begin{aligned}
&\sin p = \sin b \sin A, \\
&\sin(c - x) = \sin b \sin(C - D), \\
&\cos p \sin(C - D) = \cos A, \\
&\cos p \cos(c - x) = \cos b.
\end{aligned} \tag{B}$$

Comparing the values of $\sin p$ given in (A) and (B) gives

$$\sin a \sin B = \sin b \sin A. \tag{6}$$

Dividing both sides of the last equation in Group (B) by both sides of the last equation in Group (A) gives

$$\tan x = \frac{\cos b}{\cos a \sin c} - \cot c.$$

The combination of the second, third and last equations in (A) gives

$$\tan x = \tan a \cos B.$$

Comparing these two values of $\tan x$ leads us to the equation

$$\cos b - \cos a \cos c = \sin a \sin c \cos B. \tag{7}$$

Likewise, if the perpendicular $p$ falls outside the triangle and adds the arc $x$ to the arc $c$, and the angle $D$ to the angle $C$, there arise two right spherical triangles. The edges of one of these triangles are $p$, $x$, $a$ and the opposite angles $\pi - B$, $D$, $\frac{\pi}{2}$, and the edges of the other triangle are $p$, $c + x$, $b$ and the opposite angles $A$, $C + D$, $\frac{\pi}{2}$.

Applying Equations (5) to the first triangle gives

$$\begin{aligned} &\sin p = \sin a \sin B, \\ &\sin x = \sin a \sin D, \\ &-\cos B = \cos p \sin D, \\ &\cos D = \cos x \sin B, \\ &\cos a = \cos p \cos x. \end{aligned} \tag{C}$$

The second triangle, whose edges are $p$, $c + x$, $b$ and whose opposite angles are $A$, $C + D$, $\frac{\pi}{2}$, gives in the same manner

$$\begin{aligned} &\sin p = \sin b \sin A, \\ &\sin(c + x) = \sin b \sin(C + D), \\ &\cos A = \cos p \sin(C + D), \\ &\cos(C + D) = \cos(c + x) \sin A, \\ &\cos b = \cos p \cos(c + x). \end{aligned} \tag{D}$$

Comparing the two values of $\sin p$ in (C) and (D) gives Equation (6) again. We deduce from Equations (C) and (D) that

$$\tan x = \cot c - \frac{\cos b}{\cos a \sin c},$$

and from Equations (C) that

$$\tan x = -\tan a \cos B.$$

Comparing these two values of $\tan x$ leads us again to Equation (7), which is thus proved in complete generality for all spherical triangles, in the same way as Equation (6).

Equation (7) gives, by replacing some letters with others, the following two equations:

$$\begin{aligned} \cos a - \cos b \cos c &= \sin b \sin c \cos A, \\ \cos c - \cos a \cos b &= \sin a \sin b \cos C. \end{aligned}$$

Multiplying both sides of the last equation by $\cos b$, adding the product to the first one and dividing the sum by $\sin b$, we obtain

$$\cos a \sin b = \sin c \cos A + \sin a \cos b \cos C.$$

Substituting $\sin c$ by its value

$$\sin c = \frac{\sin C}{\sin A} \sin a$$

extracted from Equation (6) and dividing by $\sin a$, we find

$$\cot a \sin b = \cot A \sin C + \cos b \cos C. \tag{8}$$

Replacing $\sin b$ by its value

$$\sin a \frac{\sin B}{\sin A}$$

and then multiplying both sides of the equation by $\sin A$, we obtain

$$\cos a \sin B = \cos b \cos C \sin A + \sin C \cos A,$$

from which we deduce, by alteration of some letters,

$$\cos b \sin A = \cos a \cos C \sin B + \sin C \cos B.$$

Eliminating $\cos b$ from the last two equations leads us to the following equation:

$$\cos a \sin B \sin C = \cos B \cos C + \cos A. \tag{9}$$

Equations (6), (7), (8) and (9) are those that are usually given in spherical geometry and which are proven using ordinary geometry.[42]

It follows that spherical trigonometry stays the same, whether we adopt the hypothesis that the sum of the three angles of any rectilinear triangle is equal to two right angles, or whether we adopt the converse hypothesis, that is, that this sum is always less than two right angles.[43] This truly does not occur in rectilinear trigonometry.

Before proving the equations that express, in Pangeometry, the relations between the edges and the angles of an arbitrary rectilinear triangle, we shall investigate, for any segment $x$, the form of the function that we denoted up to now by $\Pi(x)$.

Consider for that purpose a right rectilinear triangle, whose edges are $a$, $b$, $c$ and whose opposite angles are $\Pi(\alpha)$, $\Pi(\beta)$, $\frac{\pi}{2}$. Let us extend $c$ beyond the vertex of the angle $\Pi(\beta)$ and let us make this extension equal to $\beta$ (see Figure 9 (a)).

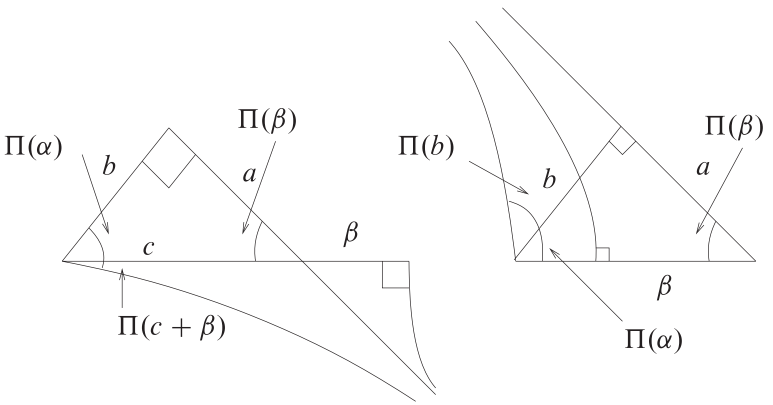
The perpendicular to $\beta$, erected at the endpoint of this segment on the side of the angle which is vertically opposite to the angle $\Pi(\beta)$, is parallel to $a$ and to its extension beyond the vertex of $\Pi(\beta)$. Let us also draw from the vertex of $\Pi(\alpha)$ a line parallel to this same extension of $a$.

The angle that this line makes with $c$ is $\Pi(c + \beta)$, the angle that it makes with $b$ is $\Pi(b)$, and we have the equation

$$\Pi(b) = \Pi(c + \beta) + \Pi(\alpha). \tag{$\Pi$}$$

[42]The formulae are contained in Euler's works, see e.g. [46] and [48].

[43]At the end of §35 of his *Geometrische Untersuchungen*, Lobachevsky drew a similar conclusion: "Hence spherical trigonometry is not dependent upon whether in a rectilinear triangle the sum of the three angles is equal to two right angles or not" (Halsted's translation in [26]).




(a) (b)

Figure 9

If we now take a segment of length $\beta$ starting at the vertex of the angle $\Pi(\beta)$, and on the edge $c$ itself, and if we erect at the endpoint of $\beta$ a perpendicular to $\beta$ on the same side as the angle $\Pi(\beta)$, then this line is parallel to the extension of $a$ beyond the vertex of the right angle. Let us draw from the vertex of the angle $\Pi(\alpha)$ a line parallel to this last perpendicular. Consequently, this line is also parallel to the second extension of $a$. The angle that this parallel makes with $c$ is in any case $\Pi(c-\beta)$, and the angle that it makes with $b$ is $\Pi(b)$ (see Figure 9 (b)). Consequently,

$$\Pi(b) = \Pi(c-\beta) - \Pi(\alpha). \qquad (\Pi')$$

It is easy to convince oneself that this equation is valid not only if $c > \beta$, but also if $c = \beta$ or if $c < \beta$. Indeed, if $c = \beta$, we have, on the one hand, $\Pi(c-\beta) = \Pi(0) = \frac{\pi}{2}$, and on the other hand, the perpendicular to $c$ drawn from the vertex of the angle $\Pi(\alpha)$ is parallel to $a$. From that, it follows that $\Pi(b) = \frac{\pi}{2} - \Pi(\alpha)$, which agrees with our equation.

If $c < \beta$, the endpoint of the line $\beta$ falls beyond the vertex of the angle $\Pi(\alpha)$, at a distance equal to $\beta - c$. The perpendicular to $\beta$ at this endpoint of $\beta$ is parallel to $a$ and to the line drawn from the vertex of the angle $\Pi(\alpha)$ parallel to $a$. From that, it follows that the two adjacent angles that this parallel makes with $c$ are, for the acute one, $\Pi(\beta - c)$, and for the obtuse one, $\Pi(\alpha) + \Pi(b)$. Now since the sum of two adjacent angles is always equal to two right angles, we obtain

$$\Pi(\beta - c) + \Pi(\alpha) + \Pi(b) = \pi,$$

or, equivalently,

$$\Pi(b) = \pi - \Pi(\beta - c) - \Pi(\alpha).$$

But from the definition of the function $\Pi(x)$, we have

$$\pi - \Pi(\beta - c) = \Pi(c - \beta),$$

which gives

$$\Pi(b) = \Pi(c - \beta) - \Pi(\alpha),$$

which is the equation that we found above, and which is therefore proven in all cases.

The two equations $(\Pi)$ and $(\Pi')$ can be replaced by the following equations:

$$\Pi(b) = \frac{1}{2}\Pi(c + \beta) + \frac{1}{2}\Pi(c - \beta),$$

$$\Pi(\alpha) = \frac{1}{2}\Pi(c - \beta) - \frac{1}{2}\Pi(c + \beta).$$

But Equation (3) gives

$$\cos \Pi(c) = \frac{\cos \Pi(b)}{\cos \Pi(\alpha)}.$$

Substituting in this equation $\Pi(b)$ and $\Pi(\alpha)$ by their values, we obtain

$$\cos \Pi(c) = \frac{\cos\left(\frac{1}{2}\Pi(c + \beta) + \frac{1}{2}\Pi(c - \beta)\right)}{\cos\left(\frac{1}{2}\Pi(c - \beta) - \frac{1}{2}\Pi(c + \beta)\right)}.$$

From this equation, we deduce

$$\tan^2 \frac{1}{2}\Pi(c) = \tan \frac{1}{2}\Pi(c - \beta) \tan \frac{1}{2}\Pi(c + \beta).$$

Since the segments $c$ and $\beta$ can vary independently from each other in any right rectilinear triangle, we can set successively, in the last equation, $c = \beta, c = 2\beta, \ldots, c = n\beta$, and we conclude from the equations thus deduced that in general, for any segment $c$ and for any positive integer $n$, we have

$$\tan^n \frac{1}{2}\Pi(c) = \tan \frac{1}{2}\Pi(nc).$$

It is easy to prove that this equation also holds for negative or fractional $n$, from which it follows, choosing the length unit in such a way that

$$\tan \frac{1}{2}\Pi(1) = e^{-1},$$

where $e$ is the basis of natural logarithms, that for any segment $x$,

$$\tan \frac{1}{2}\Pi(x) = e^{-x}.$$

This expression gives $\Pi(x) = \frac{\pi}{2}$ for $x = 0$, $\Pi(x) = 0$ for $x = \infty$, and $\Pi(x) = \pi$ for $x = -\infty$, which agrees with what we have adopted and proved above.

The value found for $\tan \frac{1}{2}\Pi(x)$ gives, for any segment $x$,[44]

$$\sin \Pi(x) = \frac{2}{e^x + e^{-x}},$$

$$\cos \Pi(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

and for two arbitrary segments $x$, $y$,

$$\sin \Pi(x+y) = \frac{\sin \Pi(x) \sin \Pi(y)}{1 + \cos \Pi(x) \cos \Pi(y)},$$

$$\begin{aligned}
\sin \Pi(x-y) &= \frac{\sin \Pi(x) \sin \Pi(y)}{1 - \cos \Pi(x) \cos \Pi(y)},\\
\cos \Pi(x+y) &= \frac{\cos \Pi(x) + \cos \Pi(y)}{1 + \cos \Pi(x) \cos \Pi(y)},\\
\cos \Pi(x-y) &= \frac{\cos \Pi(x) - \cos \Pi(y)}{1 - \cos \Pi(x) \cos \Pi(y)},\\
\tan \Pi(x+y) &= \frac{\sin \Pi(x) \sin \Pi(y)}{\cos \Pi(x) + \cos \Pi(y)}.
\end{aligned}$$

Equations (2), (3) and (4) that we found for a right spherical triangle also hold for a right rectilinear triangle whose edges are $a$, $b$, $c$ with opposite angles $\Pi(\alpha)$, $\Pi(\beta)$ and $\frac{\pi}{2}$. Thus, replacing $\Pi(\alpha)$ by $A$ and $\Pi(\beta)$ by $B$, we obtain for any right rectilinear triangle whose edges are $a$, $b$, $c$ and in which $A$ is the angle opposite to $a$, $B$ the angle opposite to $b$, and $\frac{\pi}{2}$ the angle opposite to $c$, the following equations:

$$\left.\begin{aligned}
\sin \Pi(a) \cos A &= \sin B,\\
\sin \Pi(c) \cos A &= \cos \Pi(b),\\
\cos \Pi(c) \cos B &= \cos \Pi(a).
\end{aligned}\right\} \tag{10}$$

To these equations we also add the following equation which was proved above:

$$\sin \Pi(a) \sin \Pi(b) = \sin \Pi(c). \tag{11}$$

By a permutation of the letters, the first equation in (10) can be written as

$$\sin \Pi(b) \cos B = \sin A.$$

Substituting the value of $\cos B$, extracted from the third equation, into (10), we obtain

$$\sin \Pi(b) \cos \Pi(a) = \sin A \cos \Pi(c).$$

[44] There is an equivalent formula in Lobachevsky's first published memoir, the *Elements of geometry* (1829) ([95], p. 29 of the German translation), $\cot \Pi(x) = \frac{e^x - e^{-x}}{2}$.

Eliminating $\sin\Pi(b)$ from this equation by using Equation (11), we obtain

$$\tan\Pi(c) = \sin A \tan\Pi(a). \tag{12}$$

Now let $a$, $b$, $c$ be the three edges of an arbitrary rectilinear triangle and let $A$, $B$, $C$ be the angles opposite to these edges. Let us drop from the vertex of the angle $C$ a perpendicular $p$ on the edge $c$.

If $p$ falls in the interior of the triangle, in such a way that it divides the angle $C$ into two angles $D$ and $C - D$ and the edge $c$ into two parts, $x$ opposite to $D$ and $c - x$ opposite to $C - D$, we obtain two right rectilinear triangles. The edges of one of these triangles are $p$, $x$, $b$ and their opposite angles are $A$, $D$, $\frac{\pi}{2}$. The edges of the second triangle are $p$, $c - x$, $a$ and their opposite angles are $B$, $C - D$, $\frac{\pi}{2}$ (see Figure 10).

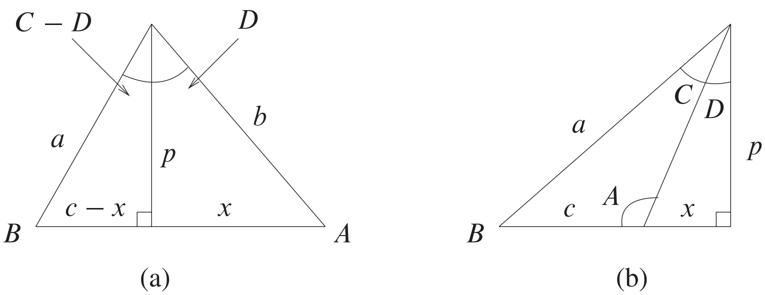



Figure 10

Applying Equation (12) to the first triangle gives

$$\tan\Pi(b) = \sin A \tan\Pi(p).$$

Applying the same equation to the second triangle, we deduce in the same way

$$\tan\Pi(a) = \sin B \tan\Pi(p).$$

From this we conclude

$$\sin A \tan\Pi(a) = \sin B \tan\Pi(b). \tag{13}$$

Applying Equations (10) and (11) to the first triangle gives

$$\cos\Pi(b)\cos A = \cos\Pi(x),$$

$$\sin\Pi(x)\sin\Pi(p) = \sin\Pi(b).$$

Applying these equations to the second triangle gives

$$\sin\Pi(p)\sin\Pi(c - x) = \sin\Pi(a).$$

Replacing $\sin \Pi(c-x)$ in the last equation by its value extracted from the general formula that we found above for $\sin \Pi(x-y)$, we obtain

$$\frac{\sin \Pi(a)}{\sin \Pi(p)} = \frac{\sin \Pi(c) \sin \Pi(x)}{1 - \cos \Pi(c) \cos \Pi(x)}$$

from which we deduce, using the following substitutions,

$$\sin \Pi(p) = \frac{\sin \Pi(b)}{\sin \Pi(x)},$$

$$\cos \Pi(x) = \cos \Pi(b) \cos A,$$

the equation

$$1 - \cos \Pi(b) \cos \Pi(c) \cos A = \frac{\sin \Pi(b) \sin \Pi(c)}{\sin \Pi(a)}. \tag{14}$$

Equations (13) and (14) can be checked for $A = \frac{\pi}{2}$, in which case the perpendicular $p$ coincides with the edge $b$, since in that case Equation (13) amounts to Equation (12) and Equation (14) amounts to Equation (11), that is, to equations that have been proved for any right rectilinear triangle.

If the perpendicular $p$ falls outside the triangle, on the line extending $c$, and if it adds a segment $x$ to the segment $c$ and an angle $D$ to the angle $C$, then we get two right triangles, the edges of one of them being $p, x, b$ and their opposite angles $(\pi - A)$, $D$, $\frac{\pi}{2}$, and the edges of the other one being $p$, $c + x$, $a$ and their opposite angles $B$, $C + D$, $\frac{\pi}{2}$.

Applying Equation (12) to the first of these triangles gives

$$\tan \Pi(b) = \sin A \tan \Pi(p).$$

From the second triangle, we obtain in the same way

$$\tan \Pi(a) = \sin B \tan \Pi(p).$$

Eliminating $\tan \Pi(p)$ from the last two equations, we again find Equation (13). Applying Equations (13) and (11) to the first triangle gives

$$-\cos \Pi(b) \cos A = \cos \Pi(x),$$

$$\sin \Pi(b) = \sin \Pi(x) \sin \Pi(p).$$

From the second triangle we likewise deduce

$$\sin \Pi(a) = \sin \Pi(p) \sin \Pi(c + x).$$

Replacing $\sin \Pi(c + x)$ in this equation with its value extracted from the general formula that we found above for $\sin \Pi(x + y)$, we have

$$\frac{\sin \Pi(a)}{\sin \Pi(p)} = \frac{\sin \Pi(c) \sin \Pi(x)}{1 + \cos \Pi(c) \cos \Pi(x)}.$$

Making the following substitutions in this equation,

$$\sin \Pi(p) = \frac{\sin \Pi(b)}{\sin \Pi(x)}; \quad \cos \Pi(x) = -\cos \Pi(b) \cos A,$$

we obtain

$$\frac{\sin \Pi(a)}{\sin \Pi(b)} = \frac{\sin \Pi(c)}{1 - \cos \Pi(b) \cos \Pi(c) \cos A},$$

an equation which is identical to Equation (14).

Equations (13) and (14) are thus proved for any rectilinear triangle.

Equation (14) gives, by mutation of letters:

$$1 - \cos \Pi(x) \cos \Pi(a) \cos B = \frac{\sin \Pi(c) \sin \Pi(a)}{\sin \Pi(b)}.$$

Multiplying both sides of this equation with both sides of Equation (14), we obtain

$$\begin{aligned} 1 - \cos \Pi(c) \cos \Pi(a) \cos B - \cos \Pi(b) \cos \Pi(c) \cos A \\ + \cos \Pi(a) \cos \Pi(b) \cos^2 \Pi(c) \cos A \cos B = \sin^2 \Pi(c), \end{aligned}$$

or, equivalently,

$$\begin{aligned} \cos^2 \Pi(c) - \cos \Pi(c) \cos \Pi(a) \cos B - \cos \Pi(b) \cos \Pi(c) \cos A \\ + \cos \Pi(a) \cos \Pi(b) \cos^2 \Pi(c) \cos A \cos B = 0. \end{aligned}$$

Eliminating in this equation the common factor $\cos \Pi(c)$, we obtain

$$\begin{aligned} \cos \Pi(c) + \cos \Pi(a) \cos \Pi(b) \cos \Pi(c) \cos A \cos B \\ - \cos \Pi(a) \cos B - \cos \Pi(b) \cos A = 0. \end{aligned}$$

Likewise, we find

$$\begin{aligned} \cos \Pi(a) + \cos \Pi(a) \cos \Pi(b) \cos \Pi(c) \cos B \cos C \\ - \cos \Pi(b) \cos C - \cos \Pi(c) \cos B = 0. \end{aligned}$$

Let us multiply both sides of this equation by $\cos A$ and then subtract this product from the product of the equation preceding it by $\cos C$. We obtain:

$$\cos \Pi(a) (\cos A + \cos B \cos C) = \cos \Pi(c) (\cos C + \cos A \cos B).$$

Taking the squares of both sides of this equation and then dividing each of them by $\cos^2 \Pi(c)$, the equation becomes

$$\frac{\cos^2 \Pi(a)}{\cos^2 \Pi(c)} (\cos A + \cos B \cos C)^2 = (\cos C + \cos A \cos B)^2.$$

But Equation (13) gives

$$\frac{1}{\cos^2 \Pi(c)} = 1 + \frac{\sin^2 A}{\sin^2 C} \tan^2 \Pi(a).$$

If we substitute $\dfrac{1}{\cos \Pi(c)}$ in the penultimate equation by its value given by the last equation, we obtain

$$\cos^2 \Pi(a) + \frac{\sin^2 A}{\sin^2 C} \sin^2 \Pi(a) = \left( \frac{\cos C + \cos B \cos A}{\cos A + \cos B \cos C} \right)^2$$

and then

$$\sin^2 \Pi(a) \left( 1 - \frac{\sin^2 A}{\sin^2 C} \right) = \frac{\sin^2 B(\sin^2 C - \sin^2 A)}{(\cos A + \cos B \cos C)^2}.$$

Dividing both sides of this equation by $\sin^2 C - \sin^2 A$ and extracting square roots, we find

$$\sin \Pi(a) = \frac{\sin B \sin C}{\cos A + \cos B \cos C}$$

with no ambiguity for the sign, since both sides of the last equation are positive. Indeed, $\Pi(a) < \frac{\pi}{2}$, $B < \pi$, $C < \pi$, from which it follows that the sines of these angles are positive. Then,

$$\cos A + \cos(B + C) = 2 \cos \frac{1}{2}(A + B + C) \cos \frac{1}{2}(B + C - A),$$

and since $A + B + C < \pi$,

$$\cos \frac{1}{2}(A + B + C)$$

and

$$\cos \frac{1}{2}(B + C - A)$$

are positive.

Adding to each side of the last equation the positive number $\sin B \sin C$ we get

$$\cos A + \cos B \cos C > 0.$$

Thus, in any rectilinear triangle we have

$$\cos A + \cos B \cos C = \frac{\sin B \sin C}{\sin \Pi(a)}. \tag{15}$$

Multiplying both sides of Equation (14) with both sides of the following equation, which follows from it by alteration of the letters,

$$1 - \cos \Pi(a) \cos \Pi(b) \cos C = \frac{\sin \Pi(a) \sin \Pi(b)}{\sin \Pi(c)}, \tag{16}$$

we obtain

$$(1 - \cos\Pi(a)\cos\Pi(b)\cos C)\ (1 - \cos\Pi(b)\cos\Pi(c)\cos A) = \sin^2\Pi(b)$$

which, after developing the left-hand side, gives

$$\begin{gathered}\cos^2\Pi(b) - \cos\Pi(a)\cos\Pi(b)\cos C - \cos\Pi(b)\cos\Pi(c)\cos A \\ + \cos^2\Pi(b)\cos\Pi(a)\cos\Pi(c)\cos A\cos C = 0,\end{gathered}$$

or, equivalently, dividing by $\cos\Pi(b)$,

$$\begin{gathered}\cos\Pi(b) + \cos\Pi(a)\cos\Pi(b)\cos\Pi(c)\cos A\cos C \\ - \cos\Pi(a)\cos C - \cos\Pi(c)\cos A = 0.\end{gathered} \tag{17}$$

Now we find, from Equation (13),

$$\cos\Pi(c) = \frac{\sin\Pi(c)\sin C}{\sin A}\cot\Pi(a).$$

In this equation, we can substitute for $\sin\Pi(c)$ its value extracted from Equation (16), which gives

$$\cos\Pi(c) = \frac{\sin\Pi(b)\cos\Pi(a)\sin C}{(1 - \cos\Pi(a)\cos\Pi(b)\cos C)\sin A}.$$

Substituting this value of $\cos\Pi(c)$ into Equation (17) gives

$$\cot A\sin C\sin\Pi(b) + \cos C = \frac{\cos\Pi(b)}{\cos\Pi(a)}. \tag{18}$$

Let us collect together Equations (13), (14), (15) and (18) that express the dependence between the sides and the angles of an arbitrary rectilinear triangle, in order to make their use easier:[45]

$$\left.\begin{aligned}&\sin A\tan\Pi(a) = \sin B\tan\Pi(b),\\ &1 - \cos\Pi(b)\cos\Pi(c)\cos A = \frac{\sin\Pi(b)\sin\Pi(c)}{\sin\Pi(a)},\\ &\cos A + \cos B\cos C = \frac{\sin B\sin C}{\sin\Pi(a)},\\ &\cot A\sin C\sin\Pi(b) + \cos C = \frac{\cos\Pi(b)}{\cos\Pi(a)}.\end{aligned}\right\} \tag{19}$$

Starting with these equations, Pangeometry becomes an analytic geometry, and thus it forms a complete and distinct geometric theory. Equations (19) are useful for

[45] Equations (19) are contained in the same form in Lobachevsky's previous works, e.g. in his *Elements of geometry* (1829) [95], p. 21 of Engel's German translation [45], and in his *Geometrische Untersuchungen zur Theorie der Parallellinien* (1840) [100], §37.

representing curves by equations in terms of the coordinates of their points, and for calculating lengths and areas of curves, and areas and volumes of bodies, as I showed in the 1829 *Scientific Memoirs of the University of Kazan*.[46]

It has been remarked above[47] that Pangeometry leads to ordinary geometry if we assume that the segments are infinitely small. We can now check this assertion.

For any infinitely small segment $x$, we can admit the following approximate values:

$$\begin{aligned} \cot \Pi(x) &= x, \\ \sin \Pi(x) &= 1 - \tfrac{1}{2}x^2, \\ \cos \Pi(x) &= x. \end{aligned}$$

If we consider the edges of the triangle as infinitely small quantities of the first order and if we neglect the infinitely small quantities of order grater than two, Equations (19), after substitution of $\sin \Pi(a)$, $\sin \Pi(b)$ etc. by their approximate values, take the form

$$\begin{aligned} &b \sin A = a \sin B, \\ &a^2 = b^2 + c^2 - 2bc \cos A, \\ &\cos A + \cos(B + C) = 0, \\ &a \sin(A + C) = b \sin A. \end{aligned}$$

The first two of these equations are the well-known equations of ordinary trigonometry. The last two give

$$A + B + C = \pi.$$

In order to give an example of the representation of curves by equations relating the coordinates of their points, let us denote by $y$ the length of the perpendicular dropped from a point on a circle of radius $r$ on a fixed diameter of that circle, and $x$ the portion of that diameter between the centre and the foot of the perpendicular $y$. Applying Equation (11) to the right triangle whose edges are $x, y, r$ gives

$$\sin \Pi(x) \sin \Pi(y) = \sin \Pi(r) \tag{20}$$

which is the equation of a circle given in the $(x, y)$-coordinates (Figure 11). If we make the convention of measuring $x$ starting from the endpoint of a diameter, Equation (20) becomes

$$\sin \Pi(r - x) \sin \Pi(y) = \sin \Pi(r),$$

or, equivalently,

$$2(e^r + e^{-r}) = (e^{r-x} + e^{-r+x})(e^y + e^{-y}).$$

[46] cf. Lobachevsky's *Elements of geometry* [95].

[47] This remark was already made by Lobachevsky in his *Elements of geometry* (1829) [95], p. 22 of Engel's translation [45], and repeated in his *Geometrische Untersuchungen zur Theorie der Parallellinien* (1840) [100], §37 .

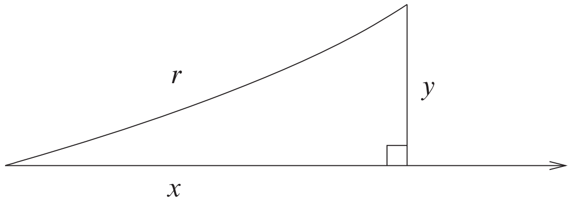



Figure 11. In this figure, $y$ is the length of the perpendicular dropped from a point on a circle of radius $r$ on a fixed diameter, and $x$ is the distance of the foot of this perpendicular to the centre of the circle. The equation of the circle is then: $\sin\Pi(x)\sin\Pi(y) = \sin\Pi(r)$.

If we divide this equation by $e^r$, and if we then set $r = \infty$, we obtain the following identity, which is the equation of the limit circle:

$$2 = (e^y + e^{-y})e^{-x}$$

or, equivalently,

$$\sin\Pi(y) = \tan\frac{1}{2}\Pi(x).$$

It follows from the definition of the limit circle that two axes of that limit circle drawn from the two endpoints of the same chord have the same slope with respect to that chord. This property can serve as a definition of the limit circle, and we can deduce from it the equation of that limit circle, by considering the triangle whose edges are $x$, $y$ and the limit circle chord $2a$. The angles of this triangle are $\Pi(a) - \Pi(y)$, opposite to $x$, $\Pi(a)$ opposite to $y$ and $\frac{\pi}{2}$ opposite to $2a$ (Figure 12).

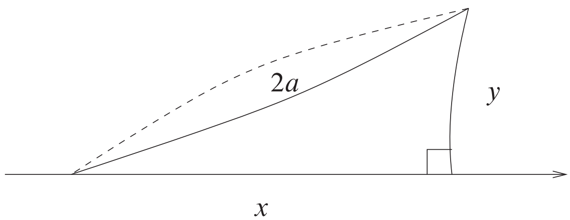



Figure 12. This figure is referred to in the construction of the equation of a limit circle, $\sin\Pi(y) = \tan\frac{1}{2}\Pi(x)$. The limit circle is represented in dashed lines.

According to Equations (10) and (11), we have in this triangle

$$\begin{aligned}
&\sin\Pi(x)\sin\Pi(y) = \sin\Pi(2a),\\
&\sin\Pi(x)\cos(\Pi(a) - \Pi(y)) = \sin\Pi(a),\\
&\sin\Pi(y)\cos\Pi(a) = \sin(\Pi(a) - \Pi(y)).
\end{aligned}$$

The last equation gives

$$2\tan\Pi(y) = \tan\Pi(a) \tag{21}$$

and the first equation can be written as

$$\sin \Pi(x) \sin \Pi(y) = \frac{\sin^2 \Pi(a)}{1 + \cos^2 \Pi(a)}.$$

Replacing $\sin^2 \Pi(a)$ and $1 + \cos^2 \Pi(a)$ in this equation by their values in terms of $\tan^2 \Pi(a)$, and introducing the value of $\tan^2 \Pi(a)$ extracted from Equation (21), we obtain

$$\sin \Pi(x) \sin \Pi(y) = \frac{2 \tan^2 \Pi(y)}{1 + 2 \tan^2 \Pi(y)}$$

and then

$$\sin \Pi(x) = \frac{2 \sin \Pi(y)}{1 + \sin^2 \Pi(y)},$$

from which we deduce

$$2 \cos^2 \frac{1}{2} \Pi(x') = \frac{(1 + \sin \Pi(y))^2}{1 + \sin^2 \Pi(y)}$$

and

$$2 \sin^2 \frac{1}{2} \Pi(x') = \frac{(1 - \sin \Pi(y))^2}{1 + \sin^2 \Pi(y)}.$$

Dividing both sides of the last equation by both sides of the one before it and extracting square roots, we obtain

$$\tan \frac{1}{2} \Pi(x') = \frac{1 - \sin \Pi(y)}{1 + \sin \Pi(y)},$$

hence

$$\sin \Pi(y) = \frac{1 - \tan \frac{1}{2} \Pi(x')}{1 + \tan \frac{1}{2} \Pi(x')}.$$

The right-hand side of the last equation can be put in the following form:

$$\frac{\cos \frac{1}{2} \Pi(x') - \sin \frac{1}{2} \Pi(x')}{\cos \frac{1}{2} \Pi(x') + \sin \frac{1}{2} \Pi(x')}$$

or, equivalently,

$$\frac{\sin\left(\frac{1}{4}\pi - \frac{1}{2}\Pi(x')\right)}{\cos\left(\frac{1}{4}\pi - \frac{1}{2}\Pi(x')\right)} = \frac{\sin \frac{1}{2} \Pi(x)}{\cos \frac{1}{2} \Pi(x)} = \tan \frac{1}{2} \Pi(x)$$

and consequently

$$\sin \Pi(y) = \tan \frac{1}{2} \Pi(x)$$

as already found.

To give an example of the computation of the length of a curve, let us search for the expression of the length of the circumference of a circle of radius $r$. Let us draw two radii, with central angle $\frac{2\pi}{n}$, where $n$ is an integer. Let us drop from the endpoint of one of these two radii a perpendicular $p$ on the other radius. The larger $n$ is, the more the product $np$ will be close to the length of the circumference of the circle. The right triangle which has $r$ as hypotenuse, $p$ as one of the sides of the hypotenuse and $\frac{2\pi}{n}$ as angle opposite to $p$ (Figure 13) satisfies Equation (13):

$$\sin\frac{2\pi}{n}\tan\Pi(p) = \tan\Pi(r).$$

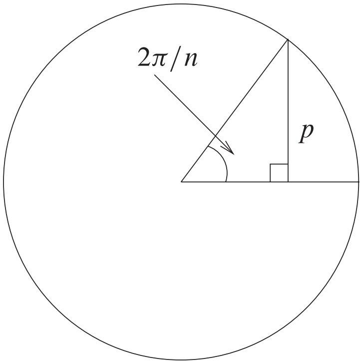



Figure 13. Computing the circumference of a circle, as the limit of the value of $np$ as $n \to \infty$.

But it is known that

$$\lim\left(n\sin\frac{2\pi}{n}\right) = 2\pi$$

as $n = \infty$, whereas

$$\frac{1}{n}\tan\Pi(p) = \frac{2}{n(e^p - e^{-p})}$$

and

$$n(e^p - e^{-p}) = 2np,$$

with an approximation which becomes more and more accurate as $n$ increases, and, consequently, as $p$ decreases. From this, we have

$$\text{Circumference}\,(r) = np = 2\pi\cot\Pi(r)$$

or, equivalently,

$$\text{Circumference}\,(r) = (e^r - e^{-r})\pi,$$

which gives, for very small $r$,

$$\text{Circumference}\,(r) = 2\pi r,$$

as in ordinary geometry.

Let us also determine the limit circle arc $s$ in coordinates. Let $y$ be the perpendicular dropped from the endpoint of the arc $s$ on the axis passing by the other endpoint, and let $x$ be the portion of this axis bounded by the vertex of the arc and the foot of the perpendicular. Let $c$ be the chord of the arc $s$ (Figure 14). Likewise, let $c_1, c_2, c_3, \ldots$ be the chords of the arcs $\frac{1}{2}s$, $\frac{1}{2^2}s$, $\frac{1}{2^3}s, \ldots$. We already proved (Equation (21)) that

$$\cot \Pi(y) = 2 \cot \Pi\left(\frac{1}{2}c\right).$$

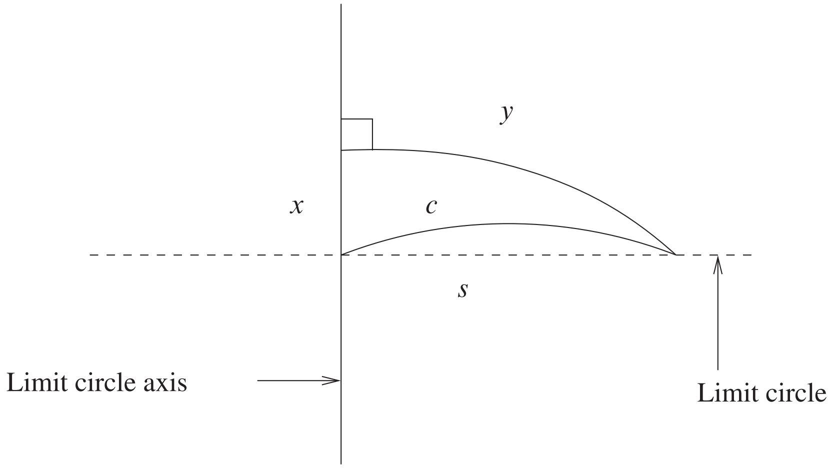



Figure 14. The figure refers to the computation of the length of a limit circle arc $s$. We find $s = \cot \Pi(y)$.

Similarly, we have

$$\cot \Pi\left(\frac{1}{2}c\right) = 2 \cot \Pi\left(\frac{1}{2}c_1\right),$$

$$\cot \Pi\left(\frac{1}{2}c_1\right) = 2 \cot \Pi\left(\frac{1}{2}c_2\right),$$

$$\cot \Pi\left(\frac{1}{2}c_2\right) = 2 \cot \Pi\left(\frac{1}{2}c_3\right)$$

and in general, for any positive integer $n$,

$$\cot \Pi\left(\frac{1}{2}c_{n-1}\right) = 2 \cot \Pi\left(\frac{1}{2}c_n\right)$$

from which we conclude

$$\cot \Pi(y) = 2^{n+1} \cot \Pi\left(\frac{1}{2}c_n\right).$$

If the integer $n$ is very large and if, consequently, the segment $c_n$ is very small, we obtain

$$2^{n+1} \cot \Pi\left(\frac{1}{2}c_n\right) = 2^n c_n.$$

Since

$$2^n c_n = s \quad \text{for } n = \infty,$$

it follows that

$$s = \cot \Pi(y). \tag{22}$$

Let us also determine the limit circle arc $s$ in terms of the portion $t$ of the tangent at the vertex of the axis drawn from an endpoint of the arc $s$, comprised between the contact point and the intersection point between the tangent and the axis drawn from the other endpoint of the arc $s$ (Figure 15). In other words, let us determine the function

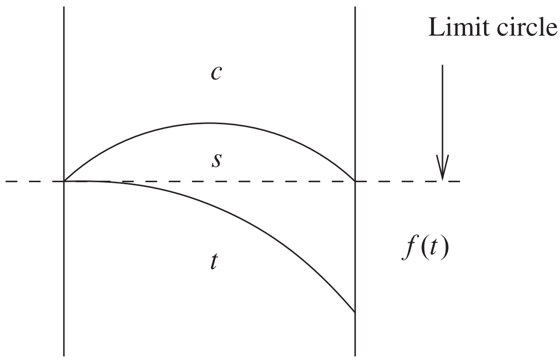



Figure 15. The length of the limit circle arc $s$ is $\cos \Pi(t)$. In this figure are drawn the straight segment joining the endpoints of the limit circle arc, and a straight segment tangent to this limit circle arc. The vertical lines represent two axes through the endpoints of the limit circle arc.

that we earlier called $L(t)$. In the triangle whose edges are $c, t, f(t)$ and with opposite angles $\Pi(t)$, $\pi - \Pi(\frac{1}{2}c)$, $\frac{\pi}{2} - \Pi(\frac{1}{2}c)$, we find, applying Equation (13),

$$\sin \Pi(t) \tan \Pi(c) = \sin \Pi\left(\frac{1}{2}c\right) \tan \Pi(t).$$

But we already saw (in Equation 21) that

$$\tan \Pi\left(\frac{1}{2}c\right) = 2 \tan \Pi(y).$$

Combining this with the remark that

$$\tan \Pi(c) = \frac{\sin^2 \Pi\left(\frac{1}{2}c\right)}{2 \cos \Pi\left(\frac{1}{2}c\right)},$$

we obtain

$$\cos \Pi(t) = 2 \cot \Pi\left(\frac{1}{2}c\right),$$

or, equivalently, using Equation (22),

$$\cos \Pi(t) = s = L(t).$$

The equation of a straight line has a rather complicated form, if we want it to be general, and if we want it to represent the straight line for any possible position of that line with respect to the coordinate axes.

Let us drop from a fixed point of the straight line a perpendicular $a$ on the $x$-axis and let us call $L$ the angle that this perpendicular makes with the line. Let us denote by $y$ the perpendicular dropped on the $x$-axis from another point on the given line, whose distance to the first point is $l$. Finally, let $x$ be the portion of the $x$-axis bounded by these two perpendiculars. Let us draw a straight line between the vertex of $a$ and the foot of $y$ and let $r$ be the length of the portion of this line bounded by these two points. We obtain two triangles, one which is a right triangle with edges $a$, $x$, $r$ and opposite angles $A$, $X$, $\frac{\pi}{2}$, and the other with edges $y$, $r$, $l$ and opposite angles $L - X$, $C$, $\frac{\pi}{2} - A$ (see Figure 16).

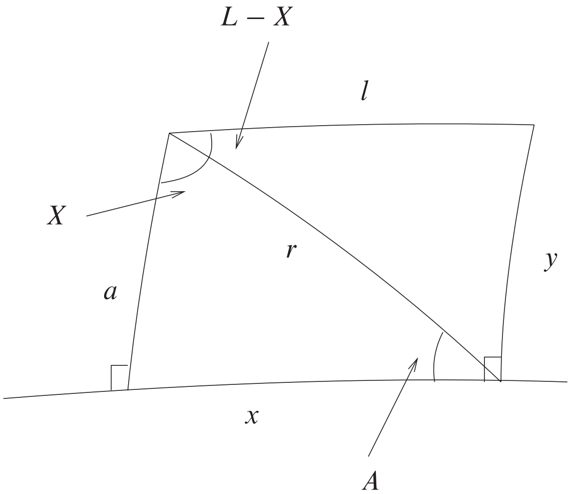



Figure 16

Applying Equations (10) and (11) to the first triangle gives

$$\begin{aligned}
&\sin \Pi(x) \sin \Pi(a) = \sin \Pi(r),\\
&\sin \Pi(x) \cos X = \sin A,\\
&\sin \Pi(a) \cos A = \sin X,\\
&\cos \Pi(r) \cos A = \cos \Pi(x),\\
&\cos \Pi(r) \cos X = \cos \Pi(a).
\end{aligned}$$

From these equations, we deduce

$$\begin{aligned}
&\tan A = \tan \Pi(x) \cos \Pi(a), \\
&\tan \Pi(r) = \tan \Pi(x) \sin \Pi(a) \cos A, \\
&\tan X = \tan \Pi(a) \cos \Pi(x), \\
&\cos \Pi(x) = \cos \Pi(r) \cos A, \\
&\sin X = \sin \Pi(a) \cos A.
\end{aligned}$$

Applying the last equation in (19) to the second triangle gives

$$\cot(L - X) \cos A \sin \Pi(r) + \sin A = \frac{\cos \Pi(r)}{\cos \Pi(y)}$$

from which it follows that

$$\begin{aligned}
\cos \Pi(y) &= \frac{\cos \Pi(r)}{\cot(L - X) \cos A \sin \Pi(r) + \sin A} \\
&= \frac{\cos \Pi(r)(\tan L - \tan X)}{(1 + \tan L \tan X) \cos A \sin \Pi(a) \sin \Pi(x) + \sin A(\tan L - \tan X)}.
\end{aligned}$$

Substituting the value of $\tan \Pi(x)$ in this equation, we obtain

$$\begin{aligned}
\cos \Pi(y) = \cos \Pi(r)\, &(\tan L - \tan \Pi(a) \cos \Pi(x)) \\
&\div (1 + \tan L \tan \Pi(a) \cos \Pi(x)) \cos A \sin \Pi(a) \sin \Pi(x) \\
&\quad + \sin A\, (\tan L - \tan \Pi(a) \cos \Pi(x))\,.
\end{aligned}$$

Substituting the value of $\cos \Pi(r)$ in this equation we find, after simplifying,

$$\cos \Pi(y) = \frac{\cos \Pi(a)}{\sin \Pi(x)} - \sin \Pi(a) \cot \Pi(x) \cot L. \tag{23}$$

If the given line is parallel to the $x$-axis, we have $L = \Pi(a)$ and Equation (23) takes the following form:

$$\cos \Pi(y) = \frac{\cos \Pi(a)}{\sin \Pi(x)} - \frac{\cos \Pi(a)}{\tan \Pi(x)}$$

or, equivalently,

$$\cos \Pi(y) = \cos \Pi(a) e^{-x}. \tag{24}$$

If we denote by $s$ and $s'$ the lengths of two limit circle arcs bounded by the $x$-axis and the line parallel to that axis such that the first arc, $s$, is tangent to $a$ at the foot of $a$ and the second arc, $s'$, is tangent to $y$ at the foot of $y$, we obtain, from what we have already proved,

$$\begin{aligned}
s &= \cos \Pi(a), \\
s' &= \cos \Pi(y),
\end{aligned}$$

which gives

$$s' = se^{-x},$$

where $x$ is the distance between the two arcs $s$ and $s'$. This equation shows that the constant $E$ which we introduced above to denote the constant ratio of two limit circle arcs bounded by two parallel lines whose distance is equal to 1, is equal to $e$, that is, the basis of natural logarithms.

If we set $a = 0$ in Equation (23) and if we replace $L$ by $\pi - L$, we obtain

$$\cos \Pi(y) = \cot \Pi(x) \cot L,$$

which is therefore the equation of a line passing through the coordinate origin $x$ and making an angle $L$ with the $x$-axis. This is consistent with Equation (10).

Now consider a quadrilateral having two edges, $a$ and $y$, that are perpendicular to a third edge, $x$. Let $c$ be the fourth edge and $\varphi$ the angle made by $a$ and $c$, while the angle between $c$ and $y$ is a right angle.[48] Let us draw the diagonal $r$ that joins the vertex of the angle $\varphi$ to the vertex of the opposite right angle. This diagonal divides the quadrilateral into two right triangles (see Figure 17). The edges of one of these two triangles are $a$, $x$, $r$ and its opposite angles are $A$, $X$, $\frac{\pi}{2}$. The edges of the other triangle are $y$, $c$, $r$ and its opposite angles are $\varphi - X$, $\frac{\pi}{2} - A$, $\frac{\pi}{2}$.

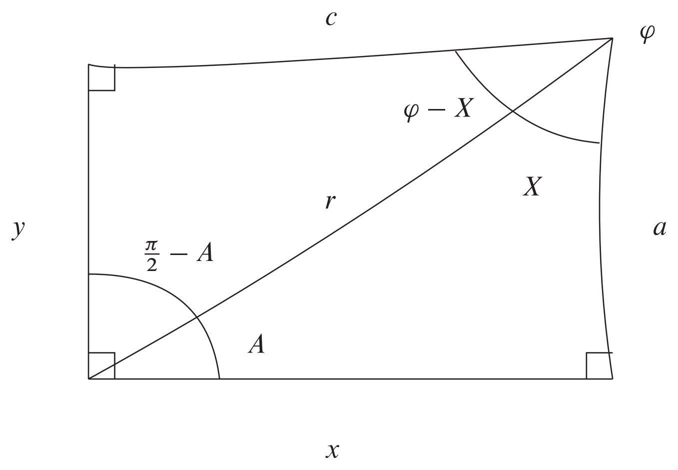



Figure 17

[48]Such a quadrilateral with three right angles is usually called a *Lambert quadrilateral* or an *Ibn al-Haytham-Lambert quadrilateral*, see [121] and and [131]. The study of these quadrilaterals, in particular the monotonicity properties of their side lengths in terms of other parameters, is of paramount importance in hyperbolic geometry.

Applying Equations (10), (11) and (13) to the first triangle gives

$$\left.\begin{aligned}\sin\Pi(r) &= \sin\Pi(a)\sin\Pi(x),\\ \sin A\tan\Pi(a) &= \sin X\tan\Pi(x),\\ \cos\Pi(r)\cos A &= \cos\Pi(x),\\ \cos\Pi(r)\cos X &= \cos\Pi(a).\end{aligned}\right\}\tag{G}$$

Likewise, the second triangle gives the following equations:

$$\left.\begin{aligned}\sin\Pi(y)\sin\Pi(c) &= \sin\Pi(r),\\ \sin\Pi(y)\cos(\varphi-X) &= \cos A,\\ \cos\Pi(r)\cos(\varphi-X) &= \cos\Pi(c),\\ \cos\Pi(r)\sin A &= \cos\Pi(y).\end{aligned}\right\}\tag{H}$$

Equation (12), applied to the first triangle, gives

$$\left.\begin{aligned}\tan\Pi(r) &= \sin X\tan\Pi(x),\\ \tan\Pi(r) &= \sin A\tan\Pi(a)\end{aligned}\right\}\tag{K}$$

whereas the application of the same equation to the second triangle gives

$$\left.\begin{aligned}\tan\Pi(r) &= \sin(\varphi-X)\tan\Pi(y),\\ \tan\Pi(r) &= \cos A\tan\Pi(c).\end{aligned}\right\}\tag{L}$$

Replacing $\sin\Pi(r)$ in the second equation in (K) by its value obtained from Equations (G), we find

$$\cos\Pi(r) = \frac{\sin\Pi(x)\cos\Pi(a)}{\sin A}.$$

Substituting this value of $\cos\Pi(r)$ into the last equation in (H), we obtain

$$\cos\Pi(y) = \sin\Pi(x)\cos\Pi(a).\tag{25}$$

Dividing both members of the last equation in (H) by both members of the third equation in (G) gives

$$\tan A = \frac{\cos\Pi(y)}{\cos\Pi(x)}.$$

Substituting the value of $\cos\Pi(y)$ that we have just found into this equation, we obtain

$$\tan A = \tan\Pi(x)\cos\Pi(a).$$

Dividing both members of the second equation in (G) by both members of the last equation in the same group gives

$$\frac{\tan X\tan\Pi(x)}{\cos\Pi(r)} = \frac{\sin A\tan\Pi(a)}{\cos\Pi(a)}.$$

Substituting the value of $\sin A$ obtained from the last equation in (H) into this equation, we obtain

$$\tan X = \frac{\cos \Pi(y) \tan \Pi(a)}{\cos \Pi(a)} \cot \Pi(x).$$

Finally, replacing $\cos \Pi(y)$ in this equation by the value found above, we obtain

$$\tan X = \cos \Pi(x) \tan \Pi(a).$$

The combination of the second equation in (H) with the first equation in (L) gives

$$\tan(\varphi - X) \frac{\tan \Pi(y)}{\sin \Pi(y)} = \frac{\tan \Pi(r)}{\cos A}$$

or, equivalently,

$$\tan(\varphi - X) = \frac{\cos \Pi(y) \tan \Pi(r)}{\cos A}$$

and, replacing $\tan \Pi(r)$ by its value given by the second equation in (K),

$$\tan(\varphi - X) = \tan A \tan \Pi(a) \cos \Pi(y).$$

Replacing $\tan A$ and $\tan X$ by the values we found above, this equation takes the form

$$\tan \varphi = \frac{\tan \Pi(a)}{\cos \Pi(x)}. \tag{26}$$

This equation shows that $x$ is always real if the angle $\varphi$ is greater than $\Pi(a)$ and smaller than a right angle, or if $\pi - \varphi > \Pi(a)$, $\pi - \varphi < \frac{\pi}{2}$. The value of $\cos \Pi(x)$ is positive if $\frac{\pi}{2} > \varphi > \Pi(a)$ and consequently $x$ is also positive. But if $\frac{\pi}{2} > \pi - \varphi > \Pi(a)$, then the value of $\cos \Pi(x)$ becomes negative and the segment $x$ is situated on the other side of the perpendicular $a$.

This shows that if two straight lines, situated in the same plane, do not intersect no matter how we extend them, and if the two lines are not parallel, then they must be perpendicular to the same straight line. All pairs of straight lines that are in the same plane, and are neither parallel nor perpendicular to a common straight line must necessarily intersect. The straight lines that are in the same plane and that necessarily intersect each other if they are sufficiently extended are therefore only the pairs of straight lines that satisfy the following property: at each point on these lines, the angle made by the line passing by that point and the perpendicular dropped from that point on the other line is smaller than the angle of parallelism corresponding to the length of this perpendicular. Using the preceding results, it is possible to simplify greatly the general equation of a straight line (23) in the case where the straight line described by that equation does not intersect the $x$-axis.

Let $a$ be the perpendicular dropped on the $x$-axis from a fixed point that is arbitrarily chosen on the given line. Let $L$ be the angle made by this perpendicular and the line, and among the two angles we choose the one that is situated on the side of positive $x$'s. Let us first find a segment $l$ satisfying

$$\cot \Pi(l) = \tan \Pi(a) \cot L.$$

This is always possible, provided that $L > \Pi(a)$, that is, provided that the given line does not intersect the $x$-axis. Let us transport this segment $l$ to the $x$-axis starting from the coordinate origin, on the side of positive or negative $x$'s depending on the sign of $l$. Then, let us erect at the endpoint of the segment $l$ a perpendicular to the $x$-axis, and let us extend it until it meets the given line. Let $b$ be the portion of this perpendicular that is bounded by the given line and the $x$-axis. The angle that this perpendicular makes with the given line is, according to Equation (26), a right angle (Figure 18). If we now take the foot of the perpendicular $b$ as the coordinate origin, we have, from Equation (25),

$$\cos \Pi(b) = \cos \Pi(y) \sin \Pi(x), \tag{27}$$

which is the general equation of a line that does not intersect the $x$-axis. We can set $y = a$ in this equation and $x = -l$ at the same time, which gives

$$\cos \Pi(b) = \cos \Pi(a) \sin \Pi(l).$$

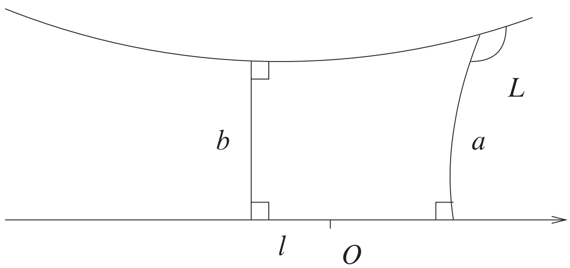


Figure 18

Replacing $\cos \Pi(b)$ and $\sin \Pi(l)$ by their values, this equation takes the form

$$\cos \Pi(y) \sin \Pi(x) = \cos \Pi(a) \sqrt{1 - \tan^2 \Pi(a) \cot^2 L}.$$

The right-hand side of this equation becomes imaginary as soon as

$$\tan \Pi(a) \cot L > 1,$$

that is, for any line that intersects the $x$-axis.

Using the preceding discussion, we can solve the problem of finding the distance between two points in the plane, whose positions are determined by their rectangular coordinates $x$, $y$ and $x'$, $y'$. Let us set, for brevity,

$$\Delta x = x' - x, \quad \Delta y = y' - y.$$

Let us drop a perpendicular from the vertex of $y$ to $y'$ and let us denote by $q$ the length of that perpendicular. Let $y_1$ denote the segment of $y$ bounded by the $x$-axis and the perpendicular $q$ (see Figure 19).

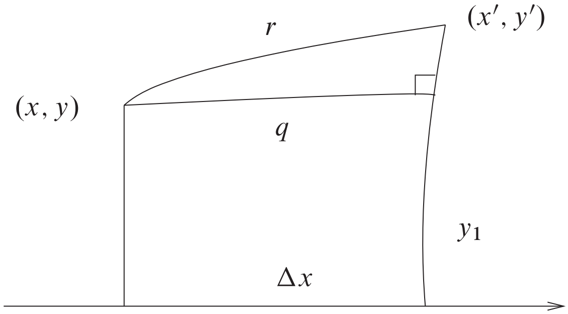



Figure 19

It follows from Equation (25) that

$$\cos \Pi(y_1) = \cos \Pi(y) \sin \Pi(\Delta x),$$
$$\cos \Pi(\Delta x) = \cos \Pi(q) \sin \Pi(y).$$

After having determined the vales of $y_1$ and $q$ from these equations, the distance between the two points, which we denote by $r$, is given by the following equation, which we deduce from Equation (11):

$$\sin \Pi(r) = \sin \Pi(y' - y_1) \sin \Pi(q).$$

If $\Delta x$ and $\Delta y$, and, consequently, $q$ and $r$, are so small that we can neglect the higher order powers of these quantities with respect to the smaller order ones, then $r$ represents the element $ds$ of a curve, and we can find the expression of this element as follows:

$$\sin \Pi(q) = 1 - \frac{1}{2}q^2,$$
$$\cos \Pi(q) = q - \frac{1}{3}q^3,$$
$$\sin \Pi(r) = 1 - \frac{1}{2}r^2,$$
$$\sin \Pi(y' - y_1) = 1 - \frac{1}{2}(y' - y_1)^2.$$

With this, we obtain

$$q = \frac{\Delta x}{\sin \Pi(y)}$$

and[49]

$$ds = \sqrt{dy^2 + \frac{dx^2}{\sin^2 \Pi(y)}}.$$

In the case where the curve is a limit circle, we have

$$\sin \Pi(y) = e^{-x}.$$

From the general expressions that we gave for $\sin \Pi(a)$, etc. in terms of $a$, we deduce

$$d\Pi(a) = -\sin \Pi(a) da.$$

By differentiating the equation of the limit circle, we find

$$\sin \Pi(y) \cos \Pi(y) dy = e^{-x} dx$$

and

$$ds = \frac{e^x dx}{\sqrt{1 - e^{-2x}}}.$$

Integrating with respect to $x$, starting from $x = 0$, we find

$$s = \sqrt{e^{2x} - 1}$$

or, in other words,

$$s = \cot \Pi(y),$$

which agrees with what we already found.

If we denote by $r$ the segment from a point on a curve to the coordinate origin and by $\varphi$ the angle made by this segment with the positive $x$-axis (Figure 20), we obtain, from Equation (12), in the triangle whose edges are $y$, $x$, $r$,

$$\tan \Pi(r) = \sin \varphi \tan \Pi(y).$$

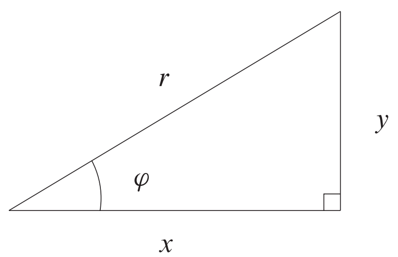



Figure 20. Polar and rectangular coordinates in the hyperbolic plane.

[49]A formula for the length element $ds$ is also contained in Lobachevsky's *Elements of geometry* [95], §29 (p. 38 of the German edition).

Taking the logarithms of the two sides of this equation and differentiating with respect to $y$, $\varphi$, $v$, we obtain

$$\frac{dr}{\cos\Pi(r)} = -\cot\varphi d\varphi + \frac{dy}{\cos\Pi(y)}.$$

From this equation, we deduce

$$dy = \left(\cot\varphi d\varphi + \frac{dr}{\cos\Pi(r)}\right)\cos\Pi(y)$$

or, substituting $\cos\Pi(y)$ by its value in terms of $r$ and $\varphi$,

$$dy = \frac{\cos\varphi\cos\Pi(r)d\varphi + \sin\varphi dr}{\sqrt{1-\cos^2\varphi\cos^2\Pi(r)}}.$$

To express $dx$ in terms of $r$ and $\varphi$, let us use (Equation (10)):

$$\cos\Pi(r)\cos\varphi = \cos\Pi(x).$$

Differentiating with respect to $r, \varphi, x$ the logarithms of the two sides of this equation, we obtain

$$\frac{\sin^2\Pi(r)dr}{\cos\Pi(r)} - \tan\varphi d\varphi = \frac{\sin^2\Pi(x)dx}{\cos\Pi(x)}$$

from which we deduce, using the equations

$$\sin\Pi(x)\sin\Pi(y) = \sin\Pi(r),$$

$$\cos\Pi(r)\cos\varphi = \cos\Pi(x),$$

the following equation, which expresses the desired value:

$$\frac{dx}{\sin\Pi(y)} = \frac{\cos\varphi\sin\Pi(r)dr - d\varphi\sin\varphi\cot\Pi(r)}{\sqrt{1-\cos^2\varphi\cos^2\Pi(r)}}.$$

From this we obtain

$$ds = \sqrt{dr^2 + d\varphi^2\cot^2\Pi(r)}.$$

Concerning the circle, assuming that the coordinate origin is at the centre, we find, since $dr = 0$,

$$ds = d\varphi\cot\Pi(r).$$

Integrating from $\varphi = 0$ to $\varphi = \frac{\pi}{2}$ and multiplying the result by 4, we find the following expression for the circumference of the circle of radius $r$:

$$2\pi\cot\Pi(r)$$

which coincides with the one we already found.

If we call $s$ a limit circle arc measured starting from the $x$-axis, the revolution of $s$ around the $x$-axis produces a portion of a limit sphere, and the endpoint of this arc describes a circle which we can determine on the limit sphere in the same manner as a circle of radius $s$ is determined in its plane in ordinary geometry. It follows that the circumference is equal to $2\pi s$. On the other hand, the circumference of the same circle considered in its plane where the perpendicular $y$, dropped from an endpoint of the arc $s$ on the axis of the limit circle which serves as $x$-axis and passes through the other endpoint, is the circle radius, is given in Pangeometry by the formula

$$2\pi \cot \Pi(y),$$

from which it follows that

$$s = \cot \Pi(y),$$

as was already proved.

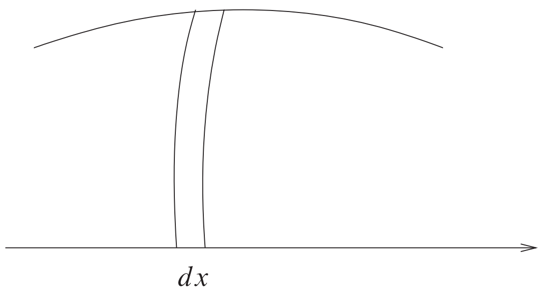


Figure 21. Computing plane area using a decomposition by infinitesimally close limit circles having the $x$-axis as an axis.

In order to find the area element of plane figures, we divide the plane by limit circles that all have the $x$-axis as an axis, in such a way that the distance between each limit circle and the next one is infinitely small and expressed by $dx$ (Figure 21). Let $s$ be the arc of one of these limit circles which is bounded by the $x$-axis and a point on a given curve whose coordinates are $(x, y)$. Then, let $s'$ be an arc of another of these limit circles which is bounded by the $x$-axis and a point of the given curve whose coordinates are $(x + dx, y + dy)$.

The infinitesimal area of the plane region bounded by $s$ and $s'$ on the one hand, and by the curve and the $x$-axis on the other hand, has the expression[50]

$$dS = \frac{es\,dx}{e-1}$$

[50] In this expression, the term $\frac{e}{e-1}$ is a normalisation, which is determined by the curvature constant of the space, or, equivalently, as Lobachevsky puts it, by the choice of a unit of length (recall the computation of the ratio of the lengths of equidistant pieces of limit circles in the first pages of this memoir, where the factor $\frac{E}{E-1}$ that appears there is here $\frac{e}{e-1}$). See the remark he makes after Equation (24).

or, using the substitution $s = \cot \Pi(y)$,

$$dS = \frac{e\,dx \cot \Pi(y)}{e-1}.$$

As an example, let us determine the area of the limit circle whose equation we found in rectangular coordinates:

$$\sin \Pi(y) = e^{-x}.$$

We have the following expression for the differential of area:

$$dS = \frac{e}{e-1} dy \cos \Pi(y) \cot \Pi(y).$$

Integrating this expression starting from $y = 0$, we find, for the area bounded by the limit circle arc, the $x$-axis, and the ordinate $y$, the expression

$$S = \frac{e}{e-1}\left(\cot \Pi(y) - \frac{1}{2}\pi + \Pi(y)\right).$$

We already saw that the area of a plane region bounded by two parallel lines indefinitely extended in their parallelism direction and a limit circle having these two parallel lines as axes has the following expression:

$$\frac{es}{e-1} = \frac{e \cot \Pi(y)}{e-1}.$$

From this, we find the expression for the area bounded by two parallel lines, one of which is perpendicular to $y$, that are drawn from the two endpoints of $y$ and that are extended indefinitely on the parallelism side:

$$\frac{1}{2}\pi - \Pi(y).$$

Using this formula, we can determine the area of a rectilinear triangle in terms of the angles of that triangle (see Figure 22). For this, let $a$, $b$, $c$ denote the edges of the triangle and let $A = \Pi(\alpha)$, $B = \Pi(\beta)$ and $\frac{\pi}{2}$ be its angles. Let us extend the hypotenuse $c$ beyond the vertex of the angle $\Pi(\beta)$ by an amount equal to $\beta$. The perpendicular to $\beta$ drawn from the endpoint of $\beta$ is parallel to the extension of $a$. The area of the plane region bounded on the one hand by these two parallels indefinitely extended on the parallelism side, and on the other hand by the segment $\beta$, has the value

$$\frac{1}{2}\pi - \Pi(\beta).$$

Now let us draw from the vertex of the angle $A$ a parallel to the perpendicular, which will therefore make with $c$ the angle $\Pi(c + \beta)$ and which will thus be parallel to the extension of $a$. Then the area of the plane region contained between $c + \beta$ and the two parallels infinitely extended on the side of parallelism is

$$\frac{1}{2}\pi - \Pi(c + \beta).$$

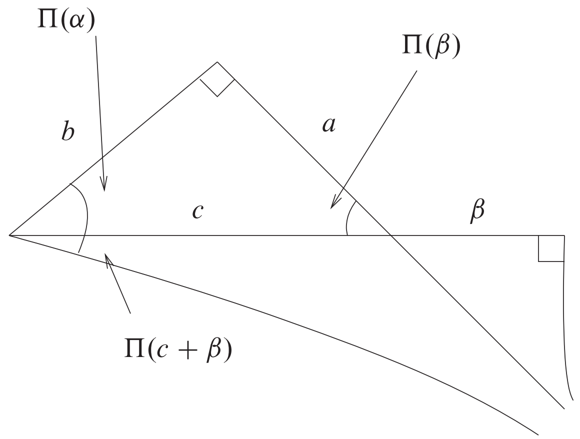



Figure 22. The quadrilateral used in the proof of the formula giving the area of a right triangle as angular deficit.

In the same way, the area of the plane region bounded by $b$, the line drawn from the vertex $A$, and the edge $a$ with its extension, is equal to

$$\frac{1}{2}\pi - \Pi(b).$$

Then, the sum of $\frac{\pi}{2} - \Pi(\beta)$ and $\frac{\pi}{2} - \Pi(b)$, from which we subtract $\frac{\pi}{2} - \Pi(c+\beta)$, is the expression of the area of the triangle. Thus, the value of this area is

$$\frac{1}{2}\pi - \Pi(b) - \Pi(\beta) + \Pi(c+\beta).$$

On the other hand, we already proved that

$$\Pi(b) = \Pi(\alpha) + \Pi(c+\beta).$$

Substituting $\Pi(b)$ in the expression of the area of the right rectilinear triangle by this value, the expression for this area becomes

$$\frac{1}{2}\pi - \Pi(\alpha) - \Pi(\beta),$$

that is, the area of a right rectilinear triangle is equal to the difference between two right angles and the sum of the three angles of the triangle.

From this, we also deduce that the area of any rectilinear triangle is equal to the excess of two right angles over the sum of the three angles of that triangle.[51] This

[51] In hyperbolic geometry, the area of a triangle is usually *defined* as the angular deficit, that is, as the difference between two right angles and the angle sum of the triangle. One can define, more generally, an *area function* on plane figures, as a function that is positive on triangles, additive, and invariant under congruence.

follows from the fact that the area of any rectilinear triangle is the sum of the areas of two right rectilinear triangles.

It is easy to deduce from what precedes that the area of any quadrilateral is equal to the excess to four right angles of the sum of the four angles of that quadrilateral, and that in general, the area of any $n$-sided polygon is equal to the excess to $(n-2)\pi$ of the angle sum of the polygon.

In particular, consider a quadrilateral, with two edges $a$ and $y$ perpendicular to a third edge $x$, and whose fourth edge $t$ is perpendicular to the edge $a$ and makes with $y$ an angle which we shall call $\omega$ (Figure 23). We already showed (Equation (25)) that there is the following relation between the constitutive elements of the quadrilateral:

$$\cos \Pi(a) = \cos \Pi(y) \sin \Pi(x).$$

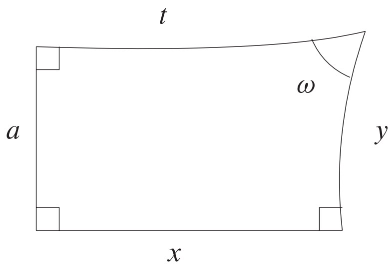



Figure 23

If we consider $x$ and $y$ as variables and $a$ as a constant, then the area of this quadrilateral can be expressed, as any planar area, and according to what we proved before, by the integral

$$\int dx \cot \Pi(y)$$

It is easy to see that the angular deficit satisfies these properties. In his *Theorie der Parallellinien*, written in 1766, J. H. Lambert already made the discovery that under the hypothesis of the failure of Euclid's postulate, with the other postulates satisfied, such an area function is unique up to a multiplicative constant. From this, it follows that in the hyperbolic plane, any natural definition of an area function is, restricted to triangles, a multiple of the angular deficit function. Lambert, who knew this fact, suspected (but never claimed) that the hypothesis of the failure of Euclid's postulate leads to a contradiction. Lambert wrote : "If it were possible under the third hypothesis to cover a large triangle with two equal triangles, then we could prove in a simple manner that for every triangle the excess from $180^{\circ}$ of the sum of its three angles is proportional to the area of the surface." [Wenn es bey der dritten Hypothese möglich wäre, mit gleichen und ähnlichen Triangeln einer grössern Triangel zu bedecken: so würde es sich auch leichte darthun lassen, dass bey jedem Triangel der Ueberschuss von $180$ Gr. über die Summe seiner drey Winkel dem Flächenraume des Triangles proportional wäre.] ([139], p. 202). Gauss also wrote on that topic, in his notes and in his correspondence. For instance, in a letter to Gerling dated 16 March 1819, [52], p. 182, Gauss wrote: "The angular deficit from $180^{\circ}$ in the plane triangle, for example, does not only get greater as the area gets greater, but it is exactly proportional to it". [Der Defect der Winkelsumme im ebenen Dreieck gegen $180^{\circ}$ ist z.B., nicht bloss desto grösser, je grösser der Flächeninhalt ist, sondern ihm genau proportional.] In a letter to Wolfgang Bolyai, dated 6 March 1832, [52], p. 220, Gauss wrote a proof of the fact that the area of a triangle is equal to its angular deficit.

which, applied to our case and after substituting $\cot \Pi(y)$ by its value, gives the value

$$\int_0 \frac{dx \cos \Pi(a)}{\sqrt{\sin^2 \Pi(x) - \cos^2 \Pi(a)}}.$$

On the other hand, this same area is, as a consequence of the theorem that expresses the area of an arbitrary plane polygon in terms of its angles,

$$= \frac{1}{2}\pi - \omega.$$

This gives

$$\frac{1}{2}\pi - \omega = \cos \Pi(a) \int_0 \frac{dx}{\sqrt{\sin^2 \Pi(x) - \cos^2 \Pi(a)}}. \tag{M}$$

The angle $\omega$ which the edge $t$ makes with the edge $y$ is given by the following (Equation (27)):

$$\tan \omega = \frac{\tan \Pi(y)}{\cos \Pi(x)}.$$

Writing, in Equation (M), $\alpha$ instead of $\Pi(a)$, and $\xi$ instead of $\Pi(x)$, the equation becomes

$$\frac{\frac{1}{2}\pi - \omega}{\cos \alpha} = \int_{\frac{\pi}{2}} \frac{d\xi}{\sin \xi \sqrt{\sin^2 \alpha - \cos^2 \xi}},$$

where $\alpha$ is a constant.

That this formula is correct can be checked by differentiating. In this way, Pangeometry gives a new method for finding approximate values of definite integrals.

Let us consider the integral

$$\int A dx$$

where $A$ is a given function of $x$. To compute the value of this integral, we must set $A = \cot \Pi(y)$ and determine the values of $y'$, $y''$, $y'''$, ... that correspond to $x'$, $x''$, $x'''$ ... taken arbitrarily between the integration limits. Then, we must compute the lengths of the chords that join the vertices from $y'$ to $y''$, from $y''$ to $y'''$ and so on, and the angles that each chord makes with the extension of the next chord. The sum of these angles gives an approximate value of the integral.

The area of the plane region bounded by a given segment and two mutually parallel lines which are drawn from the endpoints of the segment and extended indefinitely on the side of parallelism is equal to $\pi$ minus the sum of the two angles that the two parallel lines make with the given segment, since this figure can be considered as a triangle with one angle equal to zero.

The area of a plane curve can be divided by lines that are all parallel to a given line, for instance the $y$-axis. If we draw from the point of abscissa $x$ on the $x$-axis a line parallel to the $y$-axis, this line will make with the $x$-axis an angle equal to $\Pi(x)$.[52]

[52]Note that, in hyperbolic geometry, although we can cover the whole plane (as a foliation) by lines that are parallel to the $y$-axis, not all of these lines intersect the $x$-axis.

Likewise, the line drawn from the endpoint of abscissa $x + dx$ will make with the $x$-axis an angle equal to $\Pi(x + dx)$ (Figure 24). It follows that the area of the plane region bounded by $dx$ and these two parallels is equal to $d\Pi(x)$. Now let $u$ be the

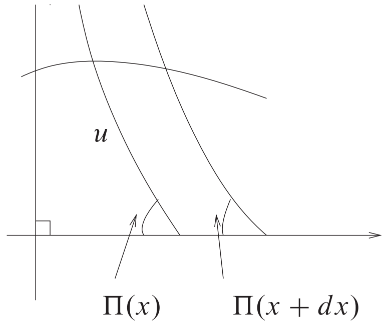



Figure 24. The area below a planar curve is computed by considering infinitesimally close lines that are parallel to the $y$-axis.

length of the portion of the first parallel bounded by the $x$-axis and the curve. The area of the region bounded by the two parallels and which is outside the given curve is, according to what we proved above,

$$-e^{-u} d\Pi(x).$$

It follows that the portion of this area that is situated between the curve and the $x$-axis, that is, the area element of the curve, is given by

$$dS = -(1 - e^{-u}) d\Pi(x).$$

To compute the area of a circle of radius $r$, we must take the general expression of the area element of a curve that we found above, that is, the expression

$$dS = dx \cot \Pi(y),$$

and substitute the value of $\cot \Pi(y)$ extracted from the equation of the circle

$$\sin \Pi(x) \sin \Pi(y) = \sin \Pi(r),$$

where the origin of rectangular coordinates is at the centre of the circle. This gives

$$dS = dx \sqrt{\frac{\sin^2 \Pi(x)}{\sin^2 \Pi(r)} - 1}.$$

Integrating from $x = 0$, we obtain

$$S = \frac{1}{\sin \Pi(r)} \arcsin \left( \frac{\cos \Pi(x)}{\cos \Pi(r)} \right) - \arcsin \left( \frac{\cot \Pi(x)}{\cot \Pi(r)} \right).$$

For $x = r$, this gives, for the area of the quarter of the circle,

$$\frac{\pi}{2 \sin \Pi(r)} - \frac{1}{2}\pi.$$

Multiplying by 4, we find the area of the circle:

$$2\pi \left( \frac{1}{\sin \Pi(r)} - 1 \right) = \pi (e^{\frac{1}{2}r} - e^{-\frac{1}{2}r})^2.$$

If $r$ is extremely small, this expression for the circle area becomes $\pi r^2$, which is the expression given by ordinary geometry.

The preceding expression of the area of the circle allows us to give the following expression for the area element of any curve:

$$dS = d\varphi \left( \frac{1}{\sin \Pi(r)} - 1 \right),$$

in which $r$ is the radius vector drawn from the coordinate origin to a point on the curve, and where $\varphi$ is the angle that this radius vector makes with a fixed line that passes through the coordinate origin.

Applying this formula to the determination of the area of a rectilinear triangle with edges $a$, $b$, $c$ and opposite angles $A$, $B$, $C$ gives, taking the angles $A$, $C$ and the edges $b$, $a$ as variables:

$$\text{Area of the triangle } = \int_0^A dA \left( \frac{1}{\sin \Pi(b)} - 1 \right).$$

The edge $b$ can be expressed in terms of $c$, $A$, $B$ using the last equation in (19):

$$\cot B \sin A \sin \Pi(c) + \cos A = \frac{\cos \Pi(c)}{\cos \Pi(b)}.$$

Let us extract from this equation the value of $\sin \Pi(b)$, and let us substitute it in the expression of the area of the triangle. We obtain:

$$\text{Area of the triangle } = \int_0^A \frac{dA}{\sqrt{1 - \dfrac{\cos^2 \Pi(c)}{(\cot B \sin A \sin \Pi(c) + \cos A)^2}}} - A.$$

But we already showed that the area of a triangle is

$$\pi - A - B - C$$

where $A$ and $B$ are given angles and $C$ is given by Equation (19):

$$\cos C + \cos A \cos B = \frac{\sin A \sin B}{\sin \Pi(c)}.$$

Comparing these two expressions of the area of a triangle gives

$$\pi - B - C = \int_0^A \frac{dA\,(\cot B \sin A \sin \Pi(c) + \cos A)}{\sqrt{(\cot B \sin A \sin \Pi(c) + \cos A)^2 - \cos^2 \Pi(c)}}.$$

For $B = \frac{\pi}{2}$, this equation gives

$$\frac{\pi}{2} - C = \int_0^A \frac{\cos A\, dA}{\sqrt{\cos^2 A - \cos^2 \Pi(c)}}.$$

After integration, this equation becomes

$$\frac{\pi}{2} - C = \arcsin\left(\frac{\sin A}{\sin \Pi(c)}\right),$$

which agrees with the equation that determines $C$.

From what precedes we can deduce two expressions for the value of the area of an arbitrary closed polygon, one of them expressed as a definite integral, and the other one depending only on the sum of the angles of the polygon. The two values of the same area are equal. This gives us a new way for finding values of several definite integrals, values which would have been difficult to find by other means.

To give still another example, let us consider a right rectilinear triangle, with $x$ and $y$ being the sides forming its right angle, and $r$ being its hypotenuse. Let $A$ be the angle opposite to $y$, and $B$ the angle opposite to $x$. Equations (10) and (11) give, for this triangle,

$$\begin{aligned}
&\sin \Pi(x) \sin \Pi(y) = \sin \Pi(r),\\
&\sin \Pi(x) \cos B = \sin A,\\
&\cos \Pi(r) \cos A = \cos \Pi(x),\\
&\cos \Pi(r) \cos B = \cos \Pi(y).
\end{aligned}$$

From these equations we deduce

$$\begin{aligned}
\cos \Pi(r) &= \frac{\cos \Pi(x)}{\cos A},\\
\sin \Pi(r) &= \sqrt{1 - \left(\frac{\cos \Pi(x)}{\cos A}\right)^2},\\
\sin \Pi(y) &= \frac{1}{\sin \Pi(x)} \sqrt{1 - \left(\frac{\cos \Pi(x)}{\cos A}\right)^2} = \sqrt{\frac{1}{\sin^2 \Pi(x)} - \frac{\cot^2 \Pi(x)}{\cos^2 A}},\\
\cot \Pi(y) &= \frac{\sin A \cos \Pi(x)}{\sqrt{\cos^2 A - \cos^2 \Pi(x)}}.
\end{aligned}$$

Substituting $\Pi(x) = \frac{\pi}{2} - \omega$ in the last equation, we obtain

$$\cot \Pi(y) = \frac{\sin A \sin \omega}{\sqrt{\cos^2 A - \sin^2 \omega}}.$$

But we already saw that the differential of the area is $dx \cot \Pi(y)$, which gives, when applied to the present situation,

$$dx \cot \Pi(y) = \sin A \frac{d\omega \tan \omega}{\sqrt{\cos^2 A - \sin^2 \omega}}.$$

From this, integrating from $\omega = 0$, which corresponds to $x = 0$, and noting that the area expressed by the integral is also equal to $\frac{\pi}{2} - A - B$, we deduce the following:

$$\frac{\pi}{2} - A - B = \sin A \int_0^{\omega} \frac{\tan \omega d\omega}{\sqrt{\cos^2 A - \sin^2 \omega}}$$

where $A$ is a constant angle and where $B$ is determined by the equation

$$\cos B = \frac{\sin A}{\cos \omega}.$$

If $\omega = \frac{\pi}{2}$, the hypotenuse becomes parallel to the edge $y$ and the angle $B$ becomes equal to zero. Thus we have in this case

$$\frac{\pi}{2} - A = \sin A \int_0^{\frac{\pi}{2} - A} \frac{d\omega \tan \omega}{\sqrt{\cos^2 A - \sin^2 \omega}}.$$

One can determine the value of a more general integral by considering the area of an arbitrary rectilinear triangle, whose edges are $a$, $b$, $c$ and whose opposite angles are $A$, $B$, $C$, by dividing this area into elements by parallel lines. Let us take the vertex of the angle $C$ as a coordinate origin, and the edge $a$ as an $x$-axis. Let $B = \Pi(\beta)$, where $\beta$ is positive if $B < \frac{\pi}{2}$ and negative if $B > \frac{\pi}{2}$. Let us draw from the point of abscissa $x$ a line $u$ parallel to the edge $c$ and let us extend this parallel until it intersects the edge $b$ (see Figure 25). The angle that this parallel makes with the segment $x$ is $\Pi(\beta - a + x)$. It follows that the angle that this parallel makes with the extension of $x$ is $\Pi(a - \beta - x)$.

If we take as the area element of the triangle the portion of this area that is bounded by two parallels $u$ that are infinitely close, we get, according to what has been shown above, the following expression for that element:

$$dS = -d\Pi(a - \beta - x)(1 - e^{-u}).$$

Let us consider $y$, $x$ and $u$ as variables and $a$ and $\beta$ as constants. Equations (19), applied to the triangle whose edges are $x$ and $u$ and where the angle between these two edges is $\Pi(\beta - a + x)$, give:

$$\cot C \sin \Pi(\beta - a + x) \sin \Pi(x) + \cos \Pi(\beta - a + x) = \frac{\cos \Pi(x)}{\cos \Pi(u)}.$$

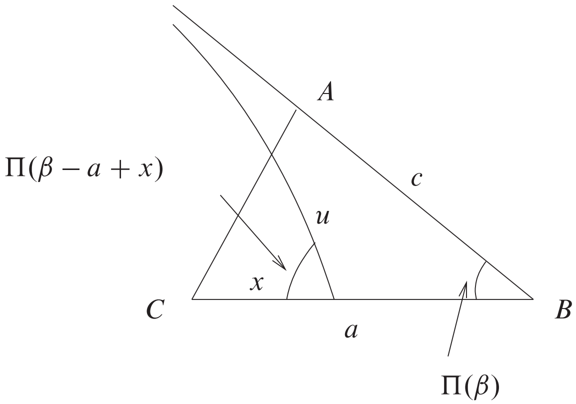



Figure 25

From this equation we deduce, setting for brevity $\Pi(\beta - a + x) = \omega$,

$$\cos \Pi(u) = \frac{\cos \Pi(x)}{\cot C \sin \omega \sin \Pi(x) + \cos \omega}$$

and then

$$e^{2u} = \frac{\cot C \sin \omega \sin \Pi(x) + \cos \omega + \cos \Pi(x)}{\cot C \sin \omega \sin \Pi(x) + \cos \omega - \cos \Pi(x)}.$$

Since

$$\sin \Pi(x) = \sin \Pi\left((\beta - a) - (\beta - a + x)\right) = \frac{\sin \Pi(\beta - a) \cos \omega}{1 - \cos \Pi(\beta - a) \cos \omega},$$

we find, in the same way,

$$\cos \Pi(x) = \frac{\cos \Pi(\beta - a) - \cos \omega}{1 - \cos \Pi(\beta - a) \cos \omega}.$$

Substituting these values of $\sin \Pi(x)$ and $\cos \Pi(x)$ in the expression for $e^{2u}$ gives

$$e^{2u} =$$

$$\frac{\cot C \sin^2 \omega \sin \Pi(\beta - a) + \cos \omega\,(1 - \cos \Pi(\beta - a) \cos \omega) + \cos \Pi(\beta - a) - \cos \omega}{\cot C \sin^2 \omega \sin \Pi(\beta - a) + \cos \omega\,(1 - \cos \Pi(\beta - a) \cos \omega) - \cos \Pi(\beta - a) + \cos \omega}$$

$$= \frac{\cot C \sin^2 \omega \sin \Pi(\beta - a) + \cos \Pi(\beta - a) \sin^2 \omega}{\cot C \sin^2 \omega \sin \Pi(\beta - a) + 2 \cos \omega - (1 + \cos^2 \omega) \cos \Pi(\beta - a)}.$$

Then, we find

$$d\Pi(a - \beta - x) = -d\Pi(\beta - a + x) = -d\omega.$$

Now comparing the two expressions for the area of a triangle gives the following equation:

$$\pi - A - B - C =$$

$$-\omega + \int_{x=0}^{x=a} d\omega \sqrt{\frac{\cot C \sin \Pi(\beta - a) \sin^2 \omega + 2\cos\omega - (1+\cos^2\omega)\cos\Pi(\beta - a)}{\cot C \sin\Pi(\beta - a)\sin^2\omega + \cos\Pi(\beta - a)\sin^2\omega}}.$$

Now if we set $\Pi(\beta - a) = \alpha$, this equation takes the form

$$\begin{aligned}(\pi - A - B - C + \alpha - \Pi(\beta)) &\sqrt{\cot C \sin\alpha + \cos\alpha} \\ &= \int_{\omega=\alpha}^{\omega=\Pi(\beta)} \frac{d\omega}{\sin\omega} \sqrt{\cot C \sin\alpha \sin^2\omega + 2\cos\omega - (1+\cos^2\omega)\cos\alpha}\end{aligned}$$

where the angles $A$, $B$ and the segment $\beta$ must be computed using the following equations:

$$\alpha = \Pi(\beta - a),$$
$$B = \Pi(\beta)$$

and

$$\cos A + \cos B \cos C = \frac{\sin B \sin C}{\sin \Pi(a)},$$

the last equation being the last equation in the set (19) applied to the triangle that we are considering.

In Pangeometry, to fix the position of a point, we can use, besides rectilinear and polar coordinates, limit circle arcs. Such a system of coordinates offers several advantages regarding the simplicity of the formulae.

Let us determine the position of a point in the plane using rectangular coordinates $x$, $y$ in such a way that $y$ is the length of the perpendicular dropped on the $x$-axis from a point of which we want to determine the position, and $x$ is the distance of the foot of the perpendicular $y$ from the coordinate origin. Let $\eta$ be the length of the limit circle arc bounded by the vertex of the perpendicular $y$ and the $x$-axis, this axis being also the axis of the limit circle, and let us denote by $\xi$ the distance from the vertex of the limit circle, situated on the $x$-axis, to the coordinate origin (Figure 26). We already saw that in this case

$$\eta = \cot \Pi(y).$$

The equation of the limit circle gives

$$e^{-(x-\xi)} = \sin \Pi(y).$$

Using these two equations, we can express $\xi$, $\eta$ in terms of $x$, $y$, and, conversely, $x$, $y$ in terms of $\xi$, $\eta$, which allows us to pass from the equation of a line expressed in the $(x, y)$-coordinates to the equation of a line expressed in the $(\xi, \eta)$-coordinates and vice-versa.

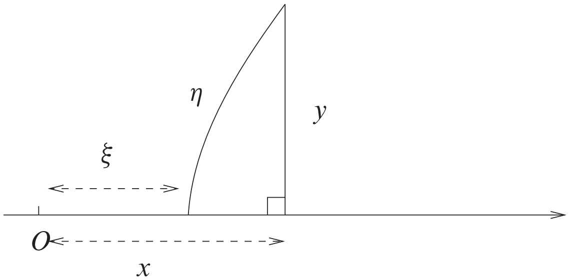



Figure 26. In this figure, $\eta$ is the length of the limit circle arc whose axis is the $x$-axis, $(x, y)$ are the rectangular coordinates, and $(\xi, \eta)$ are the limit circle coordinates of the given point.

The differential of planar areas can be expressed in terms of $\xi, \eta$ by the equation

$$d^2S = d\xi d\eta$$

where $S$ denotes area.

If we consider $S$ as a function of $x$ and $y$, we obtain

$$\frac{dS}{dx} = \frac{dS}{d\xi}.$$

Differentiating with respect to $y$, we obtain

$$\frac{d^2S}{dxdy} = \frac{1}{\sin \Pi(y)} \frac{d^2S}{d\xi d\eta} = \frac{1}{\sin \Pi(y)}$$

which agrees with what we found before.

Let us drop from a point in space a perpendicular $z$ on the $(x, y)$-plane and let us draw from this perpendicular a plane that intersects the $(x, y)$-plane in a line that is parallel to the $x$-axis. Let us take this intersection, directed towards the side of parallelism, as an axis for a limit circle that passes through the vertex of the perpendicular $z$, and let $\zeta$ be the length of the limit circle arc contained between the vertex of $z$ and this axis. We have

$$\zeta = \cot \Pi(z).$$

The length of the segment $q$ on the parallel to the $x$-axis drawn from the foot of the perpendicular $z$ and contained between the endpoint of $\zeta$ in the $(x, y)$-plane and the foot of this perpendicular (Figure 27) is given by

$$e^{-q} = \sin \Pi(z).$$

The length of the limit circle arc drawn from the foot of $z$, whose axis is the positively directed $x$-axis, and which is bounded by this point and this axis, is $\cot \Pi(y)$. The limit circle arc $\eta$, drawn from the intersection point of $\zeta$ with the $(x, y)$-plane whose

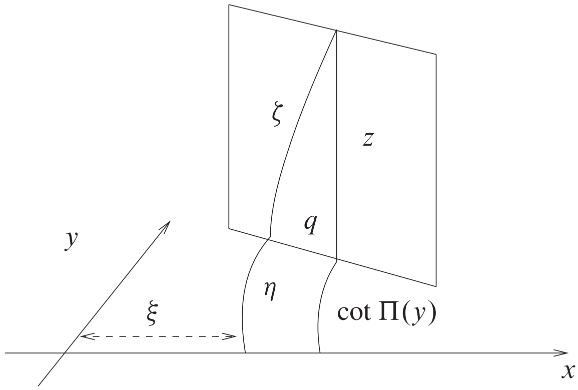



Figure 27. Using limit circle coordinates in dimension three. The vertical plane intersects the $(x, y)$-plane in a line which is parallel to the $x$-axis. The terms $\cot \Pi(y)$, $\zeta$ and $\eta$ are (lengths of) limit circle arcs.

axis is the positively directed $x$-axis, and which is bounded by this point and this axis, is, according to what we have already proved, given by the equation

$$\eta = \frac{\cot \Pi(y)}{\sin \Pi(z)}.$$

Finally, if we also denote by $\xi$ the portion of the $x$-axis comprised between the coordinate origin and the arc $\eta$, the equation of the limit circle gives

$$e^{-x+\xi+q} = \sin \Pi(y).$$

From these equations, we deduce, by first varying $z$ and then $\zeta$, which depends on it,

$$d\zeta = \frac{dz}{\sin \Pi(z)}.$$

By making only $y$ and $\eta$ vary, we obtain

$$d\eta = \frac{dy}{\sin \Pi(y) \sin \Pi(z)}.$$

Finally, by making only $\xi$ and $x$ vary, we obtain

$$d\xi = dx.$$

It remains, to complete the new geometric theory called Pangeometry, and which is based on new principles that are more general than those of ordinary geometry, to give the values of the differentials of areas of curved surfaces and of volumes of arbitrary bodies, expressed in the coordinates that determine the position of a point in space.

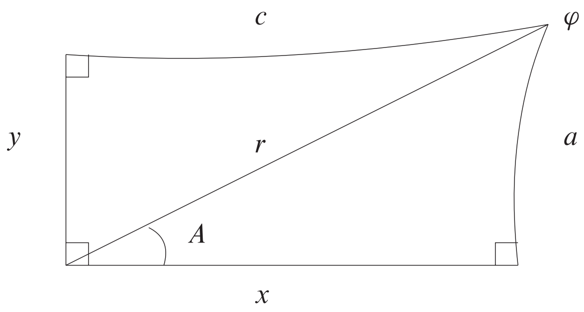



Figure 28

For this, we consider again the quadrilateral having two edges, $a$ and $y$, perpendicular to a third edge $x$, and whose fourth edge, $c$, is perpendicular to $y$ and makes with $a$ an angle $\varphi$ (Figure 28).

We already found (Equation (25)) that

$$\cos \Pi(y) = \cos \Pi(a) \sin \Pi(x).$$

Then, using Equations (10) and (11), and denoting by $r$ the diagonal drawn from the vertex of the angle $\varphi$ to the vertex of the opposite right angle, and by $A$ the angle between $x$ and $r$, we find:

$$\cos \Pi(r) \cos A = \cos \Pi(x)$$

and

$$\cos A \tan \Pi(c) = \tan \Pi(r).$$

From these two equations, we deduce

$$\cos \Pi(x) \tan \Pi(c) = \sin \Pi(r).$$

But since

$$\sin \Pi(r) = \sin \Pi(a) \sin \Pi(x),$$

we obtain

$$\tan \Pi(c) = \sin \Pi(a) \tan \Pi(x).$$

If $c$ and $x$ are small enough so that we can neglect the higher powers compared to the lower ones and take the following approximate values for $\tan \Pi(c)$ and $\tan \Pi(x)$:

$$\tan \Pi(c) = \frac{1}{c}; \quad \tan \Pi(x) = \frac{1}{x},$$

we obtain

$$c = \frac{x}{\sin \Pi(a)}. \tag{28}$$

The line $c$ joining the vertices of $a$ and of $y$ will not be perpendicular to $y$ if $a = y$ in the quadrilateral. But in that case, the line $p$ that joins the midpoint of $c$ to the midpoint of $x$ will be perpendicular to $c$ and to $x$. Thus, in Equation (27) we can replace $c$ by $\frac{1}{2}c$ and $x$ by $\frac{1}{2}x$, which will not change the form of that equation. This equation is therefore proven even in the case $a = y$, a case to which the proof that we gave above does not immediately apply.

The area of a curved surface is the sum of the areas of triangles having all of their vertices on the surface and forming a continuous lattice. The smaller the dimensions of the triangles, the more this measure will be precise. The limit of this sum, as the dimensions of the triangles decrease infinitely, from which the area can differ by a quantity less than any given quantity, is said to be the mathematical value of the area of the surface. Let us first determine the area of an arbitrary right rectilinear triangle in terms of its edges, which we denote by $a, b, c$, and let us denote the angles opposite to these edges by $\Pi(\alpha)$, $\Pi(\beta)$, $\frac{\pi}{2}$ respectively.

We already saw that in such a triangle we can substitute for

$$a, \ b, \ c, \ \alpha, \ \beta$$

the segments

$$a, \ \alpha', \ \beta, \ b', \ c$$

respectively.

Besides that, we found that

$$2\Pi(b) = \Pi(c + \beta) + \Pi(c - \beta).$$

In this equation, let us substitute $\alpha'$ for $b$, $\beta$ for $c$, and $c$ for $\beta$. We obtain

$$\pi - 2\Pi(\alpha) = \Pi(\beta + c) + \Pi(\beta - c),$$

or, equivalently,

$$2\Pi(\alpha) = \Pi(c - \beta) - \Pi(c + \beta).$$

In the same way, we find

$$2\Pi(\beta) = \Pi(c - \alpha) - \Pi(c + \alpha).$$

Interchanging the letters in the last equation, as indicated above, we obtain

$$2\Pi(c) = \Pi(\beta - b') - \Pi(\beta + b').$$

In the same way, we have

$$2\Pi(c) = \Pi(\alpha - a') - \Pi(\alpha + a')$$

from which we deduce, again by the permutation of letters mentioned above,

$$2\Pi(\beta) = \Pi(b' - a') - \Pi(b' + a').$$

Likewise, we have

$$2\Pi(\alpha) = \Pi(a' - b') - \Pi(a' + b').$$

Adding both sides of the last two equations, we obtain

$$2\Pi(\alpha) + 2\Pi(\beta) = \pi - 2\Pi(a' + b'),$$

from which we deduce that the area of a triangle $\triangle$ is given by the expression

$$\triangle = \frac{\pi}{2} - \Pi(\alpha) - \Pi(\beta) = \Pi(a' + b').$$

Then, we have

$$\tan\left(\frac{1}{2}\triangle\right) = e^{-a'}e^{-b'} = \tan\left(\frac{1}{4}\pi - \frac{1}{2}\Pi(a)\right)\tan\left(\frac{1}{4}\pi - \frac{1}{2}\Pi(b)\right).$$

From this we deduce

$$\tan\left(\frac{1}{2}\triangle\right) = \frac{e^a - 1}{e^a + 1} \cdot \frac{e^b - 1}{e^b + 1}.$$

If $a$ and $b$ are so small that we can neglect the higher powers of $a$, $b$ and $\triangle$, this formula gives

$$\triangle = \frac{1}{2}ab,$$

as in ordinary geometry.

We know that in an arbitrary right rectilinear triangle we can always choose an edge $c$ so that the perpendicular dropped from the vertex of the opposite angle $C$ in the direction of that edge falls on that edge $c$ and not on an extension of it. This perpendicular divides the edge $c$ into two parts, one of them, $x$, adjacent to the angle $A$, and the other one, $c - x$, adjacent to the angle $B$ (Figure 29). The area of this

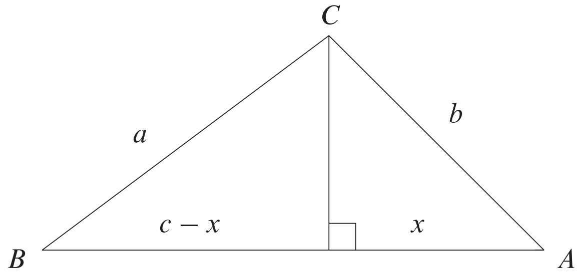



Figure 29

triangle is equal to the sum of the areas of two right triangles which are formed by this perpendicular, and it is given by the equation

$$\tan\frac{1}{2}S = \frac{\dfrac{e^x - 1}{e^x + 1} \cdot \dfrac{e^h - 1}{e^h + 1} + \dfrac{e^{c-x} - 1}{e^{c-x} + 1} \cdot \dfrac{e^h - 1}{e^h + 1}}{1 - \dfrac{e^x - 1}{e^x + 1} \cdot \dfrac{e^{c-x} - 1}{e^{c-x} + 1} \cdot \left(\dfrac{e^h - 1}{e^h + 1}\right)^2},$$

which also takes the form

$$\tan\frac{1}{2}S = \frac{(e^{2h}-1)(e^{c}-1)}{(e^{x}+e^{c-x})(e^{h}+1)^{2}+2e^{h}(e^{x}-1)(e^{c-x}-1)}.$$

This formula gives, if we neglect the higher powers of $s, h, c$ and keep the lower ones,

$$S = \frac{1}{2}ch,$$

as in ordinary geometry.

We saw that the expression of the area of a triangle in terms of the three angles $A$, $B$, $C$ of the triangle is

$$S = \pi - A - B - C.$$

Let us extract the value of $A$ in terms of $a, b, c$, from the second equation in (19). This gives the equation

$$\cos A = \frac{1-\dfrac{\sin\Pi(b)\sin\Pi(c)}{\sin\Pi(a)}}{\cos\Pi(b)\cos\Pi(c)}$$

from which we deduce

$$2\cos^{2}\frac{1}{2}A = \frac{1+\cos\Pi(b)\cos\Pi(c)-\dfrac{\sin\Pi(b)\sin\Pi(c)}{\sin\Pi(a)}}{\cos\Pi(b)\cos\Pi(c)}.$$

If we substitute in this equation

$$1+\cos\Pi(b)\cos\Pi(c)$$

by its value

$$\frac{\sin\Pi(b)\sin\Pi(c)}{\sin\Pi(b+c)},$$

the equation becomes

$$2\cos^{2}\frac{1}{2}A = \tan\Pi(b)\tan\Pi(c)\left(\frac{1}{\sin\Pi(b+c)}-\frac{1}{\sin\Pi(a)}\right).$$

Likewise, we find

$$-2\sin^{2}\frac{1}{2}A = \tan\Pi(b)\tan\Pi(c)\left(\frac{1}{\sin\Pi(b-c)}-\frac{1}{\sin\Pi(a)}\right).$$

From these two formulae, we deduce:

$$\sin^{2}A = \tan^{2}\Pi(b)\tan^{2}\Pi(c)$$
$$\times\left(-\frac{1-\cos^{2}\Pi(b)\cos^{2}\Pi(c)}{\sin^{2}\Pi(b)\sin^{2}\Pi(c)}+\frac{2}{\sin\Pi(a)\sin\Pi(b)\sin\Pi(c)}-\frac{1}{\sin^{2}\Pi(a)}\right)$$

or, equivalently,

$$\sin^2 A = -\tan^2 \Pi(b) \tan^2 \Pi(c) \\ \times \left( \frac{1}{\sin^2 \Pi(a)} + \frac{1}{\sin^2 \Pi(b)} + \frac{1}{\sin^2 \Pi(c)} - \frac{2}{\sin \Pi(a) \sin \Pi(b) \sin \Pi(c)} - 1 \right).$$

Setting, for brevity,

$$P = \sqrt{\frac{-1}{\sin^2 \Pi(a)} - \frac{1}{\sin^2 \Pi(b)} - \frac{1}{\sin^2 \Pi(c)} + \frac{2}{\sin \Pi(a) \sin \Pi(b) \sin \Pi(c)} + 1},$$

we obtain

$$\sin A = \tan \Pi(b) \tan \Pi(c) P. \tag{29}$$

We can also give to $P$ the following form:

$$P^2 = 2\left(1 + \frac{1}{\sin \Pi(a)}\right)\left(1 + \frac{1}{\sin \Pi(b)}\right)\left(1 + \frac{1}{\sin \Pi(c)}\right) \\ - \left(1 + \frac{1}{\sin \Pi(a)} + \frac{1}{\sin \Pi(b)} + \frac{1}{\sin \Pi(a)}\right)^2$$

which is symmetric in $a$, $b$, $c$.

Starting from Equation (28) and considering in that equation $P$ as an indeterminate quantity, we can prove as follows that $P$ is a symmetric function of $a$, $b$, $c$.

Let us multiply both sides of Equation (28) by $\tan \Pi(a)$, let us substitute in that equation $\sin A \tan \Pi(a)$ by its value $\sin B \tan \Pi(b)$ extracted from Equation (13), and finally let us divide by $\tan \Pi(b)$. We obtain:

$$\sin B = \tan \Pi(a) \tan \Pi(c) P.$$

Let us multiply both sides of the last equation by $\tan \Pi(b)$, let us substitute in that equation $\sin B \tan \Pi(b)$ by its value $\sin C \tan \Pi(c)$ extracted from Equation (13), and finally let us divide by $\tan \Pi(c)$. We obtain

$$\sin C = \tan \Pi(a) \tan \Pi(b) P,$$

which proves that $P$ is a symmetric function of the edges $a$, $b$, $c$.

We already found that

$$\cos A = \frac{1 - \dfrac{\sin \Pi(b) \sin \Pi(c)}{\sin \Pi(a)}}{\cos \Pi(b) \cos \Pi(c)}$$

or, equivalently,

$$\cos A = \tan \Pi(b) \tan \Pi(c) \left( \frac{1}{\sin \Pi(b) \sin \Pi(c)} - \frac{1}{\sin \Pi(a)} \right).$$

In the same way, we find

$$\cos B = \tan \Pi(c) \tan \Pi(a) \left( \frac{1}{\sin \Pi(c) \sin \Pi(a)} - \frac{1}{\sin \Pi(b)} \right)$$

and

$$\cos C = \tan \Pi(a) \tan \Pi(b) \left( \frac{1}{\sin \Pi(a) \sin \Pi(b)} - \frac{1}{\sin \Pi(c)} \right).$$

From these values of $\sin A$, $\cos A$, $\sin B$ and $\cos B$, we deduce

$$\begin{aligned}
\sin(A+B) &= \sin A \cos B + \cos A \sin B \\
&= \tan \Pi(b) \tan^2 \Pi(c) \tan \Pi(a) \cdot P \left( \frac{1}{\sin \Pi(c) \Pi(a)} - \frac{1}{\sin \Pi(b)} \right) \\
&\quad + \tan^2 \Pi(c) \tan \Pi(a) \tan \Pi(b) \cdot P \left( \frac{1}{\sin \Pi(b) \Pi(c)} - \frac{1}{\sin \Pi(a)} \right) \\
&= \tan \Pi(a) \tan \Pi(b) \tan^2 \Pi(c) \cdot P \left( \frac{1}{\sin \Pi(a)} + \frac{1}{\sin \Pi(b)} \right) \left( \frac{1}{\sin \Pi(c)} - 1 \right),
\end{aligned}$$

and finally

$$\sin(A+B) = \left( \frac{\tan \Pi(a) \tan \Pi(b) \cdot P}{\dfrac{1}{\sin \Pi(c)} + 1} \right) \left( \frac{1}{\sin \Pi(a)} + \frac{1}{\sin \Pi(b)} \right).$$

The third equation in (19) gives

$$\cos A + \cos(B+C) = \sin B \sin C \left( \frac{1}{\sin \Pi(a)} - 1 \right).$$

Let us substitute $\sin B$ and $\sin C$ in this equation by their values obtained from Equation (28). We get

$$\cos(B+C) = -\cos A + \tan \Pi(c) \tan^2 \Pi(a) \tan \Pi(b) P^2 \left( \frac{1}{\sin \Pi(a)} - 1 \right),$$

or, equivalently,

$$\cos(B+C) = -\cos A + \frac{\tan \Pi(b) \tan \Pi(c) P^2}{\dfrac{1}{\sin \Pi(a)} + 1}.$$

Using the preceding formulae, we obtain

$$\begin{aligned}
\cos(A+B+C) &= \cos A \cos(B+C) - \sin A \sin(B+C) \\
&= -\cos^2 A + \frac{\tan^2 \Pi(b) \tan^2 \Pi(c) P^2}{\dfrac{1}{\sin \Pi(a)} + 1} \left( \frac{1}{\sin \Pi(b) \sin \Pi(c)} - \frac{1}{\sin \Pi(a)} \right) \\
&\quad - \frac{\tan^2 \Pi(b) \tan^2 \Pi(c) P^2}{\dfrac{1}{\sin \Pi(a)} + 1} \left( \frac{1}{\sin \Pi(b)} + \frac{1}{\sin \Pi(c)} \right),
\end{aligned}$$

$$2\cos^2\frac{1}{2}(A+B+C)$$
$$=\sin^2 A+\frac{\tan^2\Pi(b)\tan^2\Pi(c)P^2}{\frac{1}{\sin\Pi(a)}+1}\left(\frac{1}{\sin\Pi(b)\sin\Pi(c)}-\frac{1}{\sin\Pi(a)}\right)$$
$$-\frac{\tan^2\Pi(b)\tan^2\Pi(c)P^2}{\frac{1}{\sin\Pi(a)}+1}\left(\frac{1}{\sin\Pi(b)}+\frac{1}{\sin\Pi(c)}\right)$$

and

$$2\cos^2\frac{1}{2}(A+B+C)=\tan^2\Pi(b)+\tan^2\Pi(c)P^2$$

$$+\frac{\tan^2\Pi(b)\tan^2\Pi(c)P^2}{\frac{1}{\sin\Pi(a)}+1}\left(\frac{1}{\sin\Pi(b)\sin\Pi(c)}-\frac{1}{\sin\Pi(a)}-\frac{1}{\sin\Pi(b)}-\frac{1}{\sin\Pi(c)}\right)$$

$$=\frac{\tan^2\Pi(b)\tan^2\Pi(c)P^2}{\frac{1}{\sin\Pi(a)}+1}\left(\frac{1}{\sin\Pi(b)\sin\Pi(c)}-\frac{1}{\sin\Pi(b)}-\frac{1}{\sin\Pi(c)}+1\right)$$

$$=\tan^2\Pi(a)\tan^2\Pi(b)\tan^2\Pi(c)P^2\left(\frac{1}{\sin\Pi(a)}-1\right)\left(\frac{1}{\sin\Pi(b)}-1\right)\left(\frac{1}{\sin\Pi(c)}-1\right).$$

But we already proved that the area of the triangle $\triangle$ equals $\pi-A-B-C$. Consequently,

$$\sin\frac{\triangle}{2}=\frac{1}{\sqrt{2}}\tan\Pi(a)\tan\Pi(b)\tan\Pi(c)P$$
$$\times\sqrt{\left(\frac{1}{\sin\Pi(a)}-1\right)\left(\frac{1}{\sin\Pi(b)}-1\right)\left(\frac{1}{\sin\Pi(c)}-1\right)}.$$

If $a$, $b$, $c$ are small enough so that we can set, with sufficient approximation,

$$\frac{1}{\sin\Pi(a)}=1+\frac{1}{2}a^2,\qquad \frac{1}{\sin\Pi(b)}=1+\frac{1}{2}b^2,$$
$$\frac{1}{\sin\Pi(c)}=1+\frac{1}{2}c^2,\qquad \tan\Pi(a)=\frac{1}{a}\left(1-\frac{1}{6}a^2\right),$$
$$\tan\Pi(b)=\frac{1}{b}\left(1-\frac{1}{6}b^2\right),\quad \tan\Pi(c)=\frac{1}{c}\left(1-\frac{1}{6}c^2\right),$$

we obtain

$$\sin\frac{\triangle}{2}=\frac{1}{2}\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}$$

or, neglecting the higher powers of $\triangle$,

$$\triangle = \frac{1}{4}\sqrt{2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)}.$$

Let us determine the position of a point in space by three rectangular coordinates: $z$ perpendicular to the $xy$-plane, $y$ perpendicular dropped from the foot of $z$ on the $x$-axis, and $x$ the portion of the $x$-axis comprised between the coordinate origin and the foot of $y$. Let us take three points on the curved surface of which we want to determine the area element. Let

$$x, \quad y, \quad z$$

be the coordinates of the first point, let

$$x + dx, \quad y, \quad z + \frac{dz}{dx}dx$$

be the coordinates of the second point, and let

$$x, \quad y + dy, \quad z + \frac{dz}{dy}dy$$

be the coordinates of the third point.

Let us call $t$ the distance between the vertices of two perpendiculars to the $x$-axis that are of length $y$ and which bound a segment $dx$ on that axis. Assuming that $dx$ and $dy$ are infinitely small, we have, from Equation (27),

$$t = \frac{dx}{\sin \Pi(y)}.$$

The segment joining the first two points taken on the surface forms a triangle with the two segments whose lengths are

$$\frac{dx}{\sin \Pi(y) \sin \Pi(z)} \text{ and } \frac{dz}{dx}dx.$$

Because of the smallness of the edges of that triangle, we can consider it as a triangle whose hypotenuse is the segment joining the first two points on the surface. Thus, the square of this distance is equal to

$$dx^2\left(\frac{1}{\sin^2 \Pi(y) \sin^2 \Pi(z)} + \left(\frac{dz}{dx}\right)^2\right).$$

In the same way, the square of the distance from the first point to the third is equal to

$$dy^2\left(\frac{1}{\sin^2 \Pi(z)} + \left(\frac{dz}{dy}\right)^2\right),$$

and the distance from the second point to the third is

$$\frac{dx^2}{\sin^2 \Pi(y) \sin^2 \Pi(z)} + \frac{dy^2}{\sin^2 \Pi(z)} + \left(\frac{dz}{dy}dy - \frac{dz}{dx}dx\right)^2.$$

The area of the triangle whose edges are the segments from the first point taken on the curved surface to the second, from the second to the third, and from the third to the first, and whose angle sum is approximately equal to $\pi$, because of the smallness of its edges, is given by the following, using the formula that we proved above, and the values that we found for the squares of the edges[53] :

$$\frac{d^2S}{dxdy} = \frac{1}{2\sin \Pi(z)} \sqrt{\left(\frac{dz}{dx}\right)^2 + \frac{1}{\sin^2 \Pi(y)}\left(\frac{dz}{dy}\right)^2 + \frac{1}{\sin^2 \Pi(y) \sin^2 \Pi(z)}}.$$

This is the formula for the area element of the curved surface whose equation is

$$z = f(x, y).$$

Let us apply this expression to a sphere of radius $r$. If the coordinate origin is at the centre of the sphere, the equation of the sphere gives

$$\frac{dz}{dx} = -\frac{\cos \Pi(x)}{\cos \Pi(z)},$$

$$\frac{dz}{dy} = -\frac{\cos \Pi(y)}{\cos \Pi(z)},$$

and then

$$\frac{\cos \Pi(r)}{\sin^2 \Pi(r)} \cdot \frac{\sin \Pi(y) \sin^2 \Pi(x)}{\sqrt{\sin^2 \Pi(x) \sin^2 \Pi(y) - \sin^2 \Pi(r)}} = \frac{d^2S}{d\Pi(x)d\Pi(y)}.$$

Let us multiply by $d\Pi(y)$ and let us integrate from $\sin \Pi(y) = \dfrac{\sin \Pi(r)}{\sin \Pi(x)}$ to $\Pi(y) = \frac{1}{2}\pi$. We obtain

$$\frac{dS}{d\Pi(x)} = 2\pi \sin \Pi(x) \frac{\cos \Pi(r)}{\sin^2 \Pi(r)}.$$

Let us now multiply by $d\Pi(x)$ and let us integrate from $\Pi(x) = \frac{1}{2}\pi$. We obtain

$$S = \frac{2\pi \cos \Pi(r) \cos \Pi(x)}{\sin^2 \Pi(r)},$$

[53]The same formula for $\frac{d^2S}{dxdy}$ is obtained, by different methods, in Lobachevsky's *Elements of geometry* [95], p. 42 of Engel's German translation. The factor 2 in the denominator is missing in that memoir.

which is the area of the portion of the sphere contained between two planes perpendicular to a common radius, one plane passing through the centre of the sphere and the other one at distance $x$ from the centre. To obtain the total area of the sphere, we must set $x = r$ in this formula and take the double of the result. In this way, we obtain the following expression for the total area of the sphere:

$$4\pi \cot^2 \Pi(r)$$

or, equivalently,

$$\pi(e^r - e^{-r})^2.$$

If $r$ is so small that we can neglect the higher powers of $r$, this expression becomes

$$4\pi r^2,$$

as in ordinary geometry.

Let us set

$$\cos\psi = \tan\Pi(r)\cot\Pi(y)$$

and

$$\cos\Pi(x) = \cos\Pi(r)\sin\psi\sin\varphi,$$

and let us introduce the new variables $\psi$, $\varphi$ instead of $x$, $y$ in the expression of the surface area element of the sphere of radius $r$ that we have in mind.[54]

We obtain

$$\frac{d^2S}{d\varphi d\psi} = -\frac{\cos^2\Pi(r)}{\sin\Pi(r)} \cdot \frac{\sin\psi\sqrt{1-\cos^2\Pi(r)\sin^2\psi\sin^2\varphi}}{1-\cos^2\Pi(r)\sin^2\psi}.$$

Let us multiply both sides of this equation by $8d\psi d\varphi$ and let us integrate from $\psi = 0$ to $\psi = \frac{\pi}{2}$, and from $\varphi = 0$ to $\varphi = \frac{\pi}{2}$, which will give the total surface area of the sphere. Equating this expression of the total area of the sphere with the expression of the same surface area that we found above, we deduce

$$\frac{\pi}{2\sin\Pi(r)} = \int_0^{\frac{\pi}{2}} d\psi \int_0^{\frac{\pi}{2}} \frac{\sin\psi\sqrt{1-\cos^2\Pi(r)\sin^2\psi\sin^2\varphi}}{1-\cos^2\Pi(r)\sin^2\psi} d\varphi. \qquad (30)$$

Denoting by $E(\alpha)$ the elliptic integral

$$E(\alpha) = \int_0^{\frac{\pi}{2}} \sqrt{1-\alpha^2\sin^2\varphi}\,d\varphi$$

where $\alpha$ is the constant that is under the integration sign, we obtain

$$\frac{\pi\alpha}{2\sin\Pi(r)} = \int_0^{\alpha} \frac{E(x)dx}{(1-x^2)\sqrt{\alpha^2-x^2}}.$$

[54]The same formula for $\frac{d^2S}{d\psi d\phi}$ is obtained, by different methods, in Lobachevsky's *Elements of geometry* [95], p. 43 of Engel's German translation.

Replacing, in the integral in (30), $\Pi(r)$ by $\frac{1}{2}\pi - R$, we obtain

$$\frac{1}{2}\pi R = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\sin\psi \sin R d\psi d\varphi}{\sqrt{1-\sin^2\psi \sin^2\varphi \sin^2 R}}.$$

Integrating with respect to $\psi$ between the indicated limits, we obtain

$$\pi R = \int_0^{\frac{\pi}{2}} \frac{1}{\sin\varphi} \log\left(\frac{1+\sin\varphi \sin R}{1-\sin\varphi \sin R}\right) d\varphi$$

which, when we replace $\varphi$ by $\Pi(x)$, becomes

$$\pi R = \int_0^{\infty} \log\left(\frac{e^{2x}+1+2e^x \sin R}{e^{2x}+1-2e^x \sin R}\right) dx.$$

Integrating by parts, the equation becomes

$$\frac{1}{4}\pi \frac{R}{\sin R} = \int_0^{\infty} \frac{(e^{2x}-1)e^x x dx}{e^{4x}+2e^{2x}\cos 2R+1}. \tag{31}$$

For $R = \frac{\pi}{2}$, this equation gives

$$\frac{1}{8}\pi^2 = \int_0^{\infty} \frac{e^x x dx}{e^{2x}-1}.$$

It is easy to prove that Equation (31) remains true if we replace $\cos R$ by a number greater than 1. Indeed, we have

$$\int_0^{\pi} \log\cot\frac{1}{2}\psi d\psi = 0,$$

from which it follows that for any number $a$, we have

$$\int_0^{\pi} \log(e^a \cot\frac{1}{2}\psi) d\psi = a\pi.$$

Let us transform this integral by setting $e^a \cot\frac{1}{2}\psi = e^x$. Then we obtain

$$\int_{-\infty}^{\infty} \frac{x dx}{e^{x-a}+e^{-x+a}} = \frac{1}{2}\pi a.$$

We can easily put this equation in the form

$$\int_0^{\infty} \frac{(e^x-e^{-x}) x dx}{e^{2x}+e^{2a}+e^{-2a}+e^{-2x}} = \frac{1}{2}\left(\frac{\pi a}{e^a-e^{-a}}\right)$$

from which we recover Equation (31) by replacing $a$ by $a\sqrt{-1}$.

If we take coordinates on limit circle arcs as follows, one, $\zeta$, situated in a plane passing through a perpendicular dropped from the given point on the $xy$-plane and passing by a parallel to the $x$-axis drawn from the foot of this perpendicular and having this parallel as axis; a second one, $\eta$, situated in the $xy$-plane, having the $x$-axis as axis and passing through the foot of $\zeta$, and as a third coordinate the portion $\xi$ of the $x$-axis contained between the coordinate origin and the vertex of the second arc (see Figure 27 above), then the volume element $P$ must be $d\xi d\eta d\zeta$.

Thus, we have

$$d^3P = d\xi d\eta d\zeta.$$

Let us also set $\zeta = \cot \Pi(z)$, where $z$ is the perpendicular drawn from the given point on the $xy$-plane. Then, we obtain

$$\frac{d^3P}{d\xi d\eta d\zeta} = \frac{1}{\sin \Pi(z)}.$$

From the equation of the limit circle, we deduce

$$e^{-p} = \sin \Pi(z)$$

where $p$ denotes the distance from the intersection point of the arc $\zeta$ with the $xy$-plane to the foot of the perpendicular $z$.[55] Noting that, as a consequence of the equation of the limit circle and of the length of the limit circle arc in terms of the ordinate, we have

$$\cot \Pi(y) = \eta e^{-p}$$

and

$$e^{\xi - x} = \sin \Pi(z) \sin \Pi(y),$$

we find

$$\frac{d\eta}{dy} = \frac{1}{\sin \Pi(y) \sin \Pi(z)}; \quad dx = d\xi.$$

From that, we deduce[56]

$$\frac{d^3P}{dxdydz} = \frac{1}{\sin \Pi(y) \sin^2 \Pi(z)}.$$

Multiplying this expression by $dx$ and integrating with respect to $x$, starting from $x = 0$, we find

$$\frac{d^2P}{dydz} = \frac{x}{\sin \Pi(y) \sin^2 \Pi(z)}.$$

Multiplying the same expression by $dy$ and integrating with respect to $y$, starting from $y = 0$, we obtain

$$\frac{d^2P}{dxdz} = \frac{\cot \Pi(y)}{\sin^2 \Pi(z)}.$$

[55] The variable $p$ was called $q$ before, and it was denoted by $q$ in the text illustrated by Figure 27.

[56] These formulae for $\frac{d^3P}{dxdydz}$ and $\frac{d^2P}{dydz}$ are contained in Lobachevsky's *Elements of geometry* (1829) [95], p. 52 of Engel's German translation.

Finally, multiplying by $dz$ and integrating with respect to $z$, starting from $z = 0$, we obtain

$$\frac{d^2P}{dxdy} = \frac{1}{8\sin\Pi(y)}(e^{2z} + e^{-2z} + 4z).$$

If we multiply both sides of the penultimate expression by $dxdz$ and if we first integrate with respect to $z$ until the value of $z$ extracted from the equation

$$\sin\Pi(r) = \sin\Pi(x)\sin\Pi(z)$$

and then with respect to $x$ starting from $x = 0$ until $x = r$, and if we then multiply the result by 8 to get the volume of the entire sphere, we find that this volume is equal to

$$\frac{1}{2}\pi(e^{2r} - e^{-2r} - 4r),$$

which becomes, for very small $r$, $\frac{4}{3}\pi r^3$, as in ordinary geometry.

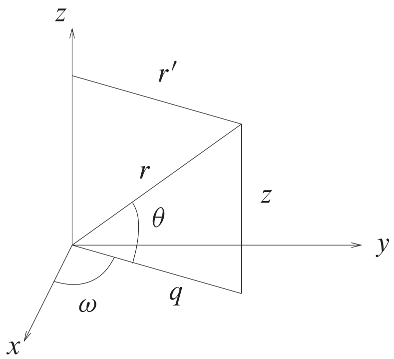



Figure 30. Polar coordinates in hyperbolic 3-space.

To express the volume element in polar coordinates, let $r$ be the distance from the coordinate origin to a point in space, whose rectangular coordinates are $x, y, z$. Let $q$ be the distance in the $xy$-plane from the origin to the foot of $z$, let $\theta$ be the angle between $r$ and $q$, and let $\omega$ be the angle between $q$ and the positive $x$-axis. Let us also set $\Pi(x) = X$, $\Pi(y) = Y$, $\Pi(z) = Z$, $\Pi(r) = R$, and $\Pi(q) = Q$. Let us draw a plane perpendicular to the $z$-axis and passing through the given point. Let $r'$ be the line drawn in that plane from the given point to the $z$-axis and let us also set $\Pi(r') = R'$. Finally, let us construct a sphere of radius $r$ whose centre coincides with the coordinate origin. The $xy$-plane will cut this sphere in a great circle whose circumference is equal, as we already proved, to

$$2\pi\cot R.$$

The portion of this circumference which is bounded by two planes that both contain the $z$-axis and that make with each other an angle $\omega$ is equal to

$$\omega\cot R.$$

The circumference of the circle produced by the intersection of the same sphere with the plane passing through the given point and which is perpendicular to the $z$-axis is equal to

$$2\pi \cot R',$$

and the portion of this circumference which is bounded by the two planes that contain the $z$-axis and that make an angle $\omega$ with each other is equal to

$$\omega \cot R'.$$

The increment of the last arc, produced by an increment $d\omega$ of the angle $\omega$, is equal to

$$d\omega \cot R'.$$

The triangle whose hypotenuse is $r$, of which one of the sides of the right angle is $r'$, and whose angle opposite to $r'$ is $\frac{\pi}{2} - \theta$, gives (using Equation (13))

$$\tan R' \cos\theta = \tan R.$$

From this we deduce

$$d\omega \cot R' = d\omega \cos\theta \cot R.$$

The circumference of the circle which is the intersection of the same sphere with a plane containing the $z$-axis is equal to

$$2\pi \cot R$$

and the length of the circle arc that corresponds to the central angle $\theta$ is

$$\theta \cot R,$$

from which it follows that the increment of that arc corresponding to an increment $d\theta$ of the angle $\theta$ is equal to

$$d\theta \cot R.$$

If all the increments are infinitely small, the volume element will be, as in ordinary geometry, expressed as the product of the three mutually perpendicular segments,

$$dr, \quad d\omega \cos\theta \cot R, \quad d\theta \cot R,$$

since this volume element can be considered as a prism. Thus, we will have the following expression for the volume element in polar coordinates:

$$dr d\omega d\theta \cos\theta \cot^2 R = d^3 P,$$

or, replacing $\cot^2 R$ by its value in terms of $r$,

$$d^3 P = \frac{1}{4} dr d\omega d\theta \cos\theta (e^r - e^{-r})^2.$$

Integrating first with respect to $r$, starting from $r = 0$, we obtain

$$d^2P = \frac{1}{8} d\omega d\theta \cos\theta (e^{2r} - e^{-2r} - 4r).$$

For the sphere centred at the coordinate origin, $r$ does not depend on $\theta$ or on $\omega$. Integrating with respect to $\omega$ from $\omega = 0$ to $\omega = 2\pi$, and with respect to $\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$, and taking the double of the result, we obtain, for the volume of the entire sphere, the expression

$$\frac{\pi}{2}(e^{2r} - e^{-2r} - 4r)$$

which is the value we have already found.

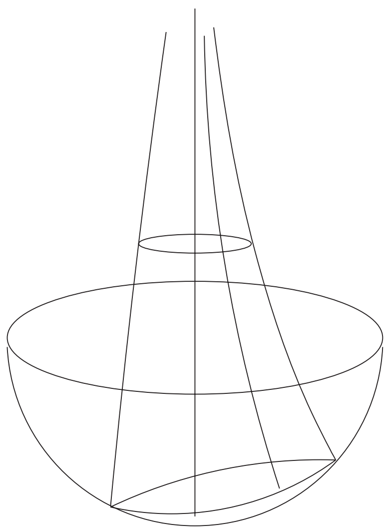


Figure 31. A conical surface, bounded by two pieces of limit spheres.

Now let us take a portion $S$ of the surface of a limit sphere bounded by a certain convex curve, and let us draw from the various points of that curve lines that are parallel to the axis of the limit sphere. These lines will form a surface which we shall call, by analogy, a conical surface, which can be indefinitely extended on both sides, but of which we shall only consider the part situated on the side of parallelism of the axes of the limit sphere. Let $S'$ be the portion of a second limit sphere whose axes are parallel to the axes of the first one and having the same direction, this portion being situated in the interior of the conical surface. (See Figure 31.) Then, $S$, $S'$ and the part of the conical surface situated between the two limit spheres enclose a volume which is finite in every direction, which we now propose to determine.

Let us call $c$ the portion of an axis of the two limit spheres which these two spheres bound, let us take consecutive lengths equal to $c$ on one of the axes of the first limit sphere that passes through one of the points of the boundary of $S$ starting from the point where this axis intersects $S'$, and let us draw from the division points limit spheres whose axes are parallel to the axes of the first two and directed in the same sense. Let $S''$, $S'''$, etc. be the consecutive portions of these limit spheres that are contained in the conical surface. It follows easily, from what has been proved previously regarding limit circle arcs that are in the same situation as the portions of limit spheres that we are now considering, that we again have

$$S' = Se^{-2c},$$
$$S'' = Se^{-4c},$$
$$S''' = Se^{-6c}$$

and so on.

Likewise, let us denote by $P$, $P'$, $P''$, $P'''$, ... the volumes bounded by the conical surface between $S$ and $S'$, between $S'$ and $S''$, ..., and let us do things in such a way that the volumes $P$, $P'$, $P''$, ... are proportional to the areas $S$, $S'$, $S''$, ...

Thus, we must have

$$P = CS$$

where $C$ depends only on $c$.

It follows that

$$P' = CS' = CSe^{-2c},$$
$$P'' = CS'' = CSe^{-4c},$$

and so on.

The sum $\sum_0^\infty P^{(n)}$ is the volume bounded by the conical surface with basis $S$ and which is indefinitely extended on the side of parallelism of the generatrices. Let $K$ be this volume. Then we have

$$K = \frac{CS}{1 - e^{-2c}}.$$

This quantity does not depend on $c$, which implies that we have

$$C = (1 - e^{-2c})A$$

where $A$ is an absolute number. Since the volume unit is arbitrary, we shall take

$$C = \frac{1}{2}(1 - e^{-2c}),$$

so that the volume $P$, which is equal to

$$P = \frac{1}{2}S(1 - e^{-2c}),$$

becomes $P = cS$ if $c$ is infinitely small, an expression which coincides with the expression of the volume of a prism of base $S$ and height $c$ in ordinary geometry.

We can also take as volume element the volume bounded by a conical surface formed by the axes of a limit sphere, these axes being drawn from all points of the boundary of a portion of this limit sphere which is infinitely small in all directions.

The large number of different expressions for the element of the same geometric quantity gives means for comparing integrals, and these means are mostly useful in the theory of definite integrals.

Having shown in what precedes in what manner we must calculate lengths of curves, areas of surfaces and volumes of bodies, we can now assert that Pangeometry is a complete geometric doctrine. A simple glimpse at Equations (19), which express the dependence that exists between the edges and the angles of rectilinear triangles, is sufficient to prove that from there, Pangeometry becomes an analytical method which replaces and generalises the analytical methods of ordinary geometry. It would be possible to start the exposition of Pangeometry from Equations (19), and even to try to substitute for these equations other equations that would express the dependencies between the angles and the edges of any rectilinear triangle. But in that case, we must prove that the new equations are compatible with the fundamental notions of geometry. Equations (19), having been deduced from these fundamental notions, are therefore necessarily compatible with them, and all other equations that we might substitute for them would lead to results that contradict these notions, unless these equations are consequences of Equations (19). Thus, Equations (19) are the basis of the most general geometry, since they do not depend on the assumption that the sum of the three angles in any rectilinear triangle is equal to two right angles.

Pangeometry, which is based on firm principles and which has been developed in the preceding pages, provides, as we saw, methods that are useful for the computation of values of various geometric quantities. At the same time, it proves that the assumption that the value of the sum of the three angles of any right rectilinear triangle is constant, an assumption which is explicitly or implicitly adopted in ordinary geometry, is not a necessary consequence of our notions of space. Only experience can confirm the truth of this assumption, for instance, by effectively measuring the sum of the three angles of a rectilinear triangle, a measure which can be done in various ways. One can measure the three angles of a rectilinear triangle constructed in an artificial plane, or the three angles of a rectilinear triangle in space. In the latter case, one must give preference to triangles whose edges are very large, since according to Pangeometry, the difference between two right angles and the three angles of a rectilinear triangle increases as the edges increase.

Let $r$ be the radius of a circle and let $A$ be a central angle whose sides bound an arc subtended by a chord of length $r$. Let us denote by $p$ the perpendicular dropped from the centre of the circle on this chord, which is divided into two equal parts by the foot of this perpendicular. Let us consider one of the two right rectilinear triangles formed by this perpendicular, the circle radii situated on the sides of the angle $A$ and the chord (Figure 32). The hypotenuse of this triangle is $r$, and the two mutually perpendicular edges are $\frac{1}{2}r$ and $p$.

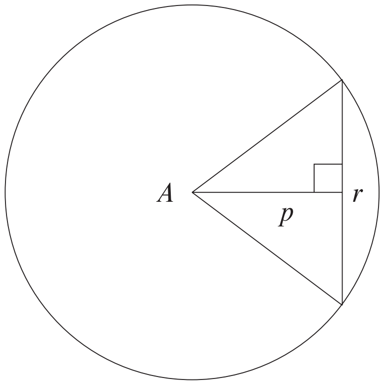



Figure 32

From the general Equation (13), we have, in that triangle:

$$\sin\frac{1}{2}A\tan\Pi\left(\frac{1}{2}r\right)=\tan\Pi(r).$$

This equation, combined with the analogous equation

$$\tan\Pi(r)=\frac{\sin^2\Pi(\frac{1}{2}r)}{2\cos\Pi(\frac{1}{2}r)}$$

gives

$$\sin\frac{1}{2}A=\frac{1}{2}\sin\Pi\left(\frac{1}{2}r\right).$$

In ordinary geometry, we have

$$A=\frac{\pi}{3}.$$

Let us assume that the effective measure gives

$$A=\frac{2\pi}{6+K},$$

where $K$ is a positive real number.

Then, we would have

$$\sin\left(\frac{\pi}{6+K}\right)=\frac{1}{2}\sin\Pi\left(\frac{1}{2}r\right).$$

If $r$ and $K$ are known, we can extract from this equation the value of $\Pi(\frac{1}{2}r)$, and from this value we can find the angle of parallelism $\Pi(x)$ of any segment $x$.

The distances between the celestial bodies provide us with a means for observing the angles of triangles whose edges are very large. Let $\alpha$ be the geocentric latitude of a fixed star at a fixed epoch, and let $\beta$ be another geocentric latitude of the same star, a latitude corresponding to an epoch where the earth is again in the plane that is perpendicular to the ecliptic, containing the first position (that is to say, the position where the latitude of the star was $\alpha$). Let $2a$ be the distance between these two positions of the earth and let $\delta$ be the angle under which this distance $2a$ is seen from the star.

If the angles $\alpha, \beta, \delta$ do not satisfy the relation

$$\alpha = \beta + \delta,$$

then this will be an indication of the fact that the sum of the three angles of that triangle differ from two right angles.

We can choose the star in such a way that $\delta$ is equal to zero, and we can always assume that there exists a segment $x$ such that

$$\Pi(x) = \alpha.$$

If $\delta = 0$, the lines drawn from the two positions of the earth to the star can be thought of as being parallel, and consequently we should have

$$\beta = \Pi(x + 2a)$$

from which it follows, using what has been proved above, that

$$\tan \frac{1}{2}\alpha = e^{-x},$$

$$\tan \frac{1}{2}\beta = e^{-x-2a}.$$

Each time observation gives, for a star with respect to which the angle denoted by $\delta$ is zero, two angles $\alpha$ and $\beta$ that are different, the last two equations will give the values of $x$ and $a$, expressed in terms of the segment taken as a unit segment in Pangeometry. Thus, having the segment $x$ that corresponds to an angle of parallelism $\Pi(x)$, we can compute the angle of parallelism $\Pi(y)$ for any given segment $y$.

# Subject index

# III.

# PANGÉOMÉTRIE

OU

# PRÉCIS DE GÉOMÉTRIE

FONDÉE

SUR UNE THÉORIE GÉNÉRALE ET RIGOUREUSE

DES

PARALLÈLES

PAR

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# PANGEOMETRIE.

—

Les notions sur lesquelles on fonde la géométrie élémentaire sont insuffisantes pour en deduire une démonstration du théorème que la somme des trois angles de tout triangle rectiligne est égale à deux angles droits, théorème de la verité duquel personne n'a douté jusqu'à présent, parcequ'on ne rencontre aucune contradiction dans les consequences qu'on en a déduites et que les mesures directes des angles des triangles rectilignes s'accordent, dans les limites des erreurs des mesures les plus parfaites, avec ce théorème.

L'insuffisance des notions fondamentales pour la démonstration de ce théorème a forcé les géomètres d'admettre explitement ou implicitement des suppositions auxiliaires, qui, quelque simples qu'elles paraissent n'en sont pas moins arbitraires et par consequent inadmissibles. Ainsi par exemple on admet, qu'un cercle de rayon infini se confond avec une ligne droite et une sphère de rayon infini avec un plan, que les angles de tout triangle rectiligne ne dépendent que du rapport des côtés et non des côtés eux mêmes, ou enfin, comme cela se fait ordinairement dans les élémens de géométrie, que par un point donné d'un plan on ne peut mener qu'une seule droite parallèle à une autre droite donnée dans le plan tandis que toutes les autres droites menées par le même point et dans le même plan doivent necessairement, étant prolongées suffisament, couper la droite donnée. On entend sous le nom de droite parallèle à une autre droite donnée une droite qui, quelque loin qu'on la prolonge des deux cotés, ne coupe jamais celle à laquelle elle est parallèle. Cette définition est par elle même insuffisante, parcequ'elle ne caracterise pas assez une seule ligne droite. On peut dire la même chose de la plupart des définitions données ordinairement dans les éléments de géométrie, car ces définitions non seulement n'indiquent pas la génération des grandeurs qu'on définit, mais ne montrent pas même que ces grandeurs peuvent exister. Ainsi on définit la ligne droite et le plan par une de leur propriétés; on dit que les lignes droites sont celles qui se confondent toujours dès qu'elles ont deux poins communs, qu'un plan est une surface avec laquelle une ligne droite se confond toujours dès que la droite a deux points communs avec elle.

Au lieu de commencer la géométrie par le plan et la ligne droite, comme on le fait ordinairement, j'ai préféré de la commencer par la sphère et le cercle dont les définitions ne sont pas sujettes au reproche d'être incomplètes puisqu'elles contiennent la génération des grandeurs qu'on définit.

En suite je définis le plan comme le lieu géométrique des intersections de sphères égales décrites autour de deux points fixes comme centres. Enfin je définis la ligne droite comme le lieu géométrique des intersections de cercles égaux situés tous dans un même plan et décrits de deux points fixes de ce plan comme centres. Ces définitions du plan et de la ligne droite acceptées, toute la théorie des plans et des droites perpendiculaires peut être exposée et démontrée avec beaucoup de simplicité et de brièveté.

Etant donné une droite et un point dans un plan, j'appelle parallèle à la droite donnée menée par le point donné la droite limite entre celles des droites menées dans le même plan par le même point et prolongées d'un coté de la perpendiculaire abaissée de ce point sur la droite donnée, qui la coupent et de celles qui ne la coupent pas.

J'ai publié une théorie complète des parallèles sous le titre »Geometrische Untersuchungen zur Theorie der Parallellinien. Berlin 1840. In der Finckeschen Buchhandlung.« Dans ce travail j'ai exposé d'abord tous les théorèmes qui peuvent être démontrés sans le secours de la théorie des parallèles. Parmi ces théorèmes, le théorème qui donne le rapport de la surface de tout triangle sphérique à la surface de la sphère entière sur laquelle il est tracé est particulièremant remarquable (Geometrische Untersuchungen § 27). Si $A$, $B$, $C$ désignent les angles d'un triangle sphérique et $\pi$ deux angles droits, le rapport de la surface de ce triangle à la surface de la sphère à laquelle il appartient sera égal au rapport de

$$\frac{1}{2}(A+B+C-\pi)$$

à quatre angles droits.

Ensuite je démontre que la somme des trois angles de tout triangle rectiligne ne peut jamais surpasser deux angles droits (Geometr. Untersuchungen. § 19) et que, si cette somme est égale à deux angles droits dans un triangle rectiligne quelquonque, elle le sera dans tous (Geometr. Untersuchungen § 20). Ainsi il n'y a que deux suppositions possibles: ou la somme des trois angles de tout triangle rectiligne est égale à deux angles droits, cette supposition donne la géométrie connue — ou dans tout triangle rectiligne cette somme est moindre que deux angles droits et cette supposition sert de base à une autre géométrie, à laquelle j'avais donné le nom de géométrie imaginaire, mais qu'il est peut être plus convenable de nommer Pangéométrie parceque ce nom désigne une théorie géométrique générale, qui comprend la géométrie ordinaire comme cas particulier. Il suit des principes adoptés dans la Pangéométrie, qu'une perpendiculaire $p$ abaissée d'un point d'une droite sur une de ses parallèles fait avec la première, deux angles, dont l'un est aigu. J'appelle cet angle, angle de parallèlisme et le coté de la première droite ou il se trouve, coté qui est le même pour tous les points de cette droite, coté du parallèlisme. Je désigne cet angle par $\Pi(p)$, puisqu'il dépend de la longueur de la perpendiculaire. Dans la géométrie ordinaire on a toujours $\Pi(p) =$ un angle droit pour toute longueur de $p$. Dans la Pangéométrie l'angle $\Pi(p)$ passe par toutes les valeurs depuis zero qui repond à $p = \infty$, jusqu'à $\Pi(p) =$ un angle droit, pour $p = o$. (Geometrische Untersuchungen § 23) Pour donner à la fonction $\Pi(p)$ une valeur analytique plus générale j'adopte, que la valeur de cette fonction pour $p$ negatif, cas auquel la définition primitive ne s'étend pas, est fixé par l'équation suivante

$$\Pi(p) + \Pi(-p) = \pi$$

ainsi pour tout angle $A > o$ et $< \pi$ on pourra trouver une ligne $p$ telle que $\Pi(p) = A$, où la ligne $p$ sera positive si $A < \frac{\pi}{2}$. Reciproquement il existe pour toute ligne $p$ un angle $A$ tel que $A = \Pi(p)$. J'appelle cercle limite le cercle dont le rayon est infini, il pourra être tracé par approximation en en construisant de la manière suivante, autant de points qu'on voudra. Prenons un point sur une droite indéfinie, nommons ce point sommet et cette droite axe du cercle limite, construisons un angle $A > o$ et $< \frac{\pi}{2}$, dont le sommet coïncide avec le sommet du cercle limite, et dont l'axe soit un des côtés, soit enfin $a$ la ligne qui donne $\Pi(a) = A$ et construisons sur le second coté de l'angle, à partir du sommet une droite $2a$, le point qui termine cette droite se trouvera sur le cercle limite; pour continner le tracé du cercle limite de l'autre côté de l'axe il faudra répéter cette construction de ce côté. Il s'ensuit que toutes les droites parallèles à l'axe du cercle limite peuvent être prises pour axes. La révolution du cercle limite autour d'un de ses axes produit une surface que je nomme sphère limite, surface qui est par consequence la limite de laquelle la sphère s'approche si le rayon croit à l'infini. Nous nommerons l'axe de revolution, et par consequence aussi toutes les droites parallèles à l'axe de révolution, axes de la sphère limite et plan diamétral tout plan qui contient un ou plusieures axes de la sphère limite. Les intersections de la sphère limite par ses plans diamétraux sont des cercles limites. Une partie de la surface de la sphère limite, limitée par trois arcs de cercle limite sera nommée triangle sphérique limite, les arcs de cercle limite seront appelés les côtés et les angles dièdres entre les plans des ces arcs angles du triangle sphérique limite. Deux droites parallèles à une troisième sont parallèles entre elles. (Geometrische Untersuchungen § 25). Il s'ensuit que tous les axes du cercle limite et de la sphère limite sont parallèles entre eux. Si trois plans se coupent deux à deux en trois droites parallèles et si l'on limite chaque plan à la partie qui est située entre ces parallèles la somme des trois angles dièdres que ces plans formeront sera égale à deux angles droits (Geometrische Untersuchungen § 28). Il suit de ce théorème que la somme des angles de tout triangle sphérique limite est égale à deux angles droits, et tout ce qu'on démontre dans la géométrie ordinaire de la proportionalité des côtés des triangles rectilignes peut par consequence être démontré de la même manière dans la Pangéométrie des triangles sphériques limites, en remplaçant seulement les droites parallèles à l'un des côtés du triangle rectiligne par des arcs de cercle limite menés par des points d'un des cotés du triangle sphérique limite et faisant tous le même angle avec ce côté. Ainsi par exemple si $p, q, r$ sont les côtés d'un triangle sphérique limite rectangle et $P, Q, \frac{\pi}{2}$ les angles opposés à ces côtés il faut adopter, de même que pour les triangles rectilignes rectangles dans la géométrie ordinaire les équations suivantes

$$p = r \sin P = r \cos Q$$
$$q = r \cos P = r \sin Q$$
$$P + Q = \frac{\pi}{2}.$$

Dans la géométrie ordinaire on démontre que la distance de deux droites parallèles est constante.

Dans la Pangéométrie au contraire la distance $p$ d'un point d'une droite à la droite parallèle diminue du coté du parallèlisme, c'est à dire du côté vers lequel est tourné l'angle de parallèlisme $\Pi(p)$.

Maintenant soient $s, s', s''$..... une serie d'arcs de cercle limite compris entre deux droites parallèles, qui servent d'axes à tous ces cercles limites, et supposons que les parties de chaque parallèle comprises entre deux arcs consecutifs soient toutes égales entre elles et égales à $x$, nommons $E$ le rapport de $s$ à $s'$

$$\frac{s}{s'} = E$$

où $E$ est un nombre plus grand que l'unité.

Supposons d'abord que $E = \frac{n}{m}$, $m, n$ etant deux nombres entiers, divisons l'arc $s$ en $m$ parties égales. Par les points de division menons des droites parallèles à l'axe des cercles limites, ces parallèles diviseront chacun des arcs $s'$, $s''$ etc. en $m$ parties égales entre elles. Soit $AB$ la première partie de $s$, $A'B'$ la première partie de $s'$, $A''B''$ la première partie de $s''$ etc. $A, A' A''$.... les points situées sur l'une des parallèles données et posons $A'B'$ sur $AB$ de manière que $A$ et $A'$ coincident et que $A'B'$ tombe sur $AB$. Répétons cette superposition $n$ fois de suite. Puisque par supposition $\frac{s}{s'} = \frac{n}{m'}$ il faudra que $nA'B' = mAB$ et que par consequence la seconde extremité de $A'B'$ coincide après la $n^{\text{ieme}}$ superposition, avec la seconde extremité de $s$, qui sera divisé en $n$ parties égales; $s' s''$.... seront aussi divisé en en $m$ parties égales chacun par les droites parallèles aux deux parallèles données. Mais si l'on imagine, qu'en faisant la superposition indiquée ci dessus $A'B'$ emporte la partie du plan limité par cet arc et les deux parallèles menées par les extremités il est clair qu'en même temps que $n$ fois $A'B'$ couvre tout l'arc $s$, $nA''B''$ couvrira tout l'arc $s'$ et ainsi de suite parceque dans ce cas les parallèles doivent coïncider dans toute leur étendue de sorte que l'on aura

$$nA''B'' = mA'B'$$

ou ce qui est la même chose

$$\frac{s'}{s''} = \frac{n}{m} = E; \quad \frac{s'}{s^{v}} = E \text{ etc.}$$

ce qu'il fallait démontrer.

Pour démontrer la même chose dans le cas que $E$ est un nombre incommensurable, on pourra employer une des méthodes usitées pour des cas semblables dans la géométrie ordinaire; j'omets ces détails pour abréger. Ainsi

$$\frac{s}{s'} = \frac{s'}{s''} = \frac{s''}{s'''} \ldots\ldots = E$$

Après quoi il n'est pas difficile de conclure que

$$s' = s\, E^{-x}$$

où $E$ est la valeur de $\frac{s}{s'}$ pour $x$, distance entre les arcs $s$, $s'$ égale à l'unité.

Il faut remarquer que ce rapport $E$ ne dépend pas de la longueur de l'arc $s$, et reste le même si les deux droites parallèles données s' éloignent ou se rapprochent l'une de l'autre. Le nombre $E$, qui est nécessairement plus grand que l'unité, ne dépend que de l'unité de longueur, qui est la distance entre deux arcs consecutifs et qui reste complètement arbitraire. La propriété que nous venons de démontrer par rapport aux arcs $s, s', s'' \ldots$, subsiste pour les aires. $P, P', P'' \ldots$, limitées par deux arcs consecutifs et les deux parallèles. On a donc.

$$P' = P\, E^{-x}$$

Si nous réunissons $n$ aires semblables $P, P', P'' \ldots\, P^{(n-1)}$ la somme sera

$$P.\frac{1 - E^{-nx}}{1 - E^{-x}}$$

Pour $n = \infty$ cette expression donne l'aire de la partie du plan entre deux droites parallèles, limitée d'un côté par l'arc $s$, et illimitée du coté du parallèlisme, et la valeur de cette aire sera

$$\frac{P}{1 - E^{-x}}$$

Si nous choisissons pour unité des aires l'aire $P$ qui répond à un arc $s$ égal aussi à l'unité et à $x = 1$ elle deviendera généralement pour un arc $s$ quelconque

$$\frac{E s}{E - 1.}$$

Dans la géométrie ordinaire le rapport désigné par $E$ est constant et égal à l'unité; il s'ensuit que dans la géométrie ordinaire deux droites parallèles sont par-tout équidistantes et que l'aire de la partie du plan située entre deux droites parallèles et limitée d'un côté seulement par une perpendiculaire commune à elles, est infinie.

Considérons à présent un triangle rectiligne rectangle dont $a, b, c$ soient les côtés et $A$, $B$, $\frac{\pi}{2}$ les angles opposés à ces côtés. Les angles $A$, $B$ peuvent être pris pour des angles de parallélisme $\Pi(\alpha)$, $\Pi(\beta)$, correspondant à des droites de longueur $\alpha$, $\beta$, positives. Convenons encore de désigner dorénavant par une lettre avec un accent une droite dont la longueur correspond à un angle de parallélisme qui est le complément à un angle droit de l'angle de parallélisme, correspondant à la droite dont la longueur est désignée par la même lettre sans accent, de manière à avoir toujours

$$\Pi(a) + \Pi(a') = \frac{1}{2}\pi$$

$$\Pi(b) + \Pi(b') = \frac{1}{2}\pi$$

Désignons par $f(a)$ la partie d'une parallèle à un axe de cercle limite interceptée entre la perpendiculaire à l'axe menée par le sommet du cercle limite et le cercle limite lui même si cette parallèle passe par un point de la perpendiculaire dont la distance au sommet est $a$ et soit enfin $L(a)$ la longueur de l'arc depuis le sommet jusqu'à cette parallèle.

Dans la géométrie ordinaire on a

$$f(a) = o$$
$$L(a) = a$$

pour toute ligne $a$.

Menons une perpendiculaire $AA'$ au plan du triangle rectangle dont les côtés ont été désignés $a, b, c$, perpendiculaire qui passe par le sommet $A$ de l'angle $\Pi(\alpha)$. Faisons passer par cette perpendiculaire deux plans dont l'un, que nous appellerons le premier plan, passe aussi par le côté $b$, et l'autre, le second plan par le côté $c$. Construisons dans le second plan la droite $BB'$ parallèle à $AA'$ qui passe par le sommet $B$ de l'angle $\Pi(\beta)$ et faisons passer un troisième plan par $BB'$ et le côté $a$ du triangle. Ce troisième plan coupera le premier en une droite $CC'$ parallèle à $AA'$. Concevons maintenant une sphère décrite du point $B$ comme centre avec un rayon arbitraire mais plus petit que $a$, sphère qui coupera conséquemment les deux côtés $a$, $c$ du triangle et la droite $BB'$ en trois points, que nous nommerons. le premier $n$; le second $m$, et le troisième $k$. Les arcs de grands cercles, intersections de cette sphère par les trois plans passant par $B$, cercles qui reunissent deux à deux les points $n, m, k$, formeront un triangle sphérique rectangle en $m$, dont les côtés seront $mn = \Pi(\beta)$, $km = \Pi(c)$, $kn = \Pi(a)$. L'angle sphérique $knm = \Pi(b)$ et l'angle $kmn$ sera droit. Les trois droites étant parallèles entre elles la somme des trois angles dièdres, que les parties des plans $AA'BB'$, $AA'CC'$, $BB'CC'$ situées entre les droites $AA'$, $BB'$, $CC'$ forment entre elles, sera égal à deux droits. Il s'ensuit que le troisième angle du triangle sphérique sera $mkn = \Pi(\alpha')$. On voit donc qu'à tout triangle rectiligne rectangle dont les côtés sont $a, b, c$ et les angles opposés $\Pi(\alpha)$, $\Pi(\beta)$, $\frac{\pi}{2}$ correspond un triangle sphérique rectangle dont les côtés sont $\Pi(\beta)$, $\Pi(c)$, $\Pi(a)$ et les angles opposés $\Pi(\alpha')$, $\Pi(b)$, $\frac{\pi}{2}$. Construisons un autre triangle rectiligne rectangle dont les cotés perpendiculaires entre eux soient $\alpha'$, $a$, dont l'hypotenuse soit $g$, dont $\Pi(\lambda)$ soit l'angle opposé au coté $a$ et $\Pi(\mu)$ l'angle opposé au côté $\alpha'$. Passons de ce triangle au triangle sphérique qui lui correspond de la même manière que le triangle sphérique $kmn$ correspond au triangle $ABC$. Les côtés de ce triangle sphérique seront conséquemment

$$\Pi(\mu),\ \Pi(g),\ \Pi(a)$$

et les angles opposés

$$\Pi(\lambda'),\ \Pi(\alpha'),\ \frac{\pi}{2}$$

et il aura ses parties égales aux parties correspondantes du triangle sphérique $k\,m\,n$, car les côtés de ce dernier étaient

$$\Pi(c)\ \ \Pi(\beta)\ \ \Pi(a)$$

et les angles opposés

$$\Pi(b),\ \Pi(\alpha'),\ \frac{\pi}{2}$$

ce qui montre, que ces triangles sphériques ont leurs hypotenuses égales et un angle adjacent égal.

Il s'ensuit que

$$\mu = c;\ g = \beta;\ b = \lambda'$$

et ainsi l'existence d'un triangle rectiligne rectangle avec les côtés

$a \quad b \quad c$ et les angles opposés

$$\Pi(\alpha)\ \ \Pi(\beta)\ \ \frac{\pi}{2}$$

suppose l'existence d'un autre triangle rectiligne rectangle avec les côtés

$a \quad \alpha' \quad \beta$ et les angles opposés

$$\Pi(b')\ \ \Pi(c)\ \ \frac{\pi}{2}.$$

On exprime la même chose en disant, que si

$$a,\ b,\ c,\ \alpha,\ \beta$$

sont les parties d'un triangle rectiligne rectangle

$$a,\ \alpha'\ \beta,\ b'\ c$$

seront les parties correspondantes d'un autre triangle rectiligne rectangle. Si nous construisons une sphère limite, dont la perpendiculaire $AA'$ au plan du triangle rectiligne rectangle donné soit un axe et dont le point $A$ soit le sommet, nous aurons un triangle situé sur la sphère limite et produit par son intersection avec les trois plans conduits par les trois côtés du triangle donné. Désignons les trois côtés de ce triangle sphérique limite par $p, q, r$ de manière que $p$ soit l'intersection de la sphère limite par le plan qui passe par $a$, $q$ l'intersection de la sphère par le plan qui passe par $b$, $r$ l'intersection de la sphère limite par le plan qui passe par $c$; les angles opposés à ces côtés seront: $\Pi(\alpha)$ opposé à $p$, $\Pi(\alpha')$ opposé à $q$ et un angle droit opposé à $r$, D'après les conventions adoptés ci dessus $q = L(b)$ $r = L(c)$. La sphère limite coupera la droite $CC'$ en un point, dont la distance à $C$ sera, d'après ces mêmes conventions, $f(b)$; de la même manière nous aurons $f(c)$ pour la distance du point d'intersection de la sphère limite avec la droite $BB'$ au point $B$.

Il est facile à voir qu'on aura

$$f(b) + f(a) = f(c)$$

Dans le triangle dont les côtés sont les arcs de cercle limite $p, q, r$ nous aurons

$$p = r \sin \Pi(\alpha);\ q = r \cos \Pi(\alpha)$$

En multipliant la première de ces deux équations par $E^{f(b)}$ il viendra

$$p\, E^{f(b)} = r \sin \Pi(\alpha).\ E^{f(b)}.$$

Mais

$$p\, E^{f(b)} = L(a)$$

et par conséquence

$$L(a) = r \sin \Pi(\alpha).\ E^{f(b)}$$

De la même manière on a

$$L(b) = r \sin \Pi(\beta)\, E^{f(a)}$$

En même temps $q = r \cos \Pi(\alpha)$, ou ce qui est la même chose $L(b) = r \cos \Pi(\alpha)$. La comparaison des deux valeurs de $L(b)$ donne l'équation

$$\cos \Pi(\alpha) = \sin \Pi(\beta).\ E^{f(a)} \qquad (1)$$

En substituant $b'$ à $\alpha$ et $c$ à $\beta$ sans changer $a$, ce qui est permis d'après ce qui a été démontré plus haut, nous aurons.

$$\cos \Pi(b') = \sin \Pi(c).\ E^{f(a)}$$

ou puisque
$$\Pi(b) + \Pi(b') = \frac{\pi}{2}$$

$$\sin \Pi(b) = \sin \Pi(c)\ E^{f(a)}$$

De la même manière on doit avoir

$$\sin \Pi(a) = \sin \Pi(c)\ E^{f(b)}$$

Multiplions la dernière equation par $E^{f(a)}$ et substituons $f(c)$ à la place de $f(b) + f(a)$; cela donnera

$$\sin \Pi(a)\ E^{f(a)} = \sin \Pi(c)\ E^{f(c)}$$

Mais comme dans un triangle rectiligne rectangle les côtés perpendiculaires peuvent varier de manière à laisser l'hypothenuse constante, nous pouvons poser dans cette équation $a = o$ sans changer $c$, cela donnera, en remarquant que $f(o) = o$ et $\Pi(o) = \frac{\pi}{2}$,

$$1 = \sin \Pi(c)\ E^{f(c)} \qquad \text{ou}$$

$$E^{f(c)} = \frac{1}{\sin \Pi(c)}$$

pour toute ligne $c$.

Prenons maintenant l'équation (1)

$$\cos \Pi(\alpha) = \sin \Pi(\beta)\ E^{f(a)}$$

et substituons y $\dfrac{1}{\sin \Pi(a)}$ à la place de $E^{f(a)}$, elle prendra la forme suivante

$$\cos \Pi(\alpha) \sin \Pi(a) = \sin \Pi(\beta) \qquad (2)$$

En y changeant $\alpha, \beta$, en $b', c$, sans changer $a$, nous trouvons

$$\sin \Pi(b) \sin \Pi(a) = \sin \Pi(c)$$

L'équation (2) donne en y changeant les lettres

$$\cos \Pi(\beta) \sin \Pi(b) = \sin \Pi(\alpha)$$

Si nous changeons dans cette équation $\beta, b, \alpha$ en $c, \alpha', b'$ il viendra

$$\cos \Pi(c) \cos \Pi(\alpha) = \cos \Pi(b) \qquad (3)$$

De la même manière nous aurons

$$\cos \Pi(c) \cos \Pi(\beta) = \cos \Pi(a) \qquad (4)$$

Les équations (2), (3), (4) se rapportent à un triangle sphérique rectangle, dont nous désignerons dans la suite les cotés par $a, b, c$ et les angles $A, B$ opposés aux côtés $a, b$ et $\frac{\pi}{2}$ opposé à $c$. Dans les équations citées nous pouvons mettre $a$ à la place de $\Pi(\beta)$, $b$ à la place de $\Pi(c)$, $c$ à la place de $\Pi(a)$, $\frac{\pi}{2} - A$ à la place $\Pi(\alpha)$, $B$ à la place de $\Pi(b)$ de cette manière les équations citées deviennent.

$$
\begin{aligned}
&\sin A \sin c = \sin a. \\
&\cos b \sin A = \cos B. \qquad\qquad (5) \\
&\cos a \cos b = \cos c.
\end{aligned}
$$

Les équations (5) se rapportent à un triangle sphérique rectangle, tel qu'il peut être déduit d'un triangle rectiligne rectangle, et dont les cotés ne peuvent par consequence surpasser $\frac{\pi}{2}$. Ajoutons que, si nous menons un arc de grand cercle par le sommet de l'angle $A$ perpendiculairement au côté $b$, cet arc coupera l'arc $a$ ou son prolongement de manière que chacun des arcs, depuis le point d'intersection jusqu'à $b$ sera $= \frac{\pi}{2}$ et l'angle de ces arcs sera $b$. Après cela il n'est pas difficile de conclure, que dans triangle sphérique rectangle si

$c < \frac{\pi}{2}$ on devra avoir $a < \frac{\pi}{2}$; $A < \frac{\pi}{2}$

si $c = \frac{\pi}{2}$ on devra avoir $a = \frac{\pi}{2}$; $A = \frac{\pi}{2}$

enfin si $c > \frac{\pi}{2}$ on devra avoir $a > \frac{\pi}{2}$; $A > \frac{\pi}{2}$

Il s'ensuit que si nous supposons $a > \frac{\pi}{2}$, il faudra supposer en même temps $c > \frac{\pi}{2}$; $A > \frac{\pi}{2}$. Si nous prolongeons dans ce cas les cotés $a$, $c$ au délà du coté $b$ jusqu'à leur point d'intersection nous aurons un autre triangle sphérique rectangle dont les côtés seront $\pi - a$, $b$, $\pi - c$ et les angles opposés $\pi - A$, $B$, $\frac{\pi}{2}$, c'est à dire un triangle auquel les équations (5) seront applicables. Mais les équations (5) ne changent pas de forme si l'on y substitue $(\pi - a)$ à $a$,

$(\pi - c)$ à $c$ et $(\pi - A)$ à $A$, ce qui démontre que les équations (5) s'appliquent à tout triangle sphérique rectangle.

Passons à un triangle sphérique quelconque, dont les côtés soient $a, b, c$ et les angles opposés $A, B, C$ sans supposer qu'un des angles soit droit, parceque les équations (5) sont démontrées pour ce cas là.

Abaissons du sommet de l'angle $C$ un arc de grand cercle $p$ perpendiculaire au coté $c$. Il peut y avoir les cas suivants: ou la perpendiculaire $p$ tombe dans l'interieur du triangle, divise l'angle $C$ en deux parties $D, C - D$, et le côté $c$ en deux parties, $x$ opposé à $D$, et $c - x$ opposé à $C - D$, ou cette perpendiculaire tombe hors du triangle et ajoute un angle $D$ à l'angle $C$ et un arc $x$ au côté $c$.

Dans le premier cas le triangle sphérique donné sera la somme de deux triangles sphériques rectangles. Les côtés d'un de ces triangles seront $p, x, a$ les angles opposés $B, D, \frac{\pi}{2}$; dans l'autre les côtés seront $p, c - x, b$ les angles opposés $A, C - D, \frac{\pi}{2}$. L'application des équations (5) au premier triangle donne

$$\left.\begin{aligned} \sin p &= \sin a \sin B \\ \sin x &= \sin a \sin D \\ \cos p \sin D &= \cos B \\ \cos x \sin B &= \cos D \\ \cos a &= \cos p \cos x. \end{aligned}\right. \qquad (\mathrm{A})$$

Le second triangle fournit de la même manière

$$\left.\begin{aligned} \sin p &= \sin b \sin A \\ \sin (c - x) &= \sin b \sin (C - D) \\ \cos p \sin (C - D) &= \cos A \\ \cos p \cos (c - x) &= \cos b. \end{aligned}\right. \qquad (\mathrm{B})$$

La comparaison des deux valeur de $\sin p$ en $(A)$, $(B)$ donne ensuite

$$\sin a \sin B = \sin b \sin A \qquad (6)$$

La dernière des équations $(B)$ étant divisée par la dernière des équations $(A)$ donne

$$\operatorname{tang} x = \frac{\cos b}{\cos a \cos c} - \operatorname{cotg} c$$

mais la combinaison de la seconde, de la troisième et de la dernière des équations ($A$) donne

$$\operatorname{tang} x = \operatorname{tang} a \cos B.$$

La comparaison de ces deux valeurs de tang $x$ nous conduit à l'équation suivante

$$\cos b - \cos a \cos c = \sin a \sin c \cos B \qquad (7)$$

Si la perpendiculaire $p$ tombe hors du triangle et ajoute l'arc $x$, à l'arc $c$, et l'angle $D$ à l'angle $C$ il se formera de même deux triangles sphériques rectangles. Les côtés de l'un de ces deux triangles seront $p, x, a$ et les angles opposés $\pi - B, D, \frac{\pi}{2}$, les côtés de l'autre seront $p, c + x, b$ les angles opposés $A, C + D, \frac{\pi}{2}$.

L'application des équations (5) au premier triangle donne

$$\left.\begin{aligned} \sin p &= \sin a \sin B \\ \sin x &= \sin a \sin D \\ -\cos B &= \cos p \sin D \\ \cos D &= \cos x \sin B \\ \cos a &= \cos p \cos x. \end{aligned}\right\} \qquad (C)$$

le second triangle, dont $p, c + x, b$ sont les côtés et $A, C + D, \frac{\pi}{2}$ les angles opposés, fournit de la même manière

$$\begin{aligned} &\sin p = \sin b \sin A \\ &\sin (c + x) = \sin b \sin (C + D) \\ &\cos A = \cos p \sin (C + D) \\ &\cos (C + D) = \cos (c + x) \sin A \\ &\cos b = \cos p \cos (c + x). \end{aligned} \qquad (D)$$

La comparaison des deux valeurs de sin $p$ en ($C$), ($D$) donne de nouveau l'équation (6); nous tirons des équations ($C$), ($D$)

$$\operatorname{tang} x \operatorname{cotg} c - \frac{\cos b}{\cos a \cos c}$$

et des équations ($C$)

$$\operatorname{tang} x = \operatorname{tang} a \cos B.$$

La comparaison de ces deux valeurs de tang $x$ nous mene de nouveau à l'équat. (7) qui est ainsi démontrée de même que l'équation (6) l'a été ci dessus pour tous les triangles sphériques en général.

L'équation (7) donne par des changemens de lettres les deux suivantes

$$\cos a - \cos b \cos c = \sin b \sin c \cos A$$
$$\cos c - \cos a \cos b = \sin a \sin b \cos C.$$

En multipliant la dernière par $\cos b$, en ajoutant le produit à la première et en divisant la somme par $\sin b$

$$\cos a \sin b = \sin c \cos A + \sin a \cos b \cos C;$$

en y substituant à la place de $\sin c$ sa valeur

$$\sin c = \frac{\sin C}{\sin A} \sin a$$

conformément à l'équat. (6) et en divisant ensuite par $\sin a$ il viendra

$$\operatorname{cotang} a \sin b = \operatorname{cotang} A \sin C + \cos b \cos C. \tag{8}$$

En y remplaçant $\sin b$ par sa valeur

$$\sin a \frac{\sin B}{\sin A},$$

et en multipliant ensuite l'équation par $\sin A$, il vient:

$$\cos a \sin B = \cos b \cos C \sin A + \sin C \cos A$$

d'où nous tirons par un changement de lettres

$$\cos b \sin A = \cos a \cos C \sin B + \sin C \cos B.$$

L'élimination de $\cos b$ des deux dernières équations nous conduit à l'équation suivante.

$$\cos a \sin B \sin C = \cos B \cos C + \cos A \tag{9}$$

Les équations (6), (7), (8), (9) sont les mêmes qu'on donne ordinairement dans la trigonométrie sphérique et qu'on démontre à l'aide de la géométrie ordinaire.

Il suit de ce qui précède que la trigonométrie sphérique reste la même, soit qu'on adopte la supposition que la somme des trois angles de tout triangle rectiligne est égale à deux angles droits, soit qu'on adopte la supposition contraire que cette somme est toujours moindre que 2 droits; ce qui est très remarquable et n'a pas lieu pour la trigonométrie rectiligne. Avant que de démontrer les équations qui expriment, dans la Pangéométrie, les rélations entre les côtés et les angles de tout triangle rectiligne, nous allons chercher pour toute ligne $x$ la forme de la fonction que nous avons désigné jusqu'à présent par $\Pi(x)$. Considérons pour cela un triangle rectiligne rectangle, dont les côtés sont $a, b, c$ et les angles opposés $\Pi(\alpha), \Pi(\beta), \frac{\pi}{2}$; prolongeons $c$ au de là du sommet de l'angle $\Pi(\beta)$

et faisons le prolongement égal à $\beta$. La perpendiculaire à $\beta$, élevée à l'extrémité de cette ligne et du côté de l'angle opposé par le sommet à $\Pi(\beta)$ sera parallèle à $a$ et à son prolongement au delà du sommet de $\Pi(\beta)$. Menons encore par le sommet de $\Pi(\alpha)$ une droite parallèle à ce même prolongement de $a$.

L'angle que cette droite fera avec $c$ sera $\Pi(c+\beta)$ et l'angle qu'elle fera avec $b$ sera $\Pi(b)$ et on aura l'équation

$$\Pi(b) = \Pi(c+\beta) + \Pi(\alpha) \tag{$\Pi$}$$

Si nous prenons la longueur $\beta$ à partir du sommet de l'angle $\Pi(\beta)$ sur le côté $c$ lui même et que nous élevons à l'extrémité de $\beta$ une perpendiculaire à $\beta$ du côté de l'angle $\Pi(\beta)$, cette droite sera parallèle au prolongement de $a$ au delà du sommet de l'angle droit. Menons par le sommet de l'angle $\Pi(\alpha)$ une droite parallèle à cette dernière perpendiculaire, qui sera conséquemment aussi parallèle au second prolongement de $a$. L'angle de cette parallèle avec $c$ sera dans tous les cas $\Pi(c-\beta)$ et l'angle qu'elle fait avec $b$ sera $\Pi(b)$, par conséquent

$$\Pi(b) = \Pi(c-\beta) - \Pi(\alpha) \tag{$\Pi'$}$$

Il est facile de se convaincre que cette équation est vraie non seulement si $c > \beta$, mais aussi si $c = \beta$ et si $c < \beta$. En effet si $c = \beta$, on a d'un côté $\Pi(c-\beta) = \Pi(o) = \frac{\pi}{2}$, de l'autre côté la perpendiculaire à $c$ menée par le sommet de l'angle $\Pi(\alpha)$ devient parallèle à $a$ d'où il suit que $\Pi(b) = \frac{\pi}{2} - \Pi(\alpha)$, ce qui s'accorde avec notre équation.

Si $c < \beta$ l'extrémité de la ligne $\beta$ tombera au delà du sommet de l'angle $\Pi(\alpha)$ à une distance égale à $\beta - c$. La perpendiculaire à $\beta$ à cette extrémité de $\beta$ sera parallèle à $a$ et à la droite menée par le sommet de l'angle $\Pi(\alpha)$ parallèlement à $a$, d'où il suit que les deux angles adjacents que cette parallèle fait avec $c$ seront, l'aigu égal à $\Pi(\beta - c)$, l'obtus égal a $\Pi(\alpha) + \Pi(b)$. Mais la somme de deux angles adjacents est toujours égale à deux droits ainsi

$$\Pi(\beta - c) + \Pi(\alpha) + \Pi(b) = \pi$$

ou

$$\Pi(b) = \pi - \Pi(\beta - c) - \Pi(\alpha)$$

Mais d'après la définition de la fonction $\Pi(x)$

$$\pi - \Pi(\beta - c) = \Pi(c - \beta)$$

ce qui donne

$$\Pi(b) = \Pi(c-\beta) - \Pi(\alpha)$$

c'est à dire l'équation trouvée plus haut, qui est ainsi démontrée pour tous les cas.

Les deux équations $(\Pi)$, $(\Pi')$ peuvent être remplacées par les deux suivantes

$$\Pi(b) = \tfrac{1}{2}\Pi(c+\beta) + \tfrac{1}{2}\Pi(c-\beta)$$
$$\Pi(\alpha) = \tfrac{1}{2}\Pi(c-\beta) - \tfrac{1}{2}\Pi(c+\beta)$$

Mais l'équation (3) nous donne

$$\cos\Pi(c) = \frac{\cos\Pi(b)}{\cos\Pi(\alpha)};$$

en substituant dans cette équation à la place de $\Pi(b)$, $\Pi(\alpha)$ leurs valeurs, il vient

$$\cos\Pi(c) = \frac{\cos\left\{\tfrac{1}{2}\Pi(c+\beta) + \tfrac{1}{2}\Pi(c-\beta)\right\}}{\cos\left\{\tfrac{1}{2}\Pi(c-\beta) - \tfrac{1}{2}\Pi(c+\beta)\right\}}$$

De cette équation nous déduisons la suivante

$$\operatorname{tang}^2\tfrac{1}{2}\Pi(c) = \operatorname{tang}\tfrac{1}{2}\Pi(c-\beta)\operatorname{tang}\tfrac{1}{2}\Pi(c+\beta)$$

Les lignes $c$ et $\beta$ pouvant varier indépendemment l'une de l'autre dans tout triangle rectiligne rectangle, nous pouvons poser successivement dans la dernière équation $c=\beta$, $c=2\beta$, .... $c=n\beta$, et nous concluons des équations ainsi déduites, qu'en général pour toute ligne $c$ et pour tout nombre entier positif $n$

$$\operatorname{tang}^n\tfrac{1}{2}\Pi(c) = \operatorname{tang}\tfrac{1}{2}\Pi(nc)$$

Il est facile de démontrer la vérité de cette équation pour $n$ négatif ou fractionnaire, d'où il suit, qu'en choisissant l'unité de longueur telle qu'on ait

$$\operatorname{tang}\tfrac{1}{2}\Pi(1) = e^{-1}$$

où $e$ est la base des logarithmes Népériens, on aura pour toute ligne $x$

$$\operatorname{tang}\tfrac{1}{2}\Pi(x) = e^{-x}$$

Cette expression donne $\Pi(x) = \frac{\pi}{2}$ pour $x=o$ et $\Pi(x)=o$ pour $x=\infty$, $\Pi(x)=\pi$ pour $x=-\infty$ conformément à ce que nous avons adopté et démontré plus haut.

La valeur trouvée pour $\operatorname{tang}\tfrac{1}{2}\Pi(x)$ donne pour toute ligne $x$.

$$\sin\Pi(x) = \frac{2}{e^{x}+e^{-x}}$$

$$\cos\Pi(x) = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$$

et pour deux lignes arbitraires, $x, y$

$$\sin \Pi (x+y) = \frac{\sin \Pi (x) \sin \Pi (y)}{1 + \cos \Pi (x) \cos \Pi (y)}$$

$$\sin \Pi (x-y) = \frac{\sin \Pi (x) \sin \Pi (y)}{1 - \cos \Pi (x) \cos \Pi (y)}$$

$$\cos \Pi (x+y) = \frac{\cos \Pi (x) + \cos \Pi (y)}{1 + \cos \Pi (x) \cos \Pi (y)}$$

$$\cos \Pi (x-y) = \frac{\cos \Pi (x) - \cos \Pi (y)}{1 - \cos \Pi (x) \cos \Pi (y)}$$

$$\operatorname{tang} \Pi (x+y) = \frac{\sin \Pi (x) \sin \Pi (y)}{\cos \Pi (x) + \cos \Pi (y)}.$$

Les équations (2), (3), (4) que nous avons trouvées pour un triangle sphérique rectangle se rapportent aussi à un triangle rectiligne rectangle dont les côtés sont $a, b, c$ et les angles opposés $\Pi(\alpha)$, $\Pi(\beta)$ et $\frac{\pi}{2}$. Donc en remplaçant $\Pi(\alpha)$ par $A$, $\Pi(\beta)$ par $B$ nous aurons pour tout triangle rectiligne rectangle, dont les côtés sont $a, b, c$ et où $A$ est l'angle opposé à $a$, $B$ opposé à $b$ et $\frac{\pi}{2}$ opposé à $c$, les équations suivantes:

$$\left.\begin{aligned} \sin \Pi (a) \cos A &= \sin B \\ \sin \Pi (c) \cos A &= \cos \Pi (b) \\ \cos \Pi (c) \cos B &= \cos \Pi (a) \end{aligned}\right\} \qquad (10)$$

A ces équations nous ajoutons encore l'équation que voici et qui a été aussi démontrée plus haut

$$\sin \Pi (a) \sin \Pi (b) = \sin \Pi (c) \qquad (11)$$

La première des équations (10) peut, en y échangeant les lettres entre elles, être écrite ainsi

$$\sin \Pi (b) \cos B = \sin A.$$

en y substituant la valeur de $\cos B$, tirée de la troisième des équations (10) il vient

$$\sin \Pi (b) \cos \Pi (a) = \sin A \cos \Pi (c)$$

en éliminant de cette équation $\sin \Pi (b)$ au moyen de l'équation (11) nous aurons

$$\operatorname{tang} \Pi (c) = \sin A \operatorname{tang} \Pi (a). \qquad (12)$$

Soient maintenant $a, b, c$ les côtés d'un triangle rectiligne quelquonque et $A, B, C$ les angles opposés à ces côtés. Abaissons du sommet de l'angle $C$ une perpendiculaire $p$ sur le côté $c$. Si $p$ tombe dans l'interieur du triangle, de manière à diviser l'angle $C$ en deux angles $D$ et $C - D$ et le côté $c$ en deux parties, $x$ opposé à $D$, $c - x$ opposé à $C - D$, il se formera deux triangles rectilignes rectangles. Les côtés de l'un des ces triangles seront $p, x, b$, les angles opposés $A, D, \frac{\pi}{2}$, les côtés de l'autre seront $p$, $c - x$, $a$ et les angles opposés $B, C - D, \frac{\pi}{2}$.

L'application de l'équation (12) au premier de ces triangles donne

$$\operatorname{tang} \Pi(b) = \sin A \operatorname{tang} \Pi(p)$$

du second de ces triangles nous tirons de la même manière

$$\operatorname{tang} \Pi(a) = \sin B \operatorname{tang} \Pi(p)$$

d'où nous concluons

$$\sin A \operatorname{tang} \Pi(a) = \sin B \operatorname{tang} \Pi(b). \tag{13}$$

L'application des équations (10) et (11) au premier triangle fournit

$$\cos \Pi(b) \cos A = \cos \Pi(x)$$
$$\sin \Pi(x) \sin \Pi(p) = \sin \Pi(b)$$

le second triangle donne

$$\sin \Pi(p) \sin \Pi(c - x) = \sin \Pi(a).$$

En substituant dans cette dernière équation au lieu de $\sin \Pi(c - x)$ sa valeur tirée de la formule générale trouvée plus haut pour $\sin \Pi(x-y)$, il vient

$$\frac{\sin \Pi(a)}{\sin \Pi(p)} = \frac{\sin \Pi(c) \sin \Pi(x)}{1 - \cos \Pi(c) \cos \Pi(x)}$$

d'où nous déduisons en substituant

$$\sin \Pi(p) = \frac{\sin \Pi(b)}{\sin \Pi(x)}$$
$$\cos \Pi(x) = \cos \Pi(b) \cos A$$

l'équation suivante

$$1 - \cos \Pi(b) \cos \Pi(c) \cos A = \frac{\sin \Pi(b) \sin \Pi(c)}{\sin \Pi(a)} \tag{14}$$

Les équations (13), (14) se vérifient d'elles mêmes si $A = \frac{\pi}{2}$ où la perpendiculaire $p$ se confond avec le côté $b$, car dans ce cas l'équation (13) se réduit à l'équation (12) et l'équation (14) à l'équation (11) équations qui ont été démontrées pour tout triangle rectiligne rectangle. Si la perpendiculaire $p$ tombe hors du triangle sur le prolongement de $c$, et ajoute une ligne $x$ à la ligne $c$ et un angle $D$ à l'angle $C$, il se forme deux triangles rectangles, les côtés de l'un sont $p, x, b$ et les angles opposés $(\pi - A), D, \frac{\pi}{2}$, les côtés de l'autre seront $p, c + x, a$ et les angles opposés $B, C + D, \frac{\pi}{2}$.

L'application de l'équation (12) au premier de ces triangles donne

$$\operatorname{tang} \Pi(b) = \sin A \operatorname{tang} \Pi(p).$$

Du second triangle nous tirons de la même manière

$$\operatorname{tang} \Pi(a) = \sin B \operatorname{tang} \Pi(p).$$

En éliminant $\operatorname{tang} \Pi(p)$ des deux dernières équations on trouve de nouveau l'équation (13). L'application des équations (13), (11) au premier triangle fournit

$$-\cos \Pi(b) \cos A = \cos \Pi(x)$$
$$\sin \Pi(b) = \sin \Pi(x) \sin \Pi(p);$$

du second triangle nous tirons de la même manière

$$\sin \Pi(a) = \sin \Pi(p) \sin \Pi(c + x).$$

En remplaçant dans cette équation $\sin \Pi(c + x)$ par sa valeur tirée de la formule générale trouvée plus haut pour $\sin \Pi(x + y)$, on a

$$\frac{\sin \Pi(a)}{\sin \Pi(p)} = \frac{\sin \Pi(c) \sin \Pi(x)}{1 + \cos \Pi(c) \cos \Pi(x)}$$

En substituant dans cette équation

$$\sin \Pi(p) = \frac{\sin \Pi(b)}{\sin \Pi(x)}; \ \cos \Pi(x) = -\cos \Pi(b) \cos A$$

il vient

$$\frac{\sin \Pi(a)}{\sin \Pi(b)} = \frac{\sin \Pi(c)}{1 - \cos \Pi(b) \cos \Pi(c) \cos A}$$

équation identique à l'équation (14).

Les équations (13), (14) sont ainsi démontrées pour tout triangle rectiligne

L'équation (14) donne par un changement de lettres

$$1 - \cos \Pi(c) \cos \Pi(a) \cos B = \frac{\sin \Pi(c) \sin \Pi(a)}{\sin \Pi(b)}$$

En multipliant cette équation membre à membre avec l'équation (14) nous trouvons

$$1 - \cos \Pi(c) \cos \Pi(a) \cos B - \cos \Pi(b) \cos \Pi(c) \cos A$$
$$+ \cos \Pi(a) \cos \Pi(b) \cos^2 \Pi(c) \cos A \cos B = \sin^2 \Pi(c)$$

ou $\cos^2 \Pi(c) - \cos \Pi(c) \cos \Pi(a) \cos B - \cos \Pi(b) \cos \Pi(c) \cos A$

$$+ \cos \Pi(a) \cos \Pi(b) \cos^2 \Pi(c) \cos A \cos B = o.$$

En supprimant dans cette équation le facteur commun $\cos \Pi(c)$ nous avons

$$\cos \Pi(c) + \cos \Pi(a) \cos \Pi(b) \cos \Pi(c) \cos A \cos B - \cos \Pi(a) \cos B$$
$$- \cos \Pi(b) \cos A = o.$$

De la même manière nous trouvons

$$\cos \Pi(a) + \cos \Pi(a) \cos \Pi(b) \cos \Pi(c) \cos B \cos C - \cos \Pi(b) \cos C$$
$$- \cos \Pi(c) \cos B = o.$$

Multiplions cette équation par $\cos A$ et retranchons le produit du produit de l'équation précédente par $\cos C$ nous aurons:

$$\cos \Pi(a) \left\{ \cos A + \cos B \cos C \right\} = \cos \Pi(c) \left\{ \cos C + \cos A \cos B \right\}$$

Élevant les deux membres de cette équation au carré et divisant après par $\cos^2 \Pi(c)$, elle prend la forme suivante

$$\frac{\cos^2 \Pi(a)}{\cos^2 \Pi(c)} \left\{ \cos A + \cos B \cos C \right\}^2 = \left\{ \cos C + \cos A \cos B \right\}^2$$

Mais l'équation (13) donne

$$\frac{1}{\cos^2 \Pi(c)} = 1 + \frac{\sin^2 A}{\sin^2 C} \operatorname{tang}^2 \Pi(a)$$

Si nous substituons dans l'avant-dernière équation au lieu de $\frac{1}{\cos \Pi(c)}$ sa valeur donnée par la dernière, il vient

$$\cos^2 \Pi(a) + \frac{\sin^2 A}{\sin^2 C} \sin^2 \Pi(a) = \left\{ \frac{\cos C + \cos B \cos A}{\cos A + \cos B \cos C} \right\}^2$$

et ensuite

$$\sin^2 \Pi(a) \left\{ 1 - \frac{\sin^2 A}{\sin^2 C} \right\} = \frac{\sin^2 B (\sin^2 C - \sin^2 A)}{(\cos A + \cos B \cos C)^2}$$

Divisant les deux membres de cette équation par $\sin^2 C - \sin^2 A$ et extrayant la racine carrée il viendra

$$\sin \Pi(a) = \frac{\sin B \sin C}{\cos A + \cos B \cos C}$$

sans ambiguité de signe, parceque les deux membres de la dernière équation sont tous les deux positifs. En effet $\Pi(a) < \frac{\pi}{2}$, $B < \pi$, $C < \pi$, d'où il suit que les sinus de ces angles sont positifs; ensuite

$$\cos A + \cos(B + C) = 2 \cos \tfrac{1}{2}(A + B + C) \cos \tfrac{1}{2}(B + C - A)$$

Mais $A + B + C < \pi$, par conséquence

$$\cos \tfrac{1}{2}(A + B + C)$$

sera positif ainsi que

$$\cos \tfrac{1}{2}(B + C - A),$$

en ajoutant à chacun des deux membres de la dernière équation le nombre positif $\sin B \sin C$ nous trouvons

$$\cos A + \cos B \cos C > o$$

Ainsi dans tout triangle rectiligne

$$\cos A + \cos B \cos C = \frac{\sin B \sin C}{\sin \Pi(a)} \tag{15}$$

La multiplication de l'équation (14) membre à membre par l'équation suivante, qui en résulte par un changement de lettres

$$1 - \cos \Pi(a) \cos \Pi(b) \cos C = \frac{\sin \Pi(a) \sin \Pi(b)}{\sin \Pi(c)} \tag{16}$$

donne

$$\{1 - \cos \Pi(a) \cos \Pi(b) \cos C\} \{1 - \cos \Pi(b) \cos \Pi(c) \cos A\} = \sin^2 \Pi(b)$$

à laquelle on peut après l'exécution de la multiplication indiquée dans le premier membre donner la forme suivante:

$$\cos^2 \Pi(b) - \cos \Pi(a) \cos \Pi(b) \cos C - \cos \Pi(b) \cos \Pi(c) \cos A$$
$$+ \cos^2 \Pi(b) \cos \Pi(a) \cos \Pi(c) \cos A \cos C = o.$$

ou en divisant par $\cos \Pi(b)$

$$\cos \Pi(b) + \cos \Pi(a) \cos \Pi(b) \cos \Pi(c) \cos A \cos C - \cos \Pi(a) \cos C$$
$$- \cos \Pi(c) \cos A = o. \tag{17}$$

Mais nous trouvons d'après l'équation (13)

$$\cos \Pi(c) = \frac{\sin \Pi(c) \sin C}{\sin A} \operatorname{cotang} \Pi(a)$$

Dans cette équation nous pouvons substituer à $\sin \Pi(c)$ sa valeur tirée de l'équation (16); après quoi

$$\cos \Pi(c) = \frac{\sin \Pi(b) \cos \Pi(a) \sin C}{\{1 - \cos \Pi(a) \cos \Pi(b) \cos C\} \sin A.}$$

La substitution de cette valeur de cos $\Pi(c)$ dans l'équation (17) nous donne

$$\operatorname{cotang} A \sin C \sin \Pi(b) + \cos C = \frac{\cos \Pi(b)}{\cos \Pi(a)} \tag{18}$$

Réunissons les équations (13), (14), (15), (18) qui expriment les dépendances entre les côtés et les angles de tout triangle rectiligne, pour en faciliter l'application

$$\left.\begin{aligned} &\sin A \operatorname{tang} \Pi(a) = \sin B \operatorname{tang} \Pi(b) \\ &1 - \cos \Pi(b) \cos \Pi(c) \cos A = \frac{\sin \Pi(b) \sin \Pi(c)}{\sin \Pi(a)} \\ &\cos A + \cos B \cos C = \frac{\sin B \sin C}{\sin \Pi(a)} \\ &\operatorname{cotang} A \sin C \sin \Pi(b) + \cos C = \frac{\cos \Pi(b)}{\cos \Pi(a)} \end{aligned}\right\} \tag{19}$$

A commencer par ces équations la Pangéométrie devient géométrie analytique et forme de cette manière une théorie géométrique complète et distincte. Les équations (19) servent à représenter les lignes courbes par des équations entre les coordonnées de leurs points; à calculer la longueur et les aires des courbes, les surfaces et les volumes des corps, comme je l'ai montré dans les mémoires scientifiques de l'université de Kasan pour l'année 1829.

Il a été remarqué plus haut que la Pangéométrie donne la géométrie ordinaire si nous supposons les lignes infiniment petites. Nous pouvons maintenant vérifier cette assertion.

Pour toute ligne $x$ infiniment petite nous pouvons admettre les valeurs approchées suivantes:

$$\begin{aligned} \operatorname{cotang} \Pi(x) &= x \\ \sin \Pi(x) &= 1 - \tfrac{1}{2}x^2 \\ \cos \Pi(x) &= x. \end{aligned}$$

Si nous regardons les côtés du triangle comme des infiniment petits du premier ordre et que nous négligeons les infiniment petits d'un ordre supérieur au second, les équations (19) prendront, après la substitution des valeurs approchées de $\sin \Pi(a)$, $\sin \Pi(b)$ etc. la forme suivante:

$$\begin{aligned} &b \sin A = a \sin B \\ &a^2 = b^2 + c^2 - 2bc \cos A \\ &\cos A + \cos(B + C) = o \\ &a \sin(A + C) = b \sin A. \end{aligned}$$

Les deux premières de ces équations sont les équations connues de la trigonometrie ordinaire. Les deux dernières donnent

$$A + B + C = \pi.$$

Nommons, pour donner un exemple de la représentation des lignes courbes par des équations entre les coordonnées de leur points, $y$ la longueur de la perpendiculaire abaissée d'un point de la circonférence d'un cercle de rayon $r$ sur un diamètre fixe de ce cercle, et $x$ la partie de ce diamètre entre le centre et le pied de la perpendiculaire $y$. L'application de l'équation (11) au triangle rectangle dont les côtés sont $x, y, r$ donne

$$\sin \Pi(x) \sin \Pi(y) = \sin \Pi(r) \tag{20}$$

ce qui est l'équation d'un cercle entre les coordonnées rectangles $x, y$. Si nous convenons de compter $x$ à partir d'une extrémité du diamètre, l'équation (20) devient

$$\sin \Pi(r - x) \sin \Pi(y) = \sin \Pi(r)$$

ou bien $$2\,(e^{r} + e^{-r}) = (e^{r-x} + e^{-r+x})\,(e^{y} + e^{-y});$$

si nous divisons cette équation par $e^{r}$, et que nous posons après $r = \infty$, nous aurons l'équation suivante, qui est l'équation du cercle limite

$$2 = (e^{y} + e^{-y})\,e^{-x}$$

ou $$\sin \Pi(y) = \operatorname{tang} \tfrac{1}{2} \Pi(x).$$

Il suit de la définition du cercle limite que deux axes du cercle menés par les deux extrémités d'une même corde, sont également inclinés sur cette corde, propriété qui pourrait servir de définition au cercle limite et de laquelle on peut aussi déduire l'équation de cette courbe en considérant le triangle dont les côtés sont $x$, $y$ et la corde $2a$ du cercle limite; les angles de ce triangle seront $\Pi(a) - \Pi(y)$ opposé à $x$, $\Pi(a)$ opposé à $y$ et $\frac{\pi}{2}$ opposé à $2a$. Conformément aux équations (10), (11) on a dans ce triangle

$$\begin{aligned}
&\sin \Pi(x) \sin \Pi(y) = \sin \Pi(2a) \\
&\sin \Pi(x) \cos \left\{ \Pi(a) - \Pi(y) \right\} = \sin \Pi(a) \\
&\sin \Pi(y) \cos \Pi(a) = \sin \left\{ \Pi(a) - \Pi(y) \right\}.
\end{aligned}$$

La dernière équation donne

$$2 \operatorname{tang} \Pi(y) = \operatorname{tang} \Pi(a) \tag{21}$$

et la première peut être écrite ainsi

$$\sin \Pi(x) \sin \Pi(y) = \frac{\sin^2 \Pi(a)}{1 + \cos^2 \Pi(a)}$$

En substituant dans cette équation au lieu de $\sin^2 \Pi(a)$, $1 + \cos^2 \Pi(a)$ leurs valeurs en $\operatorname{tang}^2 \Pi(a)$ et en introduisant la valeur de $\operatorname{tang}^2 \Pi(a)$ tirée de l'équation (21) il vient

$$\sin \Pi(x) \sin \Pi(y) = \frac{2 \operatorname{tang}^2 \Pi(y)}{1 + 2 \operatorname{tang}^2 \Pi(y)}$$

et ensuite

$$\sin \Pi(x) = \frac{2 \sin \Pi(y)}{1 + \sin^2 \Pi(y)}$$

d'où nous déduisons

$$2 \cos^2 \tfrac{1}{2} \Pi(x') = \frac{\{1 + \sin \Pi(y)\}^2}{1 + \sin^2 \Pi(y)}$$

$$2 \sin^2 \tfrac{1}{2} \Pi(x') = \frac{\{1 - \sin \Pi(y)\}^2}{1 + \sin^2 \Pi(y)}$$

En divisant la dernière de ces équations par l'avant-dernière et extrayant la racine carrée, nous aurons

$$\operatorname{tang} \tfrac{1}{2} \Pi(x') = \frac{1 - \sin \Pi(y)}{1 + \sin \Pi(y)}$$

d'où $$\sin \Pi(y) = \frac{1 - \operatorname{tang} \frac{1}{2} \Pi(x')}{1 + \operatorname{tang} \frac{1}{2} \Pi(x')}$$

Le second membre de cette équation peut prendre la forme suivante:

$$\frac{\cos \frac{1}{2} \Pi(x') - \sin \frac{1}{2} \Pi(x')}{\cos \frac{1}{2} \Pi(x') + \sin \frac{1}{2} \Pi(x')}$$

ou $$\frac{\sin \{ \frac{1}{4} \pi - \frac{1}{2} \Pi(x') \}}{\cos \{ \frac{1}{4} \pi - \frac{1}{2} \Pi(x') \}} = \frac{\sin \frac{1}{2} \Pi(x)}{\operatorname{sos} \frac{1}{2} \Pi(x)} = \operatorname{tang} \tfrac{1}{2} \Pi(x)$$

et par conséquence

$$\sin \Pi(y) = \operatorname{tang} \tfrac{1}{2} \Pi(x)$$

comme nous avons trouvé plus haut.

Pour donner un exemple de la rectification des courbes, cherchons l'expression de la longueur d'une circonférence de cercle de rayon $r$. Menons deux rayons, dont l'angle an centre soit $\frac{2\pi}{n}$, où $n$ désigne un nombre entier. Abaissons de l'extrémité d'un de ces deux rayons une perpendiculaire $p$ sur l'autre. Le produit $np$ différera d'autant moins de la longueur de la circonférence du cercle que $n$ est plus grand. Le triangle rectangle, dont $p$ est une cathète, $r$ l'hypotenuse et $\frac{2\pi}{n}$ l'angle opposé a $p$ donne (équat. 13).

$$\sin\frac{2\pi}{n}\operatorname{tang}\Pi(p)=\operatorname{tang}\Pi(r)$$

Mais il est connu que

$$\lim\left\{n\sin\frac{2\pi}{n}\right\}=2\pi$$

pour $n=\infty$, tandis que

$$\frac{1}{n}\operatorname{tang}\Pi(p)=\frac{2}{n(e^{p}-e^{-p})}$$

et

$$n(e^{p}-e^{-p})=2np$$

avec une approximation d'autant plus grande, que $n$ est plus grand et conséquement $p$ est plus petit. Après quoi

$$\text{circonférence }(r)=np=2\pi\operatorname{cotg}\Pi(r)$$

c'est à dire $\quad$ $\text{circonférence }(r)=(e^{r}-e^{-r})\pi$

ce qui donne pour $r$ très petit

$$\text{circonférence }(r)=2\pi r$$

comme dans la géométrie ordinaire.

Déterminons encore l'arc $s$ de cercle limite au moyen des coordonnées: $y$ perpendiculaire abaissée d'une extrémité de l'arc $s$ sur l'axe menée par l'autre extrémité, $x$ partie de cette axe comprise entre le sommet de l'arc et le pied de la perpendiculaire. Soit $c$ la corde de l'arce $s$; soient de même $c_1, c_2, c_3 \ldots\ldots$ les cordes des arcs $\frac{1}{2}s, \frac{1}{2^2}s, \frac{1}{2^3}\ldots\ldots$ Nous avons démontré plus haut (équat. 21) que

$$\operatorname{cotg}\Pi(y)=2\operatorname{cotg}\Pi(\tfrac{1}{2}c)$$

semblablement on a

$$\operatorname{cotg}\Pi(\tfrac{1}{2}c)=2\operatorname{cotg}\Pi(\tfrac{1}{2}c_1)$$

$$\operatorname{cotg}\Pi(\tfrac{1}{2}c_1)=2\operatorname{cotg}\Pi(\tfrac{1}{2}c_2)$$

$$\operatorname{cotg}\Pi(\tfrac{1}{2}c_2)=2\operatorname{cotg}\Pi(\tfrac{1}{2}c_3)$$

et généralement pour tout nombre $n$ entier positif

$$\operatorname{cotg}\Pi(\tfrac{1}{2}c_{n-1})=2\operatorname{cotg}\Pi(\tfrac{1}{2}c_n)$$

d'où nous concluons

$$\operatorname{cotg}\Pi(y)=2^{n+1}\operatorname{cotg}\Pi(\tfrac{1}{2}c_n)$$

si n est un nombre très grand et $c_n$ par conséquent une ligne très petite nous aurons

$$2^{n+1}\operatorname{cotg}\Pi(\tfrac{1}{2}c_n)=2^{n}c_n$$

Mais $$2^n c_n = s \text{ pour } n = \infty$$
d'où il suit que
$$s = \operatorname{cotg} \Pi(y) \tag{22}$$

Déterminons encore l'arc $s$ de cercle limite au moyen de la partie $t$ de la tangente au sommet de l'axe menée par une extrémité de l'arc $s$, comprise entre le point de contact et l'intersection de la tangente et de l'axe menée par l'autre extrémité de l'arc $s$, c'est-à-dire déterminons la fonction que nous avons auparavant désignée par $L(t)$. Dans le triangle dont les côtés sont $c, t, f(t)$ et les angles opposés $\Pi(t)$, $\pi - \Pi(\frac{1}{2}c)$, $\frac{\pi}{2} - \Pi(\frac{1}{2}c)$ nous trouvons en appliquant l'équation (13)
$$\sin \Pi(t) \operatorname{tang} \Pi(c) = \sin \Pi(\tfrac{1}{2}c) \operatorname{tang} \Pi(t).$$
Mais nous avons vu que (équation 21)
$$\operatorname{tang} \Pi(\tfrac{1}{2}c) = 2 \operatorname{tang} \Pi(y)$$
à quoi nous ajouterons la remarque que
$$\operatorname{tang} \Pi(c) = \frac{\sin^2 \Pi(\frac{1}{2}c)}{2\cos \Pi(\frac{1}{2}c)}$$
et il vient
$$\cos \Pi(t) = 2 \operatorname{cotg} \Pi(\tfrac{1}{2}c)$$
c'est-à-dire, en vertu de l'équation (22)
$$\cos \Pi(t) = s = L(t).$$

L'équation de la ligne droite a une forme assez compliquée, si l'on veut qu'elle soit générale et qu'elle représente la ligne droite, quelle que soit sa position par rapport aux axes des coordonnées. Abaissons d'un point fixe de la droite donnée une perpendiculaire $a$ sur l'axe des $x$ et nommons $L$ l'angle que cette perpendiculaire fait avec la droite Nommons encore $y$ la perpendiculaire abaissée sur l'axe des $x$ d'un autre point de la droite donnée dont $l$ soit la distance au premier point; soit enfin $x$ la partie de l'axe des $x$ comprise entre les deux perpendiculaires. Menons une ligne droite par le sommet de $a$ et le pied de $y$ et soit $r$ la longueur de la partie de cette droite comprise entre ces deux points. Il se formera deux triangles l'un rectangle avec les côtés $a, x, r$ et les angles opposés $A, X, \frac{\pi}{2}$ l'autre avec les côtés $y, r, l$ et les angles opposés $L - X$, $C, \frac{\pi}{2} - A$.

L'application des équations (10), (11) au premier de ces triangles donne:

$$\sin \Pi(x) \sin \Pi(a) = \sin \Pi(r)$$
$$\sin \Pi(x) \cos X = \sin A$$
$$\sin \Pi(a) \cos A = \sin X$$
$$\cos \Pi(r) \cos A = \cos \Pi(x)$$
$$\cos \Pi(r) \cos X = \cos \Pi(a).$$

De ces équations nous tirons

$$\tang A = \tang \Pi(x) \cos \Pi(a)$$
$$\tang \Pi(r) = \tang \Pi(x) \sin \Pi(a) \cos A$$
$$\tang X = \tang \Pi(a) \cos \Pi(x)$$
$$\cos \Pi(x) = \cos \Pi(r) \cos A$$
$$\sin X = \sin \Pi(a) \cos A.$$

L'application de la dernière des equations (19) au second triangle fournit:

$$\operatorname{cotg}(L - X) \cos A \sin \Pi(r) + \sin A = \frac{\cos \Pi(r)}{\cos \Pi(y)}$$

d'où il suit que

$$\cos \Pi(y) = \frac{\cos \Pi(r)}{\operatorname{cotg}(L - X) \cos A \sin \Pi(r) + \sin A}$$
$$= \frac{\cos \Pi(r) \{ \tang L - \tang X \}}{\{ 1 + \operatorname{tg} L \operatorname{tg} X \} \cos A \sin \Pi(a) \sin \Pi(x) + \sin A \{ \operatorname{tg} L - \operatorname{tg} X \}}$$

en substituant dans cette équation au lieu de tang $X$ sa valeur il vient

$$\cos \Pi(y) =$$
$$\frac{\cos \Pi(r) \{ \tang L - \tang \Pi(a) \cos \Pi(x) \}}{\{ 1 + \operatorname{tg} L \operatorname{tg} \Pi(a) \cos \Pi(x) \} \cos A \sin \Pi(a) \sin \Pi(x) + \sin A \{ \operatorname{tg} L - \operatorname{tg} \Pi(a) \cos \Pi(x) \}}$$

Substituons dans cette équation au lieu de $\cos \Pi(r)$, sa valeur; nous trouverons toute réduction faite que

$$\cos \Pi(y) = \frac{\cos \Pi(a)}{\sin \Pi(x)} - \sin \Pi(a) \operatorname{cotg} \Pi(x) \operatorname{cotg} L. \tag{23}$$

Si la droite donnée est parallèle à l'axe des $x$ on aura $L = \Pi(a)$ et l'équation (23) prendra la forme suivante:

$$\cos \Pi(y) = \frac{\cos \Pi(a)}{\sin \Pi(x)} - \frac{\cos \Pi(a)}{\tang \Pi(x)}$$

où $$\cos \Pi(y) = \cos \Pi(a)\, e^{-x} \tag{24}$$

Si nous désignons par $s, s'$ les longueurs de deux arcs de cercle limite compris entre l'axe des $x$ et la droite parallèle à cette axe et dont le premier $s$ soit tangent à $a$ au pied de $a$ et le second tangent à $y$ au pied de $y$ nous aurons d'après ce que nous avons démontré

$$s = \cos \Pi(a)$$
$$s' = \cos \Pi(y)$$

après quoi

$$s' = s\, e^{-x}$$

où $x$ est la distance entre les deux arcs $s$ et $s'$. Cette équation montre que la constante $E$, introduite plus haut pour désigner le rapport constant de deux arcs de cercle limite compris entre deux parallèles, dont la distance est égale à l'unité, est égale à $e$, c'est-à-dire à la base des logarithmes Népériens.

Si nous posons dans l'équat. (23) $a = o$ et si nous posons $\pi - L$ à la place de $L$ nous aurons

$$\cos \Pi(y) = \operatorname{cotg} \Pi(x) \operatorname{cotg} L$$

ce qui est par conséquence l'équation d'une droite qui passe par l'origine des coordonnées $x$ et fait un angle $L$ avec l'axe des $x$, ce qui s'accorde avec l'équation (10).

Considérons maintenant un quadrilatère, dont deux côtés $a, y$ sont perpendiculaires au troisième côté $x$. Soit $c$ le quatrième côté et $\varphi$ l'angle entre $a$ et $c$ tandis que l'angle entre $c$ et $y$ est droit. Menons la diagonale $r$ qui passe par le sommet de l'angle $\varphi$ et par le sommet de l'angle droit opposé. Cette diagonale divise le quadrilatère en deux triangles rectangles. Les côtés de l'un de ces deux triangles sont $a, x, r$ et les angles opposés $A, X, \frac{\pi}{2}$; les côtés de l'autre sont $y, c, r$ et les angles opposés $\varphi - X, \frac{\pi}{2} - A, \frac{\pi}{2}$.

L'application des équations (10), (11), (13) au premier de ces triangles donne

$$\left.\begin{aligned}
&\sin \Pi(r) = \sin \Pi(a) \sin \Pi(x)\\
&\sin A \operatorname{tang} \Pi(a) = \sin X \operatorname{tang} \Pi(x)\\
&\cos \Pi(r) \cos A = \cos \Pi(x)\\
&\cos \Pi(r) \cos X = \cos \Pi(a)
\end{aligned}\right\} \quad (G)$$

le second triangle fournit de la même manière les équations suivantes:

$$\left.\begin{aligned} &\sin \Pi(y) \sin \Pi(c) = \sin \Pi(r) \\ &\sin \Pi(y) \cos(\varphi - X) = \cos A \\ &\cos \Pi(r) \cos(\varphi - X) = \cos \Pi(c) \\ &\cos \Pi(r) \sin A = \cos \Pi(y). \end{aligned}\right\} \quad (H)$$

L'équation (12) appliquée au premier triangle donne

$$\left.\begin{aligned} &\operatorname{tang} \Pi(r) = \sin X \operatorname{tang} \Pi(x) \\ &\operatorname{tang} \Pi(r) = \sin A \operatorname{tang} \Pi(a) \end{aligned}\right\} \quad (K)$$

tandis que l'application de la même équation au second triangle fournit

$$\left.\begin{aligned} &\operatorname{tang} \Pi(r) = \sin(\varphi - X) \operatorname{tang} \Pi(y) \\ &\operatorname{tang} \Pi(r) = \cos A \operatorname{tang} \Pi(c) \end{aligned}\right\} \quad (L)$$

En substituant dans la seconde des équations (K) pour $\sin \Pi(r)$ la valeur tirée des équations (G) nous trouvons

$$\cos \Pi(r) = \frac{\sin \Pi(x) \cos \Pi(a)}{\sin A}.$$

La substitution de cette valeur de $\cos \Pi(r)$ dans la dernière des équations (H) donne

$$\cos \Pi(y) = \sin \Pi(x) \cos \Pi(a). \tag{25}$$

En divisant membre à membre la dernière des équations (H) par la troisième des équations (G) il vient

$$\operatorname{tang} A = \frac{\cos \Pi(y)}{\cos \Pi(x)}$$

substituons dans cette équation la valeur que nous venons de trouver pour $\cos \Pi(y)$ à la place de $\cos \Pi(y)$, nous aurons

$$\operatorname{tang} A = \operatorname{tang} \Pi(x) \cos \Pi(a).$$

La division membre à membre, de la seconde des équations (G) par la dernière de ces mêmes équations produit

$$\frac{\operatorname{tang} X \operatorname{tang}(x)}{\cos \Pi(r)} = \frac{\sin A \operatorname{tang} \Pi(a)}{\cos \Pi(a)}$$

En substituant dans cette équation à la place de $\sin A$ sa valeur tirée de la dernière des équations (H) on trouve

$$\operatorname{tang} X = \frac{\cos \Pi(y) \operatorname{tang} \Pi(a)}{\cos \Pi(a)} \operatorname{cotg} \Pi(x)$$

Remplaçant enfin dans cette équation $\cos \Pi(y)$ par sa valeur trouvée plus haut, il vient

$$\tang X = \cos \Pi(x) \tang \Pi(a).$$

La combinaison de la seconde des équations (H) avec la première des équations (L) donne encore

$$\tang(\varphi - X) \frac{\tang \Pi(y)}{\sin \Pi(y)} = \frac{\tang \Pi(r)}{\cos A}$$

ou

$$\tang(\varphi - X) = \frac{\cos \Pi(y) \tang \Pi(r)}{\cos A}$$

et si nous substituons la valeur de $\tang \Pi(r)$ donnée par la seconde des équations (K)

$$\tang(\varphi - X) = \tang A \tang \Pi(a) \cos \Pi(y).$$

Cette équation prend, en y substituant à la place de $\tang A$, $\tang X$, leurs valeurs trouvées plus haut, la forme suivante:

$$\tang \varphi = \frac{\tang \Pi(a)}{\cos \Pi(x)}. \tag{26}$$

Cette équation montre que $x$ est toujours réelle si l'angle $\varphi$ est plus grand que $\Pi(a)$ et plus petit qu'un angle droit ou si $\pi - \varphi > \Pi(a)$, $\pi - \varphi < \frac{\pi}{2}$. La valeur de $\cos \Pi(x)$ est positive si $\frac{\pi}{2} > \varphi > \Pi(a)$ et la ligne $x$ est par conséquence aussi positive; mais si $\frac{\pi}{2} > \pi - \varphi > \Pi(a)$, la valeur de $\cos \Pi(x)$ devient négative et la ligne $x$ est située de l'autre côté de la perpendiculaire $a$.

Cela démontre que si deux droites, situées dans un même plan, ne se rencontrent pas quelque loin qu'on les prolonge sans être pourtant parallèles, elles doivent être toutes les deux perpendiculaires à une même droite; toutes les paires de droites qui étant dans le même plan ne sont ni parallèles ni perpendiculaires à une même droite doivent nécessairement se couper. Les droites qui étant dans un même plan se coupent nécessairement après un prolongement suffisant ne seront dont que celles, pour chaque point desquelles l'angle, que la droite passant par ce point fait avec la perpendiculaire abaissé de ce point sur l'autre droite, est plus petit que l'angle de parallèlisme correspondant à la longueur de cette perpendiculaire. A l'aide des résultats précédents il est possible de simplifier beaucoup l'équation générale de la ligne droite (23) dans le cas ou la droite à laquelle l'équation appartient ne coupe pas l'axe des $x$.

Soit $a$ la perpendiculaire abaissée sur l'axe des $x$ d'un point fixe, mais arbitraire pris sur la droite donnée, $L$ celui des deux angles entre cette perpendiculaire et la droite, qui est situé du côté des $x$ positifs. Cherchons d'abord une ligne $l$ telle que

$$\cos \Pi(l) = \operatorname{tang} \Pi(a) \operatorname{cotg} L$$

ce qui est toujours possible tant que $L > \Pi(a)$, c'est-à-dire tant que la droite ne coupe pas l'axe des $x$. Portons cette droite $l$ sur l'axe des $x$ à partir de l'origine des coordonnées du côté des $x$ positifs ou négatifs selon le signe de $l$. Érigeons à l'extrémité de la ligne $l$ une perpendiculaire à l'axe des $x$, prolongeons la jusque à ce qu'elle rencontre la ligne donnée et soit $b$ la partie de cette perpendiculaire comprise entre la droite donnée et l'axe des $x$. L'angle sous lequel cette perpendiculaire rencontrera la droite donnée doit être droit d'après l'équation (26). Si nous convenons maintenant de prendre le pied de la perpendiculaire $b$ pour origine des coordonnées, nous aurons d'après l'équation (25)

$$\cos \Pi(b) = \cos \Pi(y) \sin \Pi(x) \tag{27}$$

ce qui est l'équation générale d'une droite qui ne coupe pas l'axe des $x$. Nous pouvons poser dans cette équation $y = a$ et en même temps $x = -l$ ce qui donne

$$\cos \Pi(b) = \cos \Pi(a) \sin \Pi(l);$$

cette équation prend, si l'on y substitue à la place $\cos \Pi(b)$, $\sin \Pi(l)$, leurs valeurs, la forme suivante

$$\cos \Pi(y) \sin \Pi(x) = \cos \Pi(a) \sqrt{1 - \operatorname{tang}^2 \Pi(a) \operatorname{cotg}^2 L}.$$

Le second membre de cette équation devient imaginaire aussitôt que $\operatorname{tang} \Pi(a) \operatorname{cotg} L > 1$, c'est-à-dire pour toute droite qui coupe l'axe des $x$. A l'aide de ce qui précède nous pouvons résoudre le problème de trouver la distance de deux points, dont la position dans le plan est déterminée par leurs coordonnées rectangulaires $x, y$ et $x', y'$. Posons pour abréger

$$\triangle x = x' - x, \ \triangle y = y' - y.$$

Abaissons une perpendiculaire du sommet de $y$ sur $y'$ et désignons la longueur de cette perpendiculaire par $q$, tandis que $y_1$ désigne la partie de $y'$ comprise entre l'axe des $x$ et la perpendiculaire $q$.

En conséquence de l'équation (25) nous aurons

$$\cos \Pi(y_1) = \cos \Pi(y) \sin \Pi(\triangle x)$$
$$\cos \Pi(q) = \cos \Pi(\triangle x) \sin \Pi(y_1).$$

Après avoir déterminé les valeurs de $y_1$, $q$ à l'aide de ces équations, la distance cherchée des deux points, désignons la par $r$, sera donnée par l'équation suivante, qui se tire de l'équation (11).

$$\sin \Pi(r) = \sin \Pi(y' - y_1) \sin \Pi(q).$$

Si $\triangle x$ et $\triangle y$ et par conséquence $q, r$ sont très petits de sorte qu'on puisse négliger les puissances supérieures de ces quantités devant les inférieures, $r$ représentera l'élément $ds$ d'une ligne courbe à l'expression duquel on parvient en prenant

$$\sin \Pi(q) = 1 - \tfrac{1}{2} q^2$$
$$\cos \Pi(q) = q - \tfrac{1}{3} q^3$$
$$\sin \Pi(r) = 1 - \tfrac{1}{2} r^2$$
$$\sin \Pi(y' - y_1) = 1 - \tfrac{1}{2} (y' - y_1)^2$$

après quoi il vient

$$q = \frac{\triangle x}{\sin \Pi(y)}$$

$$ds = \sqrt{dy^2 + \frac{dx^2}{\sin \Pi(y)}}$$

Pour le cercle limite on a

$$\sin \Pi(y) = e^{-x}.$$

Des expressions générales qui déterminent $\sin \Pi(a)$ etc. en fonction de $a$ et qui ont été données plus haut, on tire

$$d\,\Pi(a) = -\sin \Pi(a)\, da$$

après quoi on trouve en différentiant l'équation du cercle limite

$$\sin \Pi(y) \cos \Pi(y)\, dy = e^{-x}$$

et $$ds = \frac{dx\, e^{x}}{\sqrt{1 - e^{-2x}}};$$

en intégrant par rapport à $x$ depuis $x = o$ on trouve

$$s = \sqrt{e^{2x} - 1}$$

ou autrement $$s = \cot \Pi(y)$$

comme nous l'avons trouvé plus haut. Si nous désignons par $r$ la distance d'un point d'une ligne courbe à l'origine des coordonnées et par $\varphi$ l'angle que cette distance $r$ fait avec l'axe des $x$ positifs, nous aurons dans le triangle, dont les côtés sont $y, x, r$ d'après l'équation (12)

$$\tan \Pi(r) = \sin \varphi \tan \Pi(y).$$

En prenant les logarithmes des deux membres de cette équation et en différentiant par rapport à $y, \varphi, v$, il vient

$$\frac{dr}{\cos \Pi(r)} = -\operatorname{cotg} \varphi \, d\varphi + \frac{dy}{\cos \Pi(y)}.$$

De cette équation nous tirons

$$dy = \left\{ \operatorname{cotg} \varphi \, d\varphi + \frac{dr}{\cos \Pi(r)} \right\} \cos \Pi(y)$$

ou en $y$ substituant à la place de $\cos \Pi(y)$ sa valeur en $r$ et $\varphi$

$$dy = \frac{\cos \varphi \cos \Pi(r) \, d\varphi + \sin \varphi \, dr}{\sqrt{1 - \cos^2 \varphi \cos^2 \Pi(r)}}.$$

Pour exprimer $dx$ en $r$ et $\varphi$ prenons (équat. 10)

$$\cos \Pi(r) \cos \varphi = \cos \Pi(x).$$

La différentiation par rapport à $r, \varphi, x$ des logarithmes des deux membres de cette équation fournit

$$\frac{\sin^2 \Pi(r) \, dr}{\cos \Pi(r)} - \operatorname{tang} \varphi \, d\varphi = \frac{\sin^2 \Pi(x) \, dx}{\cos \Pi(x)}$$

d'où nous tirons, à l'aide des équations

$$\sin \Pi(x) \sin \Pi(y) = \sin \Pi(r)$$
$$\cos \Pi(r) \cos \varphi = \cos \Pi(x),$$

l'équation suivante, qui exprime la valeur cherchée

$$\frac{dx}{\sin \Pi(y)} = \frac{\cos \varphi \sin \Pi(r) \, dr - d\varphi \sin \varphi \operatorname{cotg} \Pi(r)}{\sqrt{1 - \cos^2 \varphi \cos^2 \Pi(r)}}$$

après quoi

$$ds = \sqrt{dr^2 + d\varphi^2 \operatorname{cotg}^2 \Pi(r)}.$$

Pour le cercle, en supposant que l'origine des coordonnées est au centre nous trouvons, puisque $dr = o$,

$$ds = d\varphi \operatorname{cotg} \Pi(r);$$

en intégrant depuis $\varphi = o$ jusqu'à $\varphi = \frac{\pi}{2}$ et en multipliant le résultat par 4 nous trouvons l'expression suivante de la circonférence du cercle de rayon $r$

$$2\pi \operatorname{cotg} \Pi(r)$$

qui coïncide avec celle que nous avons trouvée plus haut.

Si nous appelons $s$ un arc de cercle limite compté depuis l'axe des $x$, la révolution de $s$ autour de l'axe des $x$ produira une partie de sphère limite et l'extrémité de cet arc décrira une circonférence de cercle, qui se détermine sur la sphère limite de la même manière qu'une circonférence de rayon $s$ est déterminée dans son plan dans la géométrie ordinaire, d'où il suit que la circonférence doit être égale à $2\pi s$. De l'autre côté la circonférence du même cercle considérée dans son plan où la perpendiculaire $y$, abaissée d'une extrémité de l'arc $s$ sur l'axe de cercle limite qui sert d'axe des $x$ et passe par l'autre extrémité, est le rayon du cercle, sera donnée dans la Pangéométrie par la formule

$$2\pi \operatorname{cotg} \Pi(y)$$

d'où il suit que

$$s = \operatorname{cotg} \Pi(y)$$

comme il à été démontré auparavant.

Pour trouver l'élément des aires planes nous divisons le plan par des cercles limites qui tous ont pour axe l'axe des $x$ de manière que la distance de chaque cercle limite au suivant soit infiniment petite et puisse être exprimée par $dx$. Soit $s$ l'arc d'un de ces cercles limites compris entre l'axe des $x$ et un point d'une ligne courbe donnée dont les coordonnées soient $x, y$. Soit encore $s'$ l'arc d'un autre de ces ces cercles limites compris entre l'axe des $x$ et un point de la courbe donnée, déterminé par les coordonnées $x + dx$, $y + dy$.

La partie infiniment petite du plan comprise entre $s$ et $s'$ d'un côté et entre la courbe et l'axe des $x$ de l'autre côté aura pour expression

$$ds = \frac{es\,dx}{e - 1}$$

ou, en substituant $s = \operatorname{cotg} \Pi(y)$,

$$ds = \frac{e\,dx \operatorname{cotg} \Pi(y)}{e - 1}.$$

Comme exemple déterminons l'aire du cercle limite pour lequel nous avons trouvé l'équation en coordonnées rectangulaires

$$\sin \Pi(y) = e^{-x}$$

à l'aide de laquelle nous trouvons l'expression suivante de la différentielle de l'aire cherchée

$$ds = \frac{e}{e - 1} dy \cos \Pi(y) \operatorname{cotg} \Pi(y).$$

En intégrant cette expression depuis $y = o$, nous trouvons pour l'aire comprise entre l'arc de cercle limite l'axe des $x$ et l'ordonnée $y$

$$s = \frac{e}{e-1}\left\{\operatorname{cotg} \Pi(y) - \tfrac{1}{2}\pi + \Pi(y)\right\}.$$

Nous avons vu que la partie d'un plan comprise entre deux droites parallèles, prolongées indéfiniment du côté du parallélisme et limitée par un arc $s$ de cercle limite auquel les deux parallèles servent d'axes, a pour expression

$$\frac{es}{e-1} = \frac{e \operatorname{cotg} \Pi(y)}{e-1}.$$

Après quoi nous trouvons pour l'aire comprise entre deux droites parallèles, dont l'une perpendiculaire à $y$, menées par les deux extrémités de $y$ et prolongées indéfiniment du côté du parallélisme, la formule

$$\tfrac{1}{2}\pi - \Pi(y).$$

A l'aide de cette formule nous pouvons déterminer l'aire d'un triangle rectiligne en fonction des angles de ce triangle. Soient pour cela les côtés du triangle $a, b, c$ et les angles opposés $A = \Pi(\alpha)$, $B = \Pi(\beta)$, $\frac{\pi}{2}$; prolongeons l'hypotenuse $c$ au delà du sommet de l'angle $\Pi(\beta)$ et faisons le prolongement égal à $\beta$. La perpendiculaire à $\beta$ menée par l'extrémité de $\beta$ sera parallèle au prolongement du côté $a$ et l'aire de la partie du plan comprise entre ces deux parallèles prolongées indéfiniment du côté du parallélisme et limitée de l'autre côté par la ligne $\beta$ aura pour valeur

$$\tfrac{1}{2}\pi - \Pi(\beta)$$

Si nous menons maintenant par le sommet de l'angle $A$ une parallèle à la perpendiculaire qui sera par conséquence inclinée sur $c$ sous l'angle $\Pi(c+\beta)$ et sera aussi parallèle au prolongement de $a$, la valeur de la partie du plan entre $c+\beta$ et les deux parallèles prolongées à l'infini du côté du parallélisme sera

$$\tfrac{1}{2}\pi - \Pi(c+\beta).$$

De la même manière la partie du plan entre $b$, la droite menée par le sommet de $A$ et le côté $a$ avec son prolongement est

$$= \tfrac{1}{2}\pi - \Pi(b).$$

Après quoi la somme de $\frac{\pi}{2} - \Pi(\beta)$ et de $\frac{\pi}{2} - \Pi(b)$ diminuée de $\frac{\pi}{2} - \Pi(c+\beta)$ sera l'expression de l'aire du triangle, qui aura ainsi pour valeur

$$\tfrac{1}{2}\pi - \Pi(b) - \Pi(\beta) + \Pi(c+\beta).$$

Mais nous avons démontré que

$$\Pi(b) = \Pi(\alpha) + \Pi(c+\beta).$$

En substituant dans l'expression de l'aire du triangle rectiligne rectangle cette valeur à la place de $\Pi(b)$, l'expression de cette aire prend la forme suivante

$$\tfrac{1}{2}\pi - \Pi(\alpha) - \Pi(\beta)$$

c'est-à-dire que l'aire d'un triangle rectiligne rectangle est égale à la différence entre deux angles droits et la somme des trois angles du triangle; d'où il suit encore que l'aire de tout triangle rectiligne est égale à l'excés de deux angles droits sur la somme des trois angles du triangle. Cela suit de ce que l'aire de tout triangle rectiligne est la somme des aires de deux triangles rectilignes rectangles.

Il est facile de déduire de ce qui précède, que l'aire de tout quadrilatère est égale à l'excés de quatre angles droits sur la somme des quatre angles du quadrilatère et en général que l'aire de tout polygone de $n$ côtés est égale à l'excés de $(n-2)\pi$ sur la somme des angles du polygone.

Considérons en particulier un quadrilatère, dont deux côtés $a, y$ sont tous les deux perpendiculaires au troisieme côté $x$, et dont le quatrième côté $t$ est perpendiculaire au côté $a$ et fait avec $y$ un angle que nous désignons par $\omega$. Nous avons démontré plus haut (équation 25) qu'entre les parties constituantes d'un tel quadrilatère il existe l'équation suivante

$$\cos \Pi(a) = \cos \Pi(y) \sin \Pi(x).$$

Si nous considérons $x, y$ comme variables et $a$ comme constante, l'aire de ce quadrilatère s'exprime, comme toute aire plane, d'après ce qui a été démontré plus haut, par l'intégrale

$$\int dx \operatorname{cotg} \Pi(y)$$

qui appliquée au cas qui nous occupe, donne, en substituant la valeur de $\operatorname{cotg} \Pi(y)$, la valeur suivante de l'aire

$$\int_0 \frac{dx \cos \Pi(a)}{\sqrt{\sin^2 \Pi(x) - \cos^2 \Pi(a)}}$$

tandis que cette même aire est, en conséquence du théorème qui exprime l'aire de tout polygone plan en fonction des angles

$$= \tfrac{1}{2}\pi - \omega$$

ce qui donne

$$\tfrac{1}{2}\pi - \omega = \cos \Pi(a) \int_0 \frac{dx}{\sqrt{\cos^2 \Pi(x) - \cos^2 \Pi(a)}}. \qquad \text{(M)}$$

L'angle $\omega$, que le côté $t$ fait avec le côté $y$, est donné par l'équation suivante (équation 26)

$$\operatorname{tang} \omega = \frac{\operatorname{tang} \Pi(y)}{\cos \Pi(x)};$$

nous écrivons dans l'équation (M) $\alpha$ au lieu de $\Pi(a)$ et $\xi$ au lieu de $\Pi(x)$, elle deviendera:

$$\frac{\tfrac{1}{2}\pi - \omega}{\cos \alpha} = \int_{\frac{\pi}{2}} \frac{d\xi}{\sin \xi \sqrt{\sin^2 \alpha - \sin^2 \xi}}$$

où $\alpha$ est une quantité constante.

La justesse de la valeur trouvée pour cette intégrale peut etre vérifiée par la différentiation. La Pangéométrie indique ainsi une nouvelle méthode pour trouver les valeurs approchées des intégrales définies.

Soit donnée l'intégrale

$$\int A\, dx$$

où $A$ est une fonction donnée de $x$; pour calculer la valeur de cette intégrale il faut poser $A = \operatorname{cotg} \Pi(y)$ et déterminer les valeurs de $y', y'', y'''$ etc. qui correspondent à $x', x'', x'''\ldots$ prises arbitrairement dans les limites de l'intégration; après il faut calculer la longueur des cordes qui réunissent les sommets de $y'$ à $y''$, de $y''$ à $y'''$ etc. et ainsi de suite, et les angles que chaque corde fait avec le prolongement de la corde suivante. La somme de ces angles donnera la valeur approchée de l'intégrale.

L'aire de la partie du plan, comprise entre une droite donnée et deux droites parallèles entre elles, menées par les extrémités de la droite donnée et prolongées indéfiniment du côté du parallèlisme sera égal à $\pi$ moins la somme des deux angles que les deux parallèles font avec la droite donnée, parce que cette figure peut-être regardée comme un triangle dont un des angles serait nul.

L'aire d'une courbe plane peut-être divisée en éléments par des droites toutes parallèles à une droite donnée par exemple à l'axe des $y$. Si nous menons par l'extrémité de l'abscisse $x$ une droite parallèle à l'axe des $y$, cette droite fera avec l'axe des $x$ un angle $= \Pi(x)$; la droite menée par l'extrémité de l'abscisse $x + dx$ fera de même avec l'axe des $x$ un angle égal à $\Pi(x + dx)$, d'où il suit que l'aire de la partie du plan comprise entre $dx$ et ces deux parallèles sera égale à $-d\,\Pi(x)$. Soit maintenant $u$ la longueur de la partie de la première parallèle comprise entre l'axe des $x$ et la courbe, la partie de l'aire comprise entre les deux parallèles qui est hors de la courbe donnée sera d'après ce qui à été démontré plus haut:

$$-e^{-u}\,d\,\Pi(x)$$

d'où il suit que la partie de cette aire qui est située entre la courbe et l'axe des $x$, c'est-à-dire l'élément de l'aire de la courbe, aura pour expression:

$$dS = -(1-e^{-u})\,d\,\Pi(x).$$

Pour calculer l'aire d'un cercle de rayon $r$, il faut dans l'expression générale de l'élément de l'aire d'une courbe trouvée auparavant, expression qui était

$$dS = dx \cot g\,\Pi(y)$$

substituer la valeur de $\cot g\,\Pi(y)$ tirée de l'équation du cercle

$$\sin\Pi(x)\sin\Pi(y) = \sin\Pi(r)$$

où l'origine des coordonnées rectangulaire est au centre du cercle, cela donne

$$dS = dx\sqrt{\frac{\sin^2\Pi(x)}{\sin^2\Pi(r)} - 1}\,;$$

en intégrant depuis $x = o$, nous trouvons

$$S = \frac{1}{\sin\Pi(r)}\arc\sin\left(\frac{\cos\Pi(x)}{\cos\Pi(r)}\right) - \arc\sin\left(\frac{\cot g\,\Pi(x)}{\cot g\,\Pi(r)}\right).$$

Pour $x = r$, cela donne pour l'aire du quart du cercle

$$\frac{\pi}{2\sin\Pi(r)} - \tfrac{1}{2}\pi\,;$$

en multipliant par 4 nous trouvons pour l'aire du cercle

$$2\pi\left\{\frac{1}{\sin\Pi(r)} - 1\right\} = \pi\left(e^{\frac{1}{2}r} - e^{-\frac{1}{2}r}\right)^2.$$

Si $r$ est extrêmement petit cette expression donne l'aire du cercle $= \pi r^2$, ce qui est la même expression que la géométrie ordinaire fournit pour l'aire du cercle.

L'expression précédente de l'aire du cercle nous permet de donner à l'élément de l'aire de toute ligne courbe encore l'expression suivante:

$$dS = d\varphi\left\{\frac{1}{\sin\Pi(r)} - 1\right\}$$

où $r$ est le rayon vecteur mené de l'origine des coordonnées à un point de la courbe et $\varphi$ l'angle que ce rayon vecteur fait avec une droite fixe, qui passe par l'origine des coordonnées.

L'application de cette formule à la détermination de l'aire d'un triangle rectiligne, dont les côtés sont $a, b, c$ et les angles opposés $A, B, C$ donne, si nous regardons les angles $A, C$ et les côtés $b, a$ comme variables

$$\text{l'aire du triangle} = \int_0^A dA \left\{ \frac{1}{\sin \Pi(b)} - 1 \right\}.$$

le côté $b$ s'exprime en fonction de $c, A, B$ à l'aide de la dernière des équations (19)

$$\operatorname{cotg} B \sin A \sin \Pi(c) + \cos A = \frac{\cos \Pi(c)}{\cos \Pi(b)}.$$

Tirons de cette équation la valeur de $\sin \Pi(b)$ et substituons la dans l'expression de l'aire du triangle, il viendra

$$\text{l'aire du triangle} = S = \int_0^A \frac{dA}{\sqrt{1 - \frac{\cos^2 \Pi(c)}{(\operatorname{cotg}^2 B \sin A \sin \Pi(c) + \cos A)^2}}} - A.$$

Mais il a été démontré que l'aire du triangle est

$$= \pi - A - B - C$$

où $A$ et $B$ sont des angles donnés et $C$ est donné par l'équation (19)

$$\cos C + \cos A \cos B = \frac{\sin A \sin B}{\sin \Pi(c)}.$$

La comparaison de ces deux expressions de l'aire du triangle donne ensuite:

$$\pi - B - C = \int_0^A \frac{dA \left\{ \operatorname{cotg} B \sin A \sin \Pi(c) + \cos A \right\}}{\sqrt{(\operatorname{cotg} B \sin A \sin \Pi(c) + \cos A)^2 - \cos^2 \Pi(c)}}.$$

Si $B = \frac{\pi}{2}$ cette équation donne

$$\frac{\pi}{2} - C = \int_0^A \frac{dA}{\sqrt{\cos^2 A - \cos^2 \Pi(c)}}$$

équation qui, après l'intégration, prend la forme

$$\frac{\pi}{2} - C = \text{arc}\sin\left(\frac{\sin A}{\sin \Pi(c)}\right)$$

ce qui est d'accord avec l'équation qui détermine $C$.

On peut déduire de ce qui précède deux expressions de la valeur de l'aire de tout polygone fermé, l'une exprimée par une intégrale définie, l'autre dépendant seulement de la somme des angles du polygone. Les deux valeurs de la même aire doivent être égales entre elles, on a de cette manière un nouveau moyen de trouver la valeur de beaucoup d'intégrales définies, valeurs qu'il serait souvent difficile de trouver d'une autre manière.

Pour en donner encore nn exemple considérons un triangle rectiligne rectangle, qui a pour côtés de l'angle droit $x, y$ et pour hypothenuse $r$. Soit $A$ l'angle opposé à $y$ et $B$ l'angle opposé à $x$. Les équations (10), (11) donnent pour ce triangle

$$\begin{aligned} \sin \Pi(x) \sin \Pi(y) &= \sin \Pi(r) \\ \sin \Pi(x) \cos B &= \sin A \\ \cos \Pi(r) \cos A &= \cos \Pi(x) \\ \cos \Pi(r) \cos B &= \cos \Pi(y) \end{aligned}$$

De ces équations nous déduisons

$$\cos \Pi(r) = \frac{\cos \Pi(x)}{\cos A}$$

$$\sin \Pi(r) = \sqrt{1 - \left(\frac{\cos \Pi(x)}{\cos A}\right)^2}$$

$$\sin \Pi(y) = \frac{1}{\sin \Pi(x)}\sqrt{1 - \left(\frac{\cos \Pi(x)}{\cos A}\right)^2} = \sqrt{\frac{1}{\sin^2 \Pi(x)} - \frac{\operatorname{cotg}^2 \Pi(x)}{\cos^2 A}}$$

$$\operatorname{ctg} \Pi(y) = \frac{\sin A \cos \Pi(x)}{\sqrt{\sin^2 A - \cos^2 \Pi(x)}}.$$

En substituant dans cette dernière équation $\Pi(x) = \frac{\pi}{2} - \omega$, nous trouvons

$$\operatorname{cotg} \Pi(y) = \frac{\sin A \sin \omega}{\sqrt{\sin^2 A - \sin^2 \omega}}.$$

Mais nous avons vu que la différentielle de l'aire est $dx \cot g\, \Pi(y)$ ce qui donne, étant appliqué au cas actuel

$$dx \cot g\, \Pi(y) = \sin A \frac{d\omega \tang \omega}{\sqrt{\sin^2 A - \sin^2 \omega}}$$

d'où nous concluons, en intégrant depuis $\omega = o$, ce qui correspond à $x = o$, et en remarquant que l'aire exprimée par l'intégrale est aussi exprimée par $\frac{\pi}{2} - A - B$, que

$$\frac{\pi}{2} - A - B = \sin A \int_o^\omega \frac{\tang \omega \, d\omega}{\sqrt{\sin^2 A - \sin^2 \omega}}$$

où $A$ est un angle constant tandis que $B$ est déterminée par l'équation

$$\cos B = \frac{\sin A}{\cos \omega}.$$

Si $\omega = \frac{\pi}{2}$, l'hypothenuse devient parallèle au côté $y$ et l'angle $B$ sera égal à zero. On a donc dans ce cas

$$\frac{\pi}{2} - A = \sin A \int_o^{\frac{\pi}{2}-A} \frac{d\omega \tang \omega}{\sqrt{\sin^2 A - \sin^2 \omega}}.$$

On peut déterminer la valeur d'une intégrale plus générale, en considérant l'aire d'un triangle rectiligne quelconque, dont les côtés sont $a, b, c$ et les angles opposés $A, B, C$ et en divisant cette aire en éléments par des droites parallèles entre elles. Prenons le sommet de l'angle $C$ pour origine des coordonnées et le coté $a$ pour axe des abscises $x$. Soit $B = \Pi(\beta)$ où $\beta$ est une ligne positive si $B < \frac{\pi}{2}$ et négative si $B > \frac{\pi}{2}$. Menons par l'extrémité de l'abscisse $x$ une droite $u$ parallèle au côté $c$ et prolongeons cette parallèle jusqu'à ce qu'elle coupe le côté $b$. L'angle que cette parallèle fait avec l'abscisse $x$ sera $\Pi(\beta - a + x)$ d'où il suit que l'angle que cette parallèle fait avec le prolongement de $x$ sera $\Pi(a - \beta - x)$.

Si nous prenons pour élément de l'aire du triangle la partie de cette aire qui est comprise entre deux parallèles $u$ infiniment voisines nous aurons, d'après ce qui a été démontré plus haut, l'expression suivante pour cet élément

$$dS = - d\Pi(a - \beta - x)(1 - e^{-u}).$$

Regardons $y, x$ et $u$ comme variables et $a$ et $\beta$ comme constants. Les équations (19) appliquées au triangle dont les côtés sont $x$, $u$ et l'angle entre ces deux côtés $\Pi(\beta - a + x)$ donnent

$$\operatorname{cotg} C \sin \Pi(\beta - a + x) \sin \Pi(x) + \cos \Pi(\beta - a + x) = \frac{\cos \Pi(x)}{\cos \Pi(u)};$$

de cette équation nous tirons en posant pour abréger $\Pi(\beta - a + x) = \omega$

$$\cos \Pi(u) = \frac{\cos \Pi(x)}{\operatorname{cotg} C \sin \omega \sin \Pi(x) + \cos \omega}$$

puis

$$e^{2u} = \frac{\operatorname{cotg} C \sin \omega \sin \Pi(x) + \cos \omega + \cos \Pi(x)}{\operatorname{cotg} C \sin \omega \sin \Pi(x) + \cos \omega - \cos \Pi(x)}.$$

Mais

$$\sin \Pi(x) = \sin \Pi \left\{(\beta - a) - (\beta - a + x)\right\} = \frac{\sin \Pi(\beta - a) \sin \omega}{1 - \cos \Pi(\beta - a) \cos \omega};$$

de la même manière nous trouvons

$$\cos \Pi(x) = \frac{\cos \Pi(\beta - a) - \cos \omega}{1 - \cos \Pi(\beta - a) \cos \omega}.$$

La substitution de ces valeurs de $\sin \Pi(x)$, $\cos \Pi(x)$ dans l'expression de $e^{2u}$ donne

$$e^{2u} =$$

$$\frac{\operatorname{ctg} C \sin^2 \omega \sin \Pi(\beta - a) + \cos \omega \left\{1 - \cos \Pi(\beta - a) \cos \omega\right\} + \cos \Pi(\beta - a) - \cos \omega}{\operatorname{ctg} C \sin^2 \omega \sin \Pi(\beta - a) + \cos \omega \left\{1 - \cos \Pi(\beta - a) \cos \omega\right\} - \cos \Pi(\beta - a) + \cos \omega}$$

$$= \frac{\operatorname{ctg} C \sin^2 \omega \sin \Pi(\beta - a) + \cos \Pi(\beta - a) \sin^2 \omega}{\operatorname{ctg} C \sin^2 \omega \sin \Pi(\beta - a) + 2 \cos \omega - \left\{1 + \cos^2 \omega\right\} \cos \Pi(\beta - a)}$$

et nous trouvons ensuite

$$d\, \Pi(a - \beta - x) = -d\, \Pi(\beta - a + x) = -d\omega$$

après quoi la comparaison des deux expressions de l'aire du triangle donne l'équation

$$\pi - A - B - C = -\omega +$$

$$\int_{x=0}^{x=a} d\omega \sqrt{\frac{\operatorname{cotg} C \sin \Pi(\beta - a) \sin^2 \omega + 2 \cos \omega - (1 + \cos^2 \omega) \cos \Pi(\beta - a)}{\operatorname{cotg} C \sin \Pi(\beta - a) \sin^2 \omega + \cos \Pi(\beta - a) \sin^2 \omega}}.$$

Si nous posons encore $\Pi(\beta - a) = \alpha$, cette équation prendra la forme

$$[\pi - A - B - C]\,[\operatorname{cotg} C \sin \alpha + \cos \alpha] = \int_{\omega=\alpha}^{\omega=\Pi(\beta)} \frac{d\omega}{\sin \omega} \sqrt{\operatorname{cotg} C \sin \alpha \sin^2 \omega + 2 \cos \omega - (1 + \cos^2 \omega) \cos \alpha}$$

où les angles $A, B$ et la ligne $\beta$ doivent être calculés au moyen des équations

$$\alpha = \Pi(\beta - a), \quad B = \Pi(\beta)$$

$$\cos A + \cos B \cos C = \frac{\sin B \sin C}{\sin \Pi(a)}$$

dont la dernière est la dernière des équations (19), appliquée au triangle que nous avons considéré.

On peut employer dans la Pangéométrie pour fixer la position d'un point, hormis les coordonnées rectilignes et polaires, des arcs de cercles limite et ce dernier système offre même beaucoup d'avantage sous le rapport de la simplicité des formules.

Déterminons la position d'un point dans un plan par des coordonnées rectangulaires $x, y$ de manière que $y$ soit la longueur de la perpendiculaire abaissée du point, dont on veut déterminer la position, sur l'axe des $x$ et $x$ la distance du pied de la perpendiculaire $y$ de l'origine des coordonnées. Soit $\eta$ la longueur de l'arc de cercle limite compris entre le sommet de la perpendiculaire $y$ et l'axe des $x$, qui est en même temps l'axe du cercle limite et nommons $\xi$ la distance du sommet du cercle limite, qui est situé sur l'axe des $x$, à l'origine des coordonnées. Nous avons vu que dans ce cas

$$\eta = \operatorname{cotg} \Pi(y)$$

ensuite l'équation du cercle limite donne

$$e^{-(x-\xi)} = \sin \Pi(y)$$

à l'aide de ces deux équations on peut exprimer $\xi, \eta$ en fonction de $x, y$ ou inversement $x, y$ en fonction de $\xi, \eta$, ce qui permet de passer de l'équation d'une ligne exprimée en $x, y$ à l'équation de cette même ligne exprimée en $\xi, \eta$ ou inversement.

La différentielle des aires planes s'exprime en $\xi, \eta$ par l'équation

$$d^2 S = d\xi\, d\eta$$

où $S$ est l'aire.

Si nous regardons $S$ comme fonction de $x, y$ nous avons

$$\left(\frac{dS}{dx}\right) = \frac{dS}{d\xi}$$

puis en différentiant par rapport à $y$

$$\frac{d^2 S}{dx\, dy} = \frac{1}{\sin \Pi(y)} \frac{d^2 S}{d\xi\, d\eta} = \frac{1}{\sin \Pi(y)}$$

ce qui s'accorde avec ce que nous avons trouvé plus haut.

Abaissons d'un point dans l'espace une perpendiculaire $z$ sur le plan des coordonnées $x, y$ et menons par cette perpendiculaire un plan qui coupe le plan $x, y$ en une droite parallèle à l'axe des $x$. Prenons cette intersection, dirigée du côté du parallèlisme pour axe d'un cercle limite qui passe par le sommet de la perpendiculaire $z$ et soit $\zeta$ la longeur de l'arc de ce cercle limite compris entre le sommet de $z$ et cet axe. On a

$$\zeta = \operatorname{cotg} \Pi(z);$$

la partie $q$ de la parallèle à l'axe des $x$ menée par le pied de la perpendiculaire $z$ et comprise entre le sommet de $\zeta$ et le pied de cette perpendiculaire, sera donné par l'équation

$$e^{-q} = \sin \Pi(z).$$

L'arc de cercle limite mené par le pied de $z$ de manière à avoir l'axe des $x$ dirigé du côté des $x$ positifs pour axe et compris entre ce point et cet axe aura pour longueur $\operatorname{cotg} \Pi(y)$ et l'arc $\eta$ de cercle limite, mené par le point d'intersection de $\zeta$ avec le plan des $x, y$ ayant l'axe des $x$ dirigé du côté des $x$ positifs pour axe et compris entre ce point et cet axe sera, en vertu de ce qui a été démontré, donné par l'équation

$$\eta = \frac{\operatorname{cotg} \Pi(y)}{\sin \Pi(z)}.$$

Si nous appelons encore $\xi$ la partie de l'axe des $x$ comprise entre l'origine des coordonnées et l'arc $\eta$, l'équation du cercle limite donne

$$e^{-x+\xi+q} = \sin \Pi(y).$$

De ces équations nous tirons en ne faisant varier d'abord que $z$ et $\zeta$ qui en dépend

$$d\zeta = \frac{dz}{\sin \Pi(z)}.$$

En ne faisant varier que $y$ et $\eta$, il vient

$$d\eta = \frac{dy}{\sin \Pi(y) \sin \Pi(z)}.$$

Enfin en ne faisant varier que $\xi$ et $x$, il vient

$$d\xi = dx.$$

Il ne reste, pour compléter la nouvelle théorie géométrique désignée Pangéométrie, et qui est basée sur des nouveaux principes plus généraux que ceux de la géométrie ordinaire, qu'à donner les valeurs des différentielles de l'aire d'une surface courbe et du volume d'un corps quelconque éxprimées à l'aide de coordonnées qui déterminent la position d'un point dans l'espace.

Considérons dans ce but de nouveau le quadrilatère dont deux côtés $a, y$ sont perpendiculaires au troisième $x$ et dont le quatrieme côté $c$ est perpendiculaire à $y$ et fait avec $a$ l'angle $\varphi$.

Nous avons trouvé (équation 25)

$$\cos \Pi(y) = \cos \Pi(a) \sin \Pi(x).$$

Puis nous trouvons à l'aide des équations (10), (11) en nommant $r$ la diagonale menée du sommet de l'angle $\varphi$ au sommet de l'angle droit opposé et $A$ l'angle entre $x$ et $r$

$$\cos \Pi(r) \cos A = \cos \Pi(x)$$

$$\cos A \operatorname{tang} \Pi(c) = \operatorname{tang} \Pi(r).$$

De ces deux équations nous tirons

$$\cos \Pi(x) \operatorname{tang} \Pi(c) = \sin \Pi(r).$$

Mais

$$\sin \Pi(r) = \sin \Pi(a) \sin \Pi(x)$$

et par conséquence

$$\operatorname{tang} \Pi(c) = \sin \Pi(a) \operatorname{tang} \Pi(x).$$

Si $c$ et $x$ sont si petits qu'on puisse négliger les puissances supérieures devant les inférieures et prendre pour valeurs approchées de $\operatorname{tang} \Pi(c)$, $\operatorname{tang} \Pi(x)$ les suivantes

$$\operatorname{tang} \Pi(c) = \tfrac{1}{c}; \ \operatorname{tang} \Pi(x) = \tfrac{1}{x}$$

on trouve

$$c = \frac{x}{\sin \Pi(a)}. \qquad (27)$$

La droite $c$ qui joint les sommet de $a$ et de $y$ ne sera pas perpendiculaire à $y$ si $a = y$ dans le quadrilatère. Dans ce cas la droite $p$ qui joint le milieu de $c$ au milieu de $x$ sera perpendiculaire à $c$

et à $x$. Nous pouvons donc remplacer dans l'équation (27) $c$ par $\frac{1}{2}c$ et $x$ par $\frac{1}{2}x$ ce qui ne change pas la forme de cette équation. Elle est ainsi démontrée même pour le cas $a = y$, cas auquel la démonstration donnée plus haut n'est pas immédiatement applicable.

Les aires des surfaces courbes ont pour mesure la somme des aires des triangles, qui forment un reseau continu dont tous les sommets sont situés sur la surface. Cette mesure sera d'autant plus exacte que les dimensions des triangles seront plus petites.

La limite de laquelle cette somme s'approche indéfiniment si les dimensions des triangles diminuent indéfiniment et de laquelle elle peut différer d'une grandeur moindre que toute grandeur donnée est dite la valeur mathématique de l'aire de la surface Déterminons d'abord l'aire d'un triangle rectiligne rectangle en fonction des côtés, que nous désignerons par $a, b, c$, nommons les angles opposés à ces côtés respectivement $\Pi(\alpha)$, $\Pi(\beta)$, $\frac{\pi}{2}$.

Nous avons vu qu'on peut, dans un tel triangle, substituer à

$$a, b, c, \alpha, \beta$$

les lignes

$$a, \alpha', \beta, b', c$$

respectivement.

Outre cela nous avons trouvé que

$$2\,\Pi(b) = \Pi(c+\beta) + \Pi(c-\beta);$$

substituons dans cette équation $\alpha'$ à $b, \beta$, à $c$ et $c$ à $\beta$, il viendra

$$\pi - 2\,\Pi(\alpha) = \Pi(\beta + c) + \Pi(\beta - c)$$

ou

$$2\,\Pi(\alpha) = \Pi(c-\beta) - \Pi(c+\beta).$$

De la même manière nous trouvons

$$2\,\Pi(\beta) = \Pi(c-\alpha) - \Pi(c+\alpha).$$

En échangeant dans cette dernière équation les lettres comme il a été dit plus haut on aura

$$2\,\Pi(c) = \Pi(\beta - b') - \Pi(\beta + b')$$

De la même manière on a

$$2\,\Pi(c) = \Pi(\alpha - a') - \Pi(a + a')$$

d'où nous déduisons par l'échange de lettres indiqué plus haut

$$2\,\Pi(\beta) = \Pi(b' - a') - \Pi(b' + a')$$

De la même manière on a

$$2\,\Pi(\alpha) = \Pi(a' - b') - \Pi(a' + b');$$

en ajoutant les deux dernières équations nous trouvons

$$2\,\Pi(\alpha) + 2\,\Pi(\beta) = \pi - 2\,\Pi(a' + b'),$$

après quoi l'aire du triangle $\triangle$ est donnée par l'expression suivante:

$$\triangle = \frac{\pi}{2} - \Pi(\alpha) - \Pi(\beta) = \Pi(a' + b')$$

et en suite

$$\operatorname{tang} \tfrac{1}{2}\triangle = e^{-a'} e^{-b'} =$$
$$= \operatorname{tang}\left[\tfrac{1}{4}\pi - \tfrac{1}{2}\Pi(a)\right] \operatorname{tang}\left[\tfrac{1}{4}\pi - \tfrac{1}{2}\Pi(b)\right]$$

d'où nous tirons enfin

$$\operatorname{tang} \tfrac{1}{2}\triangle = \frac{e^a - 1}{e^a + 1} \cdot \frac{e^b - 1}{e^b + 1}.$$

Si $a$ et $b$ sont très petits, de sorte qu'on puisse négliger les puissances supérieures de $a$, $b$ et $\triangle$ cette formule donne

$$\triangle = \tfrac{1}{2} ab$$

comme dans la géométrie ordinaire. On sait qu'on peut toujours choisir dans un triangle rectiligne quelconque le côté $c$ de manière, que la perpendiculaire abaissée du sommet de l'angle opposé $C$ sur la direction de ce côté, tombe sur le côté $c$ lui même et non pas sur son prolongement; cette perpendiculaire divise le côté $c$ en deux parties, l'une $x$ adjacente à l'angle $A$, l'autre $c - x$ adjacente à l'angle $B$. L'aire $S$ de ce triangle sera égale à la somme des aires des deux triangles rectangles formés par cette perpendiculaire et sera donnée par l'équation

$$\operatorname{tang} \tfrac{1}{2} S = \frac{\dfrac{e^x - 1}{e^x + 1} \cdot \dfrac{e^h - 1}{e^h + 1} + \dfrac{e^{c-x} - 1}{e^{c-x} + 1} \cdot \dfrac{e^h - 1}{e^h + 1}}{1 - \dfrac{e^x - 1}{e^x + 1} \cdot \dfrac{e^{c-x} - 1}{e^{c-x} + 1} \cdot \left(\dfrac{e^h - 1}{e^h + 1}\right)^2}$$

équation à laquelle on peut donner la forme

$$\operatorname{tang} \tfrac{1}{2} S = \frac{(e^{2h} - 1)(e^c - 1)}{(e^c + 1)(e^h + 1)^2 + 2e^h (e^x - 1)(e^{c-x} - 1)};$$

cette formule donne, si l'on néglige les puissances supérieures de $s$, $h$, $c$ vis à vis des inférieures

$$S = \tfrac{1}{2} ch$$

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comme dans la géométrie ordinaire. Nous avons vu que l'expression de l'aire d'un triangle en fonction des trois angles $A$, $B$, $C$ du triangle était

$$S = \pi - A - B - C.$$

Tirons la valeur de $A$ en fonction de $a$, $b$, $c$ de la seconde des équations (19) Cela donne l'équation

$$\cos A = \frac{1 - \dfrac{\sin \Pi(b) \sin \Pi(c)}{\sin \Pi(a)}}{\cos \Pi(b) \cos \Pi(c)}$$

de laquelle il suit

$$2 \cos^2 \tfrac{1}{2} A = \frac{1 + \cos \Pi(b) \cos \Pi(c) - \dfrac{\sin \Pi(b) \sin \Pi(c)}{\sin \Pi(a)}}{\cos \Pi(b) \cos \Pi(c)}.$$

Si nous substituons dans cette formule à la place de

$$1 + \cos \Pi(b) \cos \Pi(c)$$

sa valeur

$$\frac{\sin \Pi(b) \sin \Pi(c)}{\sin \Pi(b+c)}$$

elle prend la forme

$$2 \cos^2 \tfrac{1}{2} A = \operatorname{tang} \Pi(b) \operatorname{tang} \Pi(c) \left\{ \frac{1}{\sin \Pi(b+c)} - \frac{1}{\sin \Pi(a)} \right\};$$

de la même manière on trouve

$$-2 \sin^2 \tfrac{1}{2} A = \operatorname{tang} \Pi(b) \operatorname{tang} \Pi(c) \left\{ \frac{1}{\sin \Pi(b-c)} - \frac{1}{\sin \Pi(a)} \right\};$$

de ces deux formules nous déduisons

$$\sin^2 A = \operatorname{tang}^2 \Pi(b) \operatorname{tang}^2 \Pi(c) \left\{ - \frac{1 - \cos^2 \Pi(b) \cos^2 \Pi(c)}{\sin^2 \Pi(b) \sin^2 \Pi(c)} + \frac{2}{\sin \Pi(a) \sin \Pi(b) \sin \Pi(c)} - \frac{1}{\sin^2 \Pi(a)} \right\}$$

ou

$$\sin^2 A = - \operatorname{tang}^2 \Pi(b) \operatorname{tang}^2 \Pi(b) \left\{ \frac{1}{\sin^2 \Pi(a)} + \frac{1}{\sin^2 \Pi(b)} + \frac{1}{\sin^2 \Pi(c)} - \frac{2}{\sin \Pi(a) \sin \Pi(b) \sin \Pi(c)} - 1 \right\}.$$

En posant pour abreger

$$P = \sqrt{\frac{-1}{\sin^2 \Pi(a)} - \frac{1}{\sin^2 \Pi(b)} - \frac{1}{\sin^2 \Pi(c)} + \frac{2}{\sin \Pi(a) \sin \Pi(b) \sin \Pi(c)} + 1}$$

on a $$\sin A = \operatorname{tang} \Pi(b) \operatorname{tang} \Pi(c)\, P. \tag{28}$$

On peut aussi donner à $P$ la forme suivante:

$$P^2 = 2\left\{1 + \frac{1}{\sin \Pi(a)}\right\}\left\{1 + \frac{1}{\sin \Pi(b)}\right\}\left\{1 + \frac{1}{\sin \Pi(c)}\right\}$$
$$- \left\{1 + \frac{1}{\sin \Pi(a)} + \frac{1}{\sin \Pi(b)} + \frac{1}{\sin \Pi(c)}\right\}^2$$

symétrique par rapport à $a, b, c$

En partant de l'équation (28) et en y regardant $P$ comme une quantité indéterminée on peut prouver de la manière suivante que $P$ doit être une fonction symétrique par rapport à $a, b, c$.

Multiplions l'équation (28) par $\operatorname{tang} \Pi(a)$, substituons y à $\sin A \operatorname{tang} \Pi(a)$ sa valeur $\sin B \operatorname{tang} \Pi(b)$ tirée de l'équation (13) et divisons après par $\operatorname{tang} \Pi(b)$, il viendra

$$\sin B = \operatorname{tang} \Pi(a) \operatorname{tang} \Pi(c)\, P.$$

Multiplions cette dernière équation par $\operatorname{tang} \Pi(b)$, substituons y à $\sin B \operatorname{tang} \Pi(b)$ sa valeur $\sin C \operatorname{tang} \Pi(c)$ tirée de l'équation (13) et divisons après par $\operatorname{tang} \Pi(c)$, nous aurons

$$\sin C = \operatorname{tang} \Pi(a) \operatorname{tang} \Pi(b)\, P,$$

cela démontre que $P$ est une fonction symétrique des côtés $a, b, c$.

Nous avons dejà trouvé

$$\cos A = \frac{1 - \dfrac{\sin \Pi(b) \sin \Pi(c)}{\sin \Pi(a)}}{\cos \Pi(b) \cos \Pi(c)}$$

ou ce qui est la même chose

$$\cos A = \operatorname{tang} \Pi(b) \operatorname{tang} \Pi(c)\left\{\frac{1}{\sin \Pi(b) \sin \Pi(c)} - \frac{1}{\sin \Pi(a)}\right\};$$

de la même manière on trouve

$$\cos B = \operatorname{tang} \Pi(c) \operatorname{tang} \Pi(a)\left\{\frac{1}{\sin \Pi(c) \sin \Pi(a)} - \frac{1}{\sin \Pi(b)}\right\}$$

$$\cos C = \operatorname{tang} \Pi(a) \operatorname{tang} \Pi(b)\left\{\frac{1}{\sin \Pi(a) \sin \Pi(b)} - \frac{1}{\sin \Pi(c)}\right\}.$$

De ces valeurs de $\sin A$, $\cos A$, $\sin B$, $\cos B$ nous déduisons

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$= \operatorname{tang} \Pi(b) \operatorname{tang}^2 \Pi(c) \operatorname{tang}(a) P \left\{ \frac{1}{\sin \Pi(c) \sin \Pi(a)} - \frac{1}{\sin \Pi(b)} \right\}$$

$$+ \operatorname{tang}^2 \Pi(c) \operatorname{tang} \Pi(a) \operatorname{tang} \Pi(b) P \left\{ \frac{1}{\sin \Pi(b) \sin \Pi(c)} - \frac{1}{\sin \Pi(a)} \right\}$$

$$= \operatorname{tang} \Pi(a) \operatorname{tang} \Pi(b) \operatorname{tang}^2 \Pi(c) P \left\{ \frac{1}{\sin \Pi(a)} + \frac{1}{\sin \Pi(b)} \right\} \left\{ \frac{1}{\sin \Pi(c)} - 1 \right\}$$

et enfin

$$\sin(A+B) = \frac{\operatorname{tang} \Pi(a) \operatorname{tang} \Pi(b) P}{\left\{ \frac{1}{\sin \Pi(c)} + 1 \right\}} \left\{ \frac{1}{\sin \Pi(a)} + \frac{1}{\sin \Pi(b)} \right\}.$$

La troisième des équations (19) donne

$$\cos A + \cos(B+C) = \sin B \sin C \left\{ \frac{1}{\sin \Pi(a)} - 1 \right\}$$

Substituons dans cette équation à la place de $\sin B$, $\sin C$ leurs valeurs tirées de l'équation (28), elle donnera

$$\cos(B+C) = -\cos A + \operatorname{tang} \Pi(c) \operatorname{tang}^2 \Pi(a) \operatorname{tang} \Pi(b) P^2 \left\{ \frac{1}{\sin \Pi(a)} - 1 \right\}$$

ou ce qui est la même chose

$$\cos(B+C) = -\cos A + \frac{\operatorname{tang} \Pi(b) \operatorname{tang} \Pi(c) P^2}{\frac{1}{\sin \Pi(a)} + 1}.$$

A l'aide des formules précédentes nous trouvons

$$\cos(A+B+C) = \cos A \cos(B+C) - \sin A \sin(B+C)$$

$$= -\cos^2 A + \frac{\operatorname{tang}^2 \Pi(b) \operatorname{tang}^2 \Pi(c) P^2}{\frac{1}{\sin \Pi(a)} + 1} \left\{ \frac{1}{\sin \Pi(b) \sin \Pi(c)} - \frac{1}{\sin \Pi(a)} \right\}$$

$$- \frac{\operatorname{tang}^2 \Pi(b) \operatorname{tang}^2 \Pi(c) P^2}{\frac{1}{\sin \Pi(a)} + 1} \left\{ \frac{1}{\sin \Pi(b)} + \frac{1}{\sin \Pi(c)} \right\}.$$

$$2\cos^2 \tfrac{1}{2}(A+B+C) = \sin^2 A + \frac{\operatorname{tg}^2 \Pi(b) \operatorname{tg}^2 \Pi(c) P^2}{\frac{1}{\sin \Pi(a)} + 1} \left\{ \frac{1}{\sin \Pi(b) \sin \Pi(c)} - \frac{1}{\sin \Pi(a)} \right\}$$

$$- \frac{\operatorname{tang}^2 \Pi(b) \operatorname{tang}^2 \Pi(c) P^2}{\frac{1}{\sin \Pi(a)} + 1} \left\{ \frac{1}{\sin \Pi(b)} + \frac{1}{\sin \Pi(c)} \right\}.$$

$$2\cos^2 \tfrac{1}{2}(A+B+C) = \operatorname{tang}^2 \Pi(b) \operatorname{tang}^2 \Pi(c) P^2$$

$$+\frac{\operatorname{tang}^2\Pi(b)\operatorname{tang}^2\Pi(c)P^2}{\frac{1}{\sin\Pi(a)}+1}\left\{\frac{1}{\sin\Pi(b)\sin\Pi(c)}-\frac{1}{\sin\Pi(a)}-\frac{1}{\sin\Pi(b)}-\frac{1}{\sin\Pi(c)}\right\}$$

$$=\frac{\operatorname{tang}^2\Pi(b)\operatorname{tang}^2\Pi(c)P^2}{\frac{1}{\sin\Pi(a)}+1}\left\{\frac{1}{\sin\Pi(b)\sin\Pi(c)}-\frac{1}{\sin\Pi(b)}-\frac{1}{\sin\Pi(c)}+1\right\}$$

$$=\operatorname{tg}^2\Pi(a)\operatorname{tg}^2\Pi(b)\operatorname{tg}^2\Pi(c)P^2\left(\frac{1}{\sin\Pi(a)}-1\right)\left(\frac{1}{\sin\Pi(b)}-1\right)\left(\frac{1}{\sin\Pi(c)}-1\right)$$

Mais il a été démontré que l'aire du triangle $\triangle = \pi - A - B - C$, par conséquence

$$\sin\frac{\triangle}{2} = \frac{1}{\sqrt{2}} \operatorname{tang}\Pi(a) \operatorname{tang}\Pi(b) \operatorname{tang}\Pi(c) P \times$$

$$\sqrt{\left(\frac{1}{\sin\Pi(a)}-1\right)\left(\frac{1}{\sin\Pi(b)}-1\right)\left(\frac{1}{\sin\Pi(c)}-1\right)}.$$

Si $a, b, c$ sont très petits de manière qu'on puisse poser avec une approximation suffisante

$$\frac{1}{\sin\Pi(a)} = 1+\tfrac{1}{2}a^2;\quad \frac{1}{\sin\Pi(b)} = 1+\tfrac{1}{2}b^2$$

$$\frac{1}{\sin\Pi(c)} = 1+\tfrac{1}{2}c^2;\quad \operatorname{tang}\Pi(a) = \tfrac{1}{a}(1-\tfrac{1}{2}a^2)$$

$$\operatorname{tng}\Pi(b) = \tfrac{1}{b}(1-\tfrac{1}{2}b^2);\ \operatorname{tang}\Pi(c) = \tfrac{1}{c}(1-\tfrac{1}{2}c^2)$$

il viendra

$$\sin\tfrac{1}{2}\triangle = \tfrac{1}{4}\sqrt{\frac{a^2+b^2+c^2}{2}}$$

ou en rejetant les puissances de $\triangle$ supérieures à la première

$$\triangle = \frac{1}{2\sqrt{2}}\sqrt{a^2+b^2+c^2}.$$

Déterminons la position d'un point dans l'espace par trois coordonnées rectangulaires: $z$ perpendiculaire au plan de $xy$, $y$ perpendiculaire abaissée du pied de $z$ sur l'axe des $x$ et $x$ partie de l'axe des $x$ comprise entre l'origine des coordonnées et le pied de $y$. Prenons sur la surface courbe, dont il s'agit de déterminer l'élément de l'aire, trois points et soient les coordonnées du premier point $x, y, z$, les coordonnées du second point

$$x + dx,\ y,\ z + \left(\frac{dz}{dx}\right) dx$$

et les coordonnées du troisième point

$$x,\ y + dy,\ z + \left(\frac{dz}{dy}\right) dy.$$

Nommons $t$ la distance entre les sommets de deux perpendiculaires à l'axe des $x$ égales à $y$, qui interceptent entre elles une partie $dx$ de cet axe. En supposant $dx$, $dy$ infiniment petits nous aurons en vertu de l'équation (27)

$$t = \frac{dx}{\sin \Pi(y)}.$$

La distance des deux premiers points pris sur la surface courbe forme un triangle avec les droites dont les longueurs sont:

$$\frac{dx}{\sin \Pi(y) \sin \Pi(z)},\ \left(\frac{dz}{dx}\right) dx.$$

Nous pouvons considérer ce triangle à cause de la petitesse de ses côtés comme un triangle dont l'hypoténuse est la distance entre les premiers deux points pris sur la surface. Nous aurons donc pour le carré de cette distance

$$dx^2 \left\{ \frac{1}{\sin^2 \Pi(y) \sin^2 \Pi(z)} + \left(\frac{dz}{dx}\right)^2 \right\}.$$

De la même manière nous trouvons pour le carré de la distance du premier point au troisième

$$dy^2 \left\{ \frac{1}{\sin^2 \Pi(z)} + \left(\frac{dz}{dy}\right)^2 \right\}$$

et pour la distance du second point au troisième

$$\frac{dx^2}{\sin^2 \Pi(y) \sin^2 \Pi(z)} + \frac{dy^2}{\sin^2 \Pi(z)} + \left\{ \left(\frac{dz}{dy}\right) dy - \left(\frac{dz}{dx}\right) dx \right\}^2.$$

L'aire du triangle, dont les côtés sont les distances du premier point pris sur la surface courbe au second, du second au troisième et du troisième au premier et la somme des trois angles duquel sera sensiblement égale à $\pi$, à raison de la petitesse des côtés, sera en vertu de la formule démontrée plus haut et des valeurs que nous avons trouvées pour les carrés de ses côtés

$$\frac{d^2S}{dx\,dx}=\frac{1}{2\sin\Pi(z)}\sqrt{\left(\frac{dz}{dx}\right)^2+\frac{1}{\sin^2\Pi(y)}\left(\frac{dz}{dy}\right)^2+\frac{1}{\sin^2\Pi(y)\sin^2\Pi(z)}}$$

ce qui est l'élément de l'aire de la surface courbe dont l'équation est

$$z=f(x,y).$$

Appliquons cette expression à une sphère de rayon $r$. Si l'origine des coordonnées est au centre de la sphère, l'équation de la sphère donnera:

$$\left(\frac{dz}{dx}\right)=-\frac{\cos\Pi(x)}{\cos\Pi(z)}$$

$$\left(\frac{dz}{dy}\right)=-\frac{\cos\Pi(y)}{\cos\Pi(z)}$$

et ensuite

$$\frac{\cos\Pi(r)}{\sin^2\Pi(r)}\cdot\frac{\sin\Pi(y)\sin^2\Pi(x)}{\sqrt{\sin^2\Pi(x)\sin^2\Pi(y)-\sin^2\Pi(r)}}=\frac{d^2S}{d\,\Pi(x)\,d\,\Pi(y)}.$$

Multiplions par $d\,\Pi(y)$ et intégrons depuis $\sin\Pi(y)=\dfrac{\sin\Pi(r)}{\sin\Pi(x)}$ jusqu'à $\Pi(y)=\frac{1}{2}\pi$, il viendra:

$$\frac{dS}{d\,\Pi(x)}=2\pi\sin\Pi(x)\frac{\cos\Pi(r)}{\sin^2\Pi(r)}.$$

Multiplions encore par $d\,\Pi(x)$ et intégrons depuis $\Pi(x)=\frac{1}{2}\pi$ nous aurons:

$$S=\frac{2\pi\cos\Pi(r)\cos\Pi(x)}{\sin^2\Pi(r)}$$

ce qui est l'aire de la surface du segment de sphère compris entre deux plans perpendiculaires à un même rayon, dont l'un passe par le centre de la sphère et l'autre à une distance $x$ du centre. Pour avoir l'aire de la surface de la sphère entière il faut mettre dans cette formule $x=r$ et doubler le résultat. De cette manière on a pour l'aire de la surface de la sphère entière l'expression

$$4\pi\cot\mathrm{g}^2\,\Pi(r)$$

ou

$$\pi(e^r-e^{-r})^2;$$

si $r$ est si petit qu'on puisse rejeter les puissances supérieures de $r$, cette expression se reduit à

$$4\pi r^2$$

comme dans la géométrie ordinaire.

Posons

$$\cos\psi = \operatorname{tang} \Pi(r) \operatorname{cotg} \Pi(y)$$

$$\cos \Pi(x) = \cos \Pi(r) \sin\psi \sin\varphi$$

et introduisons les nouvelles variables $\psi$, $\varphi$ au lieu de $x, y$ dans l'expression de l'élément de la surface de la sphère de rayon $r$ que nous avons en vue.

Nous trouvons

$$\frac{d^2 S}{d\varphi\, d\psi} = -\frac{\cos \Pi(r)}{\sin \Pi(r)} \frac{\sin\psi \sqrt{1 - \cos^2 \Pi(r) \sin^2\psi \sin^2\varphi}}{1 - \cos^2 \Pi(r) \sin^2\psi}.$$

Multiplions cette équation par $8\, d\psi\, d\varphi$ et intégrons depuis $\psi = o$ jusqu'à $\psi = \frac{\pi}{2}$, depuis $\varphi = o$, jusqu'à $\varphi = \frac{\pi}{2}$, ce qui nous donnera la surface de la sphère entière. En égalant cette expression de la surface de la sphère entière à l'expression de la même surface que nous avons trouvé plus haut, nous concluons que

$$\frac{\pi}{\sin \Pi(r)} = \int_o^{\frac{\pi}{2}} d\psi \int_o^{\frac{\pi}{2}} d\varphi \frac{\sin\psi \sqrt{1 - \cos^2 \Pi(r) \sin^2\psi \sin^2\varphi}}{1 - \cos^2 \Pi(r) \sin^2\psi}. \tag{30}$$

Si nous désignons par $E(\alpha)$ l'intégrale elliptique

$$E(\alpha) = \int_o^{\frac{\pi}{2}} d\varphi \sqrt{1 - \alpha^2 \sin^2\varphi}$$

où $\alpha$ est la constante qui se trouve sous le signe intégral nous avons

$$\frac{\pi\alpha}{\sin \Pi(r)} = \int_o^{\alpha} \frac{dx\, E(x)}{(1 - x^2)\sqrt{\alpha^2 - x^2}}.$$

En posant dans l'intégrale (30) $\frac{1}{2}\pi - R$ à la place de $\Pi(r)$ il vient

$$\tfrac{1}{2}\pi R = \int_o^{\frac{\pi}{2}} \int_o^{\frac{\pi}{2}} \frac{d\psi\, d\varphi \sin\psi \sin R}{\sqrt{1 - \sin^2\psi \sin^2\varphi \sin^2 R}}.$$

Effectuant l'intégration par rapport à $\psi$ dans les limites indiquées nous trouvons

$$\pi R = \int_o^{\frac{\pi}{2}} \frac{d\varphi}{\sin\varphi} \log\left(\frac{1 + \sin\varphi \sin R}{1 - \sin\varphi \sin R}\right)$$

ce qui, en mettant $\Pi(x)$ à la place de $\varphi$, prend la forme

$$\pi R = \int_0^\infty dx \log \left\{ \frac{e^{2x} + 1 + 2e^x \sin R}{e^{2x} + 1 - 2e^x \sin R} \right\}.$$

L'integration par parties reduit cette équation à

$$\tfrac{1}{2} \pi \frac{R}{\sin R} = \int_0^\infty \frac{(e^{2x} - 1)\, e^x .\, x\, dx}{e^{4x} + 2e^{2x} \cos 2R + 1}. \tag{31}$$

Pour $R = \frac{\pi}{2}$ cette équation donne

$$\tfrac{1}{8} \pi^2 = \int_0^\infty \frac{e^x\, x\, dx}{e^{2x} - 1}.$$

Il est facile de démontrer que l'équation (31) reste vraie quand même on met à la place de $\cos R$ un nombre plus grand que l'unité.

En effet on a

$$\int_0^\pi d\psi \log \operatorname{cotg} \tfrac{1}{2} \psi = o$$

d'où il suit que pour tout nombre $a$

$$\int_0^\pi d\psi \log (e^a \operatorname{cotg} \tfrac{1}{2} \psi) = a\pi.$$

Transformons cette intégrale en posant $e^a \operatorname{cotg} \frac{1}{2} \psi = e^x$, il viendra

$$\int_{-\infty}^{+\infty} \frac{x\, dx}{e^{x-a} + e^{-x+a}} = \tfrac{1}{4} \pi a.$$

On peut donner facilement à cette équation la forme suivante

$$\int_0^\infty \frac{(e^x - e^{-x})\, x\, dx}{e^{2x} + e^{2a} + e^{-2a} + e^{-2x}} = \tfrac{1}{2} \left( \frac{\pi a}{e^a - e^{-a}} \right)$$

d'où l'on revient à l'équation (31) en remplaçant $a$ par $a\sqrt{-1}$. Si nous prenons pour coordonnées des arcs de cercle limite, l'un $\zeta$ situé dans un plan passant par une perpendiculaire abaissée du point donnée sur le plan $xy$ et par une parallèle à l'axe des $x$ menée par le pied de cette perpendiculaire et ayant cette parallèle pour axe, l'autre $\eta$ situé dans le plan $xy$ ayant l'axe des $x$ pour axe et passant par le pied de $\zeta$ et si nous prenons pour troisième coordonnée la partie $\xi$ de l'axe des $x$ comprise entre l'origine des coordonnées et le sommet du second arc, l'élément du volume $P$ doit être $d\xi\, d\eta\, d\zeta$.

On a donc

$$d^3 P = d\xi \, d\eta \, d\zeta.$$

Posons encore $\zeta = \operatorname{cotg} \Pi(z)$ où $z$ est la perpendiculaire abaissée du point donné sur le plan $xy$, nous aurons

$$\frac{d^3 P}{d\zeta \, d\eta \, dz} = \frac{1}{\sin \Pi(z)}.$$

De l'équation du cercle limite nous tirons

$$e^{-p} = \sin \Pi(z)$$

où $p$ désigne la distance du point d'intersection de l'arc $\zeta$ avec le plan $xy$ au pied de la perpendiculaire $z$. En remarquant que, en conséquence de l'équation du cercle limite et de la valeur de l'arc de cercle limite en fonction de l'ordonnée, on a

$$\operatorname{cotg} \Pi(y) = \eta \, e^{-p}$$

$$e^{\xi - x} = \sin \Pi(z) \sin \Pi(y)$$

on trouve

$$\frac{d\eta}{dy} = \frac{1}{\sin \Pi(y) \sin \Pi(z)}; \quad dx = d\xi;$$

d'où il suit que

$$\frac{d^3 P}{dx \, dy \, dz} = \frac{1}{\sin \Pi(y) \sin^2 \Pi(z)}.$$

En multipliant cette expression par $dx$ et en intégrant par rapport à $x$ depuis $x = o$, nous trouvons

$$\frac{d^2 P}{dy \, dz} = \frac{x}{\sin \Pi(y) \sin^2 \Pi(z)};$$

en multipliant la même expression par $dy$ et en intégrant par rapport à $y$ depuis $y = o$

$$\frac{d^2 P}{dx \, dz} = \frac{\operatorname{cotg} \Pi(y)}{\sin^2 \Pi(z)}$$

et enfin en multipliant par $dz$ et en intégrant par rapport à $z$ depuis $z = o$

$$\frac{d^2 P}{dx \, dy} = \frac{1}{8 \sin \Pi(y)} \left\{ e^{2z} + e^{-2z} + 4z \right\}.$$

Si l'on multiplie l'avant dernière de ces expressions par $dx \, dz$ que l'on intègre d'abord par rapport à $z$ jusqu'à la valeur de $z$ tirée de l'équation

$$\sin \Pi(r) = \sin \Pi(x) \sin \Pi(z)$$

et puis par rapport à $x$ depuis $x = o$ jusqu'à $x = r$ et que l'on multiplie le resultat par 8 pour avoir le volume de la sphère entiere, on trouvera le volume de la sphère entière $= \frac{1}{2}\pi \left\{ e^{2r} - e^{-2r} - 4r \right\}$ ce qui donne pour $r$ très petit $\frac{4}{3}\pi r^3$, comme dans la géométrie ordinaire.

Soit pour exprimer l'élément de volume en coordonnées polaires, $r$ la distance de l'origine des coordonnées à un point de l'espace, dont les coordonnées rectangulaires sont $x, y, z$. Nommons $q$ la droite menée dans le plan $xy$ de l'origine des coordonnées au pied de $z$, $t$ l'angle entre $r$ et $q$, $\omega$ l'angle de $q$ et de l'axe des $x$ positifs. Posons encore $\Pi(x) = X$, $\Pi(y) = Y$, $\Pi(z) = Z$, $\Pi(r) = R$, $\Pi(q) = Q$. Menons un plan perpendiculaire à l'axe des $z$ et qui passe par le point donné. Soit $r'$ la droite menée dans ce plan du point donné à l'axe des $z$ et posons encore $\Pi(r') = R'$. Construisons enfin une sphère de rayon $r$ dont le centre coïncide avec l'origine des coordonnées. Le plan $xy$ coupera cette sphère dans un grand cercle dont la circonférence sera égale, d'après ce qui a été démontré plus haut, à

$$2\pi \operatorname{cotg} R$$

la partie de cette circonférence interceptée par deux plans qui passent tous les deux par l'axe des $z$ et inclines l'un sur l'autre sous l'angle $\omega$ doit être

$$\omega \operatorname{cotg} R.$$

La circonférence de cercle, produite par l'intersection de la même sphère par le plan qui passe par le point donné et qui est perpendiculaire à l'axe des $z$ sera égale à

$$2\pi \operatorname{cotg} R'$$

et la partie de cette circonférence, interceptée par les deux plans qui passent par l'axe des $z$ et sont inclinés l'un sur l'autre sous l'angle $\omega$ doit être

$$\omega \operatorname{cotg} R'.$$

L'acroissement de ce dernier arc, produit par l'accroissement $d\omega$ de l'angle $\omega$ doit être

$$d\omega \operatorname{cotg} R'.$$

Le triangle, dont l'hypoténuse est $r$, l'un des côtés de l'angle droit $r'$ et dont l'angle opposé à $r'$ est $\frac{\pi}{2} - \theta$, donne (d'après l'équation 13)

$$\operatorname{tang} R' \cos\theta = \operatorname{tang} R$$

d'où il suit que

$$d\omega \operatorname{cotg} R' = d\omega \cos\theta \operatorname{cotg} R.$$

La circonférence du cercle qui est l'intersection de la même sphère par un plan mené par l'axe des $z$ est égale à

$$2\pi \operatorname{cotg} R$$

et l'arc de ce cercle qui correspond à l'angle $\theta$ au centre doit être

$$\theta \operatorname{cotg} R$$

d'où il suit que l'accroissement de cet arc qui correspond à un accroissement $d\theta$ de l'angle $\theta$ doit être

$$d\theta \operatorname{cotg} R.$$

Si tous les acroissements sont infiniment petits l'élément du volume sera, comme dans la géométrie ordinaire, exprimé par le produit des trois lignes perpendiculaires entre elles

$$dr,\ d\omega \cos\theta \operatorname{cotg} R,\ d\theta \operatorname{cotg} R$$

par-ce qu'il peut être consideré comme un prisme; on aura donc l'expression suivante de l'élément de volume en coordonnées polaires

$$dr\, d\omega\, d\theta \cos\theta \operatorname{cotg}^2 R = d^3 P$$

ou en substituant pour $\operatorname{cotg}^2 R$ sa valeur en $r$

$$d^3 P = \tfrac{1}{4}\, dr\, d\omega\, d\theta \cos\theta\, (e^r - e^{-r})^2.$$

En intégrant d'abord par rapport à $r$ depuis $r = o$ il vient

$$d^2 P = \tfrac{1}{8}\, d\omega\, d\theta \cos\theta\, (e^{2r} - e^{-2r} - 4r).$$

Pour la sphère dont le centre est à l'origine des coordonnées $r$ ne depend pas de $\theta$ et $\omega$. En intégrant par rapport à $\omega$ depuis $\omega = o$, jusqu'à $\omega = 2\pi$ et par rapport à $\theta$ depuis $\theta = o$ juqu'à $\theta = \frac{\pi}{2}$ et en doublant le résultat il vient pour le volume de la sphère entière $\frac{\pi}{2}(e^{2r} - e^{-2r} - 4r)$ comme plus haut.

Prenons maintenant une partie $S$ de la d'une surface sphère limite terminée par un contour rentrant sur lui même, menons par les différents points de ce contour des droites parallèles à l'axe de la sphère, ils f rmeront une surface que nous nommerons par analogie conique et qui s'étend indéfiniment des deux côtés, mais dont nous ne considérons que la partie située du côté de parallèlisme des axes de la sphère limite. Soit $S'$ la partie d'une seconde sphère limite, dont les axes sont parallèles aux axes de la première et dirigés en même sens, partie qui est située dans l'intérieur de la surface conique. $S, S'$ et la partie de la surface conique située entre les deux sphères limites renferment un volume fini en tout sens que nous nous proposons de déterminer. Nommons $c$ la partie d'un axe des deux sphères interceptée entre elles, appliquons une longueur égale à $c$ plusieurs fois sur un des axes de la première sphère qui passe par un des point du contour de $S$ à partir du point ou cet axe perce $S'$ et menons par les points de division des sphères limites, dont les axes soient parallèles aux axes des deux premières et dirigés en même sens. Soient $S''$, $S'''$ etc. les parties de ces sphères limites consécutives comprises dans la surface conique. Il suit facilement de ce qui a été démontré plus haut par rapport aux arcs de cercle limite située comme le sont les parties de sphère limite que nous considérons maintenant qu'on aura toujours

$$S' = S\,e^{-2c}$$
$$S'' = S\,e^{-4c}$$
$$S''' = S\,e^{-6c} \quad \text{et ainsi de suite.}$$

Nommons de même $P$, $P'$, $P''$, $P'''$ etc. les volumes interceptés par la surface conique entre $S, S'$; entre $S', S''$ et ainsi de suite et faisons attention à ce que les volumes $P$, $P'$, $P''$ etc. doivent être proportionnels aux surfaces $S, S', S''$ etc.

Nous devons donc avoir

$$P = C\,S$$

où $C$ est une fonction de $c$ seule; il suit de la que

$$P' = C\,S' = C\,S\,e^{-2c}$$
$$P'' = C\,S'' = C\,S\,e^{-4c} \quad \text{et ainsi de suite.}$$

La somme $\sum\limits_{c}^{\infty} P^{(n)}$ sera donc le volume compris dans la surface conique, dont la base est $S$ et qui est indéfiniment prolongée du côté du parallélisme des génératrices. Soit $K$ ce volume, nous aurons

$$K = \frac{C\,S}{1 - e^{-2c}};$$

cette grandeur ne doit pas dépendre de $c$, ce qui exige qu'on ait

$$C = (1 - e^{-2c})\,A$$

où $A$ est un nombre absolu, et comme l'unité de volume est arbitraire nous prendrons

$$C = \tfrac{1}{2}(1 - e^{-2c})$$

dans le but que le volume $P$, étant

$$P = \tfrac{1}{2}\,S\,(1 - e^{-2c})$$

devienne $P = cS$ si $c$ est infiniment petit, expression qui coïncide avec l'expression du volume d'un prisme de base $S$ et de hauteur $c$ dans la géométrie ordinaire. On peut encore prendre pour l'élément de volume le volume compris dans une surface conique formée par les axes d'une surface de sphère limite, axes qui sont menés par tous les points du contour d'une partie de cette surface infiniment petite dans tout sens.

Le grand nombre d'expressions différentes pour l'élément de la même grandeur géométrique donne des moyens pour la comparaisons des intégrales, moyens qui sont surtout utiles dans la théorie des intégrales définies.

Ayant montré dans ce qui précède de quelle manière il faut calculer la longueur des lignes courbes, l'aire des surfaces et le volume des corps, il nous est permis d'affirmer que la Pangéométrie est une doctrine géométrique complète. Un simple coup d'oeil sur les équations (19) qui expriment la dépendance existante entre les côtés et les angles des triangles rectilignes est suffisant pour démontrer qu'à partir de là la Pangéométrie devient une méthode analytique qui remplace et généralise les méthodes analytiques de la géométrie ordinaire. On pourrait commencer l'exposition de la Pangéométrie par les équations (19) et même essayer de substituer à ces équations d'autres équations qui exprimeraient les dépendances entre les angles et les côtés de tout triangle réctiligne; mais dans ce dernier cas, il faudrait démontrer que ces nouvelles équations s'accordent avec les notions fondamentales de la géométrie. Les équations (19) ayant été déduites de ces notions fondamentales s'accordent donc nécessairement avec elles et toutes les équations qu'on pourrait vouloir leur substituer doivent, si ces équations ne sont pas une suite des équations (19), conduire à des résultats contraires à ces notions. Ainsi les équations (19) sont la base de la géométrie la plus générale puisqu'elles ne dépendent pas de la supposition que la somme des trois angles de tout triangle rectiligne est égale à deux angles droits.

La Pangéométrie, qui est fondée sur des principes certains et qui a été développée dans ce qui précède donne, comme on a vu, des méthodes propres à calculer la valeur des différentes grandeurs géométriques et démontre en même temps que la supposition, que la valeur de la somme des trois angles de tout triangle rectiligne est constante, supposition adoptée explicilement ou implicitement dans la géométrie ordinaire, n'est pas une conséquence nécessaire de nos notions sur l'espace. Il n'y a que l'expérience qui puisse confirmer la vérité de cette supposition, par exemple par la mesure effective des trois angles d'un triangle rectiligne, mesure qui peut être effectuée de différentes manières. On peut mesurer les trois angles d'un triangle rectiligne construit sur un plan artificiel ou les trois angles d'un triangle rectiligne dans l'espace. Dans ce dernièr cas on devra préférer les triangles dont les côtés sont très grands, puisque d'après la Pangéométrie, la différence entre deux angles droits et la somme des trois angles d'un triangle rectiligne est d'autant plus grande que les côtés sont plus grands.

Soit $r$ le rayon d'un cercle, $A$ un angle au centre dont les côtés comprennent un arc soustendu par une chorde égale à $r$. Nommons $p$ la perpendiculaire abaissée du centre du cercle sur cette chorde, qui est divisée en deux parties égales par le pied de la perpendiculaire. Considérons un des deux triangles rectilignes rectangles formés par cette perpendiculaire, les rayons du cercle situés sur les côtés de l'angle $A$ et la chorde, triangle dont l'hypoténuse sera $r$ et les côtés perpendiculaires entre eux $\frac{1}{2}r, p$.

D'après l'équation générale (13) on aura dans ce triangle

$$\sin \tfrac{1}{2} A \operatorname{tang} \Pi(\tfrac{1}{2} r) = \operatorname{tang} \Pi(r)$$

équation qui, combinée avec l'équation indentique

$$\operatorname{tang} \Pi(r) = \frac{\sin^2 \Pi(\frac{1}{2} r)}{2 \cos \Pi(\frac{1}{2} r)}$$

donne

$$\sin \tfrac{1}{2} A = \tfrac{1}{2} \sin \Pi(\tfrac{1}{2} r).$$

Dans la géométrie ordinaire on a

$$A = \tfrac{\pi}{3}.$$

Supposons que la mesure effective donne

$$A = \frac{2\pi}{6 + K}$$

où $K$ est un nombre positif.

On devra donc avoir

$$\sin\left(\frac{\pi}{6+K}\right) = \tfrac{1}{2} \sin \Pi(\tfrac{1}{2} r).$$

Si $r$ et $K$ sont donnés ou peut tirer de cette équation la valeur de $\Pi(\frac{1}{2} r)$ à l'aide de quoi on peut trouver l'angle de parallèlisme $\Pi(x)$ pour toute ligne $x$. Les distances entre les corps célestes nous fournissent le moyen d'observer les angles de triangles dont les côtés sont très grands. Soit $\alpha$ la latitude géocentrique d'une étoile fixe à une époque fixe et $\beta$ une autre latitude géocentrique de la même étoile, latitude qui correspond à l'époque où la terre se trouve de nouveau dans le plan perpendiculaire à l'écliptique, mené par sa première position (c'est-à-dire la position où la latitude de l'étoile était $\alpha$); soit $2a$ la distance entre ces deux positions de la terre et $\delta$ l'angle sous lequel est vu la distance $2a$ de l'étoile.

Si les angles $\alpha, \beta, \delta$ ne satisfont pas à la rélation

$$\alpha = \beta + \delta$$

ce sera un signe que la somme des trois angles de ce triangle diffère de deux angles droits

On peut choisir l'étoile de manière que $\delta$ soit égal à zéro et on pourra toujours supposer qu'il existe une ligne $x$ telle que

$$\Pi(x) = \alpha.$$

Si $\delta = o$ les droites menées des deux positions de la terre à l'étoile peuvent être censées parallèles et par conséquence on devra avoir

$$\beta = \Pi(x + 2a)$$

d'où il suit, d'après ce qui a été démontré plus haut que

$$\operatorname{tang} \tfrac{1}{2} \alpha = e^{-x}$$

$$\operatorname{tang} \tfrac{1}{2} \beta = e^{-x-2a}.$$

Toutes les fois que l'observation aura donné pour une étoile par rapport à laquelle l'angle désigné par $\delta$ est zero, deux angles $\alpha, \beta$ différents les deux dernières équations donneront $x$ et $a$ exprimés au moyen de la ligne prise pour unité dans la Pangéométrie. Ayant ainsi la ligne $x$ qui correspond à un angle de parallèlisme $\Pi(x)$ on pourra calculer l'angle de parallèlisme $\Pi(y)$ pour toute ligne $y$ donnée.

## ERRATA.

Page 291 ligne 4 en remontant au lieu de $\text{tang}\, a \cos B$ lisez $-\text{tang}\, a \cos B$
Page 291 ligne 6 en remontant au lieu de $\text{tg}\, x \cot c -$ lisez $\text{tg}\, x = \cot c -$
Page 303 ligne 13 en remontant au lieu de $\frac{1}{2^3}$ lisez $\frac{1}{2^3} s$
Page 307 ligne 4 en remontant au lieu de $\text{tg}\, X \text{tg}\, (x)$ lisez $\text{tg}\, X \text{tg}\, \Pi(x)$
Page 310 ligne 15 en descendant au lieu de $\frac{dx^2}{\sin \Pi(y)}$ lisez $\frac{dx^2}{\sin^2 \Pi(y)}$
Page 310 ligne 11 en remontant au lieu de $e^{-\infty}$ lisez $e^{-\infty}\, dx$
Page 315 ligne 2 en descendant au lieu de $\cos^2 \Pi(x)$ lisez $\sin^2 \Pi(x)$
Page 315 ligne 8 en descendant au lieu de $\sin \frac{1}{2} \alpha$ lisez $\sin^2 \alpha$
Page 333 ligne 4 en descendant au lieu de $\frac{1}{2}\pi$ lisez $\frac{1}{4}\pi$.

# УЧЕНЫЯ ЗАПИСКИ,

ИЗДАВАЕМЫЯ

## ИМПЕРАТОРСКИМЪ КАЗАНСКИМЪ УНИВЕРСИТЕТОМЪ.

## 1855.

### КНИЖКА I.

КАЗАНЬ,

ВЪ УНИВЕРСИТЕТСКОЙ ТИПОГРАФІИ.

1856.

# ПАНГЕОМЕТРІЯ.

*Заслуж. Профессора Н. И. Лобачевскаго.*

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Понятія, на которыхъ основываютъ начала геометріи недостаточны чтобъ отсюда вывести доказательство теоремы: сумма трехъ угловъ прямолинейнаго треугольника равна двумъ прямымъ; теорема, въ справедливости которой никто до сихъ поръ не сомнѣвался, потому что не встрѣчаютъ ни какого противорѣчія въ заключеніяхъ, которыя отсюда выводятся и потому что измѣреніе угловъ, въ прямолинейныхъ треугольникахъ согласуется въ предѣлахъ ошибокъ самыхъ точныхъ измѣреній съ этой теоремой. Недостаточность начальныхъ понятій для доказательства приведенной теоремы принудила геометровъ допускать прямо или косвенно вспомогательныя положенія, которыя какъ ни просты кажутся, тѣмъ не менѣе произвольны и слѣдовательно допущены быть не могутъ. Такъ напр. принимаютъ: что кругъ съ безконечно великимъ полупоперечникомъ переходитъ въ прямую линію, а сфера съ безконечно великимъ полупоперечникомъ въ плоскость; что углы прямолинейнаго треугольника зависятъ только отъ содержанія боковъ, но не отъ самыхъ боковъ, или наконецъ, какъ это обыкновенно принимаютъ въ началахъ геометріи, что изъ данной точки въ плоскости не можно провести болѣе, одной прямой параллельной съ данной прямою въ той же плоскости, тогда какъ всѣ другія прямыя, проведенныя изъ той же точки и въ той же плоскости, должны необходимо по достаточномъ продолженіи пересѣкать данную прямую. Подъ линіею параллельной другой, разумѣютъ прямую линію, которая сколько бы не продолжалась въ обѣ стороны, никогда не встрѣчаетъ ту, съ которой она параллельна. Это опредѣленіе само по себѣ недостаточно, потому что оно не указываетъ на единственную линію. Тоже можно сказать о большой части опредѣленій, даваемыхъ въ началахъ геометріи, потому что эти опредѣленія, не только не указываютъ на происхожденіе геометрической величины, которую хотятъ опредѣлить, но даже не доказываютъ, что такія величины существовать могутъ. Такимъ образомъ опредѣляютъ прямую линію и плоскость нѣкоторыми ихъ свойствами; говорятъ что прямыя линіи суть тѣ, которыя сливаются, какъ скоро у нихъ двѣ общія точки; что плоскость есть такого рода поверхность, въ которой прямая линія лежитъ вся, какъ скоро проведена чрезъ двѣ точки взятыя на плоскости. Вмѣсто того, чтобы начинать геометрію прямой линіею и плоскостью, какъ это дѣлаютъ обыкновенно, я предпочелъ начать сферой и кругомъ, которыхъ опредѣленіе не подлежитъ упреку въ неполнотѣ, потому что въ этихъ опредѣленіяхъ заключается способъ какимъ образомъ эти величины происходятъ. Потомъ я опредѣляю плоскость, какъ поверхность гдѣ пересѣкаются равныя сферы, описанныя около двухъ постоянныхъ точекъ. Наконецъ опредѣляю прямую линію, какъ пересѣченіе равныхъ круговъ въ плоскости, описанныхъ около двухъ постоянныхъ точекъ той же плоскости. Допустивъ такія опредѣленія, вся теорія прямыхъ и плоскостей перпендикулярныхъ можетъ быть изложена строго съ легкостью и краткостью. Прямую проведенную изъ данной точки въ плоскости, я называю параллельною къ данной прямой въ той же плоскости, какъ скоро она составляетъ границу между тѣми прямыми, проведенными изъ той же точки въ той же плоскости, которыя пересѣкаютъ данную прямую по достаточному продолженію и тѣхъ которыя не пересѣкаютъ сколько бы не продолжались. Ту сторону, въ которой пересѣченіе происходитъ, я называю стороною параллельности. Я издалъ полную теорію параллельныхъ подъ заглавіемъ «Geometrische Untersuchungen zur Theorie der Parallellinien. Berlin 1840. In der Finke'schen Buchhandlung.»

Въ этомъ сочиненіи я изложилъ доказательства всѣхъ предложеній, въ которыхъ не нужно прибѣгать къ помощи параллельныхъ линій. Между этими предложеніями то, которое даетъ отношеніе поверхности сферическаго треугольника ко всей сферѣ, особенно достойно замѣчанія. (Geometr. Untersuchungen § 27.) Если $A$, $B$, $C$ означаютъ углы сферическаго треугольника, то содержаніе поверхности этого сферическаго треугольника къ поверхности всей сферы, которой онъ принадлежитъ, будетъ равно содержанію

$$\tfrac{1}{2}(A+B+C-\pi),$$

къ четыремъ прямымъ угламъ. Здѣсь $\pi$ означаетъ два прямыхъ угла.

Потомъ я доказываю, что сумма трехъ угловъ въ прямолинейномъ треугольникѣ не можетъ быть болѣе двухъ прямыхъ угловъ (Geometr. Unters. § 19), и если эта сумма равна двумъ прямымъ угламъ въ какомъ нибудь прямолинейномъ треугольникѣ, то она должна быть такова во всѣхъ прямолинейныхъ треугольникахъ (Geometr. Unters. § 20). Итакъ два только предпололоженія возможны: или сумма трехъ угловъ во всякомъ прямолинейномъ треугольникѣ равна двумъ прямымъ угламъ. — это предположеніе составляетъ обыкновенную геометрію—или во всякомъ прямолинейномъ треугольникѣ эта сумма менѣе двухъ прямыхъ и это послѣднее предположеніе служитъ основаніемъ особой геометріи, которой я далъ названіе воображаемой геометріи, но которую можетъ быть приличнѣе назвать пангеометріей, потому что это названіе означаетъ геометрію въ обширномъ видѣ, гдѣ обыкновенная геометрія будетъ частный случай.

Изъ принятыхъ началъ въ пангеометріи слѣдуетъ, что перпендикулъ $p$, опущенный изъ одной точки прямой линіи на параллельную къ ней, дѣлаетъ съ первой линіей два угла, изъ которыхъ одинъ острый. Я называю этотъ уголъ угломъ параллельности, а сторону первой изъ этихъ прямыхъ линій, гдѣ острый уголъ находится, и которая остается таже для всѣхъ точекъ на прямой, стороной параллельности. Я означаю этотъ уголъ $\Pi(p)$, потому что онъ зависитъ отъ длины перпендикула $p$. Въ обыкновенной геометріи $\Pi(p)=\frac{\pi}{2}$ для всякаго $p$. Въ пангеометріи углу $\Pi(p)$ принадлежатъ всѣ значенія отъ $o$, которому соотвѣтствуетъ $p=\infty$, до $\frac{\pi}{2}$ которому соотвѣтствуетъ $p=o$. (Geometr. Unters. § 23). Чтобы функціи $\Pi(p)$ дать аналитическое значеніе болѣе общее, я принимаю, что для отрицательныхъ $p$, случай на который первое опредѣленіе не распространяется, значеніе этой функціи дается уравненіемъ: $\Pi(p)+\Pi(-p)=\pi$.

Итакъ для всякаго угла $A>o$ и $<\pi$, можно найти линію $p$ такъ, что $\Pi(p)=A$, гдѣ линія $p$ положительна если $A<\frac{\pi}{2}$. Обратно для всякой линіи $p$ существуетъ уголъ $A$ такъ, что $A=\Pi(p)$.

Я называю предѣльнымъ кругомъ такой кругъ, котораго полупоперечникъ принимаю безконечно великимъ; онъ можетъ быть начерченъ постепенно точками къ нему принадлежащими. Беремъ точку на данной прямой, называемъ эту точку вершиной, а самую прямую—осью предѣльнаго круга; построимъ на этой прямой уголъ $A \geqq o$ и $\leqq \frac{\pi}{2}$, вершина котораго совпадаетъ съ вершиною предѣльнаго круга, и котораго одна изъ сторонъ совпадаетъ съ осью, пусть наконецъ $a$ линія, которая даетъ $\Pi(a)=A$ и кладемъ на другой бокъ угла $A$ отъ вершины прямую $2a$, конецъ этой прямой будетъ лежать на предѣльномъ кругѣ; чтобы продолжить предѣльный кругъ по другую сторону оси, надобно повторить тоже построеніе на этой сторонѣ оси. Отсюда видно, что всѣ прямыя параллельныя съ осью предѣльнаго круга могутъ равно почитаться за оси предѣльнаго круга.

Обращеніе предѣльнаго круга около одной изъ его осей производитъ поверхность, которую я называю *предѣльная сфера*, поверхность, которая слѣдовательно будетъ граница приближенія для сферы съ возрастаніемъ полупоперечниковъ до безконечности.

Мы назовемъ ось обращенія, а слѣдовательно и всѣ линіи параллельныя съ осью

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обращенія, осью предѣльной сферы, а поперечной плоскостію будемъ называть плоскость, въ которой лежитъ одна или нѣсколько осей обращенія. Пересѣченіе предѣльной сферы съ поперечной плоскостью даетъ предѣльный кругъ. Часть поверхности на предѣльной сферѣ, ограниченная тремя дугами предѣльнаго круга, называется предѣльнымъ сферическимъ треугольникомъ, дуги предѣльнаго круга будутъ называться боками предѣльнаго сферическаго треугольника, а плоскостные углы между плоскостей, гдѣ лежатъ дуги предѣльнаго круга будутъ называться углами предѣльнаго сферическаго треугольника.

Двѣ прямыя параллельныя третьей параллельны между собою (Geometr. Unters. § 25). Отсюда слѣдуетъ, что всѣ оси предѣльнаго круга и предѣльной сферы параллельны между собою.

Если три плоскости пересѣкаются по двѣ въ трехъ параллельныхъ прямыхъ, и если каждая плоскость ограничена между двухъ параллельныхъ, то сумма трехъ плоскостныхъ угловъ, которые эти плоскости составляютъ по порядку одна съ другой, равна двумъ прямымъ угламъ. (Geometr. Unters. § 28).

Изъ этаго предложенія слѣдуетъ, что сумма трехъ угловъ въ предѣльномъ сферическомъ треугольникѣ равна двумъ прямымъ угламъ, и что всё то, что въ обыкновенной геометріи доказываютъ о содержаніи боковъ прямолинейнаго треугольника, можетъ быть повторено и доказано въ Пангеометріи для боковъ предѣльнаго сферическаго треугольника, стоитъ только замѣнить параллельныя прямыя съ однимъ изъ боковъ треугольника, дугами предѣльнаго круга, проведенными чрезъ точки на одномъ изъ боковъ предѣльнаго сферическаго треугольника подъ однимъ угломъ съ этимъ бокомъ. Такъ напр. если $p$, $q$, $r$. бока предѣльнаго сферическаго прямоугольнаго треугольника и $P$, $Q$, $\frac{\pi}{2}$ углы противъ этихъ боковъ, то надобно принимать также какъ для прямоугольнаго прямолинейнаго треугольника въ обыкновенной геометріи слѣдующія уравненія.

$$p = r \sin P = r \cos Q$$

$$q = r \cos P = r \sin Q$$

$$P + Q = \frac{\pi}{2}.$$

Въ обыкновенной геометріи доказываютъ, что разстояніе двухъ параллельныхъ прямыхъ вездѣ одинаково, въ Пангеометріи напротивъ, разстояніе $p$ точекъ одной прямой до параллельной съ нею уменьшается на сторонѣ параллельности, т. е. на сторонѣ въ которую обращенъ уголъ параллельности $\Pi(p)$. Теперь пусть $S$, $S'$ $S''$,. и т. д. рядъ дугъ предѣльныхъ круговъ между двухъ параллельныхъ прямыхъ которыя служатъ осями этимъ предѣльнымъ кругамъ, предположимъ, что части параллельныхъ между дугъ послѣдовательныхъ, равны между собою и измѣряются взаимнымъ разстояніемъ двухъ послѣдовательныхъ дугъ $x$; называемъ $E$ содержаніе $S$ къ $S'$ когда $x$ равно единицѣ длины

$$\frac{S}{S'} = E$$

гдѣ $E$ число положительное и болѣе единицы. Пусть теперь число $E$ выражается содержаніемъ двухъ цѣлыхъ чиселъ $n$ къ $m$, такъ что $E=\frac{n}{m}$; раздѣляемъ дугу $S$ на $m$ равныхъ частей; чрезъ точки дѣленія ведемъ параллельныя съ осями предѣльныхъ круговъ. Эти параллельныя раздѣлятъ дуги $S'$, $S''$...каждую на $m$ равныхъ частей. Переносимъ площадь между дугъ $S'$, $S''$ на площадь между дугъ $S, S'$ и полагая дугу $S'$ на $S$, а слѣдовательно дугу $S''$ на $S'$. повторяемъ дугу $\frac{S'}{m}$; она должна укладываться $n$ разъ въ дугѣ $S$.

Параллельность линій заставляетъ дугу $\frac{S''}{m}$ укладываться также $n$ разъ по дугѣ $S'$; слѣдовательно,

$$\frac{S}{S'}=\frac{S'}{S''}.$$

Чтобы доказать тоже самое въ случаѣ несоизмѣримости числа $E$, можно прибѣгнуть къ тѣмъ же способамъ, которые обыкновенно употребляются въ геометріи въ подобныхъ случаяхъ; я для краткости выпускаю эти подробности. Итакъ,

$$\frac{S}{S'}=\frac{S'}{S''}=\frac{S''}{S'''}=\ldots\ldots=E.$$

Это требуетъ, чтобы для всякой линіи $x$

$$S'=S\,E^{-x}$$

гдѣ $E$ представляетъ число равное содержанію $S$ къ $S'$ когда $x=1$.

Надобно замѣтить, что содержаніе $E$ не зависитъ отъ длины дуги $S$ и остается тоже, если двѣ данныя прямыя параллельныя сближаются и удаляются другъ отъ друга, оставаясь параллельными. Число $E$, которое необходимо болѣ единицы, зависитъ только отъ той прямой линіи, которая выбрана за единицу въ измѣреніи прямыхъ линій и которая измѣряетъ разстояніе между двухъ послѣдовательныхъ дугъ, и можетъ быть взята произвольно. Свойство которое мы доказали въ отношеніи къ дугамъ $S, S' S''$.....остается и для площадей $P, P' P''$.....ограниченныхъ послѣдовательными дугами и параллньными прямыми. Итакъ

$$P'=P.\,E^{-x}.$$

Если мы соединяемъ $n$ подобныхъ площадей сряду, то сумма будетъ

$$P\frac{1-E^{-nx}}{1-E^{-x}}.$$

Для $n=\infty$, это выраженіе даетъ площадь между двухъ параллельныхъ ограниченной съ одной стороны дугой $S$ и неограниченной со стороны параллелизма; значеніе этой площади будетъ слѣдовательно:

$$\frac{P}{1-E^{-x}}.$$

Если избираемъ за единицу площадей ту площадь, которая отвѣчаетъ дугѣ $S=1$ и разстоянію $x=1$, то найденное выраженіе для площади дѣлается для всякой дуги $S$

$$=\frac{E.\ S}{E-1}.$$

Въ обыкновенной геометріи число $E$ постоянно и равно единицѣ; послѣ чего въ обыкновенной геометріи двѣ прямыя параллельныя стоятъ повсюду равно другъ отъ друга, а площадь между двумя параллельными, ограниченная только съ одной стороны общимъ перпендикуломъ къ нимъ, безконечно велика.

Разматриваемъ теперь прямолинейный прямоугольный треугольникъ, котораго $a$, $b$, $c$, бока, а $A$, $B$, $\frac{\pi}{2}$ углы противоположные этимъ бокамъ. Острые углы $A$, $B$ могутъ быть принимаемы за углы параллельности $\Pi\,(\alpha)$, $\Pi\,(\beta)$ соотвѣтственные двумъ прямымъ $\alpha$, $\beta$ положительнымъ; условимся еще означать удареніемъ надъ буквой ту прямую, для которой уголъ параллельности служитъ дополненіемъ до прямаго угла, углу параллельности, который отвѣчаетъ прямой, означенный тою же буквой безъ ударенія. Такимъ образомъ.

$$\Pi\,(\alpha)+\Pi\,(\alpha')=\frac{\pi}{2}$$

$$\Pi\,(b)+\Pi\,(b')=\frac{\pi}{2}.$$

Представляемъ себѣ перпендикулъ $a$ къ оси предѣльнаго круга въ вершинѣ этой оси. Чрезъ вершину перпендикула $a$ ведемъ прямую, параллельную съ осью на сторонѣ параллельности. Означаемъ $f\,(a)$ часть этой параллельной между вершиною перпендикула и дугою, означаемъ $L\,(a)$ длину дуги предѣльнаго круга отрѣзанную параллельной линіей къ вершинѣ предѣльнаго круга. Въ обыкновенной геометріи $L\,(a)=a$; $f\,(a)=o$ для всякой линіи $a$.

Возставляемъ перпендикулъ $AA'$ къ плоскости прямолинейнаго и прямоугольнаго треугольника, котораго бока назвали $a$, $b$, $c$. и пусть перпендикулъ $AA'$ проходитъ чрезъ вершину угла $A$. Проводимъ чрезъ этотъ перпендикулъ двѣ плоскости: одну, которую назовемъ первой чрезъ бокъ $b$, а другую, которую назовемъ второй плоскостію чрезъ бокъ $C$.

Ведемъ во второй плоскости чрезъ вершину $B$ угла $\Pi\,(\beta)$ прямую $BB'$, параллельную съ $AA'$. Третью плоскость ведемъ чрезъ $BB'$ и чрезъ сторону $a$ треугольника.

Эта третья плоскость пересѣчетъ плоскость первую въ прямой $CC'$, параллельной съ $AA'$. Воображаемъ теперь сферу, описанную около точки $B$ какъ центра произвольнымъ полупоперечникомъ менѣе чѣмъ $a$. Эта сфера пересѣчетъ слѣдовательно стороны $a$, $c$ треугольника и прямую $BB'$ въ трехъ точкахъ, которыя означимъ: 1-ю $n$. 2-ю $m$ 3-ю $k$. Дуги большаго круга, которыя происходятъ отъ пересѣченія сферы съ тремя плоскостями проходящими чрезъ $B$ и которыя соединяютъ точки $n$, $m$, $k$, по двѣ, составятъ сферическій треугольникъ прямоугольный въ $m$, и котораго бока будутъ $mn = \Pi(\beta)$; $km = \Pi(c)$; $kn = \Pi(a)$. Уголъ $knm$ сферическаго треугольника $= \Pi(b)$ а уголъ $kmn = \frac{\pi}{2}$. Три прямыя $A, A_1$, $BB'$, $CC'$, будучи параллельными между собою производятъ сумму трехъ плоскостныхъ угловъ $= \pi$. Отсюда слѣдуетъ что третій уголъ $mkn$ сферическаго треугольника $= \Pi(a')$. Итакъ всякому прямолинейному прямоугольному треугольнику, котораго бока $a, b, c$ съ противоположными углами $\Pi(\alpha), \Pi(\beta), \frac{\pi}{2}$ отвѣчаетъ треугольникъ сферическій прямоугольный, котораго бока $\Pi(\beta)$, $\Pi(c)$, $\Pi(\alpha)$, съ противоположными углами $\Pi(\alpha')$ $\Pi(b)$, $\frac{\pi}{2}$. Построимъ еще прямолинейный прямоугольный треугольникъ, котораго перпендикулярные бока будутъ $\alpha'$, $a$, гипотенуза пусть будетъ $g$, пусть $\Pi(\lambda)$ уголъ противъ бока $a$ и $\Pi(\mu)$ уголъ противъ бока $\alpha'$.

Переходимъ отъ этого треугольника къ сферическому, подобнымъ образомъ какъ перешли отъ прямолинейнаго треугольнаго $ABC$ къ сферическому треугольнику $kmn$. Бока новаго сферическаго треугольника будутъ

$$\Pi(\mu), \Pi(g), \Pi(\alpha) \text{ съ противопол. углами}$$
$$\Pi(\lambda'), \Pi(\alpha'), \frac{\pi}{2}$$

Въ немъ будутъ части соотвѣтственно равны частямъ сферическаго треугольника $kmn$, потому что бока этого послѣдняго были $\Pi(c)$, $\Pi(\beta)$, $\Pi(\alpha)$ съ противоположными углами

$$\Pi(b), \Pi(\alpha'), \frac{\pi}{2}$$

Это доказываетъ, что эти два сферическіе прямоугольные треугольники имѣютъ гипотнузы равные и одинъ изъ прилежащихъ угловъ равный, а потому одинаковы между собою. Отсюда слѣдуетъ

$$\mu = c;\ g = \beta;\ b = \lambda'$$

Итакъ существованіе прямолинейнаго прямоугольнаго треугольника съ боками.

$$a, \quad b, \quad c \quad \text{и съ противоположными углами}$$
$$\Pi(\alpha), \Pi(\beta), \frac{\pi}{2}$$

предполагаетъ существованіе прямолинейнагонпрямоугольнаго треугольника, котораго бока $a$, $\alpha'$ $\beta$ съ противоположными углами

$$\Pi(b'), \Pi(c), \frac{\pi}{2}$$

Тоже самое можно выразить, сказавъ, что если

$$a, b, c, \alpha, \beta$$

части прямолинейнаго прямоугольнаго треугольника, то

$$a, \alpha', \beta, b' \; c$$

будутъ соотвѣтственныя части другаго прямолинейнаго прямоугольнаго треугольника. Если воображаемъ предѣльную сферу чрезъ точку $A$ даннаго прямолинейнаго прямоугольнаго треугольника, которой сферѣ перпендикулъ $AA'$ къ плоскости этого треугольника служитъ осью, а точка $A$ вершиною, то мы получимъ предѣльный сферическій треугольникъ отъ пересѣченія предѣльной сферы съ тремя плоскостями, проведенными чрезъ прямыя $AA'$, $BB'$, $CC'$.

Означимъ бока этого предѣльнаго сферическаго треугольника $p, q, r$. такимъ образомъ что $p$ будетъ пересѣченіе предѣльной сферы плоскостью, которая проходитъ чрезъ $a$; $q$ пересѣченіемъ предѣльной сферы съ плоскостью, которая проходитъ чрезъ $b$, и $r$ пересѣченіемъ сферы съ плоскостью, которая проходитъ чрезъ $c$; углы противъ этихъ боковъ будутъ: $\Pi(\alpha)$ противъ $p$; $\Pi(\alpha')$ противъ $q$; $\frac{\pi}{2}$ противъ $r$. Согласно съ вышепринятымъ означеніемъ здѣсь $q = L(b)$; $r = L(c)$. Предѣльная сфера пересѣчетъ прямую $CC'$ въ точкѣ, которой разстояніе отъ $c$ будетъ согласно съ тѣмъ же означеніемъ $f(b)$, подобнымъ образомъ будетъ $f(c)$ разстояніе точки пересѣченія предѣльной сферы съ прямою $BB$, отъ точки $B$. Легко видѣть, что

$$f(b) + f(a) = f(c)$$

Въ треугольникѣ, котораго бока предѣльныя дуги $p, q, r$ будетъ

$$p = r \sin \Pi(\alpha); \; q = r \cos \Pi(\alpha).$$

Умножая первое изъ этихъ двухъ уравненій на $E^{f(b)}$ получимъ:

$$p\, E^{f(b)} = r \sin \Pi(\alpha).\, E^{f(b)}; \text{ но}$$

$$p\, E^{f(b)} = L(a),$$

а слѣдовательно $$L(a) = r \sin \Pi(\alpha)\; E^{f(b)}.$$

Такимъ же образомъ будетъ

$$L(b) = r \sin \Pi(\beta)\, E^{f(a)}$$

Вмѣстѣ съ тѣмъ

$$q = r \cos \Pi(\alpha)$$

или что все равно: $$L(b) = r \cos \Pi(\alpha)$$

Сравненіе двухъ этихъ значеній $L(b)$ даетъ уравненіе

$$\cos \Pi(\alpha) = \sin \Pi(\beta).\, E^{f(a)} \qquad (1)$$

перемѣняя здѣсь $\alpha$ на $b'$, $\beta$ на $c$ и оставляя $a$ безъ перемѣны, какъ это дозволяется послѣ того, что было доказано выше, получимъ

$$\cos \Pi(b') = \sin \Pi(c).\, E^{f(a)}$$

или такъ какъ

$$\Pi(b) + \Pi(b') = \frac{\pi}{2}$$

$$\sin \Pi(b) = \sin \Pi(c).\, E^{f(a)}$$

Такимъ же образомъ должно быть

$$\sin \Pi(a) = \sin \Pi(c).\, E^{f(b)}.$$

Умножаемъ послѣднее уравненіе на $E^{f(a)}$ и ставимъ сюда, $f(c)$ на мѣсто $f(a) + f(b)$; такъ находимъ

$$\sin \Pi(a).\, E^{f(a)} = \sin \Pi(c).\, E^{f(c)}.$$

Но какъ въ прямолинейномъ прямоугольномъ треугольникѣ перпендикулярные бока могутъ измѣняться, тогда какъ гипотенуза остается таже, то можемъ въ этомъ уравненіи полагать $a = o$ не перемѣняя $c$; это даетъ, если къ тому замѣтимъ, что $f(o) = o$ и $\Pi(o) = \frac{\pi}{2}$,

$$1 = \sin \Pi(c).\, E^{f(c)}$$

или

$$E^{f(c)} = \frac{1}{\sin \Pi(c)}$$

для всякой линіи $c$.

Теперь возьмемъ уравненіе (1)

$$\cos \Pi(\alpha) = \sin \Pi(\beta).\, E^{f(a)}$$

и поставимъ сюда $\frac{1}{\sin \Pi(a)}$ на мѣсто $E^{f(a)}$, оно приметъ такой видъ

$$\cos \Pi(\alpha).\, \sin \Pi(a) = \sin \Pi(\beta). \tag{2}$$

Перемѣняя здѣсь $\alpha$, $\beta$ на $b'$, $c$ оставляя $a$ безъ перемѣны получимъ,

$$\sin \Pi(b).\, \sin \Pi(a) = \sin \Pi(c).$$

Уравненіе (2) даетъ съ перемѣной въ немъ буквъ

$$\cos \Pi(\beta).\, \sin \Pi(b) = \sin \Pi(\alpha).$$

Если въ этомъ уравненіи $\beta$, $b$, $a$ перемѣняемъ на $c$, $\alpha'$, $b'$, получимъ,

$$\cos \Pi(c).\, \cos \Pi(\alpha) = \cos \Pi(b). \tag{3}$$

Такимъ же образомъ находимъ

$$\cos \Pi(c).\, \cos \Pi(\beta) = \cos \Pi(a). \tag{4}$$

Уравненіе 2, 3, 4 относятся къ сферическому прямоугольному треугольнику, котораго бока впередъ будемъ означать $a$, $b$, $c$, и гдѣ $A$, $B$ означимъ углы противъ боковъ $a$, $b$, а $\frac{\pi}{2}$ противъ $c$. Въ сказанныхъ уравненіяхъ можемъ ставить $a$ на мѣсто $\Pi(\beta)$, $b$ на мѣсто $\Pi(c)$, $c$ на мѣсто $\Pi(a)$, $\frac{\pi}{2} - A$ на мѣсто $\Pi(a)$; $B$ на мѣсто $\Pi(b)$.

Такимъ образомъ уравненія дѣлаются

$$\left.\begin{aligned} \sin A.\, \sin c &= \sin a \\ \cos b.\, \sin A &= \cos B \\ \cos a.\, \cos b &= \cos c \end{aligned}\right\} \tag{5}$$

Уравненія (5) относятся къ сферическому треугольнику, каковый можетъ быть произведенъ изъ прямолинейнаго прямоугольнаго треугольника и котораго слѣдовательно бока не могутъ превосходить $\frac{\pi}{2}$. Прибавимъ сюда, что если ведемъ чрезъ вершину угла $A$ дугу большаго круга перпендикулярно къ боку $b$, то эта дуга встрѣтитъ или бокъ $a$ или его продолженіе такимъ образомъ, что каждая изъ двухъ дугъ отъ точки пересѣченія до стороны $b$ будетъ $= \frac{\pi}{2}$, а уголъ между этими дугами будетъ $b$.

Послѣ сего не трудно заключить, что во всякомъ прямоугольномъ сферическомъ треугольникѣ если

$$c < \frac{\pi}{2} \text{ то должно быть } a < \frac{\pi}{2} \text{ и } A < \frac{\pi}{2},$$

если

$$c = \frac{\pi}{2} \text{ то должно быть } a = \frac{\pi}{2} \text{ и } A = \frac{\pi}{2},$$

наконецъ если

$$c > \frac{\pi}{2} \text{ то должно быть } a > \frac{\pi}{2} \text{ и } A > \frac{\pi}{2};$$

отсюда слѣдуетъ что предположивъ $a > \frac{\pi}{2}$ надобно предположить вмѣстѣ съ тѣмъ $c > \frac{\pi}{2}$, $A > \frac{\pi}{2}$. Если продолжаемъ въ этомъ случаѣ бока $a$, $c$ по другую сторону бока $b$ до ихъ взаимнаго пересѣченія, то получимъ другой сферическій прямоугольный треугольникъ, котораго бока $\pi - a$, $b$, $\pi - c$, а противоположныя углы $\pi - A$, $B$, $\frac{\pi}{2}$ треугольникъ, къ которому примѣняются уравненія (5). Но уравненія (5) не перемѣняютъ своего вида, если здѣсь вставимъ $\pi - a$ на мѣсто $a$, $\pi - c$ на мѣсто $c$ и $\pi - A$ на мѣсто $A$. Это доказываетъ, что уравненія (5) относятся ко всѣмъ вообще сферическимъ прямоугольнымъ треугольникамъ.

Переходимъ ко всякому сферическому треугольнику гдѣ бока $a$, $b$, $c$ съ противоположными углами $A$, $B$, $C$, не предполагая между углами $A$, $B$, $C$, прямого, потому что уравненія (5) доказаны для такого случая.

Опускаемъ изъ вершины угла $C$ дугу большаго круга $p$ перпендикулярно къ боку $c$, могутъ встрѣтится два случая: 1) Или перпендикулъ $p$ пройдетъ внутри треугольника раздѣляя уголъ $C$ на двѣ части $D$, $C - D$, а такъ же бокъ $c$ на двѣ части $x$ противъ $D$, и $c - x$ противъ $C - D$. 2) Или перпендикулъ $p$ пройдетъ внѣ треугольника, присоединяя къ углу $C$ уголъ $D$ и къ боку $c$ дугу $x$.

Въ первомъ случаѣ данный сферическій треугольникъ будетъ сумма двухъ сферическихъ прямоугольныхъ треугольниковъ. Въ одномъ изъ этихъ двухъ треугольниковъ бока будутъ $p, x, a$ съ противоположными углами $B$, $D$, $\frac{\pi}{2}$ въ другомъ бока $p, c - x, b$ съ противоположными углами $A$, $C - D$, $\frac{\pi}{2}$. Примѣняя сюда къ первому треугольнику уравненія (5) получимъ

$$\left.\begin{aligned} \sin p &= \sin a \sin B \\ \sin x &= \sin a \sin D \\ \cos p \sin D &= \cos B \\ \cos x \sin B &= \cos D \\ \cos a &= \cos p \cos x. \end{aligned}\right\} \tag{A}$$

Другой треугольникъ даетъ подобнымъ образомъ

$$\left.\begin{aligned} \sin p &= \sin b \sin A \\ \sin (c - x) &= \sin b \sin (C - D) \\ \cos p \sin (C - D) &= \cos A \\ \cos p \cos (c - x) &= \cos b. \end{aligned}\right\} \tag{B}$$

Сравненіе двухъ значеній $\sin p$ изъ $(A)$ и $(B)$ даетъ

$$\sin a \sin B = \sin b \sin A \tag{6}$$

Послѣднее изъ уравненій $(B)$ будучи раздѣлена на послѣднее изъ уравненій $(A)$ даетъ

$$\operatorname{tang} x = \frac{\cos b}{\cos a \sin c} - \operatorname{cotg} c.$$

Между тѣмъ соединеніе втораго съ третьимъ и послѣднимъ изъ уравненій $(A)$ даетъ

$$\operatorname{tang} x = \operatorname{tang} a \cos B.$$

Сравненіе двухъ значеній для $\operatorname{tang} x$ приводитъ насъ къ уравненію,

$$\cos b - \cos a \cos c = \sin a \sin c \cos B. \tag{7}$$

Если перпендикулъ падаетъ внѣ даннаго сферическаго треугольника, присоединяя такимъ образомъ дугу $x$ къ боку $c$ и уголъ $D$ къ углу $C$, то произойдутъ какъ и прежде два сферическія прямоугольныя треугольника. Бока одного изъ этихъ двухъ треугольниковъ будутъ $p$, $x$, $a$ съ противоположными углами $\pi - B$, $D$, $\frac{\pi}{2}$; бока другаго треуголь-

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ника будутъ $p, c+x, b$ съ противоположными углами $A, C+D, \frac{\pi}{2}$. Примѣненіе уравненій (5) къ первому изъ этихъ треугольниковъ даетъ:

$$\left.\begin{aligned} \sin p &= \sin a \sin B \\ \sin x &= \sin a \sin D \\ -\cos B &= \cos p \sin D \\ \cos D &= \cos x \sin B \\ \cos a &= \cos p \cos x. \end{aligned}\right\} \qquad (C$$

Второй треугольникъ, котораго бока $p, c+x, b$ съ противоположными углами $A, C+D, \frac{\pi}{2}$ даетъ такимъ же образомъ,

$$\left.\begin{aligned} \sin p &= \sin b \sin A \\ \sin (c+x) &= \sin b \sin (C+D) \\ \cos A &= \cos p \sin (C+D) \\ \cos (C+D) &= \cos (c+x) \sin A \\ \cos b &= \cos p \cos (c+x). \end{aligned}\right\} \qquad (D)$$

Сравненіе двухъ значеній для $\sin p$ изъ уравненій $(C)$ и $(D)$ снова даетъ уравненіе (6).

Изъ уравненій $(C)$ и (D) находимъ

$$\tang x = \cotg c - \frac{\cos b}{\cos a \sin c}$$

Изъ уравненій $(C)$ находимъ еще

$$\tang x = -\tang a \cos B.$$

Сравненіе двухъ значеній для $\tang x$ приводитъ насъ снова къ уравненію (7), которое такимъ образомъ какъ и уравненіе (6) доказано для всѣхъ сферическихъ треугольниковъ вообще.

Уравненіе (7) съ перемѣною въ немъ буквъ даетъ еще два слѣдующія

$$\cos a - \cos b \cos c = \sin b \sin c \cos A$$

$$\cos c - \cos a \cos b = \sin a \sin b \cos C.$$

Умножая послѣднее уравненіе на $\cos b$, присоединяя произведеніе къ первому и раздѣляя сумму на $\sin b$ получимъ

$$\cos a \sin b = \sin c \cos A + \sin a \cos b \cos C.$$

Вставляя сюда значеніе $\sin c$

$$\sin c = \frac{\sin C}{\sin A} \cdot \sin a,$$

согласно съ уравненіемъ (6), потомъ раздѣляя на $\sin a$ получимъ

$$\cotang a \sin b = \cotang A \sin C + \cos b \cos C. \qquad (8)$$

Замѣняя здѣсь $\sin b$ его значеніемъ

$$\sin a \frac{\sin B}{\sin A},$$

потомъ умножая уравненіе на $\sin A$, получимъ

$$\cos a \sin B = \cos b \cos C \sin A + \sin C \cos A,$$

откуда получимъ съ перемѣною буквъ

$$\cos b \sin A = \cos a \cos C \sin B + \sin C \cos B.$$

Изключеніе $\cos b$ изъ двухъ послѣднихъ уравненій приводитъ насъ къ уравненію:

$$\cos a \sin B \sin C = \cos B \cos C + \cos A. \tag{9}$$

Уравненія (6), (7), (8); (9) тѣже, какія обыкновенно даютъ въ сферической тригонометріи, и которыя доказываютъ помощію обыкновеновенной геометріи. Итакъ сферическая тригонометрія остается въ томъ же видѣ, будетъ ли допущено предположеніе, что сумма трехъ угловъ въ прямолинейномъ треугольникѣ всегда равна $\pi$ или предположенію что она $< \pi$, это весьма замѣчательно въ отношеніи къ сферической тригонометріи но неподтверждается для тригонометріи прямолинейной.

Прежде нежели приступимъ къ выводу уравненій, которыя выражаютъ въ пангеометріи зависимость боковъ и угловъ всякаго прямолинейнаго треугольника, мы займемся изслѣдованіемъ, какова должна быть, функція $\Pi(x)$ для всякой линіи $x$. Съ этой цѣлью разсматриваемъ прямолинейный треугольникъ, котораго бока $a, b, c$ съ противоположными углами $\Pi(\alpha), \Pi(\beta), \frac{\pi}{2}$; продолжаемъ бокъ $c$ за вершину угла $\Pi(\beta)$ и дѣлаемъ продолженіе равное $\beta$.

Въ концѣ линіи $\beta$ ставимъ перпендикулъ, который и будетъ параллельный съ бокомъ $a$ и съ продолженіемъ этого бока за вершину угла $\Pi(\beta)$. Ведемъ еще чрезъ вершину угла $\Pi(\alpha)$ параллельную къ тому же продолженію бока $a$. Уголъ, который эта прямая дѣлаетъ съ бокомъ $c$ будетъ $\Pi(c+\beta)$, уголъ, который она дѣлаетъ съ $b$ будетъ $\Pi(b)$. Такимъ образомъ составится уравненіе

$$\Pi(b) = \Pi(c+\beta) + \Pi(\alpha) \tag{$\Pi$}$$

Если линію $\beta$ кладемъ отъ вершины угла $\Pi(\beta)$ на самый бокъ $c$ и къ линіи $\beta$ ведемъ въ концѣ перпендикулъ на сторонѣ угла $\Pi(\beta)$, то эта прямая будетъ параллельною къ продолженію $a$ за вершину прямаго угла. Ведемъ чрезъ вершину угла $\Pi(\alpha)$ параллельную къ послѣднему перпендикулу, которая будетъ слѣдовательно также параллелe ко второму продолженію бока $a$, дѣлая уголъ $\Pi(c-\beta)$ съ бокомъ $c$, а $\Pi(b)$ съ бокомъ $b$; итакъ

$$\Pi(b) = \Pi(c-\beta) - \Pi(\alpha). \tag{$\Pi'$}$$

Легко увѣриться, что это уравненіе справедливо не только для $c$ болѣе $\beta$ но также для $c=\beta$ и $c<\beta$. Дѣйствительно если $c=\beta$, то $\Pi(c-\beta) = \Pi(o) = \frac{\pi}{2}$ съ другой стороны перпендикулъ къ $c$ чрезъ вершину угла $\Pi(\alpha)$ дѣлается параллельнымъ къ $a$, откуда слѣдуетъ что $\Pi(b) = \frac{\pi}{2} - \Pi(\alpha)$, что согласно съ нашимъ уравненіемъ. Если $c < \beta$, то конецъ линіи $\beta$ упадетъ за вершину угла $\Pi(\alpha)$ на разстояніи $\beta - c$ отъ вершины; перпендикулъ къ $\beta$ въ концѣ этой линіи будетъ параллеленъ съ $a$ и съ прямой проведенною чрезъ вершину угла $\Pi(\alpha)$ параллельно къ $a$; откуда слѣдуетъ что два смѣжные угла, которые эта параллельная дѣлаетъ съ бокомъ $c$ острый равенъ $\Pi(\beta - c)$, а тупой равенъ $\Pi(\alpha) + \Pi(b)$. Сумма двухъ смежныхъ угловъ составляетъ $\pi$, слѣдовательно

$$\Pi(\beta - c) + \Pi(\alpha) + \Pi(b) = \pi$$

или

$$\Pi(b) = \pi - \Pi(\beta - c) - \Pi(\alpha).$$

Но согласно съ опредѣленіемъ функціи $\Pi(x)$

$$\pi - \Pi(\beta - c) = \Pi(c - \beta)$$

послѣ чего

$$\Pi(b) = \Pi(c - \beta) - \Pi(\alpha),$$

тоже уравненіе, какое было найдено выше и которое такимъ образомъ доказано для всѣхъ случаевъ.

Два уравненія ($\Pi$) и ($\Pi'$) могутъ быть замѣнены еще такими

$$\Pi(b) = \tfrac{1}{2}\Pi(e + \beta) + \tfrac{1}{2}\Pi(c - \beta)$$
$$\Pi(\alpha) = \tfrac{1}{2}\Pi(c - \beta) - \tfrac{1}{2}\Pi(c - \beta);$$

но уравненіе (3) даетъ

$$\cos\Pi(c) = \frac{\cos\Pi(b)}{\cos\Pi(\alpha)}.$$

Вставляя въ это уравненіе на мѣсто $\Pi(b)$ и $\Pi(\alpha)$ ихъ значенія находимъ:

$$\cos\Pi(c) = \frac{\cos\left\{\frac{1}{2}\Pi(c + \beta) + \frac{1}{2}\Pi(c - \beta)\right\}}{\cos\left\{\frac{1}{2}\Pi(c - \beta) - \frac{1}{2}\Pi(c + \beta)\right\}}.$$

Изъ этого уравненія мы выводимъ слѣдующее

$$\operatorname{tang}^2\tfrac{1}{2}\Pi(c) = \operatorname{tang}\tfrac{1}{2}\Pi(c - \beta)\operatorname{tang}\tfrac{1}{2}\Pi(c + \beta).$$

Такъ какъ линіи $c, \beta$ могутъ измѣняться независимо одна отъ другой во всякомъ прямолинейномъ прямоугольномъ треугольникѣ, то полагая постепенно въ послѣднемъ уравненіи $c = \beta$; $c = 2\beta$, $c = 3\beta$ ..... $c = n\beta$, мы выводимъ изъ этихъ уравненій для всякаго цѣлаго положительнаго числа $n$

$$\operatorname{tang}^n\tfrac{1}{2}\Pi(c) = \operatorname{tang}\tfrac{1}{2}\Pi(nc).$$

Легко доказать справедливость этого уравненія для чиселъ $n$ отрицательныхъ или дробныхъ. Отсюда слѣдуетъ что взявъ за единицу прямыхъ линій такую, которая даетъ:

$$\operatorname{tang}\tfrac{1}{2}\Pi(1) = e^{-1}$$

гдѣ $e$ основаніе неперовыхъ логариѳмовъ, получимъ для всякой линіи $x$

$$\operatorname{tang} \tfrac{1}{2} \Pi(x) = e^{-x}$$

Это выраженіе даетъ $\Pi(x) = \frac{1}{2}\pi$ для $x = o$, и $\Pi(x) = o$ для $x = \infty$, $\Pi(x) = \pi$ для $x = -\infty$, согласно съ тѣмъ, что было принято и доказано выше. Значеніе найденное для $\operatorname{tang} \frac{1}{2} \Pi(x)$ даетъ для всякой линіи $x$

$$\sin \Pi(x) = \frac{2}{e^x + e^{-x}}$$

$$\cos \Pi(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

и для двухъ произвольныхъ линій $x, y$

$$\sin \Pi(x+y) = \frac{\sin \Pi(x) \sin \Pi(y)}{1 + \cos \Pi(x) \cos \Pi(y)}$$

$$\sin \Pi(x-y) = \frac{\sin \Pi(x) \sin \Pi(y)}{1 - \cos \Pi(x) \cos \Pi(y)}$$

$$\cos \Pi(x+y) = \frac{\cos \Pi(x) + \cos \Pi(y)}{1 + \cos \Pi(x) \cos \Pi(y)}$$

$$\cos \Pi(x-y) = \frac{\cos \Pi(x) - \cos \Pi(y)}{1 - \cos \Pi(x) \cos \Pi(y)}$$

$$\operatorname{tang} \Pi(x-y) = \frac{\sin \Pi(x) \sin \Pi(y)}{\cos \Pi(x) + \cos \Pi(y)}.$$

Уравненія (2), (3), (4), найденныя для сферическихъ прямоугольныхъ треугольниковъ, относятся также къ прямолинейному прямоугольному треугольнику котораго бока $a$, $b$, $c$ съ противоположными углами $\Pi(\alpha)$, $\Pi(\beta)$, $\frac{\pi}{2}$. Итакъ поставляя $A$ на мѣсто $\Pi(\alpha)$ $B$ на мѣсто $\Pi(\beta)$ получимъ для всякаго прямолинйнаго прямоугольнаго треугольника, котораго бока $a, b, c$, и гдѣ $A$ уголъ противъ $a$, $B$ уголъ противъ $b$ и $\frac{\pi}{2}$ уголъ противъ $c$, слѣдующія уравненія:

$$\left.\begin{aligned} \sin \Pi(a) \cos A &= \sin B \\ \sin \Pi(c) \cos A &= \cos \Pi(b) \\ \cos \Pi(c) \cos A &= \cos \Pi(a). \end{aligned}\right\} \tag{10}$$

Къ этимъ уравненіямъ присоединимъ еще слѣдующее, которое также было доказано выше

$$\sin \Pi(a) \sin \Pi(b) = \sin \Pi(c). \tag{11}$$

Первая изъ уравненій (10) съ перемѣною въ немъ буквъ можетъ быть еще написана такъ:

$$\sin \Pi(b) \cos B = \sin A.$$

Вставляя сюда значеніе $\cos B$ изъ третьяго уравненія (10) получимъ:

$$\sin \Pi(b) \cos \Pi(a) = \sin A \cos \Pi(c).$$

Исключая изъ этого уравненія $\sin \Pi(b)$ помощью уравненія (11) получимъ:

$$\operatorname{tang} \Pi(c) = \sin A \operatorname{tang} \Pi(a). \tag{12}$$

Пусть теперь $a, b, c$ будутъ бока вообще какого нибудь прямолинейнаго треугольника, потомъ $A, B, C$ углы противъ этихъ боковъ. Опускаемъ перпендикулъ $p$ изъ вершины угла $C$ на бокъ $c$. Если $p$ упадетъ внутри треугольника и раздѣлитъ уголъ $C$ на два угла $D$ и $C - D$ и бокъ $c$ на двѣ части: $x$ противъ $D$, $c - x$ противъ $C - D$, то произойдутъ два прямолинейныя прямоугольныя треугольника. Въ одномъ бока будутъ $p, x, b$ съ противоположными углами $A, D, \frac{\pi}{2}$; въ другомъ бока будутъ $p, c - x, a$ съ противоположными углами $B, C - D, \frac{\pi}{2}$. Примѣненіе уравненія (12) къ первому изъ этихъ двухъ треугольниковъ даетъ:

$$\operatorname{tang} \Pi(b) = \sin A \operatorname{tang} \Pi(p).$$

Второй изъ эихъ двухъ треугольниковъ даетъ такимъ же образомъ:

$$\operatorname{tang} \Pi(a) = \sin B \operatorname{tang} \Pi(p);$$

откуда заключимъ:

$$\sin A \operatorname{tang} \Pi(a) = \sin B \operatorname{tang} \Pi(b). \tag{13}$$

Примѣненіе уравненій (10) и (11) къ первому изъ двухъ треугольниковъ даетъ:

$$\cos \Pi(b) \cos A = \cos \Pi(x),$$

$$\sin \Pi(x) \sin \Pi(p) = \sin \Pi(b).$$

Второй треугольникъ даетъ:

$$\sin \Pi(p) \sin \Pi(c - x) = \sin \Pi(a).$$

Вставляя въ это послѣднее уравненіе на мѣсто $\sin \Pi(c - x)$ его значеніе, взятое изъ общей формулы, найденной выше для $\sin \Pi(x - y)$, получимъ:

$$\frac{\sin \Pi(a)}{\sin \Pi(p)} = \frac{\sin \Pi(c) \sin \Pi(x)}{1 - \cos \Pi(c) \cos \Pi(x)};$$

откуда мы выводимъ, вставляя

$$\sin \Pi(p) = \frac{\sin \Pi(b)}{\sin \Pi(x)}$$

$$\cos \Pi(x) = \cos \Pi(b) \cos A$$

слѣдующее уравненіе

$$1 - \cos \Pi(b) \cos \Pi(c) \cos A = \frac{\sin \Pi(b) \sin \Pi(c)}{\sin \Pi(a)}. \tag{14}$$

Уравненія (13) (14) повѣряются сами собой, для $A=\frac{\pi}{2}$ когда перпендикулъ $p$ совпадаетъ съ бокомъ $b$, потому что въ этомъ случаѣ уравненіе (13) дѣлается уравненіемъ (12) а уравненіе (14) переходитъ въ уравненіе (11), уравненія доказанныя для всякаго прямолинейнаго прямоугольнаго треугольника.

Если перпендикулъ $p$ падаетъ внѣ треугольника на продолженіе бока $c$, прибавляя линію $x$ къ боку $c$ и уголъ $D$ къ углу $C$, то образуются два прямоугольныя треугольника. Бока одного будутъ $p$, $x$, $b$ съ противоположными углами $(\pi - A), D, \frac{\pi}{2}$, бока другаго будутъ $p, c+x$, $a$ съ противоположными углами $B$, $C+D, \frac{\pi}{2}$. Примѣненіе уравненія (12) къ первому изъ этихъ двухъ треугольниковъ даетъ:

$$\operatorname{tang} \Pi(b) = \sin A \operatorname{tang} \Pi(p).$$

Изъ другаго треугольника мы выводимъ такимъ же образомъ:

$$\operatorname{tang} \Pi(a) = \sin B \operatorname{tang} \Pi(p).$$

Исключая $\operatorname{tang} \Pi(p)$ изъ двухъ послѣднихъ уравненій снова находимъ уравненіе (13). Примѣненіе уравненій (10) и (11) къ первому изъ двухъ треугольниковъ даетъ:

$$-\cos \Pi(b) \cos A = \cos \Pi(x)$$
$$\sin \Pi(b) = \sin \Pi(x) \sin \Pi(p).$$

Изъ втораго треугольника мы находимъ такимъ же образомъ:

$$\sin \Pi(a) = \sin \Pi(p) \sin \Pi(c+x)$$

Замѣняя въ этомъ уравненіи $\sin \Pi(c+x)$ его значеніемъ, взятымъ изъ общей формулы, найденной выше для $\sin \Pi(x+y)$ получимъ:

$$\frac{\sin \Pi(a)}{\sin \Pi(p)} = \frac{\sin \Pi(c) \sin \Pi(x)}{1+\cos \Pi(c) \cos \Pi(x)}.$$

Вставляя сюда:

$$\sin \Pi(p) = \frac{\sin \Pi(b)}{\sin \Pi(x)}$$
$$\cos \Pi(x) = -\cos \Pi(b) \cos A$$

получимъ:

$$\frac{\sin \Pi(a)}{\sin \Pi(b)} = \frac{\sin \Pi(c)}{1-\cos \Pi(b) \cos \Pi(c) \cos A},$$

уравненіе тожественное съ уравненіемъ (14).

Таким образомъ уравненія (13) (14) доказаны для всякаго прямолинейнаго треугольника.

Уравненіе (14) съ перемѣною буквъ даетъ:

$$1 - \cos \Pi(c) \cos \Pi(a) \cos B = \frac{\sin \Pi(c) \sin \Pi(a)}{\sin \Pi(b)}$$

Умножая это уравненіе почленно на уравненіе (14) получимъ:

$$1 - \cos \Pi(c) \cos \Pi(a) \cos B - \cos \Pi(b) \cos \Pi(c) \cos A + \cos \Pi(a) \cos \Pi(b) \cos^2 \Pi(c) \cos A \cos B = \sin^2 \Pi(c)$$

или:

$$\cos^2 \Pi(c) - \cos \Pi(c) \cos \Pi(a) \cos B - \cos \Pi(b) \cos \Pi(c) \cos A + \cos \Pi(a) \cos \Pi(b) \cos^2 \Pi(c) \cos A \cos B = 0.$$

Откидывая отъ этаго уравненія общій множитель $\cos \Pi(c)$ получимъ:

$$\cos \Pi(c) - \cos \Pi(a) \cos B - \cos \Pi(b) \cos A + \cos \Pi(a) \cos \Pi(b) \cos \Pi(c) \cos A \cos B = 0.$$

Такимъ же образомъ находимъ:

$$\cos \Pi(a) - \cos \Pi(b) \cos C - \cos \Pi(c) \cos B + \cos \Pi(a) \cos \Pi(b) \cos \Pi(c) \cos B \cos C = 0.$$

Умножаемъ это уравненіе на $\cos A$ и вычитаемъ произведеніе изъ произведенія предыдущаго уравненія на $\cos C$, получимъ:

$$\cos \Pi(a) \left\{\cos A + \cos B \cos C\right\} = \cos \Pi(c) \left\{\cos C + \cos A \cos B\right\}.$$

Возвышая обѣ части этого уравненія въ квадратъ и раздѣляя потомъ на $\cos^2 \Pi(c)$ оно приметъ слѣдующій видъ:

$$\frac{\cos^2 \Pi(a)}{\cos^2 \Pi(c)} \left\{\cos A + \cos B \cos C\right\}^2 = \left\{\cos C + \cos A \cos B\right\}^2.$$

Между тѣмъ уравненіе (13) даетъ:

$$\frac{1}{\cos^2 \Pi(c)} = 1 + \frac{\sin^2 A}{\sin^2 C} \operatorname{tang}^2 \Pi(a).$$

Если мы въ предпослѣднее уравненіе вставимъ вмѣсто $\frac{1}{\cos^2 \Pi(c)}$ его значеніе изъ послѣдняго уравненія, то получимъ:

$$\cos^2 \Pi(a) + \frac{\sin^2 A}{\sin^2 C} \sin^2 \Pi(a) = \left\{\frac{\cos C + \cos A \cos B}{\cos A + \cos B \cos C}\right\}^2$$

потомъ:

$$\sin^2 \Pi(a) \left\{1 - \frac{\sin^2 A}{\sin^2 C}\right\} = \frac{\sin^2 B (\sin^2 C - \sin^2 A)}{(\cos A + \cos B \cos C)^2}.$$

Раздѣляя это уравненіе на $\sin^2 C - \sin^2 A$ и извлекая корень квадратный, находимъ:

$$\sin \Pi(a) = \frac{\sin B \sin C}{\cos A + \cos B \cos C}.$$

безъ обоюдности въ знакахъ, потому что члены послѣдняго уравненія оба положительны. Дѣйствительно $\Pi(a) < \frac{\pi}{2}$; $B < \pi$; $C < \pi$, откуда слѣдуетъ, что синусы этихъ угловъ положительны, потомъ:

$$\cos A + \cos(B + C) = 2\cos\tfrac{1}{2}(A + B + C)\cos\tfrac{1}{2}(B + C - A),$$

но $A+B+C<\pi$, слѣдовательно $\cos\frac{1}{2}(A+B+C)$ положителенъ, такъ же, какъ $\cos\frac{1}{2}(B+C-A)$; прикладывая къ обоимъ членамъ послѣдняго уравненія положительное число $\sin B \sin C$ находимъ: $\cos A + \cos B\cos C > O$. Итакъ во всякомъ прямолинейномъ треугольникѣ:

$$\cos A + \cos B\cos C = \frac{\sin B\sin C}{\sin\Pi(a)} \tag{15}$$

Умноженіе уравненія (14) почленно на слѣдующее уравненіе, которое отсюда происходитъ съ перемѣною буквъ

$$1 - \cos\Pi(a)\cos\Pi(b)\cos C = \frac{\sin\Pi(a)\sin\Pi(b)}{\sin\Pi(c)} \tag{16}$$

даетъ:

$$\{1 - \cos\Pi(a)\cos\Pi(b)\cos C\}\ \{1 - \cos\Pi(b)\cos\Pi(c)\cos A\} = \sin^2\Pi(b),$$

которому послѣ сдѣланнаго умноженія можно дать слѣдующій видъ:

$$\cos^2\Pi(b) - \cos\Pi(a)\cos\Pi(b)\cos C - \cos\Pi(b)\cos\Pi(c)\cos A + \cos^2\Pi(b)\cos\Pi(a)\cos\Pi(c)\cos A\cos C = O$$

или раздѣляя на $\cos\Pi(b)$

$$\cos\Pi(b) - \cos\Pi(a)\cos C - \cos\Pi(c)\cos A + \cos\Pi(a)\cos\Pi(b)\cos\Pi(c)\cos A\cos C = O. \tag{17}$$

Но мы находимъ согласно съ уравненіемъ (13).

$$\cos\Pi(c) = \frac{\sin\Pi(c)\sin C}{\sin A}\operatorname{cotang}\Pi(a);$$

въ этомъ уравненіи мы можемъ вставить вмѣсто $\sin\Pi(c)$, его значеніе, взятое изъ уравненія (16), получимъ:

$$\cos\Pi(c) = \frac{\sin\Pi(b)\cos\Pi(a)\sin C}{\{1 - \cos\Pi(a)\cos\Pi(b)\cos C\}\sin A}.$$

Вставляя это значеніе $\cos\Pi(c)$ въ уравненіе (17) получимъ:

$$\operatorname{cotang} A\sin C\sin\Pi(b) + \cos C = \frac{\cos\Pi(b)}{\cos\Pi(a)} \tag{18}$$

Соединяемъ уравненія (13), (14), (15) и (18), которые выражаютъ зависимость угловъ и боковъ всякаго прямолинейнаго треугольника, такимъ образомъ чтобы облегчить ихъ примѣненіе

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$$\left.\begin{aligned}
&\sin A \operatorname{tang} \Pi(a) = \sin B \operatorname{tang} \Pi(b)\\
&1 - \cos \Pi(b) \cos \Pi(c) \cos A = \frac{\sin \Pi(b) \sin \Pi(c)}{\sin \Pi(a)}\\
&\cos A + \cos B \cos C = \frac{\sin B \sin C}{\sin \Pi(a)}\\
&\operatorname{cotang} A \sin C \sin \Pi(b) + \cos C = \frac{\cos \Pi(b)}{\cos \Pi(a)}
\end{aligned}\right\}$$

Съ этими уравненіями Пангеометрія переходитъ въ Аналитику и такимъ образомъ составляетъ особенную полную геометрическую теорію; уравненія (19) служатъ для представленія кривыхъ линій уравненіями между координатами ихъ точекъ, для вычисленія длины и площадей кривыхъ, поверхности и объема тѣлъ, какъ я показалъ это въ ученыхъ запискахъ Казанскаго Университета за 1829 годъ.

Выше было замѣчено, что Пангеометрія переходитъ въ обыкновенную геометрію съ предположеніемъ линіи чрезвычайно малыхъ. Теперь мы можемъ повѣрить это заявленіе. Для всякой линіи $x$ чрезвычайно малой, можемъ довольствоваться приближенными значеніями

$$\operatorname{cotang} \Pi(x) = x$$
$$\sin \Pi(x) = 1 - \tfrac{1}{2} x^2$$
$$\cos \Pi(x) = x.$$

Если мы разсматриваемъ бока въ треугольникѣ какъ безконечно малыя перваго порядка, и если мы пренебрегаемъ безконечно малыми величинами далѣе втораго порядка, то уравненія (19) послѣ вставленія приближенныхъ значеній $\sin \Pi(a)$, $\sin \Pi(b)$ и проч. примутъ слѣдующій видъ:

$$b \sin A = a \sin B$$
$$a^2 = b^2 + c^2 - 2\, bc \cos A$$
$$\cos A + \cos (B + C) = O$$
$$a \sin (A + C) = b \sin A.$$

Первыя два изъ этихъ уравненій извѣстны въ обыковенной тригонометріи. Послѣнія два даютъ

$$A + B + C = \pi$$

Чтобы дать примѣръ, какимъ образомъ кривыя линіи опредѣляются помощью координатъ ихъ точекъ, называемъ $y$ перпендикулъ, опущенный отъ точки на кругѣ къ одному изъ его неподвижныхъ поперечниковъ, котораго величину означаемъ $2\,r$, слѣдовательно подъ $r$ разумѣемъ полупоперечникъ круга.

Назовемъ еще $x$ часть этого поперечника отъ центра до перпендикула $y$. Примѣненіе уравненія (11) къ прямоугольному треугольнику, котораго бока $x, y, r$ даетъ

$$\sin \Pi(x) \sin \Pi(y) = \sin \Pi(r) \tag{20}$$

что и составляетъ уравненіе круга между прямыхъ перпендикулярныхъ координатъ $x$, $y$. Если мы считаемъ $x$ отъ конца поперечника въ кругѣ, то уравненіе (20) приметъ видъ:

$$\sin \Pi (r - x) \sin \Pi (y) = \sin \Pi (r)$$

или что все равно:

$$2 (e^{r} + e^{-r}) = (e^{r-x} + e^{-r+x}) (e^{y} + e^{-y})$$

Если мы раздѣляемъ это уравненіе на $e^{r}$ и если потомъ положимъ $r = \infty$, то получимъ уравненіе для предѣльнаго круга:

$$2 = (e^{y} + e^{-y}) e^{-x}$$

или

$$\sin \Pi (y) = \operatorname{tang} \tfrac{1}{2} \Pi (x).$$

Изъ опредѣленія предѣльнаго круга слѣдуетъ, что двѣ оси предѣльнаго круга, проведеныя чрезъ концы одной хорды, бываютъ наклонены къ этой хордѣ подъ равными углами, свойство, которое можетъ быть принято за опредѣленіе предѣльнаго круга, и откуда можно тоже вывести уравненіе этой кривой, разсматривая треугольникъ, котораго бока $x$, $y$ и хорда $2a$ предѣльнаго круга. Углы этого треугольника будутъ: $\Pi (a) - \Pi (y)$ противъ $x$, $\Pi (a)$ противъ $y$ и $\frac{\pi}{2}$ противъ $2a$. Согласно съ уравненіями (10) (11) въ этомъ треугольникѣ будетъ:

$$\sin \Pi (x) \sin \Pi (y) = \sin \Pi (2a)$$
$$\sin \Pi (x) \cos \{\Pi (a) - \Pi (y)\} = \sin \Pi (a)$$
$$\sin \Pi (y) \cos \Pi (a) = \sin \{\Pi (a) - \Pi (y)\}.$$

Послѣднее уравненіе даетъ:

$$2 \operatorname{tang} \Pi (y) = \operatorname{tang} \Pi (a), \tag{21}$$

а первое уравненіе можетъ быть написано такъ:

$$\sin \Pi (x) \sin \Pi (y) = \frac{\sin^2 \Pi (a)}{1 + \cos^2 \Pi (a)}$$

Вставляя въ это уравненіе вмѣсто $\sin^2 \Pi (a)$, $1 + \cos^2 \Pi (a)$ ихъ значенія, выраженныя посредствомъ $\operatorname{tang}^2 \Pi (a)$, и вводя сюда значеніе $\operatorname{tang}^2 \Pi (a)$, изъ уравненія (21) находимъ:

$$\sin \Pi (x) \sin \Pi (y) = \frac{2 \operatorname{tang}^2 \Pi (y)}{1 + 2 \operatorname{tang}^2 \Pi (y)}$$

потомъ

$$\sin \Pi (x) = \frac{2 \sin \Pi (y)}{1 + \sin^2 \Pi (y)}$$

откуда выводимъ:

$$2 \cos^2 \tfrac{1}{2} \Pi(x') = \frac{\{1 + \sin \Pi(y)\}^2}{1 + \sin^2 \Pi(y)}$$

$$2 \sin^2 \tfrac{1}{2} \Pi(x') = \frac{\{1 - \sin \Pi(y)\}^2}{1 + \sin^2 \Pi(y)}$$

Раздѣляя послѣднее изъ этихъ уравненій на предпослѣднее и извлекая квадратный корень получимъ:

$$\operatorname{tang} \tfrac{1}{2} \Pi(x') = \frac{1 - \sin \Pi(y)}{1 + \sin \Pi(y)}$$

отсюда

$$\sin \Pi(y) = \frac{1 - \operatorname{tang} \tfrac{1}{2} \Pi(x')}{1 + \operatorname{teng} \tfrac{1}{2} \Pi(x')}.$$

Второй части этого уравненія можно дать еще видъ:

$$\frac{\cos \tfrac{1}{2} \Pi(x') - \sin \tfrac{1}{2} \Pi(x')}{\cos \tfrac{1}{2} \Pi(x') + \operatorname{snn} \tfrac{1}{2} \Pi(x)}$$

$$\frac{\sin \{\tfrac{1}{4}\pi - \tfrac{1}{2} \Pi(x')\}}{\cos \{\tfrac{1}{4}\pi - \tfrac{1}{2} \Pi(x')\}} = \frac{\sin \tfrac{1}{2} \Pi(x)}{\cos \tfrac{1}{2} \Pi(x)} = \operatorname{tang} \tfrac{1}{2} \Pi(x)$$

а слѣдовательно

$$\sin \Pi(y) = \operatorname{tang} \tfrac{1}{2} \Pi(x)$$

какъ найдено было выше.

Чтобы дать примѣръ какъ вычисляется длина кривой линіи, ищемъ длину окружности круга, котораго полупоперечникъ $r$. Ведемъ два полупоперечника, которыхъ уголъ при центрѣ пусть будетъ $\frac{2\pi}{n}$ гдѣ $n$ означаетъ цѣлое число. Опускаемъ отъ конца одного изъ полупоперечниковъ перпендикулъ $p$ на другой полупоперечникъ. Произведеніе $np$ будетъ тѣмъ менѣе разниться съ длиною окружности, чѣмъ число $n$ болѣе.

Прямоугольный треугольникъ, гдѣ $p$ одинъ изъ катетовъ, $r$ гипотенуза, а $\frac{2\pi}{n}$ уголъ противъ $p$, даетъ (уравненіе 13).

$$\sin \frac{2\pi}{n} \operatorname{tang} \Pi(p) = \operatorname{tang} \Pi(r)$$

Но извѣстно, что

$$\left\{ n \sin \frac{2\pi}{n} \right\} = 2\pi \text{ для } n = \infty,$$

между тѣмъ какъ

$$\frac{\operatorname{tang} \Pi(p)}{n} = \frac{2}{n(e^{p} - e^{-p})}.$$

и

$$n(e^{p} - e^{-p}) = 2np$$

съ точностію тѣмъ большею, чѣмъ число $n$ болѣе, а слѣдовательно $p$ менѣе. Послѣ чего длина окружности $r = np = 2\pi \operatorname{cotg} \Pi(r)$ или длина окружности $r = \pi(e^{r} - e^{-r})$; это даетъ для $r$ чрезвычайно малаго числа длина окружности $r = 2\pi r$, какъ въ обыкновенной геометріи.

Опредѣляемъ еще дугу $s$ предѣльнаго круга посредствомъ координатъ, $y$ перпендикула, опущенаго отъ одного конца дуги $s$ на ось, проведенную чрезъ другой конецъ, а $x$ часть этой оси между вершиною и перпендикуломъ $y$. Пусть $c$ хорда дуги $s$, пусть также $c_1$, $c_2$, $c_3$ и т. д. хорды дугъ $\frac{1}{2}s, \frac{1}{2^2}s, \frac{1}{2^3}\ldots\ldots$ Мы доказали выше (уравненіе 21), что:

$$\operatorname{cotg} \Pi(y) = 2 \operatorname{cotg} \Pi(\tfrac{1}{2}c)$$

Подобнымъ образомъ будетъ:

$$\operatorname{cotg} \Pi(\tfrac{1}{2}c) = 2 \operatorname{cotg} \Pi(\tfrac{1}{2}c_1)$$
$$\operatorname{cotg} \Pi(\tfrac{1}{2}c_1) = 2 \operatorname{cotg} \Pi(\tfrac{1}{2}c_2)$$
$$\operatorname{cotg} \Pi(\tfrac{1}{2}c_2) = 2 \operatorname{cotg} \Pi(\tfrac{1}{2}c_3)$$

и вообще для всякаго числа $n$ положительнаго и цѣлаго

$$\operatorname{cotg} \Pi(\tfrac{1}{2}c_{n-1}) = 2 \operatorname{cotg} \Pi(\tfrac{1}{2}c_n)$$

Откуда заключаемъ:

$$\operatorname{cotg} \Pi(y) = 2^{n+1} \operatorname{cotg} \Pi(\tfrac{1}{2}c_n).$$

Если $n$ число весьма великое и слѣдовательно $c_n$ число весьма малое получимъ:

$$2^{n+1} \operatorname{cotg} \Pi(\tfrac{1}{2}c_n) = 2^n c_n.$$

Но $$2^n c_n = s \text{ для } n = \infty$$

откуда слѣдуетъ что

$$s = \operatorname{cotg} \Pi(y) \tag{22}$$

Опредѣляемъ еще дугу $s$ предѣльнаго круга, помощію $t$, части касательной, проведенной къ предѣльному кругу въ вершинѣ оси чрезъ конецъ дуги $s$, между точкой прикосновенія и точкой пересѣченія касательной съ осью чрезъ другой конецъ дуги $s$, т. е. опредѣляемъ функцію $L(t)$. Въ треугольникѣ, гдѣ бока $c$, $t$, $f(t)$ съ противоположными углами $\Pi(t)$, $\pi - \Pi(\frac{1}{2}c)$, $\frac{1}{2}\pi - \Pi(\frac{1}{2}c)$, находимъ, примѣняя сюда уравненіе (13)

$$\sin \Pi(t) \operatorname{tang} \Pi(c) = \sin \Pi(\tfrac{1}{2}c) \operatorname{tang} \Pi(t)$$

но мы видѣли (уравненіе 21), что

$$\operatorname{tang} \Pi(\tfrac{1}{2}c) = 2 \operatorname{tang} \Pi(y)$$

къ чему прибавимъ еще замѣчаніе, что

$$\operatorname{tang} \Pi(c) = \frac{\sin^2 \Pi(\frac{1}{2}c)}{2 \cos \Pi(\frac{1}{2}c)}$$

Послѣ чего

$$\cos \Pi(t) = 2 \operatorname{cotg} \Pi(\tfrac{1}{2} c)$$

т. е. въ слѣдствія уравненія (22)

$$\cos \Pi(t) = s = L(t)$$

Уравненіе прямой линіи довольно сложно, если это уравненіе должно быть общимъ и относится ко всякому положенію прямой въ отношеніи къ осямъ координатъ. Опускаемъ изъ опредѣленной точки на данной прямой перпендикулъ $a$ къ оси $x$ и называемъ $L$ уголъ, который этотъ перпендикулъ дѣлаетъ съ прямой. Называемъ еще $y$ перпендикулъ, отпущенный къ оси $x$ изъ другой точки на данной прямой. Пусть $l$ разстояніе второй точки отъ первой, и пусть $x$ будетъ часть оси $x$ между двухъ перпендикуловъ. Ведему прямую $r$ отъ вершины $a$ до конца $y$; составятся два треугольника: 1) прямоугольный, котораго бока $a$, $x$, $r$ съ противоположными углами $A, X, \frac{\pi}{2}$, 2) съ боками $y, r, l$ съ противоположными углами $L - X, C, \frac{\pi}{2} - A$. Примѣненіе уравненій (10) (11) къ первому изъ этихъ треугольниковъ даетъ:

$$\begin{gathered}
\sin \Pi(x) \sin \Pi(a) = \sin \Pi(r) \\
\sin \Pi(x) \cos X = \sin A \\
\sin \Pi(a) \cos A = \sin X \\
\cos \Pi(r) \cos A = \cos \Pi(x) \\
\cos \Pi(r) \cos X = \cos \Pi(a)
\end{gathered}$$

Изъ этихъ уравненій выводимъ:

$$\begin{aligned}
&\operatorname{tang} A = \operatorname{tang} \Pi(x) \cos \Pi(a) \\
&\operatorname{tang} \Pi(r) = \operatorname{tang} \Pi(x) \sin \Pi(a) \cos A \\
&\operatorname{tang} X = \operatorname{tang} \Pi(a) \cos \Pi(x) \\
&\cos \Pi(x) = \cos \Pi(r) \cos A \\
&\sin X = \sin \Pi(a) \cos A
\end{aligned}$$

Примѣненіе послѣдняго изъ уравненій (19) ко второму треугольнику даетъ:

$$\operatorname{cotg}(L - X) \cos A \sin \Pi(r) + \sin A = \frac{\cos \Pi(r)}{\cos \Pi(y)}$$

откуда слѣдуетъ что:

$$\cos \Pi(y) = \frac{\cos \Pi(r)}{\operatorname{cotg}(L - X) \cos A \sin \Pi(r) + \sin A}$$

$$\cos \Pi(y) = \frac{\cos \Pi(r)\ (\operatorname{tang} L - \operatorname{tang} X)}{\{1 + \operatorname{tang} L \operatorname{tang} X\} \cos A \sin \Pi(a) \sin \Pi(x) + \sin A\ \{\operatorname{tang} L - \operatorname{tang} X\}}$$

Вставляя въ это уравненіе на мѣсто $\operatorname{tang} X$ его значеніе получимъ:

$$\cos \Pi(y) = \frac{\cos \Pi(r) \left\{ \operatorname{tang} L - \operatorname{tang} \Pi(a) \cos \Pi(x) \right\}}{\left\{ 1 + \operatorname{tang} L \operatorname{tang} \Pi(a) \cos \Pi(x) \right\} \cos A \sin \Pi(a) \sin \Pi(x) + \sin A \left\{ \operatorname{tang} L - \operatorname{tang} \Pi(a) \cos \Pi(x) \right\}}.$$

Вставляемъ въ это уравненіе вмѣсто $\cos \Pi(r)$ его значеніе, мы находимъ послѣ всѣхъ привѣденій что:

$$\cos \Pi(y) = \frac{\cos \Pi(a)}{\sin \Pi(x)} - \sin \Pi(a) \operatorname{cotg} \Pi(x) \operatorname{cotg} L. \tag{23}$$

Если прямая параллельна съ осью $x$, то $L = \Pi(a)$ и уравненіе (23) приметъ видъ:

$$\cos \Pi(y) = \frac{\cos \Pi(a)}{\sin \Pi(x)} - \frac{\cos \Pi(a)}{\operatorname{tang} \Pi(x)}$$

или

$$\cos \Pi(y) = \cos \Pi(a)\, e^{-x} \tag{24}$$

Если называемъ $s$, $s'$ длину двухъ дугъ предѣльнаго круга, заключенныхъ между осью $x$ и прямой съ ней параллельной и при томъ, изъ которыхъ дугъ первая $s$, касательная къ $a$ въ основаніи, а вторая касательная къ $y$ въ основаніи, то получимъ согласно съ тѣмъ, что было доказано:

$$s = \cos \Pi(a)$$
$$s' = \cos \Pi(y)$$

послѣ чего

$$s' = s\, e^{-x}$$

гдѣ $x$ разстояніе между двухъ дугъ $s$ и $s'$. Это уравненіе показываетъ, что постоянное $E$, введенное выше, чтобы означить содержаніе двухъ дугъ предѣльнаго круга между двумя параллельными, дугъ, которыхъ разстояніе равно единицѣ, равняется $e$ т. е. основанію Неперовыхъ логариѳмъ.

Если въ уравненіи (23) положимъ $a = o$ и если ставимъ $\pi - L$ на мѣсто $L$, то получимъ:

$$\cos \Pi(y) = \operatorname{cotg} \Pi(x) \operatorname{cotg} L;$$

это уравненіе принадлежитъ прямой линіи, которая проходитъ чрезъ начало координатъ $x$, дѣлая уголъ $L$ съ осью $x$, что согласуется съ уравненіемъ (10).

Разсматриваемъ четыреугольникъ, котораго два бока $a$, $y$ перпендикулярны къ третьему боку $x$. Пусть $c$ будетъ четвертый бокъ, а $\varphi$ уголъ между $a$ и $c$ тогда какъ уголъ между $c$ и $y$ прямой. Проводимъ діагоналъ $r$ изъ вершины угла $\varphi$ къ вершинѣ противоположнаго прямаго угла. Этотъ діагоналъ дѣлитъ четыреугольникъ на два прямоугольныя треугольника; бока одного изъ этихъ треугольниковъ будутъ $a$, $x$, $r$ съ противоположными углами $A$, $X$, $\frac{\pi}{2}$, бока другаго $y$, $c$, $r$ съ противоположными углами $\varphi - X$, $\frac{\pi}{2} - A$, $\frac{\pi}{2}$.

Примѣненіе уравненій (10) (11) (13) къ первому изъ этихъ треугольниковъ дастъ:

$$\left.\begin{aligned}&\sin \Pi(r) = \sin \Pi(a) \sin \Pi(x)\\&\sin A \operatorname{tang} \Pi(a) = \sin X \operatorname{tang} \Pi(x)\\&\cos \Pi(r) \cos A = \cos \Pi(x)\\&\cos \Pi(r) \cos X = \cos \Pi(a);\end{aligned}\right\}\qquad (G)$$

второй треугольникъ даетъ такимъ же образомъ слѣдующія уравненія

$$\left.\begin{aligned}&\sin \Pi(y) \sin \Pi(c) = \sin \Pi(r)\\&\sin \Pi(y) \cos(\varphi - X) = \cos A\\&\cos \Pi(r) \cos(\varphi - X) = \cos \Pi(c)\\&\cos \Pi(r) \sin A = \cos \Pi(y)\end{aligned}\right\}\qquad (H)$$

Уравненіе (12) приложенное къ первому треугольнику даетъ:

$$\left.\begin{aligned}&\operatorname{tang} \Pi(r) = \sin X \operatorname{tang} \Pi(x)\\&\operatorname{tang} \Pi(r) = \sin A \operatorname{tang} \Pi(a)\end{aligned}\right\}\qquad (K)$$

тогда какъ примѣненіе того же уравненія ко второму треугольнику даетъ:

$$\left.\begin{aligned}&\operatorname{tang} \Pi(r) = \sin(\varphi - X) \operatorname{tang} \Pi(y)\\&\operatorname{tang} \Pi(r) = \cos A \operatorname{tang} \Pi(c)\end{aligned}\right\}\qquad (L)$$

Вставляя во второе изъ уравненій (K) на мѣсто $\sin \Pi(r)$ его значеніе взятое изъ уравненій (G) находимъ:

$$\cos \Pi(r) = \frac{\sin \Pi(x) \cos \Pi(a)}{\sin A}.$$

Вставляя это значеніе $\cos \Pi(r)$ въ послѣднее изъ уравненій (H) получимъ:

$$\cos \Pi(y) = \sin \Pi(x) \cos \Pi(a). \qquad (25)$$

Раздѣляя почленно послѣднее изъ уравненій (H) на третіе изъ уравненій (G) получимъ:

$$\operatorname{tang} A = \frac{\cos \Pi(y)}{\cos \Pi(x)};$$

вставляемъ въ это уравненіе на мѣсто $\cos \Pi(y)$ его значеніе изъ (25) будетъ:

$$\operatorname{tang} A = \operatorname{tang} \Pi(x) \cos \Pi(a).$$

Дѣленіе втораго изъ уравненій (G) на послѣднее изъ этихъ же уравненій почленно даетъ:

$$\frac{\operatorname{tang} X \operatorname{tang} \Pi(x)}{\cos \Pi(r)} = \frac{\sin A \operatorname{tang} \Pi(a)}{\cos \Pi(a)}.$$

Вставляя въ это уравненіе на мѣсто $\sin A$ его значеніе, взятое изъ послѣдняго въ уравненіяхъ (H) получимъ:

$$\operatorname{tang} X = \frac{\cos \Pi(y) \operatorname{tang} \Pi(a)}{\cos \Pi(a)} \operatorname{cotg} \Pi(x).$$

Замѣняя въ этомъ уравненіи $\cos\Pi(y)$ его значеніемъ, найденнымъ выше получимъ:

$$\tang X = \cos\Pi(x)\tang\Pi(a).$$

Соединеніе втораго изъ уравненій (H) съ первымъ изъ уравненій (L) даетъ еще:

$$\tang(\varphi - X)\frac{\tang\Pi(y)}{\sin\Pi(y)} = \frac{\tang\Pi(r)}{\cos A}$$

или

$$\tang(\varphi - X) = \frac{\cos\Pi(y)\tang\Pi(r)}{\cos A},$$

а если мы вставимъ значеніе $\tang\Pi(r)$ изъ втораго въ уравненіяхъ (K) то получимъ:

$$\tang(\varphi - X) = \tang A\tang\Pi(a)\cos\Pi(y).$$

Это уравненіе, когда сюда поставится на мѣсто $\tang A$, $\tang X$ ихъ значенія, найденныя выше, приметъ слѣдующій видъ:

$$\tang\varphi = \frac{\tang\Pi(a)}{\cos\Pi(x)} \tag{26}$$

Это уравненіе показываетъ, что $x$ всегда возможно, покуда уголъ $\varphi$ больше $\Pi$ $(a)$ и менѣе $\frac{\pi}{2}$, или

$$\pi - \varphi > \Pi(a);\ \pi - \varphi < \frac{\pi}{2}.$$

Значеніе $\cos\Pi(x)$ положительно если

$$\frac{\pi}{2} > \varphi > \Pi(a)$$

и линія $x$ слѣдовательно тоже положительна.

Но эсли $\frac{\pi}{2} > \pi - \varphi > \Pi(a)$ то значеніе $\cos\Pi(x)$ дѣлается отрицательной и линія $x$ лежитъ по другую сторону перпендикула $a$.

Это доказываетъ, что если двѣ прямыя линія, лежащія въ одной плоскости не встрѣчаются, какъ бы далеко не продолжались, не будучи однакожъ паралельными, то онѣ обѣ бываютъ перпендикулярны къ одной прямой; всякія двѣ прямыя линія, которыя, находясь въ одной плоскости, ни паралельны ни перпендикулярны къ одной прямой должны необходимо пересѣкаться.

Прямыя линіи въ одной плоскости пересѣкаются взаимно по достаточному продолженію, если уголъ между одной прямой и перпендикуломъ изъ произвольной ея точки опущенный на другую прямую, менѣе нежели уголъ паралельности, которой соотвѣтствуетъ длинѣ этаго перпендикула. Съ помощію послѣднихъ предложеній можно значительно упростить общее уравненіе прямой линіи (23) въ случаѣ, когда прямая, къ которой относится уравненіе, не пересѣкаетъ оси $x$.

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Пусть $a$ будетъ перпендикулъ опущенный къ оси $x$ изъ точки постоянной, но произвольной на данной прямой. Пусть $L$ тотъ изъ двухъ угловъ между прямой и этимъ перпендикуломъ, который лежитъ на сторонѣ $x$-овъ положительныхъ.

Опредѣляемъ напередъ такъ линію $l$, чтобы:

$$\cos \Pi(l) = \tang \Pi(a) \cotg L.$$

Это всегда возможно покуда $L > \Pi(a)$ т. е. покуда прямая не пересѣкаетъ ось $x$-овъ. Полагаемъ эту линію $l$ на ось $x$-овъ отъ начала координатъ на сторонѣ $x$-овъ положительныхъ или отрицательныхъ смотря по знаку $l$. Воставляемъ на концѣ линіи $l$ перпендикулъ къ оси $x$-овъ, продолжаемъ его до пересѣченія съ данной прямой и пусть $b$ будетъ часть этаго перпендикула между данной прямой и осью $x$-овъ. Уголъ подъ которымъ этотъ перпендикулъ встрѣчаетъ данную прямую, долженъ быть прямой согласно съ уравненіемъ (26). Если теперь конецъ перпендикула $b$ принимаемъ за начало координатъ, то будемъ имѣть согласно съ уравненіемъ (25):

$$\cos \Pi(b) = \cos \Pi(y) \sin \Pi(x), \tag{27}$$

что и будетъ общее уравненіе прямой которая не пересѣкаетъ ось $x$-овъ.

Въ этомъ уравненіи можемъ полагать $y = a$ и вмѣстѣ съ тѣмъ $x = -l$; это даетъ:

$$\cos \Pi(b) = \cos \Pi(a) \sin \Pi(l)$$

если сюда поставимъ вмѣсто $\cos \Pi(b)$, $\sin \Pi(l)$ ихъ значенія, то уравненіе сдѣлается:

$$\cos \Pi(y) \sin \Pi(x) = \cos \Pi(a) \sqrt{1 - \tang^2 \Pi(a) \cotg^2 L}$$

Вторая часть въ этомъ уравненіи дѣлается воображаемой, какъ скоро $\tang \Pi(a) \cotg L > 1$ т. е. для всякой прямой, которая пересѣкаетъ ось $x$-овъ.

Послѣ того, что до сихъ поръ найдено, можемъ рѣшить задачу: опредѣлить разстояніе двухъ точекъ, которыхъ положеніе въ плоскости дано помощью ихъ прямоугольныхъ координатъ: $x, y$ и $x', y'$. Положимъ для сокращенія

$$\triangle x = x' - x; \triangle y = y' - y.$$

Опускаемъ изъ вершины $y$ перпендикулъ къ $y'$ и означаемъ длину этаго перпендикула $q$, тогда какъ $y_1$ означаетъ часть $y'$ между осью $x$-овъ и перпендикуломъ $q$.

Согласно съ уравненіемъ (25) будетъ:

$$\cos \Pi(y_1) = \cos \Pi(y) \sin \Pi(\triangle x)$$
$$\cos \Pi(q) = \cos \Pi(\triangle x) \sin \Pi(y_1)$$

Помощью этихъ уравненій опредѣливъ значенія $y_1$ и $q$, искомое разстояніе двухъ точекъ, которое означаемъ $r$, будетъ дано слѣдующимъ уравненіемъ, которое выводится изъ уравненія (11).

$$\sin \Pi(r) = \sin \Pi(y' - y_1) \sin \Pi(q).$$

Если $\triangle x$ и $\triangle y$, а слѣдовательно $q$, $r$ весьма малы, такъ что можно пренебречь ихъ высшими степенями въ сравненіи съ нисшими, то $r$ представитъ элементъ $ds$ кривой линіи, къ выраженію котораго приходимъ, взявъ:

$$\sin \Pi(q) = 1 - \tfrac{1}{2} q^2$$
$$\cos \Pi(q) = q - \tfrac{1}{3} q^3$$
$$\sin \Pi(r) = 1 - \tfrac{1}{2} r^2$$
$$\sin \Pi(y' - y_1) = 1 - \tfrac{1}{2}(y' - y_1)^2$$

послѣ чего выходитъ:

$$q = \frac{\triangle x}{\sin \Pi(y)}.$$

$$ds = \sqrt{dy^2 + \frac{dx^2}{\sin^2 \Pi(y)}}.$$

Для предѣльнаго круга:

$$\sin \Pi(y) = e^{-x}.$$

Изъ общихъ выраженій, которыя опредѣляютъ $\sin \Pi(a)$ и т. д. помощью $a$ и которыя даны выше, выходитъ:

$$d\,\Pi(a) = -\sin \Pi(a)\, da;$$

послѣ чего дифференцируя уравненіе предѣльнаго круга находимъ:

$$\sin \Pi(y) \cos \Pi(y)\, dy = e^{-x}\, dx$$

и

$$ds = \frac{dx\, e^{x}}{\sqrt{1 - e^{-2x}}}.$$

Интегрируя въ отношеніи $x$ отъ $x = o$ находимъ:

$$s = \sqrt{e^{-2x} - 1}$$

или иначе

$$s = \operatorname{cotg} \Pi(y)$$

какъ это было найдено выше.

Если означимъ $r$ разстояніе точки по круговой линіи отъ начала координатъ, а $\varphi$ уголъ, который это разстояніе $r$ дѣлаетъ съ осью $x$–овъ положительныхъ, то мы находимъ въ прямоугольномъ треугольникѣ, котораго стороны $y, x, r$ согласно съ уравненіемъ (12):

$$\operatorname{tang} \Pi(r) = \sin \varphi \operatorname{tang} \Pi(y).$$

Взявъ логариѳмы на обоихъ сторонахъ этаго уравненія и дифференцируя въ отношеніи къ $r, \varphi, y$ получимъ:

$$\frac{dr}{\cos \Pi(r)} = -\operatorname{cotg} \varphi\, d\varphi + \frac{dy}{\cos \Pi(y)}$$

Изъ этого уравненія выводимъ:

$$dy = \left\{ \cot\varphi \, d\varphi + \frac{dr}{\cos \Pi(r)} \right\} \cos \Pi(y),$$

или вставляя на мѣсто $\cos \Pi(y)$ его значеніе помощью $\varphi$ и $r$:

$$dy = \frac{\cos\varphi \cos \Pi(r)\, d\varphi + \sin\varphi \, dr}{\sqrt{1 - \cos^2\varphi \cos^2 \Pi(r)}}.$$

Чтобы выразить $dx$ помощью $r, \varphi$, беремъ уравненіе (10)

$$\cos \Pi(r) \cos\varphi = \cos \Pi(x).$$

Диффеpенцируя логариѳмы обѣихъ частей этого уравненія въ отношеніи къ $r$, $\varphi$, $x$, получимъ:

$$\frac{\sin^2 \Pi(r)\, dr}{\cos \Pi(r)} - \tan\varphi \, d\varphi = \frac{\sin^2 \Pi(x)\, dx}{\cos \Pi(x)}$$

откуда выводимъ помощью уравненій:

$$\sin \Pi(x) \sin \Pi(y) = \sin \Pi(r)$$
$$\cos \Pi(r) \cos\varphi = \cos \Pi(x)$$

слѣдующее уравненіе, которое выражаетъ искомое значеніе $dx$:

$$\frac{dx}{\sin \Pi(y)} = \frac{\cos\varphi \sin \Pi(r)\, dr - \sin\varphi \cot \Pi(r)\, d\varphi}{\sqrt{1 - \cos^2\varphi \cos^2 \Pi(r)}}$$

послѣ чего:

$$ds = \sqrt{dr^2 + d\varphi^2 \cot^2 \Pi(r)}.$$

Для круга, полагая, что начало координатъ въ центрѣ, находимъ, такъ какъ здѣсь $dr = o$:

$$ds = d\varphi \cot \Pi(r).$$

Интегрируя отъ $\varphi = o$ до $\varphi = \frac{\pi}{2}$ и умножая все потомъ на 4 находимъ слѣдующее выраженіе для окружности съ полупоперечникомъ $r$

$$2\pi \cot \Pi(r)$$

согласно съ тѣмъ, что было найдено выше.

Если называемъ $s$ дугу предѣльнаго круга отъ оси $x$-овъ, то вращеніе $s$ коло оси $x$-овъ производитъ часть предѣльной сферы, а конецъ дуги опишетъ окружность круга, которая на предѣльной сферѣ опредѣляется также, какъ окружность круга съ полупоперечникомъ $s$ опредѣляется на плоскости въ обыкновенной геометріи; отсюда слѣдуетъ, что окружность этого круга должна быть равна $2\pi s$. Съ другой стороны окружность того же круга разсматриваемая въ его плоскости, гдѣ перпендикулъ $y$, опущенный отъ конца дуги $s$

на ось предѣльнаго круга, которая служитъ осью $x$–овъ и проходитъ чрезъ другой конецъ дуги, составляетъ полупоперечникъ круга, дается въ пангеометріи выраженіемъ:

$$2\pi \operatorname{cotg} \Pi(y);$$

откуда слѣдуетъ, что $s = \operatorname{cotg} \Pi(y)$ какъ доказано было прежде.

Чтобы раздѣлить площадь на элементы, ведемъ по площади дуги предѣльнаго круга, для которыхъ осью служитъ ось $x$–овъ и такимъ образомъ, что ихъ взаимное разстояніе безконечно мало и можетъ быть названо $dx$. Пусть $s$ одна изъ этихъ дугъ предѣльнаго круга, между осью $x$–овъ и точкой на кривой линіи, которой координаты $x$, $y$; пусть $s'$, дуга другаго изъ этихъ предѣльныхъ круговъ между осью $x$–овъ и точкой на тойже кривой, которой координаты $x + dx$, $y + dy$.

Часть площади безконечно малая заключенная между $s$ и $s'$ съ одной стороны и осью $x$–овъ и кривой съ другой стороны выразится:

$$dS = \frac{es\,dx}{e - 1}$$

куда вставляя $s = \operatorname{cotg} \Pi(y)$ получимъ:

$$dS = \frac{e\,dx \operatorname{cotg} \Pi(y)}{e - 1}.$$

Для примѣра опредѣляемъ площадь предѣльнаго круга, для котораго нашли мы уравненіе въ прямоугольныхъ координатахъ:

$$\sin \Pi(y) = e^{-x},$$

помощью котораго уравненія находимъ выраженіе для элемента искомой площади:

$$dS = \frac{e}{e - 1}\, dy \cos \Pi(y) \operatorname{cotg} \Pi(y).$$

Интегрируя это выраженіе отъ $y = o$ находимъ площадь, ограниченную дугою предѣльнаго круга, осью $x$–овъ и ординатою $y$

$$S = \frac{e}{e - 1} \left\{ \operatorname{cotg} \Pi(y) - \tfrac{1}{2}\pi + \Pi(y) \right\}$$

Мы видѣли, что площадь между двухъ паралельныхъ продолжаемая до безконечности на сторонѣ паралельности и ограниченная дугою предѣльнаго круга, которой дугѣ двѣ паралельныя служатъ осями, выражается:

$$\frac{es}{e - 1} = \frac{e \operatorname{cotg} \Pi(y)}{e - 1}$$

послѣ чего находимъ площадь между двухъ прямыхъ паралельныхъ, изъ которыхъ од–

на перпендикулярна къ $y$, которые проведены чрезъ концы $y$ и продолжены до безконечности, выраженіе

$$\tfrac{1}{2}\pi - \Pi(y).$$

Помощью этого выраженія находимъ площадь прямолинейнаго прямоугольнаго треугольника, когда даны острые углы $\Pi(\alpha)$, $\Pi(\beta)$, противъ перпендикулярныхъ боковъ $a$, $b$ треугольника. Продолжаемъ гипотенузу $c$ да вершину угла $\Pi(\beta)$ и продолженіе дѣлаемъ равно $\beta$. Перпендикулъ въ концѣ $\beta$ дѣлается параллельнымъ съ продолженіемъ бока $a$. Площадь безконечно простирающаяся между этими двумя параллельными, продолженными въ сторону параллельности, и ограниченная съ другой стороны линіею $\beta$ будетъ

$$\tfrac{1}{2}\pi - \Pi(\beta).$$

Если ведемъ теперь чрезъ вершину угла $\Pi(\alpha)$ параллельную съ перпендикуломъ и которая наклонена къ боку $c$ подъ угломъ $\Pi(c+\beta)$ и будетъ вмѣсто параллельна съ продолженіемъ бока $a$, то величина площади между $c+\beta$ и двумя параллельными чрезъ концы $c+\beta$ проведенными и продолженными до безконечности на сторонѣ параллельности будетъ:

$$\tfrac{1}{2}\pi - \Pi(c+\beta)$$

Подобнымъ образомъ часть площади между бокомъ $b$, прямой проведенной чрезъ вершину $\Pi(\alpha)$ и бокомъ $a$, безконечно продолженнымъ будетъ:

$$\tfrac{1}{2}\pi - \Pi(b)$$

Послѣ чего сумма $\tfrac{1}{2}\pi - \Pi(\beta)$, $\tfrac{1}{2}\pi - \Pi(b)$ уменьшая $\tfrac{1}{2}\pi - \Pi(c+\beta)$ будетъ выраженіе площади треугольника, которую такимъ образомъ находимъ $= \tfrac{1}{2}\pi - \Pi(b) - \Pi(\beta) + \Pi(c+\beta)$. Между тѣмъ мы доказали, что $\Pi(b) = \Pi(\alpha) + \Pi(c+\beta)$, поставя отсюда въ выраженіе площади треугольника на мѣсто $\Pi(b)$ его значеніе, находимъ площадь прямоугольнаго прямолинейнаго треугольника

$$\tfrac{1}{2}\pi - \Pi(\alpha) - \Pi(\beta);$$

это значитъ, что площадь прямолинейнаго прямоугольнаго треугольника равна разности двухъ прямыхъ угловъ безъ суммы трехъ угловъ треугольника. Отсюда слѣдуетъ еще, что площадь всякаго прямолинейнаго треугольника равна избытку двухъ прямыхъ угловъ надъ суммою трехъ угловъ треугольника.

Легко вывести изъ предыдущаго, что площадь всякого четыреугольника равна избытку четырехъ прямыхъ угловъ надъ суммою черырехъ угловъ четыреугольника и вообще площадь многоугольника $n$ сторонъ равна избытку $(n-2)\pi$ надъ суммою угловъ многоугольника.

Разсматриваемъ особенно четыреугольникъ, котораго два бока $a$, $y$ оба перпендикулярны къ третьему боку $x$ и котораго четвертый бокъ $t$ перпендикуляренъ къ боку $a$ и дѣлаетъ съ $y$ уголъ, который означимъ $\omega$, Мы доказали выше (уравненіе 25), что между составными частями такого четыреугольника существуетъ уравненіе

$$\cos \Pi(a) = \cos \Pi(y) \sin \Pi(x);$$

если разсматриваемъ $x$ и $y$ переменными и $a$ постояннымъ то площадь этого четыреугольника выражается, какъ и всякая площадь, какъ доказано выше, интеграломъ

$$\int dx \operatorname{cotg} \Pi(y),$$

который, будучи приложенъ къ настоящему случаю, даетъ, когда поставимъ сюда значеніе $\operatorname{cotg} \Pi(y)$, слѣдующее выраженіе для площади

$$\int_0 \frac{dx \cos \Pi(a)}{\sqrt{\sin^2 \Pi(x) - \cos^2 \Pi(a)}},$$

тогда какъ площадь этаго четыреугольника, согласно съ тѣмъ, какъ опредѣляется площадь всякаго многоугольника помощію угловъ, $= \frac{1}{2}\pi - \omega$. Это даетъ:

$$\tfrac{1}{2}\pi - \omega = \cos \Pi(a) \int_0 \frac{dx}{\sqrt{\sin^2 \Pi(x) - \cos^2 \Pi(a)}}. \tag{M}$$

Уголъ $\omega$, который дѣлаетъ бокъ $t$ съ бокомъ $y$, опредѣляется уравненіемъ (уравненіе 26)

$$\operatorname{tang} \omega = \frac{\operatorname{tang} \Pi(y)}{\cos \Pi(x)}.$$

Если въ уравненіи (M) пишемъ $\alpha$ вмѣсто $\Pi(a)$, $\xi$ на мѣсто $\Pi(x)$ оно сдѣлается

$$\frac{\frac{1}{2}\pi - \omega}{\cos \alpha} = \int_{\frac{\pi}{2}} \frac{d\xi}{\sin \xi \sqrt{\sin^2 \alpha - \sin^2 \xi}}$$

гдѣ $\alpha$ величина постоянная.

Справедливость этаго выраженія для интеграла можетъ быть повѣрена дифференцированіемъ. Пангеометрія указываетъ также на новой способъ приближенія къ значеніямъ опредѣленныхъ интеграловъ.

Пусть требуется найти

$$\int A\,dx$$

гдѣ $A$ данная функція отъ $x$. Чтобы отыскать значеніе этого интеграла надобно положить $A = \operatorname{cotg} \Pi(y)$, потомъ опредѣлять значенія $y'$, $y''$, $y'''$ ...., которые отвѣчаютъ $x'$ $x''$, $x'''$ ....., взятыхъ произвольно между границъ интегрированія, послѣ чего надобно вычислять длину хордъ, которыми соединяются вершины $y'$ съ $y''$, $y''$ съ $y'''$ .... и уголъ, который каждая хорда дѣлаетъ съ продолженіемъ слѣдующей хорды. Сумма этихъ угловъ даетъ приближенное значеніе интеграла.

Площадь между двумя параллельными, проведенными чрезъ концы данной прямой, продолженными до безконечности на сторонѣ параллельности, и самой прямой будетъ

$= \pi$ — сумма двухъ угловъ, которые двѣ параллельныхъ дѣлаютъ съ данной прямой, потому что эта площадь можетъ быть принимаема за площадь треугольника, въ которомъ одинъ уголъ равенъ нулю.

Площадь кривой линіи можетъ быть раздѣлена на элементы прямыми параллельными къ одной и тойже данной прямой напр. къ оси $y$. Если ведемъ на конецѣ абсциссы $x$ параллельную къ оси $y$, то эта прямая дѣлаетъ съ осью $x$ уголъ $= \Pi(x)$; прямая проведенная чрезъ конецъ абсциссы $x + dx$, дѣлаетъ такимъ же образомъ съ осью $x$ уголъ $= \Pi(x + dx)$, откуда слѣдуетъ, что площадь между этими двумя параллельными и $dx$ равна $- d\,\Pi(x)$. Пусть теперь $u$ будетъ величина первой параллельной между осью $x$ и кривой; та часть площади между двумя параллельными, которая лежитъ внѣ данной кривой, будетъ въ слѣдствіе выше доказаннаго

$$- e^{-u}\, d\,\Pi(x),$$

откуда слѣдуетъ, что часть этой площади, которая лежитъ между кривою и осью $x$ т. е. элементъ площади кривой выразится

$$dS = -(1 - e^{-u})\, d\,\Pi(x).$$

Чтобы вычислить площадь круга съ полупоперечникомъ $r$ надобно въ общее выраженіе элемента площади кривой, найденной выше,

$$dS = dx \cot \Pi(y)$$

поставить значеніе $\cot \Pi(y)$ изъ уравненія круга:

$$\sin \Pi(x) \sin \Pi(y) = \sin \Pi(r)$$

гдѣ начало прямоугольныхъ координатъ въ центрѣ круга. Это даетъ

$$dS = dx \sqrt{\frac{\sin^2 \Pi(x)}{\sin^2 \Pi(r)} - 1}\,;$$

интегрируя отъ $x = o$ находимъ:

$$S = \frac{1}{\sin \Pi(r)} \operatorname{arc} \sin \left\{\frac{\cos \Pi(x)}{\cos \Pi(r)}\right\} - \operatorname{arc} \sin \left\{\frac{\cot \Pi(x)}{\cot \Pi(r)}\right\}$$

Для $x = r$ это даетъ площадь четверти круга:

$$\frac{\pi}{2 \sin \Pi(r)} - \frac{\pi}{2}\,;$$

умножая на 4, находимъ для площади цѣлаго круга

$$2\pi \left\{\frac{1}{\sin \Pi(r)} - 1\right\}$$

или что все равно:

$$\pi \left\{ e^{\frac{r}{2}} - e^{-\frac{r}{2}} \right\}^2$$

Если $r$ чрезвычайно мало, то это выраженіе даетъ площадь круга $= \pi r^2$; тоже самое выраженіе, которое обыкновенно дается для площади круга въ Геометріи.

Помощію предыдущаго выраженія площади круга можно дать элементу площади всякой кривой линіи еще такое выраженіе:

$$dS = d\varphi \left\{ \frac{1}{\sin \Pi(r)} - 1 \right\}$$

гдѣ $r$ есть радіусъ векторъ проведенный изъ начала координатъ къ точкѣ на кривой, а $\varphi$ уголъ, который этотъ радіусъ векторъ дѣлаетъ съ постоянной прямой, проходящій чрезъ начало координатъ.

Примѣненіе этого выраженіг къ вычисленію площади треугольника, котораго бока $a, b, c$ съ противоположными углами $A$, $B$, $C$ даетъ, если разсматриваемъ $A$, $C$ и стороны $a$, $b$ какъ перемѣнные.

$$\text{площадь треугольника} = \int\limits_0^A dA \left\{ \frac{1}{\sin \Pi(b)} - 1 \right\}.$$

Бокъ $b$ выражается въ функціи $c$, $A$, $B$ помощію послѣдняго уравненія (19)

$$\cot B \sin A \sin \Pi(c) + \cos A = \frac{\cos \Pi(c)}{\cos \Pi(b)}.$$

Беремъ изъ этаго уравненія значеніе $\sin \Pi(b)$ и ставимъ его въ выраженіе для площади треугольника, получаемъ:

$$\text{площадь треугольника} = \int\limits_0^A \frac{dA}{\sqrt{1 - \dfrac{\cos^2 \Pi(c)}{(\cot B \sin A \sin \Pi(c) + \cos A)^2}}} - A.$$

между тѣмъ было доказано, что площадь треугольника

$$= \pi - A - B - C$$

гдѣ $A$ и $B$ углы данные и $C$ опредѣляется уравненіемъ (19).

$$\cos C + \cos A \cos B = \frac{\sin A \sin B}{\sin \Pi(c)}.$$

Сравненіе этихъ двухъ выраженій для площади треугольника даетъ:

$$\pi - B - C = \int\limits_0^A \frac{dA \left\{ \cot B \sin A \sin \Pi(c) + \cos A \right\}}{\sqrt{(\cot B \sin A \sin \Pi(c) + \cos A)^2 - \cos^2 \Pi(c)}}$$

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Если $B = \frac{\pi}{2}$ это уравненіе даетъ:

$$\frac{\pi}{2} - C = \int\limits_0^A \frac{dA \cos A}{\sqrt{\cos^2 A - \cos^2 \Pi(c)}}$$

уравненіе, которое послѣ интегрированія принимаетъ видъ:

$$\tfrac{1}{2}\pi - C = \operatorname{arc} \sin \left( \frac{\sin A}{\sin \Pi(c)} \right)$$

что согласно съ уравненіемъ, которое опредѣляетъ $C$.

Изъ того что было доказано впереди можно вывести для выраженія площади всякаго замкнутаго многоугольника два выраженія, одно выраженное опредѣленнымъ интеграломъ, другое зависящее только отъ суммы угловъ многоугольника.

Два значенія для тойже площади будучи необходимо равными, доставляютъ способъ вычислять опредѣленные интегралы, которыхъ значеніе былобы затруднительно найти другимъ образомъ.

Чтобы представить еще примѣръ, разсматриваемъ прямолинейный прямоугольный треугольникъ, въ которомъ перпендикулярные бока, $x, y$ а гипотенуза $r$. Пусть $A$ уголъ противъ $x$, $B$ уголъ противъ $y$. Уравненія (10) (11) даютъ для такого треугольника:

$$\sin \Pi(x) \sin \Pi(y) = \sin \Pi(r)$$
$$\sin \Pi(x) \cos B = \sin A$$
$$\cos \Pi(r) \cos A = \cos \Pi(x)$$
$$\cos \Pi(r) \cos B = \cos \Pi(y).$$

Изъ этихъ уравненій выведемъ:

$$\cos \Pi(r) = \frac{\cos \Pi(x)}{\cos A}$$

$$\sin \Pi(r) = \sqrt{1 - \left(\frac{\cos \Pi(x)}{\cos A}\right)^2}$$

$$\sin \Pi(y) = \frac{1}{\sin \Pi(x)} \sqrt{1 - \left(\frac{\cos \Pi(x)}{\cos A}\right)^2} = \sqrt{\frac{1}{\sin^2 \Pi(x)} - \frac{\operatorname{cotg}^2 \Pi(x)}{\cos^2 A}}$$

$$\operatorname{cotg} \Pi(y) = \frac{\sin A \cos \Pi(x)}{\sqrt{\sin^2 A - \cos^2 \Pi(x)}}.$$

Вставляя въ это послѣднее уравненіе $\Pi(x) = \frac{\pi}{2} - \omega$ находимъ:

$$\operatorname{cotg} \Pi(y) = \frac{\sin A \sin \omega}{\sqrt{\sin^2 A - \sin^2 \omega}}$$

Между тѣмъ мы видѣли что дифференціалъ площади есть $dx \operatorname{cotg} \Pi(y)$; а это даетъ въ настоящемъ случаѣ:

$$dx \operatorname{cotg} \Pi(y) = \sin A \frac{d\omega \operatorname{tang} \omega}{\sqrt{\sin^2 A - \sin^2 \omega}},$$

откуда заключаемъ, интегрируя отъ $\omega = o$, что соотвѣтствуетъ $x = o$ и замѣчая, что площадь выраженная интеграломъ выражается также чрезъ $\frac{\pi}{2} - A - B$, что

$$\frac{\pi}{2} - A - B = \sin A \int_o^{\omega} \frac{\operatorname{tang} \omega \, d\omega}{\sqrt{\sin^2 A - \sin^2 \omega}}$$

гдѣ $A$ уголъ постоянный, а $B$ опредѣляется уравненіемъ

$$\cos B = \frac{\sin A}{\cos \omega}.$$

Если $\omega = \frac{\pi}{2} - A$ то гипотенуза дѣлается параллельною боку $y$ и уголъ $B = o$. Итакъ въ этомъ случаѣ

$$\frac{\pi}{2} - A = \sin A \int_o^{\frac{\pi}{2} - A} \frac{\operatorname{tang} \omega \, d\omega}{\sqrt{\sin^2 A - \sin^2 \omega}}.$$

Можно опредѣлить значеніе интеграла въ болѣе общемъ видѣ, разсматривая площадь прямолинейнаго треугольника, котораго бока $a$, $b$, $c$ съ противоположными углами $A, B, C$ и раздѣляя эту площадь на элементы прямыми параллельными между собою. Беремъ вершину угла $C$ за начало координатъ и сторону $a$ за ось $x$–овъ. Пусть $B = \Pi(\beta)$ гдѣ $\beta$ положительная если $B < \frac{\pi}{2}$ и отрицательная если $B > \frac{\pi}{2}$. Ведемъ чрезъ конецъ абсциссы $x$ прямую $u$ параллельную боку $c$, и продолжаемъ до пересѣченія съ бокомъ $b$. Уголъ, которой эта параллельная дѣлаетъ съ абсциссою $x$ будетъ $\Pi(\beta - a + x)$, откуда слѣдуетъ, что уголъ, который дѣлаетъ эта параллельная съ продолженіемъ $x$ будетъ $\Pi(a - \beta - x)$. Если возмемъ за элементъ площади треугольника часть этой площади, которая заключена между двумя параллельными $u$ безконечно близкими, то получимъ, въ слѣдствіе того, что было доказано выше, слѣдующее выраженіе для этаго элемента

$$dS = - d\,\Pi(a - \beta - x) \{1 - e^{-u}\}.$$

Разсматриваемъ $y, x, u$ какъ перемѣнные, а $a$ и $\beta$ какъ постоянные.

Уравненія (19) будучи примѣнены къ треугольнику котораго бока $x$, $u$, а уголъ между этими двумя боками $\Pi(\beta - a + x)$ даютъ:

$$\operatorname{cotg} C \sin \Pi(\beta - a + x) \sin \Pi(x) + \cos \Pi(\beta - a + x) = \frac{\cos \Pi(x)}{\cos \Pi(u)}$$

изъ этаго уравненія выводимъ, положивъ для краткости $\Pi(\beta - a + x) = \omega$

$$\cos \Pi(u) = \frac{\cos \Pi(x)}{\operatorname{cotg} C \sin \omega \sin \Pi(x) + \cos \omega}$$

$$e^{2u} = \frac{\operatorname{cotg} C \sin \omega \sin \Pi(x) + \cos \omega + \cos \Pi(x}{\operatorname{cotg} C \sin \omega \sin \Pi(x) + \cos \omega - \cos \Pi(x)}$$

но $$\sin \Pi(x) = \sin \Pi \Big\{ (\beta - a) - (\beta - a + x) \Big\} = \frac{\sin \Pi(\beta - a) \sin \omega}{1 - \cos \Pi(\beta - a) \cos \omega}.$$

Такимъ же образомъ находимъ:

$$\cos \Pi(x) = \frac{\cos \Pi(\beta - a) - \cos \omega}{1 - \cos \Pi(\beta - a) \cos \omega}.$$

Вставляя эти значенія $\sin \Pi(x)$, $\cos \Pi(x)$ въ выраженіе для $e^{2u}$ получимъ:

$$e^{2u} = \frac{\operatorname{cotg} C \sin^2 \omega \sin \Pi(\beta - a) + \cos \omega \,\{1 - \cos \Pi(\beta - a) \cos \omega\} + \cos \Pi(\beta - a) - \cos \omega}{\operatorname{cosg} C \sin^2 \omega \sin \Pi(\beta - a) + \cos \omega \,\{1 - \cos \Pi(\beta - a) \cos \omega\} - \cos \Pi(\beta - a) + \cos \omega}$$

$$= \frac{\operatorname{cotg} C \sin^2 \omega \sin \Pi(\beta - a) + \cos \Pi(\beta - a) \sin^2 \omega}{\operatorname{cotg} C \sin^2 \omega \sin \Pi(\beta - a) + 2 \cos \omega - \{1 + \cos^2 \omega\} \cos \Pi(\beta - a)};$$

далѣе находимъ:

$$d\,\Pi(a - \beta - x) = - d\,\Pi(\beta - a + x) = - d\omega$$

послѣ чего сравненіе двухъ выраженій для площади треугольника даетъ уравненіе:

$$\pi - A - B - C = -\omega + \int_{x=0}^{x=a} d\omega \sqrt{\frac{\operatorname{cotg} C \sin^2 \omega \sin \Pi(\beta - a) + 2 \cos \omega - (1 + \cos^2 \omega) \cos \Pi(\beta - a)}{\operatorname{cotg} C \sin^2 \omega \sin \Pi(\beta - a) + \cos \Pi(\beta - a) \sin^2 \omega}}.$$

Если положимъ еще $\Pi(\beta - a) = \alpha$ то это уравненіе принимаетъ видъ:

$$[\pi - A - B - C]\,[\operatorname{cotg} C \sin \alpha + \cos \alpha] = \int_{\omega=\alpha}^{\omega=\Pi(\beta)} \frac{d\omega}{\sin \omega} \sqrt{\operatorname{cotg} C \sin^2 \omega \sin \alpha + 2 \cos \omega - (1 + \cos^2 \omega) \cos \alpha}$$

гдѣ углы $A$, $B$ и линія $\beta$ должны быть вычислены посредствомъ уравненій:

$$\alpha = \Pi(\beta - a), \quad B = \Pi(\beta)$$

$$\cos A + \cos B \cos C = \frac{\sin B \sin C}{\sin \Pi(a)}$$

послѣднее изъ этихъ уравненій есть послѣднее изъ уравненій (19) примѣненое къ треугольнику, который разсматривали.

Чтобы опредѣлять положеніе точки въ плоскости, можно употреблять въ пангеометріи не только прямыя и полярные координаты, но также и дуги предѣльнаго круга. Даже эта послѣдняя система представляетъ свои выгоды въ отношеніи къ простотѣ выраженіи. Опредѣляемъ положеніе точки въ плоскости помощью перпендикулярныхъ координатъ $x$ и $y$ такъ, что $y$ былъ бы перпендикулъ изъ точки, положеніе которой хотятъ опредѣлять, на ось $x$–овъ, а $x$ часть оси $x$–овъ отъ конца перпендикула до начала координатъ. Пусть $\eta$ длина дуги предѣльнаго круга отъ данной точки до оси $x$–овъ, которая вмѣстѣ служитъ осью предѣльному кругу; называемъ $\xi$ разстояніе вершины предѣльнаго круга на оси $x$–овъ до начала координатъ. Мы видѣли, что въ этомъ случаѣ

$$\eta = \operatorname{cotg} \Pi(y)$$

потомъ уравненіе предѣльнаго круга даетъ

$$e^{-(x-\xi)} = \sin \Pi(y)$$

помощью этихъ двухъ уравненій можно выразить $\xi$, $\eta$ въ зависимости отъ $x$, $y$ или на оборотъ $x$, $y$ въ зависимости отъ $\xi$, $\eta$. Это дозволяетъ переходить отъ уравненія кривой въ координатахъ $x$, $y$ къ уравненію тойже кривой въ $\xi$, $\eta$ или наоборотъ. Дифференціалъ площади выражается въ $\xi$, $\eta$ уравненіемъ:

$$d^2 S = d\xi\, d\eta$$

гдѣ $S$ площадь.

Если мы разсматриваемъ $S$ какъ функцію $x$, $y$, то имѣемъ:

$$\left(\frac{dS}{dx}\right) = \frac{dS}{d\xi}$$

потомъ дифференцируя въ отношеніи къ $y$:

$$\frac{d^2 S}{dx\, dy} = \frac{1}{\sin \Pi(y)} \frac{d^2 S}{d\xi\, d\eta} = \frac{1}{\sin \Pi(y)}$$

согласно съ тѣмъ, что было найдено выше.

Опускаемъ изъ точки въ пространствѣ перпендикулъ $z$ на плоскость координатъ $x$, $y$.

Ведемъ чрезъ этотъ перпендикулъ плоскость, которая пересѣкаетъ плоскость $x$ $y$ въ прямой параллельной къ оси $x$–овъ. Принимаемъ это пересѣченіе на сторонѣ параллельности за ось предѣльнаго круга, который проходитъ чрезъ вершину перпендикула $z$ и пусть $\zeta$ длина дуги этаго предѣльнаго круга между вершиною $z$ и осью. Имѣемъ:

$$\zeta = \operatorname{cotg} \Pi(z).$$

Часть $q$ параллельной къ оси $x$–овъ проведенной чрезъ конецъ перпендикула $z$ между вершиною $\zeta$ и концомъ перпендикула $z$ будетъ дано уравненіемъ

$$e^{-q} = \sin \Pi(z).$$

Дуга предѣльнаго круга, проведенная чрезъ конецъ $z$ такъ, что ось положительныхъ $x$-овъ служитъ вмѣстѣ осью предѣльному кругу, и которая заключается между концомъ $z$ и этой осью, будетъ повеличинѣ равна $\operatorname{cotg} \Pi(y)$, а длина дуги $\eta$ предѣльнаго круга, проведенная чрезъ пересѣченіе $\zeta$ съ плоскостью $x$, $y$ и для которой ось лежитъ на стонѣ $x$ положительныхъ, между этой точкой и осью, будетъ, какъ это было доказано, дана уравненіемъ:

$$\eta = \frac{\cotg \Pi(y)}{\sin \Pi(z)}.$$

Если мы къ тому называемъ $\xi$ часть оси $x$-овъ между началомъ координатъ и дуги $r$, то уравненіе предѣльнаго круга даетъ:

$$e^{-x+\xi+q} = \sin \Pi(y).$$

Изъ этихъ уравненій выводимъ, перемѣняя напередъ только $z$ и въ зависимости отъ него $\zeta$:

$$d\zeta = \frac{dz}{\sin \Pi(z)}.$$

Перемѣняя только $y$ и $\eta$ получимъ:

$$dy = \frac{dy}{\sin \Pi(y) \sin \Pi(z)}.$$

Наконецъ перемѣняя только $\xi$ и $x$ получимъ:

$$d\xi = dx.$$

Чтобы дополнить новую теорію подъ названіемъ пангеометріи, которая основана на началахъ болѣе общихъ, нежели начала обыкновенной геометріи, остается только дать выраженіе для дифференціала поверхностей и объемомъ помощію координатъ, опредѣляющихъ положеніе точки въ пространствѣ.

Разсматриваемъ съ этою цѣлью снова четыреугольникъ, въ которомъ два бока $a$, $y$ перпендикулярны къ третьему $x$ и въ которомъ четвертый бокъ $c$ перпендикуляренъ къ $y$ дѣлая съ $a$ уголъ $\varphi$. Мы нашли (уравненіе 25):

$$\cos \Pi(y) = \cos \Pi(a) \sin \Pi(x).$$

Потомъ находимъ помощью уравненій (10) (11), называя $r$ діагоналъ между вершиною угла $\varphi$ и вершиною прямаго угла противоположнаго и $A$ уголъ между $x$ и $r$:

$$\cos \Pi(r) \cos A = \cos \Pi(x)$$
$$\cos A \tang \Pi(c) = \tang \Pi(r).$$

Изъ этихъ двухъ уравненій выводимъ:

$$\cos \Pi(x) \tang \Pi(c) = \sin \Pi(r);$$

но

$$\sin \Pi(r) = \sin \Pi(a) \sin \Pi(x)$$

слѣдовательно:

$$\tang \Pi(c) = \sin \Pi(a) \tang \Pi(x).$$

Если $c$, $x$ такъ малы, что можно пренебрегать ихъ высшими степенями въ сравненіи

съ нисшими и допустить слѣдующія приближенныя значенія для $\operatorname{tang} \Pi(c)$, $\operatorname{tang} \Pi(x)$:
$\operatorname{tang} \Pi(c) = \frac{1}{c}$; $\operatorname{tang} \Pi(x) = \frac{1}{x}$ то находимъ

$$c = \frac{x}{\sin \Pi(x)}. \tag{27}$$

Прямая $c$, которая соединяетъ вершины $a$, $y$ не будетъ перпендикулярна къ $y$ если $a = y$ въ четыреугольникѣ; въ такомъ случаѣ прямая $p$, изъ середины $x$ проведенная къ срединѣ $c$, перпендикулярна къ $x$ и $c$. Итакъ въ уравненіи (27) можемъ $c$ замѣнить $\frac{c}{2}$, а $x$ замѣнить $\frac{x}{2}$ отъ чего уравненіе не измѣняется. Такимъ образомъ это уравненіе доказывается даже для случая $a = y$, къ которому выше данное доказательство непосредственно не примѣняется. Величина кривой поверхности измѣряется суммою площадей треугольниковъ, которые смыкаются въ одну сплошную сѣть и вершины которыхъ лежатъ на поверхности; эта мѣра будетъ тѣмъ точнѣе, чѣмъ измѣряемыя треугольники менѣе.

Граница, къ которой эта сумма приближается до безконечности, когда измѣряемые треугольники уменьшаются до безконечности и съ которою она можетъ разниться менѣе, нежели всякая данная величина, называется математическая величина поверхности. Опредѣляемъ напередъ площадь прямолинейнаго прямоугольнаго треугольника помощіи 3-хъ его боковъ $a$, $b$, $c$ съ противоположными углами $\Pi(\alpha)$, $\Pi(\alpha)$ $\frac{\pi}{2}$. Мы видѣли, что въ такомъ треугольникѣ можно замѣнить линіи

$$a, b, c, \alpha, \beta$$

линіями:

$$a, \alpha', \beta, b', c$$

соотвѣтственно.

Кромѣ того мы нашли:

$$2\,\Pi(b) = \Pi(c + \beta) + \Pi(c - \beta).$$

Вставляя $\alpha'$ вмѣсто $b$, $\beta$ вмѣсто $c$, и $c$ вмѣсто $\beta$ получимъ:

$$\pi - 2\,\Pi(\alpha) = \Pi(\beta + c) + \Pi(\beta - c)$$

или

$$2\,\Pi(\alpha) = \Pi(c - \beta) - \Pi(c + \beta).$$

Такимъ же образомъ находимъ:

$$2\,\Pi(\beta) = \Pi(c - \alpha) - \Pi(c + \alpha).$$

Перемѣняя въ этомъ уравненіи буквы, какъ это было сказано, получимъ:

$$2\,\Pi(c) = \Pi(\beta - b') - \Pi(\beta + b').$$

Такимъ же образомъ находимъ:

$$2\,\Pi(c) = \Pi(\alpha - a') - \Pi(\alpha + a')$$

откуда тъ перемѣною буквъ, какъ было выше указано, выводимъ:

$$2\,\Pi(\beta)=\Pi(b'-a')-\Pi(b'+a').$$

Такимъ же образомъ получимъ:

$$2\,\Pi(\alpha)=\Pi(a'-b')-\Pi(a'+b').$$

Сложеніе двухъ послѣднихъ уравненій даетъ:

$$2\,\Pi(\alpha)+2\,\Pi(\beta)=\pi-2\,\Pi(a'+b').$$

Послѣ чего площадь треугольника $\triangle$ дана выраженіемъ:

$$\triangle=\frac{\pi}{2}-\Pi(\alpha)-\Pi(\beta)=\Pi(a'+b')$$

и потомъ:

$$\operatorname{tang}\tfrac{1}{2}\triangle=e^{-a'}:e^{-b'}=\operatorname{tang}\left\{\tfrac{1}{4}\pi-\Pi(a)\right\}\operatorname{tang}\left\{\tfrac{1}{4}\pi-\tfrac{1}{2}\Pi(b)\right\};$$

откуда наконецъ выводимъ:

$$\operatorname{tang}\tfrac{1}{2}\triangle=\frac{e^a-1}{e^a+1}\cdot\frac{e^b-1}{e^b+1}.$$

Когда $a$, $b$ весьма малы такъ, что можно пренебрѣгать высшими степенями отъ $a$, $b$, $\triangle$ то эта формула даетъ: $\triangle=\frac{1}{2}a\,b$, какъ въ обыкновенной геометріи. Въ прямолинейномъ прямоугольномъ треугольникѣ можемъ всегда выбрать одинъ уголъ $C$ такъ, чтобы изъ вершины его перпендикулъ къ противоположному боку лежалъ внутри треугольника. Этотъ перпендикулъ раздѣлитъ бокъ $c$ треугольника на двѣ части: одна $x$, прилѣжащая къ углу $A$, другая $c-x$, прилѣжащая къ углу $B$. Площадь $S$ этого треугольника будетъ равна суммѣ площадей двухъ прямоугольныхъ треугольниковъ, произведенныхъ этитъ перпендикуломъ $h$ и будетъ дана уравненіемъ:

$$\operatorname{tang}\tfrac{1}{2}S=\frac{\dfrac{e^x-1}{e^x+1}\cdot\dfrac{e^h-1}{e^h+1}+\dfrac{e^{c-x}-1}{e^{c-x}+1}\cdot\dfrac{e^h-1}{e^h+1}}{1-\dfrac{e^x-1}{e^x+1}\cdot\dfrac{e^{c-x}-1}{e^{c-x}+1}\cdot\left\{\dfrac{e^h-1}{e^h+1}\right\}^2}$$

Уравненіе, которому можно дать видъ:

$$\operatorname{tang}\tfrac{1}{2}S=\frac{(e^{2h}-1)(e^c-1)}{(e^c+1)(e^h+1)^2+2e^h(e^x-1)(e^{c-x}-1)}.$$

Это выраженіе даетъ, если пренебрѣгаемъ высшими степенями $S$, $h$, $c$ передъ нисшими:

$$S=\tfrac{1}{2}ch$$

какъ въ обыкновенной геометріи.

Мы видѣли, что площадь треугольника выражается помощью трехъ угловъ $A$, $B$, $C$ треугольника слѣдующимъ образомъ:

$$S = \pi - A - B - C.$$

Беремъ значеніе $A$ въ зависимости отъ $a$, $b$, $c$ изъ втораго уравненія (19); такъ получимъ:

$$\cos A = \frac{1 - \dfrac{\sin \Pi(b) \sin \Pi(c)}{\sin \Pi(a)}}{\cos \Pi(b) \cos \Pi(c)};$$

откуда слѣдуетъ

$$2 \cos^2 \tfrac{1}{2} A = \frac{1 + \cos \Pi(b) \cos \Pi(c) - \dfrac{\sin \Pi(b) \sin \Pi(c)}{\sin \Pi(a)}}{\cos \Pi(b) \cos \Pi(c)}.$$

Если вставимъ сюда:

$$\frac{\sin \Pi(b) \sin \Pi(c)}{\sin \Pi(b+c)}$$

вмѣсто

$$1 + \cos \Pi(b) \cos \Pi(c)$$

то это выраженіе приметъ видъ:

$$2 \cos^2 \tfrac{1}{2} A = \operatorname{tang} \Pi(b) \operatorname{tang} \Pi(c) \left\{ \frac{1}{\sin \Pi(b+c)} - \frac{1}{\sin \Pi(a)} \right\}.$$

Такимъ же образомъ находимъ:

$$-2 \sin^2 \tfrac{1}{2} A = \operatorname{tang} \Pi(b) \operatorname{tang} \Pi(c) \left\{ \frac{1}{\sin \Pi(b-c)} - \frac{1}{\sin \Pi(a)} \right\}.$$

Изъ этихъ двухъ выраженій выводимъ:

$$\sin^2 A = \operatorname{tang}^2 \Pi(b) \operatorname{tang}^2 \Pi(c) \left\{ -\frac{1 - \cos^2 \Pi(b) \cos^2 \Pi(c)}{\sin^2 \Pi(b) \sin^2 \Pi(c)} + \frac{2}{\sin \Pi(a) \sin \Pi(b) \sin \Pi(c)} - \frac{1}{\sin^2 \Pi(a)} \right\}$$

или

$$\sin^2 A = -\operatorname{tang}^2 \Pi(b) \operatorname{tang}^2 \Pi(c) \left\{ \frac{1}{\sin^2 \Pi(a)} + \frac{1}{\sin^2 \Pi(b)} + \frac{1}{\sin^2 \Pi(c)} - \frac{2}{\sin \Pi(a) \sin \Pi(b) \sin \Pi(c)} - 1 \right\}.$$

Полагая для сокращенія

$$P = \sqrt{\frac{-1}{\sin^2 \Pi(a)} - \frac{1}{\sin^2 \Pi(b)} - \frac{1}{\sin^2 \Pi(c)} + \frac{2}{\sin \Pi(a) \sin \Pi(b) \sin \Pi(c)} + 1}$$

получимъ:

$$\sin A = \operatorname{tang} \Pi(b) \operatorname{tang} \Pi(c)\, P. \qquad (28)$$

Можно также дать $P$ слѣдующій видъ симметричный въ отношеніи къ $a, b, c$.

$$P^2 = 2\left\{1 + \frac{1}{\sin \Pi(a)}\right\}\left\{1 + \frac{1}{\sin \Pi(b)}\right\}\left\{1 + \frac{1}{\sin \Pi(c)}\right\} - \left\{1 + \frac{1}{\sin \Pi(a)} + \frac{1}{\sin \Pi(b)} + \frac{1}{\sin \Pi(c)}\right\}^2.$$

Идя отъ уравненія (28) и разсматривая здѣсь $P$ какъ величину неопредѣленную, можно доказать слѣдующимъ образомъ, что $P$ должно быть симметричная функція отъ $a, b, c$. Умножая уравненіе (28) на $\operatorname{tang} \Pi(a)$ поставляемъ сюда $\sin B \operatorname{tang} \Pi(b)$ вмѣсто его значенія $\sin A \operatorname{tang} \Pi(a)$ (уравненіе 13) и раздѣлимъ потомъ на $\operatorname{tang} \Pi(b)$, выходитъ:

$$\sin B = \operatorname{tang} \Pi(a) \operatorname{tang} \Pi(c)\, P.$$

Умножаемъ это послѣднее уравненіе на $\operatorname{tang} \Pi(b)$ и вставимъ сюда $\sin C \operatorname{tang} \Pi(c)$ вмѣсто его значенія $\sin B \operatorname{tang} \Pi(b)$ (уравненіе 13) и раздѣлимъ потомъ на $\operatorname{tang} \Pi(c)$ получимъ:

$$\sin C = \operatorname{tang} \Pi(a) \operatorname{tang} \Pi(b)\, P,$$

откуда видно, что функція $P$ симметрична въ отношеніи къ $a, b, c$.

Мы уже нашли:

$$\cos A = \frac{1 - \dfrac{\sin \Pi(b) \sin \Pi(c)}{\sin \Pi(a)}}{\cos \Pi(b) \cos \Pi(c)}$$

или все равно:

$$\cos A = \operatorname{tang} \Pi(b) \operatorname{tang} \Pi(c) \left\{\frac{1}{\sin \Pi(b) \sin \Pi(c)} - \frac{1}{\sin \Pi(a)}\right\};$$

такимъ же образомъ находимъ:

$$\cos B = \operatorname{tang} \Pi(c) \operatorname{tang} \Pi(a) \left\{\frac{1}{\sin \Pi(a) \sin \Pi(c)} - \frac{1}{\sin \Pi(b)}\right\}$$

$$\cos C = \operatorname{tang} \Pi(a) \operatorname{tang} \Pi(b) \left\{\frac{1}{\sin \Pi(a) \sin \Pi(b)} - \frac{1}{\sin \Pi(c)}\right\}.$$

Изъ этихъ выраженій для $\sin A, \cos A, \sin B, \cos B$, выводимъ:

$$\begin{aligned}
\sin(A + B) &= \sin A \cos B + \cos A \sin B \\
&= \operatorname{tang} \Pi(b) \operatorname{tang}^2 \Pi(c)\, P \operatorname{tang} \Pi(a) \left\{\frac{1}{\sin \Pi(c) \sin \Pi(a)} - \frac{1}{\sin \Pi(b)}\right\} \\
&\quad + \operatorname{tang}^2 \Pi(c) \operatorname{tang} \Pi(a)\, P \operatorname{tang} \Pi(b) \left\{\frac{1}{\sin \Pi(b) \sin \Pi(c)} - \frac{1}{\sin \Pi(a)}\right\} \\
&= \operatorname{tang} \Pi(a) \operatorname{tang} \Pi(b) \operatorname{tang}^2 \Pi(c)\, P \left\{\frac{1}{\sin \Pi(a)} + \frac{1}{\sin \Pi(b)}\right\} \left\{\frac{1}{\sin \Pi(c)} - 1\right\}
\end{aligned}$$

и наконецъ:

$$\sin(A+B) = \frac{\operatorname{tang}\Pi(a)\operatorname{tang}\Pi(b)\,P}{\left\{\frac{1}{\sin\Pi(c)}+1\right\}}\left\{\frac{1}{\sin\Pi(a)}+\frac{1}{\sin\Pi(b)}\right\}.$$

Послѣднее изъ уравненіи (19) даетъ:

$$\cos A + \cos(B+C) = \sin B\sin C\left\{\frac{1}{\sin\Pi(a)}-1\right\}$$

вставляемъ сюда вмѣсто $\sin B$, $\sin C$ ихъ значенія изъ уравненія (28) получимъ:

$$\cos(B+C) = -\cos A + \operatorname{tang}\Pi(c)\operatorname{tang}^2\Pi(a)\operatorname{tang}\Pi(b)\,P^2\left\{\frac{1}{\sin\Pi(a)}-1\right\}$$

или, что все равно

$$\cos(B+C) = -\cos A + \frac{\operatorname{tang}\Pi(b)\operatorname{tang}\Pi(c)\,P^2}{\frac{1}{\sin\Pi(a)}+1}.$$

Помощью предъидущихъ уравненій находимъ:

$$\cos(A+B+C) = \cos A\cos(B+C) - \sin A\sin(B+C)$$

$$= -\cos^2 A + \frac{\operatorname{tang}^2\Pi(b)\operatorname{tang}^2\Pi(c)\,P^2}{\left\{\frac{1}{\sin\Pi(a)}+1\right\}}\left\{\frac{1}{\sin\Pi(b)\sin\Pi(c)}-\frac{1}{\sin\Pi(a)}\right\}$$

$$-\frac{\operatorname{tang}^2\Pi(b)\operatorname{tang}^2\Pi(c)\,P^2}{\left\{\frac{1}{\sin\Pi(a)}+1\right\}}\left\{\frac{1}{\sin\Pi(b)}+\frac{1}{\sin\Pi(c)}\right\}$$

$$2\cos^2\frac{(A+B+C)}{2} = \sin^2 A + \frac{\operatorname{tang}^2\Pi(b)\operatorname{tang}^2\Pi(c)\,P^2}{\left\{\frac{1}{\sin\Pi(a)}+1\right\}}\left\{\frac{1}{\sin\Pi(b)\sin\Pi(c)}-\frac{1}{\sin\Pi(a)}\right\}$$

$$-\frac{\operatorname{tang}^2\Pi(b)\operatorname{tang}^2}{\left\{\frac{1}{\sin\Pi(a)}+1\right\}}\left\{\frac{1}{\sin\Pi(b)}+\frac{1}{\sin\Pi(c)}\right\}$$

$$= \operatorname{tang}^2\Pi(b)\operatorname{tang}\Pi(c)\,P^2 + \frac{\operatorname{tang}^2\Pi(b)\operatorname{tang}^2\Pi(c)}{\left\{\frac{1}{\sin\Pi(a)}+1\right\}}\left\{\frac{1}{\sin\Pi(b)\sin\Pi(c)}-\frac{1}{\sin\Pi(a)}-\frac{1}{\sin\Pi(b)}-\frac{1}{\sin\Pi(c)}\right\}$$

$$= \frac{\operatorname{tang}^2\Pi(b)\operatorname{tang}^2\Pi(c)\,P^2}{\left\{\frac{1}{\sin\Pi(a)}+1\right\}}\left\{1+\frac{1}{\sin\Pi(b)\sin\Pi(c)}-\frac{1}{\sin\Pi(b)}-\frac{1}{\sin\Pi(c)}\right\}$$

$$= \operatorname{tang}^2\Pi(a)\operatorname{tang}^2\Pi(b)\operatorname{tang}^2\Pi(c)\,P^2\left\{\frac{1}{\sin\Pi(a)}-1\right\}\left\{\frac{1}{\sin\Pi(b)}-1\right\}\left\{\frac{1}{\sin\Pi(c)}-1\right\};$$

но было уже доказано, что площадь, треугольника $\triangle = \pi - A - B - C$, слѣдовательно

$$\sin\frac{\triangle}{2} = \frac{1}{\sqrt{2}}\operatorname{tang}\Pi(a)\operatorname{tang}\Pi(b)\operatorname{tang}\Pi(c)P \times \sqrt{\left(\frac{1}{\sin\Pi(a)} - 1\right)\left(\frac{1}{\sin\Pi(b)} - 1\right)\left(\frac{1}{\sin\Pi(c)} - 1\right)};$$

если $a, b, c$ такъ малы, что можемъ довольствоваться приближеніемъ

$$\frac{1}{\sin\Pi(a)} = 1 + \tfrac{1}{2}a^2, \frac{1}{\sin\Pi(b)} = 1 + \tfrac{1}{2}b^2, \frac{1}{\sin\Pi(c)} = 1 + \tfrac{1}{2}c^2.$$

$$\operatorname{tang}\Pi(a) = \tfrac{1}{a}(1 - \tfrac{1}{6}a^2), \operatorname{tang}\Pi(b) = \tfrac{1}{b}(1 - \tfrac{1}{6}b^2), \operatorname{tang}\Pi(c) = \tfrac{1}{c}(1 - \tfrac{1}{6}c^2)$$

получимъ:

$$\sin\frac{\triangle}{2} = \tfrac{1}{4}\sqrt{\frac{a^2 + b^2 + c^2}{2}}$$

или отбрасывая степени $\triangle$ выше первой

$$\triangle = \frac{1}{2\sqrt{2}}\sqrt{a^2 + b + c^2}$$

Опредѣляемъ положеніе точки въ пространствѣ тремя прямоугольными координатами, $z$ перпендикулярно къ плоскости $xy$, $y$ перпендикулъ, опущенный отъ конца $z$ къ оси $x$, а $x$ часть оси $x$ между началомъ координатъ и концомъ $y$.

На кривой поверхности, элементъ которой требуется опредѣлить, беремъ три точки и пусть координаты первой $x, y, z$, координаты второй $x + dx, y, z + \left(\frac{dz}{dx}\right)dx$, а координаты третій точки $x, y + dy, z + \left(\frac{dz}{dy}\right)dy$. Называемъ $t$ разстояніе между вершинами двухъ перпендикуловъ къ оси $x$-овъ, которые оба равны $y$, и между которыми заключается часть $dx$ оси $x$, предполагая $dx, dy$ безконечно малыми, получимъ основываясь на уравненіе (27)

$$t = \frac{dx}{\sin\Pi(y)}.$$

Разстояніе двухъ первыхъ точекъ на кривой поверхности составляетъ треугольникъ съ прямыми, которыхъ длина

$$\frac{dx}{\sin\Pi(y)\sin\Pi(z)}, \left(\frac{dz}{dx}\right)dx.$$

Можемъ почитать этотъ треугольникъ за прямоугольникъ при малости его боковъ, гдѣ гипотенуза будетъ разстояніе между двумя первыми точками на поверхости. Итакъ квадратъ этаго разстоянія будетъ

$$dx^2\left\{\frac{1}{\sin^2\Pi(y)\sin^2\Pi(z)} + \left(\frac{dz}{dx}\right)^2\right\}.$$

Такимъ же образомъ находимъ квадратъ разстоянія первой точки отъ третей

$$dy^2\left\{\frac{1}{\sin^2 \Pi(z)}+\left(\frac{dz}{dy}\right)^2\right\};$$

и квадратъ разстоянія второй точки отъ третей

$$\frac{dx^2}{\sin^2 \Pi(y)\sin^2 \Pi(z)}+\frac{dy^2}{\sin^2 \Pi(z)}+\left\{\left(\frac{dz}{dy}\right)dy-\left(\frac{dz}{dx}\right)dx\right\}^2.$$

Площадь треугольника, которому боками служатъ разстоянія отъ первой точки на кривой поверхности до второй, отъ второй до третей и отъ третей до первой, гдѣ и сумма угловъ равняется $\pi$ безъ чувствительной погрѣшности по причинѣ малости боковъ, будетъ равна, въ слѣдствіе доказаннаго выше и значеній, найденныхъ для квадратовъ его сторонъ

$$\frac{d^2S}{dx\,dy}=\frac{1}{2\sin \Pi(z)}\sqrt{\left(\frac{dz}{dx}\right)^2+\frac{1}{\sin^2 \Pi(y)}\left(\frac{dz}{dy}\right)^2+\frac{1}{\sin^2 \Pi(y)\sin^2 \Pi(z)}};$$

таково выраженіе для элемента поверхности, уравненіе которой

$$z=f(x,y).$$

Примѣняемъ это выраженіе къ сферѣ, котораго полупоперечникъ $r$; если начало координатъ въ центрѣ сферы, то уравненіе сферы даетъ:

$$\left(\frac{dz}{dx}\right)=-\frac{\cos \Pi(x)}{\cos \Pi(z)}$$

$$\left(\frac{dz}{dy}\right)=-\frac{\cos \Pi(y)}{\cos \Pi(z)}$$

потомъ

$$\frac{\cos \Pi(r)}{\sin^2 \Pi(r)}\cdot\frac{\sin \Pi(y)\sin^2 \Pi(x)}{\sqrt{\sin^2 \Pi(x)\sin^2 \Pi(y)-\sin^2 \Pi(r)}}=\frac{d^2S}{d\,\Pi(x)\,d\,\Pi(y)}.$$

Умножаемъ на $d\,\Pi(y)$ и интегрируемъ отъ $\sin \Pi(y)=\dfrac{\sin \Pi(r)}{\sin \Pi(x)}$, до $\Pi(y)=\dfrac{\pi}{2}$ получимъ:

$$\frac{dS}{d\,\Pi(x)}=\frac{2\pi\sin \Pi(x)\cos \Pi(r)}{\sin^2 \Pi(r)}.$$

Умножаемъ еще на $d\,\Pi(x)$ и интегрируемъ отъ $\Pi(x)=\dfrac{\pi}{2}$ получимъ:

$$S=\frac{2\pi\cos \Pi(r)\cos \Pi(x)}{\sin^2 \Pi(r)}.$$

Это представляетъ поверхность отрѣзка сферы, заключеннаго между двухъ плоскостей перпендикулярныхъ къ одному полупоперечнику, изъ которыхъ одна проходитъ чрезъ центръ сферы, а другая на разстояніе $x$ отъ центра сферы. Чтобы найти поверхность всей сферы, надо въ это послѣднее выраженіе поставить $x = r$ и значеніе всего выраженія удвоить, такимъ образомъ находимъ величину поверхности всей сферы $4\pi \cot g^2 \Pi(r)$ или $\pi (e^r - e^{-r})^2$; если $r$ такъ мало, что можно пренебречь высшія степени отъ $r$, то это выраженіе дѣлается $4\pi r^2$, какъ въ обыкновенной геометріи.

Полагаемъ

$$\cos\psi = \tang \Pi(r) \cot g \Pi(y)$$
$$\cos \Pi(x) = \cos \Pi(r) \sin\psi \sin\varphi$$

и вводимъ новыя перемѣнныя $\psi, \varphi$ вмѣсто $x, y$ въ выраженіе для элемента поверхности шара съ полупоперечникомъ $r$, о которомъ идетъ дѣло; находимъ:

$$\frac{d^2 S}{d\varphi\, d\psi} = -\frac{\cos \Pi(r)}{\sin \Pi(r)} \frac{\sin\psi \sqrt{1 - \cos^2\Pi(r) \sin^2\psi \sin^2\varphi}}{1 - \cos^2 \Pi(r) \sin^2\psi}.$$

Умножаемъ это уравненіе на $8\, d\psi\, d\varphi$ и интегрируемъ отъ $\psi = o$ до $\psi = \frac{\pi}{2}$ и отъ $\varphi = o$ до $\varphi = \frac{\pi}{2}$, такъ находимъ поверхность цѣлой сферы. Уравнивая выраженіе, которое найдемъ такимъ образомъ для поверхности цѣлой сферы съ выраженіемъ, найденнымъ выше для тойже поверхности заключаемъ, что

$$\frac{\pi}{\sin \Pi(r)} = \int_o^{\frac{\pi}{2}} d\psi \int_o^{\frac{\pi}{2}} d\varphi \; \frac{\sin\psi \sqrt{1 - \cos^2 \Pi(r) \sin^2\psi \sin^2\varphi}}{1 - \cos^2 \Pi(r) \sin^2\psi}. \tag{30}$$

Если означаемъ $E(\alpha)$ элиптическій интегралъ

$$E(\alpha) = \int_o^{\frac{\pi}{2}} d\varphi \sqrt{1 - \alpha^2 \sin^2\varphi}$$

гдѣ $\alpha$ постоянное подъ знакомъ интеграла то находимъ помощію интеграла котораго значеніе представляетъ сферу:

$$\frac{\pi\alpha}{\sqrt{1-\alpha^2}} = \int_o^{\alpha} \frac{dx\, E(x)}{(1 - x^2)\sqrt{\alpha^2 - x^2}};$$

ставя въ интегралъ (30) $\frac{\pi}{2} - R$ вмѣсто $\Pi(r)$, получимъ:

$$\frac{\pi}{2} R = \int_o^{\frac{\pi}{2}} \int_o^{\frac{\pi}{2}} \frac{d\psi\, d\varphi \sin\psi \sin R}{\sqrt{1 - \sin^2\psi \sin^2\varphi \sin^2 R}}$$

Произведя интегрированіе въ отношеніи $\psi$ въ указанныхъ границахъ находимъ:

$$\pi R = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sin\varphi} \log \left(\frac{1+\sin\varphi\sin R}{1-\sin\varphi\sin R}\right),$$

что принимаетъ видъ, когда поставимъ $\Pi(x)$ вмѣсто $\varphi$

$$\pi R = \int_0^{\infty} dx \log\left\{\frac{e^{2x}+1+2e^{x}\sin R}{e^{2x}+1-2e^{x}\sin R}\right\};$$

интегрированіе по частямъ приводитъ этотъ интегралъ къ такому,

$$\tfrac{1}{4}\pi\frac{R}{\sin R} = \int_0^{\infty} \frac{(e^{2x}-1)\,e^{x}\,x\,dx}{e^{4x}+2e^{2x}\cos 2R+1}; \tag{31}$$

для $R = \dfrac{\pi}{2}$ это уравненіе даетъ:

$$\tfrac{1}{8}\pi^2 = \int_0^{\infty} \frac{x\,dx\,e^{x}}{e^{2x}-1}.$$

Легко доказать справедливость уравненія (31) для $\cos R > 1$.

Мы въ самомъ дѣлѣ имѣемъ:

$$\int_0^{\pi} d\psi \log\operatorname{cotg}\tfrac{1}{2}\psi = o,$$

откуда слѣдуетъ, что для всякаго числа $a$

$$\int_0^{\pi} d\psi \log(e^{a}\operatorname{cotg}\tfrac{1}{2}\psi) = a\pi.$$

Перемѣняемъ это уравненіе полагая въ немъ $e^{a}\operatorname{cotg}\frac{1}{2}\psi = e^{x}$; получимъ:

$$\int_{-\infty}^{+\infty} \frac{x\,dx}{e^{x-a}+e^{-x+a}} = \tfrac{1}{2}\pi a;$$

этому уравненію легко можно дать еще такой видъ:

$$\int_0^{\infty} \frac{(e^{x}-e^{-x})\,x\,dx}{e^{2x}+(e^{2a}+e^{-2a})+e^{-2x}} = \tfrac{1}{2}\frac{\pi a}{e^{a}-e^{-a}},$$

откуда приходимъ снова къ уравненію (31) ставя $a\sqrt{-1}$ вмѣсто $a$.

Если за координаты принимаемъ дуги предѣльнаго круга, одна $\zeta$ проведенная отъ данной точки, въ плоскости пересѣкающей плоскость $xy$ въ прямой параллельной къ оси $x$ и которой дугѣ $\zeta$ эта параллеляная, продолженная на сторонѣ параллельности служитъ осью, другая $\eta$ въ плоскости $x, y$ проведенная отъ конца $\zeta$ до оси $x$-овъ, которая вмѣстѣ служитъ осью этой дугѣ, если наконецъ за третью координату принимаемъ $\xi$, отрѣзокъ оси $x$ отъ начала координатъ до конца $\eta$, то элементъ объема $P$ долженъ быть $d\xi\ d\eta\ d\zeta$. Итакъ

$$d^3 P = d\xi\, d\eta\, d\zeta.$$

Полагаемъ еще $\zeta = \cotg \Pi(z)$ гдѣ $z$ перпендикулъ отъ данной точки на плоскость $x\ y$; получимъ:

$$\left(\frac{d^3 P}{d\xi\, d\eta\, dz}\right) = \frac{1}{\sin \Pi(z)}.$$

Изъ уравненія предѣльнаго круга выводимъ

$$e^{-p} = \sin \Pi(z)$$

гдѣ $p$ разстояніе отъ пересѣченія дуги $\zeta$ съ плоскостію $xy$ до конца перпендикула $z$.

Замѣчая къ тому, что въ слѣдствіе уравненія предѣльнаго круга и выраженія дуги предѣльнаго круга въ зависимости отъ ординаты:

$$\cotg \Pi(y) = \eta\, e^{-p}$$
$$e^{\xi - x} = \sin \Pi(z).\, \sin \Pi(y)$$

находимъ:

$$\frac{d\eta}{dy} = \frac{1}{\sin \Pi(y) \sin \Pi(z)};\ dx = d\xi;$$

откуда слѣдуетъ, что

$$\left(\frac{d^3 P}{dx\, dy\, dz}\right) = \frac{1}{\sin \Pi(y) \sin^2 \Pi(z)}.$$

Умножая это выраженіе на $dx$ и интегрируя отъ $x = o$, получимъ:

$$\left(\frac{d^2 P}{dy\, dz}\right) = \frac{x}{\sin \Pi(y) \sin^2 \Pi(z)}.$$

Умножая тоже выраженіе на $dy$ и интегрируя отъ $y = o$, получимъ:

$$\left(\frac{d^2 P}{dx\, dz}\right) = \frac{\cotg \Pi(y)}{\sin^2 \Pi(z)}.$$

Умножая наконецъ на $dz$ и интегрируя отъ $z = o$ получимъ:

$$\left(\frac{d^2 P}{dx\, dy}\right) = \frac{1}{8 \sin \Pi(y)} \left\{e^{2z} + e^{-2z} + 4z\right\}.$$

Если предпослѣднее изъ этихъ выраженій умножаемъ на $dx.\ dz$ и потомъ интегрируемъ сначала въ отношеніи къ $z$ отъ $z=o$ до значенія $z$ взятого изъ уравненія $\sin \Pi(r) = \sin \Pi(x) \sin \Pi(z)$ а потомъ въ отношеніи къ $x$ отъ $x=o$ до $x=r$ и если умножаемъ выводъ на 8, чтобы получить объемъ цѣлаго шара, то находимъ объемъ цѣлаго шара $= \frac{1}{2}\pi \left\{ e^{2r} - e^{-2r} - 4r \right\}$, что для $r$ очень малаго даетъ $\frac{4}{3}\pi r^3$, какъ въ обыкновенной геометріи.

Чтобы элементъ объема выразить въ полярныхъ координатахъ называемъ $r$ разстояніе отъ начала координатъ до точки въ пространствѣ, прямоугольныя координаты которой $x, y, z$. Называемъ $q$ прямую, проведенную отъ начала координатъ до конца $z$, $\theta$ уголъ между $r$ и $q$, $\omega$ уголъ между $q$ и оси положительныхъ $x$-въ. Полагаемъ еще $\Pi(x) = X, \Pi(y) = Y, \Pi(z) = Z, \Pi(r) = R, \Pi(q) = Q$. Ведемъ чрезъ данную точку плоскость перпендикулярно къ оси $z$. Пусть $r'$ прямая проведенная въ этой плоскости отъ данной точки къ оси $z$ и полагаемъ еще $\Pi(r') = R'$.

Описываемъ около начала координатъ какъ центра сферу полупоперечникомъ $r$. Плоскость $x\ y$ пересѣкаетъ эту сферу въ большомъ кругѣ, окружность котораго, согласно доказанному выше будетъ:

$$2\pi \cot R.$$

Часть этого круга между двухъ плоскостей, проведенныхъ черезъ ось $z$-овъ и наклоненныхъ одна къ другой подъ угломъ $\omega$, должна быть

$$\omega \cot R.$$

Окружность круга отъ пересѣченія тойже сферы съ плоскостью, проходящею чрезъ данную точку и перпендикулярной къ оси $z$ будетъ:

$$2\pi \cot R'.$$

А часть этой окружности между двухъ плоскостей, проведенныхъ чрезъ ось $z$ и наклоненныхъ подъ угломъ $\omega$, должна быть:

$$\omega \cot R'$$

Измѣненіе, произведенное въ этой послѣдней дугѣ приращеніемъ угла $\omega$ на $d\omega$, должно быть

$$d\omega \cot R'.$$

Прямоугольный треугольникъ гдѣ гипотенуза $r$, одинъ изъ боковъ прямаго угла $r'$ и уголъ противъ $r'$ равенъ $\frac{\pi}{2} - \theta$ даетъ (уравненіе 31)

$$\tan R' \cos\theta = \tan R$$

откуда слѣдуетъ, что

$$d\omega \cot R' = d\omega \cot R \cos\theta.$$

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Окружность круга отъ пересѣченія тойже сферы плоскостью, проходящею чрезъ ось $z$ равна:

$$2\pi \cot R,$$

а дуга этаго круга, которая соотвѣтствуетъ углу $\theta$ при центрѣ будетъ:

$$\theta \cot R;$$

откуда слѣдуетъ что измѣненіе этой дуги, которая соотвѣтствуетъ приращенію $d\theta$ къ углу $\theta$, должна быть:

$$d\theta \cot R.$$

Если всѣ приращенія безконечно малы, то элементъ объема выражается какъ въ обыкновенной геометріи произведеніемъ трехъ линій перпендикулярныхъ между собою

$$dr,\ d\omega \cos\theta \cot R,\ d\theta \cot R,$$

потому что объемъ этотъ можетъ быть принимаемъ за призму. Итакъ элементъ объема выражается въ полярныхъ координатахъ:

$$dr\, d\omega\, d\theta \cos\theta \cot^2 R = d^3 P$$

или замѣняя $\cot^2 R$ его значеніемъ въ $r$:

$$d^3 P = \tfrac{1}{4}\, dr\, d\omega\, d\theta \cos\theta\, (e^r - e^{-r})^2.$$

Интегрируя съ начала въ отношеніи къ $r$ отъ $r = o$ получимъ:

$$d^2 P = \tfrac{1}{8}\, d\omega\, d\theta\, (e^{2r} - e^{-2r} - 4r).$$

Для сферы которой центръ въ началѣ координатъ, $r$ независитъ отъ $\theta$ и $\omega$.

Интегрируя въ отношеніи къ $\omega$ отъ $\omega = o$ до $\omega = 2\pi$ и въ отношеніи къ $\theta$ отъ $\theta = o$ до $\theta = \frac{\pi}{2}$ и умножая выводъ на 2 получимъ объемъ цѣлой сферы: $\frac{1}{2}\pi\,(e^{2r} - e^{-2r} - 4r)$ какъ выше.

Разсматриваемъ теперь часть $S$ поверхности предѣльной сферы, ограниченной замкнутой линіей, ведемъ изъ различныхъ точекъ этой пограничной линіи прямыя параллельныя къ оси сферы, такія прямыя образуютъ поверхность, которую мы назовемъ по сходству коническою, и которая простирается безконечно въ обѣ стороны, но на которой мы будемъ разсматривать только часть отъ предѣльной сферы безконечно простирающуюся въ сторону параллельности осей предѣльной сферы. Пусть $S'$ будетъ часть другой предѣльной сферы съ осями параллельными къ прежнимъ и обращеннымъ въ туже сторону, часть, которая находится внутри конической поверхности. $S, S'$ и часть конической поверхности между обѣими предѣльными сферами заключаютъ объемъ конечный во всѣ стороны, который мы предлагаемъ опредѣлить. Называемъ $c$ часть оси между двухъ предѣльныхъ сферъ. Повторяемъ прямую $c$ нѣсколько разъ на одной изъ осей предѣльныхъ сферъ, проходящихъ чрезъ одно изъ точекъ кривой, ограничивающей $S$, начиная съ той точки гдѣ эта ось пересѣкаетъ $S'$. Ведемъ чрезъ точки дѣленія предѣльныя сферы съ осями параллельными къ осямъ двухъ первыхъ и обращенными въ туже сторону. Пусть $S''S'''$ и т. д. части этихъ предѣльныхъ сферъ внутри конической поверхности. Изъ того, что было доказано о дугахъ предѣльнаго круга расположенныхъ подобно, какъ и здѣсь части предѣльной сферы, слѣдуетъ, что

$$S' = S e^{-2c}$$
$$S'' = S e^{-4c}$$
$$S''' = S e^{-6c} \text{ и т. д.}$$

Называемъ еще $P, P', P''$ и т. д. объемы внутри конической поверхности между $S, S'$ между $S'$ и $S''$ и т. д. и замѣтимъ къ тому что объемы $P, P', P''$ и т. д. должны быть пропорціональны поверхностямъ $S, S', S''$ и т. д. Итакъ должно быть $P = C S$, гдѣ $C$ зависитъ только отъ $c$, откуда слѣдуетъ, что:

$$P' = C S' = C S e^{-2c}$$
$$P'' = C S'' = C S e^{-4c}.$$

Сумма $\sum\limits_{n=0}^{\infty} P^{(n)}$ по этому будетъ объемъ внутри конической поверхности, которой основаніе $S$ и которая простирается безпредѣльно, въ сторону параллельности прямыхъ производящихъ. Пусть этотъ объемъ будетъ $K$; получимъ:

$$K = \frac{C S}{1 - e^{-2c}}.$$

Эта величина не должна зависить отъ $c$, а это требуетъ, чтобы:

$$C = (1 - e^{-2c}) A,$$

гдѣ $A$ число отвлеченное, а такъ какъ единица объема произвольна, то мы примемъ $C = \frac{1}{2}(1 - e^{-2c})$ съ тѣмъ чтобы объемъ $P$ будучи выраженъ формулой

$$P = \tfrac{1}{2} S (1 - e^{-2c})$$

обращался въ $P = c S$ для безконечно малаго $c$, выраженіе, которое согласно съ тѣмъ, какъ выражается въ обыкновенной геометріи объемъ призмы съ основаніемъ $S$ и съ высотою $c$. Можно также за элементъ объема принимать ограниченный данною поверхностью объемъ внутри конической поверхности, образуемой осями предѣльной сферы, проведенными чрезъ всѣ точки кривой, ограничивающей часть данной поверхности безконечно малой во всѣ стороны. Большое число различныхъ выраженій для элемента тойже геометрической величины доставляетъ средства для сравненія интеграловъ, средства, которыя въ особенности полезны въ теоріи опредѣленныхъ интеграловъ.

Показавъ до сихъ поръ, какимъ образомъ надобно вычислять длину кривыхъ линій, величину поверхностей и величину объема тѣлъ, мы вправѣ утверждать, что пангеометрія составляетъ ученіе геометрическое полное. Одного взгляда на уравненія которымъ выражаютъ зависимость угловъ и боковъ прямолинейныхъ треугольниковъ, достаточно, чтобы доказать, что начиная съ этихъ уравненій пангеометрія дѣлается вычисленіемъ аналитическимъ, которое замѣняетъ и обобщаетъ аналитическій способъ обыкновенной геометріи. Можно бы начать изложеніе пангеометріи уравненіями (19) и даже попытаться замѣнить эти уравненія другими, которыя бы выражали зависимость боковъ и угловъ всякаго прямолинейнаго треугольника. Но въ этомъ послѣднемъ случаѣ надо было бы доказывать, что эти новыя уравненія согласуются съ основными геометрическими понятіями. Уравненія (19) будучи выведены съ помощію этихъ основныхъ геометрическихъ понятій, необходимо съ ними согласуются. Итакъ всѣ уравненія, какими можно бы было замѣнить уравненія (19), должны, если они не будутъ слѣдствіемъ уравненія (19), привести къ заключеніямъ противнымъ основнымъ геометрическимъ понятіямъ. Итакъ уравненія (19) служатъ основаніемъ геометріи въ самомъ общемъ видѣ, потому что они независятъ отъ предположенія, что сумма трехъ угловъ во всякомъ прямолинейномъ треугольникѣ равна двумъ прямымъ.

Пангеометрія, какъ она здѣсь изложена, основанная на началахъ несомнѣнныхъ, подаетъ, какъ мы видѣли средства къ вычисленію значенія различныхъ геометрическихъ величинъ и доказываетъ вмѣстѣ, что принятое въ обыкновенной геометріи явно или скрытно предположеніе, что сумма трехъ угловъ всякаго прямолинейнаго треугольника постоянна, не есть необходимое слѣдствіе нашихъ понятій о пространствѣ. Одинъ опытъ только можетъ подтвердить истину этаго предположенія, напр. измѣреніемъ на самомъ дѣлѣ трехъ угловъ прямолинейнаго треугольника, измѣреніе которое можетъ быть произведено различнымъ образомъ. Можно измѣрять три угла въ треугольникѣ, построенномъ на искуственной плоскости, или три угла одного треугольника въ пространствѣ. Въ этомъ послѣднемъ случаѣ надобно предпочесть треугольники, которыхъ бока очень велики, потому что согласно съ теоріей пангеометріи, разность суммы трехъ угловъ треугольника съ двумя прямыми углами тѣмъ болѣе, чѣмъ бока болѣе. Пусть $r$ полупоперечникъ круга, $A$ уголъ при центрѣ между полупоперечниками противъ хорды, равной $r$. Называемъ $p$ перпендикулъ, опущенный изъ центра круга къ хордѣ $r$, которую онъ раздѣляетъ по поламъ. Разсматриваемъ одинъ изъ прямоугольныхъ треугольниковъ котораго перпендикулярные бока $p$ и $\frac{r}{2}$ а гипотенуза $r$.

Согласно съ общимъ уравненіемъ (13) въ этомъ треугольникѣ будетъ:

$$\sin \frac{A}{2} \operatorname{tang} \Pi \left(\tfrac{\zeta}{2}\right) = \operatorname{tang} \Pi (r),$$

уравненіе, которое въ соединеніи съ тожественнымъ уравненіемъ

$$\operatorname{tang} \Pi (r) = \frac{\sin^2 \Pi \left(\frac{\zeta}{2}\right)}{2 \cos \Pi \left(\frac{\zeta}{2}\right)}$$

даетъ:

$$\sin \frac{A}{2} = \tfrac{1}{2} \sin \Pi \left(\tfrac{\zeta}{2}\right).$$

Въ обыкновенной геометріи имѣемъ:

$$A = \tfrac{\pi}{3}.$$

Пусть дѣйствительное измѣреніе даетъ:

$$A = \frac{2\pi}{6+K}$$

гдѣ $K$ положительное число. Итакъ должно быть

$$\sin \left(\frac{\pi}{6+K}\right) = \tfrac{1}{2} \sin \Pi \left(\tfrac{\zeta}{2}\right).$$

Если $K$ и $r$ даны, то возможно вывести изъ этого уравненія значеніе $\Pi \left(\frac{\zeta}{2}\right)$; послѣ чего находимъ уголъ параллельности $\Pi (x)$ для всякой прямой $x$. Разстояніе между небесными тѣлами доставляетъ намъ случай наблюдать углы треугольниковъ, которыхъ бока очень велики. Называемъ $\alpha$ геоцентрическую широту неподвижной звѣзды въ данную эпоху, а $\beta$ другую геоцентрическую широту тойже звѣзды, которая отвѣчаетъ времени, когда земля снова находится въ плоскости перпендикулярной къ эклиптикѣ и проведенной чрезъ первое мѣсто звѣзды. Пусть $2a$ разстояніе между этими двумя положеніями земли, $\delta$ уголъ подъ которымъ видно разстояніе $2a$ изъ звѣзды. Если углы $\alpha$, $\beta$, $\delta$ неудовлетворяютъ уравненію $\alpha = \beta + \delta$, то это будетъ знакомъ, что сумма трехъ угловъ этого треугольника не равна двумъ прямымъ угламъ. Можно такъ выбрать звѣзду, что $\delta = o$ и можно всегда предположить, что существуетъ линія $x$ такая, для которой

$$\Pi (x) = a.$$

Если $\delta = o$, то прямыя, проведенныя отъ двухъ положеній земли къ звѣздѣ могутъ почитаться за параллельныя, въ слѣдствіе чего должно быть $\beta = \Pi (x + 2a)$, откуда слѣдуетъ согласно съ доказаннымъ выше, что

$$\operatorname{tang} \tfrac{\alpha}{2} = e^{-x}$$
$$\operatorname{tang} \tfrac{\beta}{2} = e^{-x-2a}$$

Всякій разъ, когда $\alpha$ и $\beta$ въ наблюденіяхъ звѣзды для которой $\delta = o$, окажутся различными, два послѣднія уравненія дадутъ $x$ и $a$, выраженныя посредствомъ линіи принятой за единицу въ пангеометріи. Такимъ образомъ зная уголъ параллельности $\Pi(x)$ для опредѣленной линіи $x$ можно вычислить уголъ $\Pi(y)$ для всякой линіи $y$.

# II. Lobachevsky's biography

Portrait of Lobachevsky. Copy from a portrait by L. Krukov, end of XIX century, unknown artist. Museum of History of Kazan University.

# Preface to Lobachevsky's 1886 biography

Behind the Lobachevsky geometry, there is a man, called Nikolai Ivanovich Lobachevsky, whom we remember with love and profound gratitude.

I first planned to write a short text on his life to accompany my translation of the *Pangeometry*, but then I decided to translate the (anonymous) biography, written in French, that is included in the second volume of the 1883–1886 edition of his *Collected geometric works* [103]. I like this biography. It is written in a simple style, and I think it gives a fairly good idea of the atmosphere in which Lobachevsky lived and worked. I imagine that it was written by a person (or by a group of persons) who had known him personally, and who loved him.[57] Another reason I included this biography was that it has some historical importance, and I think it deserves to be more widely known. On the other hand, being one of the earliest biographies of Lobachevsky, this text is not completely faithful to the reality, and it can usefully be updated. Indeed, there are several facts on Lobachevsky's life that remained unknown for a long period of time and on which we have more information today. For that reason, I have included several footnotes, accompanying the biography translation, to discuss or correct some of the information given. I have also collected in the present preface some remarks that need more room than is usually provided in a footnote.

At least four other biographies of Lobachevsky appeared in print in the last third of the nineteenth century. The first one was written in 1868 by E. P. Yanichevsky, a mathematics professor at Kazan University, who was also a friend of Lev Tolstoy [156]. Yanichevsky presented that biography at the 1868 opening ceremony of the University of Kazan (5 November 1868). The other three biographies were written by the first Western translators of Lobachevsky, namely, Hoüel [76] in 1870 (*Bulletin des Sciences Mathématiques*), Halsted [62] in 1895 (*American Mathematical Monthly*) and Engel [45] in 1898 as a supplement to his German translation of Lobachevsky's *Principles of geometry* and *New elements of geometry, with a complete theory of parallels*.[58] New information on Lobachevsky's life was made available in the twentieth century, due to extensive work done by several Russian scholars. One must certainly mention in this respect the names of A. V. Vasiliev (1853–1929), V. F. Kagan (1869–1953), A. A. Andronov (1901–1952), D. A. Gudkov (1918–1992), B. L. Laptev (1905–1989) and B. V. Fedorenko (1913–2007). Vasiliev, Kagan, Gudkov and Laptev were mathematicians. Andronov was a physicist, but he is also well-known among mathematicians, in particular because of his work on stability in dynamical systems theory. Fedorenko was a philologist, and he was also the founder and the first director of the Dostoyevsky Museum in St. Petersburg.

[57] S. S. Demidov told me that he thinks that the author of this biography is probably A. V. Vasiliev, who prepared the 1883–1886 edition of *Collected geometric works*. Vasiliev continued doing an extensive research work on Lobachevsky's life after he published this edition.

[58] It seems that A. V. Vasiliev helped Engel in writing Lobachevsky's biography and commenting on his works, cf. [155], p. 162.

Despite the remarkable work done by these scholars and by others, there are still several basic elements in Lobachevsky's biography that remain unknown, e.g. the number of his children, his mother's origin, and her maiden name. As a general rule, we have little information on Lobachevsky's family life. We also have a very limited knowledge on the content of his teaching on geometry.

This lack of information about Lobachevsky's life is certainly due in part to the fact that the importance of his work was realised only several years after his death. Thus, by the time scientists and historians became interested in precise information about Lobachevsky's life, several details were missing, and they are probably lost forever. Some authors also note that a big fire that took place in Kazan in 1842 (that is, 14 years before Lobachevsky's death), destroyed several documents that might have been valuable for Lobachevsky's biography (see [120]).

There is still another explanation for this lack of information.

It is believed that Lobachevsky was an illegitimate child, that he knew this fact, and that, together with his mother, he hid it all his life. One of the reasons for that is that in nineteenth-century Russia, there were severe restrictions on the education and on the career possibilities of illegitimate children. It seems that this information was first made public by I. I. Vishnevsky, based on new documents about N. I. Lobachevsky that he found in 1929, cf. [120]. This claim was taken up again by D. A. Gudkov, who taught mathematics at the University of Nizhny Novgorod (the city where Lobachevsky was born) and who conducted meticulous historical research work on Lobachevsky's life. Gudkov published his results in his book *N. I. Lobachevskii. Mysteries of the biography* [59]. This book contains several archival documents supporting the claim that Nikolai Lobachevsky, together with his two brothers, Alexander and Alexei, were the children of Sergei S. Shebarshin, a land surveyor from Nizhny Novgorod, who died early and did not marry their mother, whose first name and patronymic are Praskovya Alexandrovna and whose maiden name is unknown. Praskovya Alexandrovna was separated but not divorced from her husband Ivan Maksim, with whom she had no children. In addition to these documents, Gudkov's book contains some 80 pages of memoirs of one of N. I. Lobachevsky's sons (also called Nikolai), written in the years 1898–1899. This son of Lobachevsky wrote that his father and his grandmother never gave him any information on his grandfather. A report on Gudkov's book is contained in Polotovsky's paper [119].

Let me now discuss certain points about Lobachevsky that somehow complement the biography that is translated in this volume.

We start with Lobachevsky's birthdate.

This date has long been uncertain. There has been even a polemic over this point, cf. the beginning of G. B. Halsted's biographical article *Lobachevsky* [62]. The 1886 biography that I translated says 22 October (Old Style) 1793, which is also the date given by Engel in his biography [45]. Vucinich, in his book *Science and culture in Russia* (1963) [152] wrote: "It has only recently been established that Lobachevsky was born in 1792 rather than 1793, as thought by earlier writers". Kagan, in his book *N. Lobachevsky and his contribution to science* (1957) [79], wrote that Lobachevsky was born on the 1st of December, 1792. Rosenfeld, in his biographical article in the *Dictionary of Scientific Biography* (1970) [130], gave the date 2 December 1792. The date that is generally accepted today is the one that has been established by A. A. Andronov and his team (see Andronov's article *Where and when was Lobachevsky born?* (1956) [7]); it is 20 November (Old Style) 1792, and it coincides with the date given by Rosenfeld, 2 December (New Style) 1792.

Besides the question of the birthdate of Lobachevsky, we have to discuss another important date, namely, the date at which Lobachevsky started developing hyperbolic geometry.

A set of class-notes taken at geometry lessons delivered by Lobachevsky in the academic years 1815–1816 and 1816–1817 were published in 1909 by A. V. Vasiliev. The notes show that Lobachevsky was working at that time on a "proof" of the parallel postulate. These notes are also reported on by B. L. Laptev in his paper *Theory of parallel lines in the early works of Lobachevsky* [88]; see also Laptev's paper *Bartels and the formation of the geometric ideas of Lobachevsky* [93].

We can assume that at some point, Lobachevsky shifted from attempts for proving Euclid's parallel postulate to efforts for proving that there exists a geometry in which this postulate is not satisfied.

The date of this shift is unknown. Laptev and others (see [93]) consider that there are signs indicating that during the year 1824–1825, Lobachevsky was already seriously working on non-Euclidean geometry.

According to Laptev [93], in the year 1825, Lobachevsky went through a nervous breakdown, one of whose manifestations was a severe hostility towards several people around him. Laptev reported that since 1822, Lobachevsky had been under pressure by M. L. Magnitsky, the (very conservative) academic inspector of the Kazan district, to produce an original mathematical paper proving the seriousness of his work. This pressure was probably a factor in Lobachevsky's depression, but at the same time, it seems that it pushed him to write up new ideas on the parallel postulate question, see [36].

A precise date that has been suggested as the birthdate of non-Euclidean geometry is the 12th (Old Style; 23th New Style) of February 1826, the date at which Lobachevsky read to the Physical and Mathematical Section of Kazan University a paper, in French, entitled *Exposition succinte des principes de la géométrie avec une démonstration rigoureuse du théorème des parallèles* (A brief exposition of the principles of geometry with a rigorous proof of the theorem on parallels). The manuscript of the paper does not survive. Although the title strongly suggests that Lobachevsky was offering a (fallacious) proof of the parallel postulate, several authors have argued that the paper may have contained the first public account of non-Euclidean geometry, considering that such an interpretation is more consistent with other investigations into the foundations of geometry that Lobachevsky is known to have been conducting at the time. According to Kotelnikov, whose opinion was followed by Kagan, Laptev and Rosenfeld (see [79], [93] and [131]), this 1826 paper constitutes the first public lecture given on non-Euclidean geometry. Rosenfeld interprets the expression "a rigorous proof of the theorem on parallels" that is in the title of Lobachevsky's work as the "impossibility of experimental determination of which – the imaginary or the one in common use – describes the real world". This is consistent with the fact that the lecture was delivered after Lobachevsky carried out several computations using astronomical data, see [131], p. 208. The paper was reviewed and refused for publication, and Lobachevsky published an excerpt from it three years later in the *Kazan Messenger*, under the title *On the elements of geometry*. The foreword to the English edition of Rosenfeld's *History of non-Euclidean geometry* says: "The Russian edition of this book appeared in 1976 on the hundred-and-fiftieth anniversary of the historic day of February 23, 1826, when Lobačevskii delivered his famous lecture on the discovery of non-Euclidean geometry".

Thus, it appears that it is some time during the chaotic period 1822–1826 that Lobachevsky's ideas on the new geometry grew up in his mind.

In any case, whether the date 12 (Old Style) February 1826 is or is not the date that has to be considered as the birthdate of hyperbolic geometry, it can safely be asserted that by the year 1826 Lobachevsky knew the existence of a geometry that is different from the Euclidean one, and that the era of the absolute reign of Euclidean geometry was already over.

Another sensitive question which is touched upon in the biography that I translated concerns the relation between Lobachevsky and Gauss, or, rather, their "supposed" relation, through Bartels. The text of the biography says: "One recognises in Lobachevsky's works the influence of Bartels, who was a friend of Gauss [...] we can assume that Bartels, in communicating to his young student the ideas of his illustrious master, gave to Lobachevsky's ideas their first impetus".

Johann Martin Christian Bartels, who was Lobachevsky's teacher at Kazan University, had been Gauss's teacher at high school. He recognised Gauss's talent, and he recommended him, to the Duke of Brunswick, for a scholarship to study at the *Collegium Carolinum*. Later on, the teacher-pupil relation between Bartels and Gauss transformed into a friendship, and they corresponded for several years.

Now we come to the relation between Bartels and Lobachevsky.

It is known that in 1810, Bartels gave a course on history of mathematics at Kazan University. It is natural to assume that Lobachevsky, who was Bartels' student, followed that course. It is also known that Bartels made use in his teaching of a book written by the French historian of mathematics Jean-Etienne Montucla, which contains a detailed review of Euclid's *Elements*, as well as descriptions of several attempts by various mathematicians to prove Euclid's parallel postulate (cf. [93]).

From this, is seems natural to assume that Bartels had an influence on Lobachevsky's interest in the parallel question.

But the question of whether Lobachevsky learned anything from Bartels about Gauss's ideas on non-Euclidean geometry, and of what exactly he learned, is another question, and it has been discussed by several authors. G. B. Halsted, the celebrated translator of works by Saccheri, Lobachevsky and Bolyai into English, wrote, in several places, that this issue is completely settled, and that Gauss never transmitted to Lobachevsky, through Bartels or without the aid of Bartels, any idea or any indication on hyperbolic geometry. For instance, in his article *Non-Euclidean geometry in the Encyclopaedia Britannica* [70], Halsted considers that the two famous authors of the *Encyclopaedia* article, namely, A. N. Whitehead and B. Russell, gave an unwarranted prominence to Gauss, and he finds offensive the following statement they made: "It is not known with certainty whether Gauss influenced Lobachevski and Bolyai, but the evidence we possess is against such a view." Halsted responded that "it *is* known that he did *not*, and the evidence that we possess against such influencing is absolute and final". In the article *Supplementary report on non-Euclidean geometry* [67], Halsted, reporting on the editorial work done by P. Stäckel, wrote that the new findings "only strengthen the already existing demonstration that neither of the creators of the non-Euclidean geometry [Lobachevsky and Bolyai] owed even the minutest fraction of an idea or suggestion to Gauss". Let us also quote the following, from Halsted's report on the Engel and Stäckel edition of Lobachevsky's *Elements of geometry* and of his *New elements of geometry* [66]: "Bartels never saw Gauss after 1807, received at Kazan one letter from him in 1808, probably a mere friendly epistle containing nothing mathematical, and not another word during his entire stay there." Halsted also reported that the astronomer Otto Struve, who attended in 1835 and 1836 lectures given by Bartels at Dorpat (a city which was, at that time, part of Russia, and which is today the city of Tartu in Estonia), recalled on that occasion that he repeatedly spoke of Lobachevsky as one of the most gifted scholars in Kazan, that Bartels had already received copies of Lobachevsky's first writings on non-Euclidean geometry, but that he looked upon them "more as interesting, ingenious speculations than as a work advancing science". Vucinich, in his book [152], p. 415, quoted a biographical article on Bartels written by Ia. Depman, relating a similar story. Vucinich wrote the following: "Depman cites a letter from the astronomer O. Struve to show that when Lobachevsky's work on non-Euclidean geometry reached Bartels, the latter rejected it as unscientific. Depman considers that if Bartels had been familiar with Gauss's efforts to develop a geometry in which Euclid's parallel postulate is not satisfied, he would not have been so quick to oppose Lobachevsky's ideas". We also quote Duffy, from his paper *Nicholas Ivanovich Lobachevsky* [41]: "Bartels almost certainly did not encourage in any special way Nicholas Ivanovich to dedicate his energies to solving the problem of parallel lines. Bartels' later correspondence from Dorpat makes it clear that he does not consider the new geometry of Nicholas Ivanovich to be of much value and makes no claim to have any part in its formation". The question of what Lobachevsky learned from Gauss through Bartels is also discussed in detail by Laptev in his biographical survey on Bartels [93]. Yaglom, in his book on Klein and Lie [155], p. 51, wrote that Klein, in the first edition of his book on non-Euclidean geometry, gave undue pre-eminence to Gauss for the discovery of hyperbolic geometry, but that he somehow corrected (though not completely) his view in later editions and in his other books.[59]

[59] In his *Elementary mathematics from an advanced viewpoint* (1908) ([85], p. 176 of the English translation of Part II (Geometry)), Klein wrote: "It was Gauss who first discovered the existence of a *"non-Euclidean" geometry*, which is the name that he gave to such a geometry". [Es war Gauß, der als erster die Existenz einer "nichteuklideschen Geometrie" – so nennen wir mit ihm ein geometrisches System jener Art – entdeckte.]

Finally, we recall that Gauss discussed in his correspondence his ideas on hyperbolic geometry with a very limited number of close friends, that he asked them to remain silent about his work on the subject, and that, as far as we know, his letters to Bartels do not contain any reference to the problem of parallel lines.

Let me now make a few remarks on Lobachevsky's other achievements in science.

Talking about Lobachevsky's scientific achievements, Halsted wrote in his biography [62] that "Lobachevsky was a modern scientist of the very soundest sort, whose only misfortune was to be half-a-century ahead of the world".

It is interesting to note the following sentence in the 1886 biography: "Besides his critique of the principles of geometry, Lobachevsky occupied himself with other questions that are no less important". One may see in this sentence an indication of the fact that at the time where the biography was written, the importance of Lobachevsky's work on hyperbolic geometry was not yet fully realised. Lobachevsky was a scientist in the best sense of the word, and he made discoveries in several domains. But the importance of these discoveries is not comparable to the importance of those he made in geometry.

The 1886 biography ends with a list of eleven works Lobachevsky published on various subjects other than geometry. In this list, there is a paper on education, entitled *Discourses on the most important subjects of education* (1828), and I would like to make a few comments on that, because Lobachevsky's interest in education tells us something about his personality.

Lobachevsky was most concerned about the role of education in society. He was an advocate of the view that it should be possible to acquire a social position by means of an adequate education. This view was in contrast with the conservative approach that was dominant in nineteenth-century Russia, which gave pre-eminence to blood nobility for what concerns social position.[60] Kagan wrote that Lobachevsky was an advocate of bringing education to the mass of Russian people. To show how urgent this was, Kagan reported that at that time, the Kazan gymnasium was the only such institution for the huge population of Russia who lived between Moscow and the Pacific. Some of Lobachevsky's ideas on education must have been considered as subversive by the conservative representatives of the local government, and they may have been a factor in Lobachevsky's conflict with the authorities that eventually led to his dismissal as a rector and as a professor. In any case, during his mandate as a rector, Lobachevsky invested much of his effort and time in trying to raise the standards of education, not only at Kazan University but also in the local school. An interesting report on Lobachevsky's views on education is contained in Vucinich's book *Science in Russian culture* [152].[61]

[60] One must acknowledge at the same time that the Russian enlightenment ideas were essentially shared by the nobility intelligentsia.

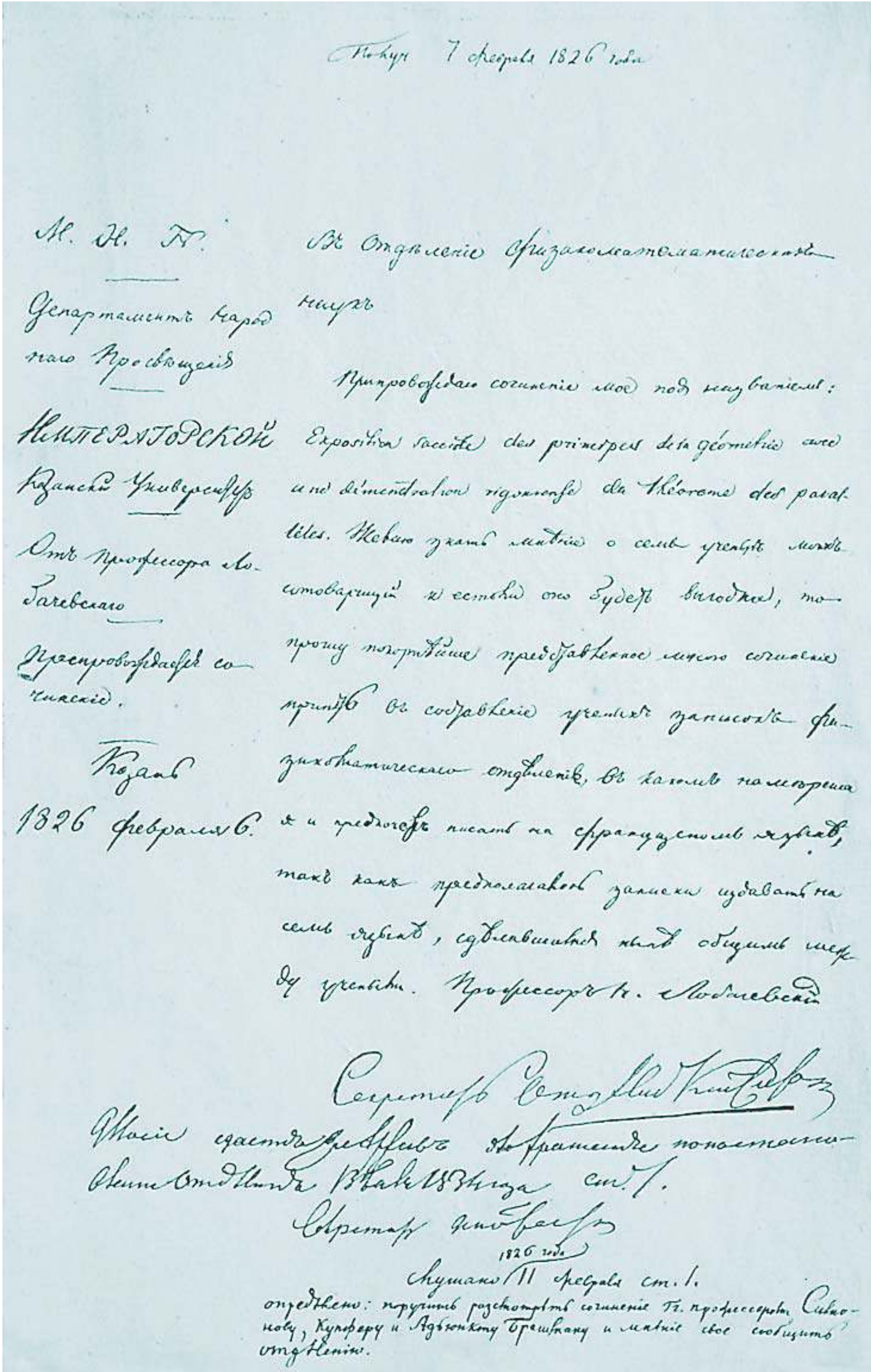
[61] It may be worth noting that E. Beltrami, who was another major figure of non-Euclidean geometry, was also profoundly involved in educational questions. Beltrami was several times a member of high-school program commissions, and he was most concerned by the quality of teaching there. We also mention that Beltrami had a political career (he was a member of the Italian Royal Senate). Another major figure of non-Euclidean geometry, F. Klein, was deeply involved in a project of transforming mathematics education in high-schools.

Talking about education, it is also worth mentioning that Lobachevsky usually wrote his papers in Russian, and that he was an advocate of the use of the Russian language in scientific writing. This was rather uncommon in nineteenth-century Russia, where most of the St. Petersburg Academicians used French for mathematical writing, considering the Russian language as inadequate in that domain. Vucinich, in [152], noted that the academician M. V. Ostrogradsky, who was a fierce opponent of Lobachevsky, wrote all his original papers in French. Vucinich quoted the following sentence by a mathematics teacher, N. D. Brashman, who was Ostrogradsky's contemporary, "If Ostrogradsky had written in the Russian language, our mathematical literature would have occupied an honoured place in Europe; but all his works addressed to the world of scholarship are written in French". Vucinich added: "In the same article, Brashman prophesied that in the future, people abroad will not read only Russian poets but also Russian geometers". See also Vucinich's article *Nikolai Ivanovich Lobachevskii: The man behind the first non-Euclidean geometry* [153] for related information.

Finally, let me mention that the 1886 biography does not give a fair account of the last years of Lobachevsky's life. Yaglom, in his book *Felix Klein and Sophus Lie* ([155], p. 57), reported that Lobachevsky, "having been for many years the rector of one of Russia's six universities, received many awards, including the highest ones. Nevertheless, he did not feel happy; he was respected only as an administrator, although he regarded himself as a scientist". Lobachevsky's son, in his memoirs published in Gudkov's book [59], painted his father as a "sullen misanthrope, unhappy in family life and almost without friends".

The information in the footnotes that are included in my translation of the biography that follows is taken from the papers and books quoted above, and from others, including the following: F. Engel's biography of Lobachevsky [45], J. Hoüel's *Notice sur la vie et les travaux de N.-I. Lobatchefsky* [76], R. Bonola's *Non-Euclidean geometry* [26], C. F. Gauss's correspondence [52], the correspondance between Beltrami and Hoüel [23], E. B. Vinberg's note on Lobachevsky on the occasion of the 200th anniversary of his birth [150], C. Houzel's article *The birth of non-Euclidean geometry* [77] and B. V. Fedorenko's *New material for the biography of N. I. Lobachevsky* [49]. Correspondance with various people mentioned in the foreword was also very useful.

In the footnotes to the biography, and also in the footnotes to the commentary that follows it, I have included short biographical notes on mathematicians which are not widely known, such as Schweikart, Taurinus, Hoüel and Barbarin. I did not include any biography on Gauss, Beltrami, Klein, Hilbert and other mathematicians of the same stature.



Получ. 7 февраля 1826 года

М. Н. П.

Департамент Народного Просвещения

ИМПЕРАТОРСКОЙ Казанский Университет

От профессора Лобачевского

Препровождает сочинение.

Казань 1826 февраля 6.

В Отделение физико-математических наук

Препровождаю сочинение мое под названием: Exposition succincte des principes de la géométrie avec une démonstration rigoureuse du théorème des parallèles. Желаю знать мнение о сем ученых моих сотоварищей и если оно будет выгодно, то прошу покорнейше представленное мною сочинение принять в составление ученых записок физико-математического отделения, в каковом намерении я и предпочел писать на французском языке, так как предполагаемое записки издавать на сем языке, сделавшемся ныне общим между учеными. Профессор Н. Лобачевский

Секретарь [illegible]

[illegible] ст. 1.

Секретарь [illegible]

1826 года

Читано 11 февраля ст. 1.

Определено: поручить рассмотреть сочинение Гг. профессорам Симонову, Купферу и Адъюнкту Брашману и мнение свое сообщить Отделению.

The original of the document on the left is in the National Archive of the Republic of Tatasrtan. A copy is conserved in the Museum of Kazan University and a photo was published in the book *N. Lobachevsky*, in the series "Outstanding scientists and graduates of Kazan State University".

M. N. P. (Ministry of People Education)

Department of People Education

Imperial University of Kazan

from professor Lobachevsky

Article Application

Kazan, 6 February 1826

7 February, 1826.

To the Section of Physical and Mathematical Sciences

I am sending you my work entitled: Exposition succinte des principes de la géométrie avec une démonstration rigoureuse du théorème des parallèles. I would like to know the opinion of my colleagues about this, and in case this opinion will be positive, I kindly ask you to accept this article to be included for publication in the Uchenye Zapiski (Scientific Memoirs). I propose to publish it in French because this language is now common for scientists.

Professor Lobachevsky.

Signature of the Secretary.

Heard on 11 February 1826. Decision: ask Professors Simonov, Kupffer, and adjunct Brashman to examine the article by M. Professor Lobachevsky and to inform the department about their opinion.

Ivan Mikhailovich Simonov was an astronomer and a geodesist. He became professor of Physics at Kazan University after he studied there. Simonov was Lobachevsky's close friend. He participated, as a leading scientist, in the Russian expedition that circumnavigated Antarctica (1819–1821). The expedition's objective, besides being geographical, was also to collect ethnographic information about people living in the Pacific Ocean.

Nikolay Brashman was a well-known geometer of Czech origin, who was newly appointed at Kazan University. He later on became professor at the University of Moscow and corresponding member of the Russian Academy of Science. He was a founding member of the Moscow Mathematical Society.

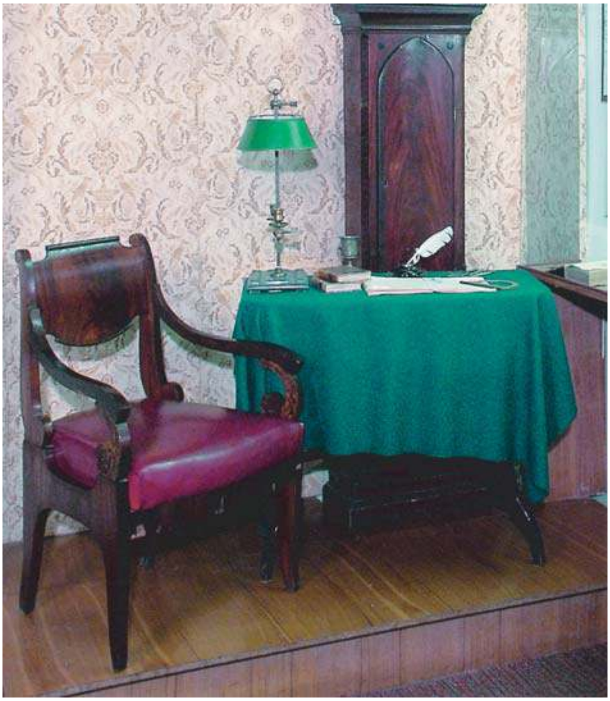
Adolph-Theodor Kupffer was a famous mineralogist. He was appointed professor at Kazan University in 1824. In 1828, he was appointed full member of the St. Petersburg Academy of Sciences.

The Lobachevsky monument in front of Kazan University:
Opening ceremony in 1896 and today. Work by Maria Dillon (1858–1932), a famous Russian sculptor who was the first female student in St. Petersburg's Academiy of Fine Arts.

The building of the old library of Kazan University.


Plaque at the residential house of Lobachevsky during the years 1827–1846.


Lobachevsky medal. Museum of History of Kazan University.



Lobachevsky's office furniture, including the table he used to work on at the University Observatory and the armchair of his Geometry office. Museum of History of Kazan University.

# Lobachevsky's biography (1886)

(Extracted from the 1883–1886 edition of the *Collected geometric works*[62])

Nikolai Ivanovich Lobachevsky was born on the 22nd of October 1793,[63] in the government of Nizhny Novgorod[64]. His father, an architect,[65] died in 1797,[66] leaving a wife and two young children,[67] with very modest income. Lobachevsky's mother, soon after the death of his father, settled in Kazan,[68] where she managed to place her sons at the city gymnasium, with a state scholarship.[69] N. I. Lobachevsky was admitted to the gymnasium in November 1802, with almost no instruction received beforehand at home. Since the period of study at the gymnasium lasted only four years, one can assume that the amount of knowledge that Lobachevsky possessed at his entrance to the university was very limited.[70] Lobachevsky passed the entrance examination, and he was admitted to the University of Kazan, again with a state scholarship, on the 14th of February 1807.[71]

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[62] *Collection complète des œuvres de N. I. Lobatcheffsky*, Tome second, 1886, Edition de l'Université Impériale de Kazan [103], pp. 3–8.

[63] It is generally accepted now that Lobachevsky's birthdate date is 20 November (Old Style) 1792, that is, 2 December (New Style) 1792, cf. Andronov's article *Where and when was Lobachevsky born*? [7], and the discussion in Polotovsky's recent paper [120].

[64] It is known, again by work of Andronov [7], that the place of birth is the *city* of Nizhny Novgorod, see [120].

[65] According to Kagan and some other authors, Lobachevsky's father was a land-surveyor. Lobachevsky himself wrote at several occasions that his father was a chief-officer, and at several others that his father was a land-surveyor, despite the fact that his legal father, Ivan Lobachevsky, was neither a chief-officer nor a land-surveyor [59]. Andronov [7] and Gudkov [59] supported the hypothesis that Lobachevsky was the illegitimate son of a land surveyor, S. S. Shebarshin, see also [120]. The paper [41] by Duffy contains sparse information on Lobachevsky's parents.

[66] Hoüel [76] wrote: around the year 1800. Duffy [41] considers that the date is unknown.

[67] According to Kagan, Rosenfeld, Gudkov, Duffy and others, there were three children: Alexander, Nikolai and Alexei. Duffy [41] reported that the oldest son, Alexander, tragically died in 1807 by drowning in the Kazanka river.

[68] Kazan is the capital of the Republic of Tatarstan, situated on the Volga river, at 720 km East of Moscow.

[69] This was not uncommon. At that time, education in Russia was usually provided at the expense of the state. Note however that Fedorenko reported that Lobachevsky's mother did not obtain public support for her children at their entrance into school, but only some time later [49].

[70] Duffy reported that Lobachevsky was frequently sick and that he often missed school, each time for several months [41].

[71] One should not fall into the mistake of thinking that the University of Kazan was just a minor university in Russia. It is true that it was a new university, since it was founded, by the emperor Alexander I, in 1805, that is, two years before Lobachevsky's admission to it. But it is the second oldest Russian university, after the University of Moscow, that was founded in 1755. The Universities of Kharkov, St. Petersburg and Kiev were founded respectively in 1808, 1819 and 1834. The tsarist intention to found a university in Kazan, that is in the Eastern part of the empire, was motivated by the desire to reinforce the relations between the Russian and the Asian cultures. Indeed, Kazan University soon became one of the most important universities in Europe for Oriental Studies. One can mention that Lev Tolstoy entered Kazan University in 1844, as a student in Turco–Arabic literature (but he left it three years later without getting any degree). For this second university of the empire, the Russian emperor aimed to ensure a good quality of teaching. For that reason, some of the

The attachment of the young student to mathematics was not immediately apparent; at least we do not see the name of N. I. Lobachevsky among the names of his fellow students that expressed, in 1808, the desire to specialise in pure mathematics.[72] It is probable that Lobachevsky was aware of the flaws in the instruction he got at the gymnasium, and for that reason he had an equal enthusiasm for the various sciences that were taught at the university. He made excellent progress in science, in comparison with all his fellow students, as can be seen from the certificate that he was awarded by the students' inspector. Disobedience and contempt of authority often brought upon Lobachevsky the dissatisfaction of the university Heads.[73] At some point, Lobachevsky was even threatened with expulsion from the university, and it was only because of Bartels' protection that he was allowed to finish his studies.[74] Bartels was the first mathematics professor at the University of Kazan, and he had a profound influence on the development of Lobachevsky's talents in mathematics.[75] The least that we can say is that Lobachevsky retained towards Bartels, until the end of his life, the feelings of the highest esteem and of the most profound gratitude.

One recognises in Lobachevsky's works the influence of Bartels, who was a friend of Gauss. We know that Gauss, in his letter of the 26th of November 1846 to the astronomer H. C. Schumacher, which concerned Lobachevsky's article *Geometrische Untersuchungen zur Theorie der Parallellinien*, said that 54 years before, he shared the same convictions concerning the possibility of a non-Euclidean geometry.[76] Thus, we

professors that were nominated there were hired from Germany. F. Engel wrote that already in the 1810s, the number of good scholars working at the Physical and Mathematical Faculty of Kazan University was higher than that in most German universities [Jedenfalls war vom Jahre 1810 ab die physico-mathematische Fakultät der Universität Kasan mit Lehrkräften so gut versorgt, wie damals kaum eine deutsche Universität], [45], p. 355.

[72]According to Kagan [79], Lobachevsky, yielding to his mother's wishes, started university by studying medicine.

[73]Kagan [79] wrote, quoting the inspector's reports in the university registers, that Lobachevsky was described as a "stubborn, relentless young man, very ambitious and manifesting signs of atheism (which was a very serious charge)."

[74]It is interesting to note that Beltrami, another major figure of hyperbolic geometry, was expelled from the University of Pavia where he was a student, for having provoked a disorder against the rector of that university, cf. [23], p. 1.

[75]Johann Christian Martin Bartels (1769–1836) was the first mathematics professor at Kazan university. He was appointed there together with five other distinguished German professors. Bartels was a friend of Gauss, of whom he had been a school-teacher. He had studied under Pfaff and Kästner, and as a geometer, he was aware of the problems raised by Euclid's parallel postulate. It is assumed that Bartels' lessons on the subject generated Lobachevsky's interest in these problems.

[76]In fact, the letter is dated 28 November 1846. Gauss's correspondence on geometry is published in Gauss's *Collected works*. Talking about Lobachevsky's *Geometrische Untersuchungen zur Theorie der Parallellinien*, Gauss wrote in that letter: "You know that for 54 years (since 1792) I had shared the same views with some additional developments of them that I do not wish to go into here; thus I have found nothing new for myself in Lobachevsky's work. But in developing the subject the author follows a road different from the one I took; Lobachevsky carried out the task in a masterly fashion and in a truly geometric spirit. I see it as a duty to call your attention to this work that is bound to give you truly exceptional pleasure". (The translation is borrowed from Rosenfeld [131], p. 220) [Sie wissen, dass ich schon seit 54 Jahren (seit 1792) dieselbe Überzeugung habe (mit einer gewissen späteren Erweiterung, deren ich hier nicht erwähnen will); materiell für mich Neues habe ich also im Lobatschewskyschen Werke nicht gefunden, aber die Entwickelung ist auf anderm Wege gemacht, als ich selbst eingeschlagen habe, und zwar von Lobatschewsky auf eine meisterhafte

can assume that Bartels, in communicating to his young student the ideas of his illustrious master, gave to Lobachevsky's ideas their first impetus.[77] In 1810, Lobachevsky received the title of bachelor, and soon after that the rank of aspirant.[78] The aspirants at that time were assistants to the professors; they read the lectures for students during illness or other absences of the professors, and they helped the professors by working with the students on practical exercises and by explaining the difficult parts of the professors' lessons.[79] But the main duty of the aspirants was to attain perfection in the sciences that they had chosen, and for that reason they were constantly in communication with their professors.

For instance, Lobachevsky used to work in Bartels' house four times a week,[80] and the first works that were studied by Lobachevsky under the guidance of Bartels were Gauss's *Disquisitiones Arithmeticae*, and the first volume of Laplace's *Celestial Mechanics*. Devoting himself with love to the development of the talents that he noticed in his young assistant, Bartels was not mistaken in his choice, and his efforts were rewarded by a success that surpassed his hope. Lobachevsky immortalised his name in the sciences and he surpassed his master by his scientific merits. It is true that neither of them lived to enjoy the eventual acclamation of Lobachevsky's work. The value of the geometric theory that was founded by Lobachevsky was acknowledged only around the year 1870, after the correspondence between Gauss and Schumacher was published,[81] and after M. Hoüel[82] provided a French translation

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Art in ächt geometrischem Geiste. Ich glaube Sie auf das Buch aufmerksam machen zu müssen, welches Ihnen gewiss ganz exquisiten Genuss gewähren wird. .....] (Gauss's *Collected works* [52], Vol. VIII, p. 238–239). Gauss had already written, on the 4th of February 1844, to his friend and former student C. L. Gerling that he had read Lobachevsky's article (Gauss's *Collected works* [52], Vol. VIII, p. 235). It seems that Lobachevsky was never aware of Gauss's letters, which were published after Gauss's death in 1855.

[77]It is now generally admitted that Bartels did not transmit to Lobachevsky any of Gauss's ideas on hyperbolic geometry, because he simply was not aware of them.

[78]I have translated the French term "bachelier" by "bachelor", and the French term "licencié" by "aspirant". Aspirant is a word that still designates a laureate of a Russian university who, after obtaining his degree, continues for a doctorate. Houzel [77] also used the word "aspirant". Rosenfeld [130] wrote that in 1812, Lobachevsky received a master's degree in physics and mathematics, and that in 1814, he became an "adjunct" in physical and mathematical sciences. Hoüel wrote that in 1811, Lobachevsky received the degree of "candidate". Kagan's translator [79] wrote that in 1814, Lobachevsky obtained a position of "assistant professor". In any case, it seems that Lobachevsky started to teach in 1812, and that his teaching included arithmetic, geometry, hydrodynamics and astronomy.

[79]Duffy mentioned that Lobachevsky, being Bartels' assistant, translated for him, in classrooms, from German into Russian [41].

[80]According to Duffy, Lobachevsky visited Bartels' house on Saturday mornings to study special topics in mathematics [41].

[81]Gauss's correspondence with Schumacher was published in the few years following Gauss's death, namely, between 1860 and 1865.

[82]Guillaume Jules Hoüel (1823–1886) is a major figure in the history of non-Euclidean geometry. G. B. Halsted wrote the following in a review of Bonola's *Non-Euclidean geometry* [69]. "The first man to so bring forth the non-Euclidean geometry that it was not stillborn, but lived and grew, was the Frenchman Hoüel, by his translations of Lobachevski in 1866 and Bolyai in 1867". Hoüel taught at the University of Bordeaux, and he was the author of geometric treatises giving a modern view on Euclid's *Elements*. It seems that Hoüel was himself working on the impossibility of proving the parallel postulate when, in 1866, he came across the writings of Lobachevsky, and became convinced of their correctness. In the same year, Hoüel

of Lobachevsky's *Beiträge zu der Theorie der Parallellinien*.

From Gauss's correspondence with Schumacher, one can see that while Lobachevsky's works were ignored by the scientific world and while their publication encountered the hostile reactions of authorities like Ostrogradsky,[83] Gauss was aware of the works of Lobachevsky and held them in proper esteem. Gauss's knowledge of the works of Lobachevsky is an indication of the relations that existed between them, either through the Royal Society of Göttingen, of which Lobachevsky was

translated from the German Lobachevsky's *Geometrische Untersuchungen zur Theorie der Parallellinien* and published it in the *Mémoires de la Société des Sciences physiques et naturelles de Bordeaux*, together with excerpts from the correspondence between Gauss and Schumacher on geometry. In 1867, Hoüel published a book in which he explained the main ideas of Lobachevsky's geometry, entitled *Essai critique sur les principes fondamentaux de la géométrie*. In 1870, he wrote a paper entitled *Note sur l'impossibilité de démontrer par une construction plane le principe de la théorie des parallèles dit Postulatum d'Euclide* [75]. Hoüel wrote that paper in response to a note by J. Carton that had been presented the year before to the French Academy of Sciences by the mathematician Joseph Bertrand, who was a member of that Academy, in which the author claimed to have proved the parallel postulate, see [21] and [3]. It is known that Hoüel learned Russian in order to translate into French some of Lobachevsky's works. Hoüel also translated into French and published in French and Italian journals works by several other non-Euclidean geometers, including Bolyai, Beltrami, Helmholtz, Riemann and Battaglini. He wrote in 1868 the first biography of Janós Bolyai. Barbarin, in his book *La géométrie non euclidienne* ([11], p. 12) reported that Hoüel, "who had an amazing working force, did not hesitate to learn all the European languages in order to make available to his contemporaries the most remarkable mathematical works." [J. Hoüel, dont la puissance de travail était prodigieuse, n'avait pas hésité à apprendre toutes les langues européennes dans le but de faire connaître à ses contemporains les œuvres mathématiques les plus remarquables.] Thus, by his works and by the translations he made, Hoüel's contribution to the diffusion of hyperbolic geometry was enormous. Besides that, being an editor of the *Mémoires de la Société des sciences physiques et naturelles de Bordeaux*, Hoüel solicited several papers on hyperbolic geometry for publication in that journal, especially after the French Academy of Sciences, in the 1870s, decided to refuse any paper on that subject (see in particular the article by Barbarin on the correspondence between Hoüel and de Tilly [13]). Beltrami had a lot of respect for Hoüel, and there was a very interesting correspondence between the two men. Sixty-five letters from Beltrami to Hoüel were published in 1998 [23]. Beltrami's famous paper *Saggio di interpretazione della geometria non-Euclidea* [16] arose from ideas that Beltrami got after reading Lobachevsky's paper *Geometrische Untersuchungen zur Theorie der Parallellinien* [100] in the French translation by Hoüel, see [23], p. 9. For a detailed survey on the influence of Hoüel's work see [27]. There is also an interesting short biographical note on Hoüel by Halsted [65].

[83] M. V. Ostrogradsky (1801–1862) rejected Lobachevsky's work, of which he was the referee, when this work was submitted to the St. Petersburg Academy of Sciences. Ostrogradsky was one of the leading mathematicians in St. Petersburg, and probably the most famous one at that time. He was a member of the St. Petersburg Academy of Sciences and of several other academies. He studied in Paris and he had relationships with some first-order mathematicians like Lagrange, Poisson and Cauchy. His fields of study included integral calculus, hydrodynamics, number theory, probability and mathematical physics. Ostrogradsky's career was very different from that of Lobachevsky. He was elected as a full member of the St. Petersburg Academy of Sciences in 1831, at the age of thirty. Before that, he had been made famous at the age of twenty-four, when Cauchy acknowledged his work in his *Mémoire sur les intégrales définies, prises entre des lignes imaginaires* (Paris, 1825). On page 2 of that memoir, Cauchy wrote: "A young Russian man, gifted with sagacity and accomplished in the art of infinitesimal calculus, Mr Ostrogradsky, who also resorted to the use of these integrals and to their transformation in ordinary integrals, gave new proofs of the formulae I recalled before, and generalised other formulae that I had presented in the nineteenth issue of the *Journal de l'École Royale Polytechnique*." [Un jeune russe, doué de beaucoup de sagacité, et très versé dans l'analyse infinitésimale, M. Ostrogradsky, ayant aussi recours à l'emploi de ces intégrales, et à leurs transformations en intégrales ordinaires, a donné une démonstration nouvelle des formules que j'ai précédemment rappelées, et généralisé d'autres formules que j'avais présentées dans le dix-neuvième cahier du *Journal de l'École Royale Polytechnique*.]

a member,[84] or also through Bartels.

This approval by the "princeps mathematicorum"[85] was the unique thing that sustained and stimulated Lobachevsky in the continuation of his works, despite the lack of concern that he encountered everywhere else. Lobachevsky tried hard to clear up his theories for the learned public, by writing several essays. After having given a brief exposition, read before the Faculty, *On the elements of geometry*,[86] he clarified his ideas in detail and in an intelligible way, addressing himself to the non-specialists, in his *New elements of geometry with a complete theory of parallel lines*[87], and he preceded them by an interesting introduction, containing a critique[88] of the proofs of Euclid's parallel postulate relative to the intersection of the perpendicular and the oblique.[89] After that, Lobachevsky proved trigonometric formulae, applying them to the resolution of triangles and to the discovery of some definite integrals, and he collected the results so obtained in his *Imaginary geometry and its applications*. Later on, he published in French his *Imaginary geometry* and in German his *New theory of parallels*. Eventually, he exposed in a logical system the results of all his geometrical works in his *Pangéométrie*, published in French and in Russian. However, all his efforts remained fruitless. Lobachevsky died without having obtained any satisfaction of seeing a fair assessment of his theory.[90]

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[84]In 1842, after Gauss's recommendation, Lobachevsky was elected foreign correspondent of the Royal Society of Göttingen, for his work on non-Euclidean geometry. According to Vinberg [150], this was the only acknowledgement of Lobachevsky's contribution to geometry in his lifetime.

[85]Soon after Gauss's death, in 1855, George V, the King of Hanover ordered a golden medal with the portrait of Gauss and with the inscription "Princeps Mathematicorum", the Prince of Mathematicians. The expression was already used to describe Gauss during his lifetime.

[86]This probably refers to Lobachevsky's 1829 paper *On the elements of geometry* [95].

[87]See [96].

[88]In that memoir, Lobachevsky started by making a critique of the "proofs" by A.-M. Legendre and by L. Bertrand of the parallel postulate, showing where their arguments fail. It was almost a necessity for Lobachevsky (who at that time was far from being considered a great mathematician), before presenting a theory that contradicted results published by first-order mathematicians, to start by pointing out the mistakes in his predecessor's arguments.

[89]The "intersection of the perpendicular and the oblique" property, stated precisely, is a form of the parallel postulate. It says that given two coplanar lines, if the first line is perpendicular to a third line and if the second line makes an oblique angle with the third one, then the first two lines necessarily intersect. This statement is close to the way the parallel postulate is formulated in Euclid's *Elements*: "That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles." (Heath's translation [72], Vol. I, p. 202). One difference between the character of this statement and the one of the more popular statement (usually called "Playfair's axiom"), that "through a given point only one parallel can be drawn to a given straight line" (cf. [72], Vol. I, p. 220) is that the former statement allows a construction, and for that reason it is more in Euclid's *Elements* spirit; see the discussion in Bonola [26], p. 19. We note for the interested reader that there is a huge literature on axioms, postulates, differences between them and so on, that goes back to Aristotle, see e.g. the *Posterior Analytics* [9], p. 73. Aristotle discussed extensively the notions that are at the basis of mathematical reasoning (axioms, definitions, hypotheses, etc.) in his *Posterior Analytics* and in his *Metaphysics*. See also Heath's *Mathematics in Aristotle* [73].

[90]We recall that János Bolyai, one of the two other founders of non-Euclidean geometry, and who died in 1860, like Lobachevsky did not witness the recognition of his work on the subject. Regarding hyperbolic geometry, the same can be said of Gauss, who died in 1855, but one must add that Gauss never sought for such credit (except in private correspondence).

On the other hand, we cannot completely blame Lobachevsky's contemporaries for their unfair evaluation of his works. In taking the path of the critique of the axioms of geometry[91] and in recognising the impossibility of proving by logical deduction Euclid's famous postulatum on the intersection of the perpendicular and the oblique, Lobachevsky took the path of an apagogical proof.[92] Assuming the possibility that the perpendicular and the oblique do not intersect for a certain value of the angle, he would have obtained a contradiction if the Euclidean postulatum was an immediate consequence of the geometrical definitions and axioms that were admitted before. But this did not happen. The results obtained by Lobachevsky seemed absurd to the common mind, but for a profound thinker they were logical and did not contradict the nature of things. It is now clear that Lobachevsky's geometric theory in two dimensions represents the geometry of a surface of constant negative curvature, having the form of a cup,[93] in the same way as spherical geometry represents the geometry of a sphere, a surface of constant positive curvature. Euclid's plane geometry represents a transition between Lobachevsky's geometry and spherical geometry. Regarding Lobachevsky's three-dimensional geometry, it has not yet received a real substratum, but it has nevertheless been useful for clearing up the foundation of the geometric method and for the extension of this method to the study of other 3-dimensional varieties that are distinct from 3-dimensional space.

For Lobachevsky's contemporaries, his theory was incomprehensible, and it seemed to contradict an axiom whose unavoidability was well-based on a prejudice that was established over thousands of years. The force of the conviction in the necessity of that axiom was so great that it was only in a private correspondence that Gauss himself expressed his assent to Lobachevsky's views.[94]

Nowadays, Lobachevsky is considered as the founder of a school of geometers who study non-Euclidean geometry, as we now call Lobachevsky's geometry. It was only a matter of at most ten years before the celebrities of the mathematical sciences took part in the development of non-Euclidean geometry and a wealth of literature began to appear.[95]

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[91]This is not correct. Lobachevsky did not criticise the axioms of geometry. On the contrary, he showed that Euclid, in the *Elements*, was right in considering the parallel postulate as an axiom rather than as a proposition. Rather, Lobachevsky criticised the mathematical establishment who thought that Euclid's parallel postulate is a consequence of the other axioms.

[92]An apagogical proof is a proof by indirect reasoning, like a proof by contradiction.

[93]The author of the biography is probably talking here about Beltrami's pseudo-sphere model. The French word "coupe" that is used here (which may designate a bowl, and which I translated by "cup") is rather vague. One can imagine a cup having the form of a funnel. The upper-half of a wine decanter may have such a form.

[94]cf. Gauss's letter to Schumacher of the 28 of November 1846, which was already mentioned in this biography.

[95]In the 1886 Russian edition of Lobachevsky's works, which contains the biography that is translated here, there is a bibliography on non-Euclidean geometry that includes 272 titles published between the years 1865 (that is, nine years after Lobachevsky's death) and 1885. This list is an updated version of the bibliography published by G. B. Halsted in his *Bibliography of hyper-space and non-Euclidean geometry* [60]. The bibliography has been superseded by D. M. Y. Sommerville's *Bibliography of non-Euclidean geometry* (1911) [138] containing a chronological catalogue of circa 4000 items, from the beginning of the work on the theory of parallels (IVth century B.C.) until the year 1911.

Besides his critique of the principles of geometry, Lobachevsky occupied himself with other questions that are no less important. His works on convergence of infinite series constitute an important contribution to the theory of series. The Algebra course, written by Lobachevsky, can be considered as a model for expository methods and for precision in definitions. In all these works, we can see his love for precise scientific definitions and for exactness in the distinction between axioms that ensue from the nature of the objects studied and, on the other hand, axioms that follow from the definitions. This distinction was soon after introduced in all the sciences by the immortal work of the French geometer and philosopher Auguste Comte.

Lobachevsky devoted most of his time to abstract sciences. But he also liked, at least equally, experimental sciences. As an example of his works in that domain, one must mention the description that he gave of his observations on the solar eclipse at Penza,[96] and the thermometers for observing the soil temperature, which were made according to Lobachevsky's ideas.

As a mathematics professor at Kazan University, Lobachevsky communicated to his auditors the results of his scientific works. The memory of his grateful students has kept a profound esteem for his lessons. Professor Popov, one of Lobachevsky's students, wrote in the biographical note dedicated to his famous master: "In the Auditorium, Professor Lobachevsky was able to be profound or enthusiastic, depending on the subject. He attached little attention to the computing mechanism, but a lot to the exactness of the notions. His lectures attracted a large attendance to the auditorium, and his lessons for a distinguished audience in which Lobachevsky developed new principles of geometry were at the highest level, in terms of profundity".

Lobachevsky's activity as a professor lasted thirty-two years, from 1814 until 1846, when Lobachevsky was appointed assistant to the trustee of Kazan's district.[97]

In holding the position of a professor of pure mathematics, Lobachevsky was led, at several times during his long career as a professor, to replace his colleagues and to teach in all branches of the mathematical sciences.

Lobachevsky was tied to Kazan University, but not only as a professor and as a scholar. For nineteen years, he was elected rector of the university, several times in succession,[98] and in the period of his activity as a rector, the university owes him the

[96]This eclipse, which took place on 8 July 1842, was observed by Lobachevsky in the Russian city of Penza. The eclipse permitted efficient observations of the sun's corona and its fiery nature, and it confirmed the idea that the sun was not habitable. Halsted, in his biographical note on Lobachevsky [62], noted that this corona was not noticed enough to receive a name until 1851, but that it had been carefully observed and minutely described by Lobachevsky.

[97]It is generally accepted that Lobachevsky was dismissed from his position as a rector. Kagan gave a different version in his book [79]. He wrote that according to the university regulations, a professor position could not last more than thirty years, that despite this fact, the University Council asked Lobachevsky to remain at that position, but that Lobachevsky declined the offer. In this respect, Kagan cites L. B. Modzalevsky, who wrote that "Lobachevsky thought it expedient that Professor A. F. Popov, his disciple, should take the University chair from him for the reason that the professorship should be represented by younger scientific personnel".

[98]Lobachevsky was the rector of Kazan University from 1827 to 1846, cf. [45], p. 382. Hoüel noted that before being a rector, Lobachevsky was, for three years, the dean of the Physico-Mathematical faculty [76]. Duffy reported that Lobachevsky was never paid for being a rector [41]. This is important information, since

construction of its finest buildings.[99] Thus, we are right in saying that Lobachevsky's life is inseparable from the history of Kazan's University.

At the end of his life, Lobachevsky lost his vision, and he continued his scientific works while he was blind. Lobachevsky's dictation of his last work, *Pangeometry*, was taken down by his students.

Lobachevsky died on the 12th of February 1856.

Besides the works that have been included in the *Collected geometric works* of Lobachevsky, he also wrote the following articles:

1. *On the resonances, or reciprocated vibrations of columns of air* – an excerpt from the paper by Wheatstone in the 1828 *Quarterly Journal* (*Kazan Messenger*, 1828, No. 11 and 12).[100]

2. *Discourses on the most important subjects of education*, read on June 5, 1828 (*Kazan Messenger*, 1828, No. 8).

3. *Algebra, or the calculus of finite quantities*, Kazan, 1833.[101]

4. *Lowering the degree in a binomial equation when the exponent minus a unit is a multiple of 8* (*Scientific Memoirs of Kazan University*, 1834).[102]

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it is known that Lobachevsky, by the end of his life, had serious financial problems.

[99] Hoüel and Kagan reported that Lobachevsky studied architecture, in order to supervise the construction of the new university buildings, cf. [76] and [79]. The paper *New documents for the biography of N. I. Lobachevsky* by Fedorenko [49] contains some of the yearly reports that Lobachevsky wrote during his mandate as a rector. We note that besides being the rector, Lobachevsky was Kazan University chief librarian, from 1825 to 1835, cf. [80] and [10]. It is reported that Lobachevsky used to go personally to St. Petersburg to buy books for the library, as well as instruments for the physics laboratory, cf. [36] and [80]. The last reference contains lists of books that Lobachevsky ordered for the library. From 25 July 1821 to 21 February 1822, Lobachevsky was on leave from Kazan University to St. Petersburg. In a report he made on the 4th of May 1822 to the University council on his stay in St. Petersburg, he made a list of books and scientific instruments he had arranged to be transferred to Kazan. The books he ordered were not only science books. We recall in this respect that Kazan University was an important centre for Oriental studies, and Lobachevsky managed to get for the library valuable ancient Arabic manuscripts of philosophers and scholars. Thanks to him, today the library possesses valuable original editions, such as the first Russian printed book of the *Apostles* (1564) and first editions of eighteen century writers like Pushkin, Griboyedov and Gogol, which Lobachevsky used to buy systematically. The book [87] contains information on Lobachevsky's positive influence on general scientific research in Kazan University. To complete the picture of Lobachevsky's untiring industry, we mention that he had between fifteen and eighteen children (according to Kagan, the exact number of his children is unknown), with the same wife. Lobachevsky got married at about the age of forty, while his wife was not yet twenty. Lobachevsky's wife came from a wealthy family. She was reputed to be careless about housekeeping (may be due to the number of children she gave birth to), and it was not unusual for Lobachevsky to do cooking and other housework (see Kagan [79]).

[100] This is a Russian translation of an excerpt from a paper by Charles Wheatstone on the theory of sound, carrying the same title, which had appeared in the *Quarterly Journal of Science, Literature and Arts*, New Series, I, pp. 175-183, London 1828.

[101] This is a 530 page book. It contains in particular several methods of algebraic equation solving. I is worth noting that one of these methods is called today the *Dandelin–Lobachevsky–Gräffe Method*. As a matter of fact, this book on algebra can be considered as the only book that Lobachevsky wrote and published, his other works being journal articles. One may also add, besides the *Algebra* book, the *Geometrische Untersuchungen zur Theorie der Parallellinien* [100], which was published as a booklet, except that this work was initially planned as a journal article, submitted to *Crelle's Journal*, but not published there, see Hoüel [76].

5. *On convergence of trigonometric series* (*Scientific Memoirs of Kazan University*, 1834).[103]

6. *Conditional equations for the motion and position of the principal axes of rotation in a solid system* (*Scientific Memoirs of Moscow University*, 1834).

7. *A method for detecting convergence of infinite series and for obtaining approximate values of functions of a large number of variables* (*Scientific Memoirs of Kazan University*, 1835, No. 2, 211–342).

8. *On the probability of some average results, obtained by repeated observations*, Crelle's Journal, Vol. 24.[104]

9. *On the convergence of infinite series* (*Scientific Memoirs of Kazan University*, 1841, No. 4).[105]

10. *The complete solar eclipse in Penza, on the 26th of June 1842* (*Journal of the Ministry of Public Education*, 1843).[106]

11. *On the value of some definite integrals* (*Kazan Memoirs*, 1852).[107]

All these pieces of works are published in Russian, except No. 8 and 9. The German translation of No. 11 has been published in Erman's Journal, *Archiv für die wissenschaftliche Kunde von Russland*.[108]

[102]This 32 page memoir is the first one published in the journal Ученые записки Казанского Императорского Университета (Uchenye zapiski Kazanskogo Imperatorskogo Universiteta, Scientific memoirs of Kazan Imperial University). This is the journal in which, twenty-one years later, Lobachevsky published his *Pangeometry*. Lobachevsky had already presented in 1813, to the Physico-Mathematical section of Kazan University, a work (which remained unpublished) on the algebraic solutions of binomial equations. This subject was also considered in his book *Algebra*.

[103]Ученые записки, 1834, No. 2. In this item and in Item 7, the French text of the biography uses the word "évanouissement" instead of "convergence", which is probably a mistake in translating from the Russian.

[104]The paper is in French: Sur la probabilité des résultats moyens, tirés des observations répétées, *Crelle's Journal*, Bd. 24, Heft 2, 164–170, Berlin 1842. The manuscript was sent to the journal in 1838. In this paper, using combinatorial analysis, Lobachevsky determined limits of probability values obtained from repeated observations.

[105]This paper is in German: Ueber die Convergenz der unendlichen Reihen, 48 pages. The memoir was submitted to *Crelle's Journal* in 1840 and was not published. Lobachevsky studied in this memoir the convergence of series of real or complex numbers, with applications to trigonometric series, and to the development of trigonometric functions in infinite products. The paper also contains the determination of certain definite integrals.

[106]This famous eclipse took place on 8 July 1842, Gregorian calendar, cf. Footnote 95.

[107]*Scientific memoirs of Kazan Imperial University*, 1852, No. 4, pp. 1–35.

[108]Georg Adolph Erman's Journal was published in Berlin. This translation appeared in 1855, in Vol. 14, p. 232–272, under the title *Ueber den Werth einiger bestimmten Integrale*.

# III. A commentary on Lobachevsky's Pangeometry

# Introduction

Let me start this commentary with a few words concerning the expressions "imaginary geometry", "pangeometry", "Lobachevsky geometry", "hyperbolic geometry", "astral geometry" and other related expressions.

Lobachevsky introduced the term "imaginary geometry" for a geometry that is more general than Euclidean geometry and which, in the modern terminology, is usually called "neutral geometry". This is a geometry that satisfies all of Euclid's axioms except the parallel postulate, and in which that postulate is neutralised; that is, it may or may not hold. This use of the term "imaginary geometry" is clearly indicated by Lobachevsky in the Introduction of his *New elements of geometry, with a complete theory of parallels* [96]. There are several possible explanations for the choice of the term "imaginary geometry". This choice may have been motivated by the use of complex numbers, which were also called imaginary numbers at that time. (Lobachevsky himself used the expression "imaginary numbers" in his book on Algebra.) The relation with complex numbers stems from the fact, remarked by Lobachevsky and others, that some hyperbolic trigonometry formulae can be obtained from those of spherical trigonometry by replacing certain quantities by the same quantities multiplied by the imaginary number $\sqrt{-1}$.[109] The reference to imaginary numbers may also stem from the fact that Lobachevsky's geometry generalises Euclidean geometry in a broad sense, in the same way complex numbers generalise real numbers. The term *imaginary* is also reminiscent of a remark made by J. H. Lambert (1728–1777) in his *Theorie der Parallellinien* [139], written in 1766. Lambert analysed quadrilaterals with three right angles[110] in neutral geometry, and he worked out the consequences of the hypotheses that the forth angle is right, obtuse or acute respectively. Lambert declared that the hypothesis of the existence of a quadrilateral with three right angles and one acute angle would hold on a "sphere of imaginary radius". Finally, in his *Elements of geometry*, Lobachevsky made the remark that even if the new geometry which he discovered does not exist in nature, it may nevertheless exist "in our imagination" ([104], Vol. I, p. 210).

[109] To say things in different words, the formulae of spherical trigonometry involve the trigonometric functions sin and cos, and the corresponding formulae of hyperbolic trigonometry are obtained from the formulae of spherical trigonometry by replacing the functions sin and cos by the hyperbolic functions sinh and cosh.

[110] These quadrilaterals are usually called *Lambert quadrilaterals*, because Lambert studied them in his *Theorie der Parallellinien*. They are sometimes also called *Ibn al-Haytham-Lambert quadrilaterals* (see [121]), since they were studied long before Lambert by al-Hassan Ibn al-Haytham (965–1039), an Arabic mathematician, who was born in Irak and who died in Egypt. Ibn al-Haytham was known in Europe by the name Alhazen. In his *Commentaries on the postulates in Euclid's Books of the Elements*, Ibn al-Haytham introduced the quadrilateral with three right angles in neutral geometry, and he analysed the three cases for the fourth angle, viz. right, acute, or obtuse. He refuted the second and third cases, by using a hypothesis he considered to be a theorem, namely, that the endpoints of a perpendicular moving on a straight line is a line (a hypothesis which we know is equivalent to the parallel postulate). Then, from the existence of quadrilaterals with four right angles, he gave a "proof" of the parallel postulate, see [132], p. 62, and the exposition in [131], p. 59. See [124] and [78] for the mathematical texts of Ibn al-Haytham. Ibn al-Haytham-Lambert quadrilaterals were studied by Lobachevsky, and they are used in the *Pangeometry*, see e.g. Equation (25), which gives a relation between the edge lengths and the angles of such a quadrilateral.

This could also be an explanation for the use of the adjective "imaginary".

Lobachevsky later introduced the term "pangeometry" for a geometry that satisfies Euclid's axioms with the exception of the parallel postulate, and in which this axiom is not satisfied. This is the geometry that is usually called today "hyperbolic geometry", that is (using more modern terminology), the geometry modelled on the complete simply connected space of constant curvature $-1$. At first sight, it is not completely clear for a reader of the *Pangeometry* whether Lobachevsky reserved the term "pangeometry" to neutral geometry (that is, to the geometry based on the Euclidean axioms with the parallel axiom neutralised), or whether this word, for him, meant exclusively hyperbolic geometry. It can be thought, on the basis of what Lobachevsky wrote when he introduced that term in the first pages of his *Pangeometry*, that the term means "neutral geometry". This appears in particular in his sentence: "This name designates a general geometric theory which includes ordinary geometry as a particular case". But later on in the text, one easily realises that the word "pangeometry" is used for results that only concern hyperbolic geometry, and not the Euclidean one. Thus, pangeometry is not a neutral geometry which is independent from the parallel postulate. Lobachevsky chose for this geometry the name "pangeometry", which designates a "general geometry",[111]

because he realised that this geometry includes the two other geometries (Euclidean and spherical), though not at the axiomatic level, but at a more geometric level. Euclidean geometry is rather considered by Lobachevsky as part of pangeometry insofar as it appears as a limiting case, at least in the following three senses:

1. Euclidean geometry is the limiting geometry of figures in hyperbolic space when these figures become infinitesimally small.
2. Euclidean geometry is the geometry of horospheres, which are limits of spheres whose radii increase towards infinity.
3. Under the axioms of hyperbolic geometry, there is an undetermined constant, which is a characteristic constant of hyperbolic space, which Lobachevsky, as well as Gauss and Bolyai, had to deal with, in particular when they worked out their trigonometric formulae. (This constant is seen *a posteriori* to be the inverse of the square of the curvature of the space considered.) Lobachevsky noticed that when the value infinity is attributed to that constant, the formulae of hyperbolic geometry become those of Euclidean geometry.

Spherical geometry is also part of pangeometry because it is the geometry of spheres in Euclidean 3-space (in the same way as it is the geometry of spheres in Euclidean 3-space).

These are the reasons why Lobachevsky considered that pangeometry includes spherical and Euclidean geometry.

One may note in this respect that Euclidean geometry is not a "pangeometry" in this sense, since (by a result obtained by David Hilbert in 1901), there is no surface

[111]The prefix "pan" became very fashionable in Russia by the beginning of the nineteenth century. The musicologist Nikolai Roslavetz (1881–1944) worked on "pantonality"; another musicologist, Ivan Wyschnegradsky (1893–1979), worked on "pansonority"; there were "pan-Russian" congresses on various subjects, and so on. (I owe this remark to Franck Jedrzejewski.)

embedded in a $C^\infty$ manner in Euclidean three-space whose intrinsic geometry is that of the Lobachevsky plane as a whole.

We also note that Lobachevsky, in his *Pangeometry* and elsewhere, refers to Euclidean geometry as "ordinary geometry".

Le us now consider the other terms.

The term "absolute geometry" was used by Janós Bolyai to denote hyperbolic geometry. Beltrami, Klein, Poincaré, and many of their contemporaries and followers used (among others) the expression "Lobachevsky geometry" (with various transliterations of the name Lobachevsky). Beltrami used, for that geometry, the expression "pseudo-spherical" geometry.[112]

The term "metageometry" was used by nineteenth century French and Belgian geometers such as J. Barbarin and P. Mansion. It denotes the theory which is common to the three geometries: Euclidean, hyperbolic and spherical, cf. [109].[113] Bertrand Russell's *Essay on the foundations of Geometry* [133] contains a chapter entitled "A short history of metageometry".

Gauss sometimes used the term "astral geometry", a term introduced in 1818 by F. K. Schweikart.[114] F. A. Taurinus[115] used the expression "logarithmic-spherical geometry" (*logarithmisch-sphärische Geometrie*) for a geometry in which the angle sum of any triangle is less than two right angles, that is, for hyperbolic geometry. He, however, thought that such a geometry cannot exist.

It seems that Klein disliked the name "non-Euclidean geometry", and he explained the reason for that in a note to his *Erlanger program* (1872) [83] (Note V at the end of the paper): "With the name non-Euclidean geometry have been associated a multitude of non-mathematical ideas, which have been as zealously cherished by some as resolutely rejected by others, but with which our purely mathematical considerations have nothing to do whatsoever. The wish to contribute towards clearer ideas in this

112In a letter sent to Hoüel (October 1869, cf. [23], p. 98), Beltrami wrote that he had first chosen the name "antispherical geometry", but that later on he decided to adopt the name "pseudo-spherical geometry", because, in his words, spherical geometry and the new geometry are "rather twin sisters than antagonistic". The term "pseudo-sphere", denoting a surface of revolution in Euclidean 3-space that has constant negative curvature obtained by rotating a tractrix, was coined by Beltrami. One must note that this surface had already been described by Gauss and later on by Minding, see [52], Vol. VIII, p. 265, and [112], but these authors did not make the relation with Lobachevsky's non-Euclidean geometry.

113Mansion, in [109], presents the first 26 propositions of Euclid's *Elements* as results in metageometry.

114Ferdinand Karl Schweikart (1780–1857) taught law at the University of Kharkov. He was also a mathematician and he transmitted to Gauss, in 1818, through Gauss's friend C. L. Gerling, a manuscript on the theory of parallels entitled *Astral geometry*, in which he claimed that there exist two geometries, the usual Euclidean one, and another one that holds for distances between stars. In a letter to H. C. Schumacher, dated 28 November 1846, Gauss wrote: "A certain Schweikart called such a geometry astral geometry, Lobachevsky calls it imaginary geometry." [Ein gewisser Schweikart nannte eine solche Geometrie Astralgeometrie, Lobatschewsky imaginäre Geometrie.] (Gauss's collected works [52], p. 238.) Gauss refers here to Schweikart's memoir *Die Theorie der Parallellinien, nebst dem Vorschlage ihrer Verbannung aus der Geometrie* [136].

115Franz Adolph Taurinus (1794–1874) was another contemporary of Gauss and Lobachevsky, and he was also Schweikart's nephew. We shall mention him in the text below, in relation with the trigonometric formulae of hyperbolic geometry.

matter has occasioned the following explanations..."[116]

Klein coined, in 1871, the expressions "elliptic geometry", "hyperbolic geometry" and "parabolic geometry" as alternative names for spherical, Lobachevsky and Euclidean geometry respectively.[117] These three geometries were considered by Klein in the general setting of projective geometry. The word "parabolic" did not really survive, the word "elliptic" is sometimes used, and the term "hyperbolic geometry" is the one that is most commonly used today for Lobachevsky geometry.

Following the current trend, I use in this commentary the term "hyperbolic geometry".

Let me also make a few comments on transliteration. Of course, such considerations are totally irrelevant from the purely mathematical point of view, but they may be of some interest for the knowledgeable reader.

My first comment concerns transcription of names. I have tried to make the transcription of Russian names uniform throughout this commentary, except in a few cases, e.g. when a name is part of a title of a book or of a paper, then I left it as it was. This concerns in particular the transliteration of the name Lobachevsky, for which there are many possibilities. For instance, it is spelled Lobacheffsky in the French edition of the Pangeometry.[118] In the first brochure (in French) issued by Kazan University at the occasion of the Lobachevsky prize, the name is spelled Lobatchefsky, which is also the spelling used by Houël in his translations and in his papers. In the papers Lobachevsky wrote in German, he transliterated his own name as Lobatschewsky, which is also the form used by Battaglini in his papers and in his Italian translations, and by Beltrami in his papers and in his correspondence.[119] Bonola, in the first edition of his *Geometria non-euclidea*, used the Italian form Lobacefski. In the German edition of Lobachevsky's works, translated by Stäckel and Engel, the name is spelled Lobatschef-

[116] [Man verknüpft mit dem Namen Nicht-Euklidische Geometrie eine Menge unmathematischer Vorstellungen, die auf der einen Seite mit eben so viel Eifer gepflegt als auf der anderen perhorrescirt werden, mit denen aber unsere rein mathematischen Betrachtungen gar Nichts zu schaffen haben. Der Wunsch, in dieser Richtung etwas zur Klärung der Begriffe beizutragen, mag die folgenden Auseinandersetzungen motiviren...]

[117] Klein, in his *Über die sogenannte Nicht-Euklidische Geometrie* [82] wrote (Stillwell's translation p. 72): "Following the usual terminology in modern geometry, these three geometries will be called *hyperbolic, elliptic or parabolic* in what follows, according as the points at infinity of a line are real, imaginary or coincident." [Einem in der neueren Geometrie gewöhnlichen Sprachgebrauche folgend, sollen diese drei Geometrien bezüglich als *hyperbolische* oder *elliptische* oder *parabolische* Geometrie im nachstehenden bezeichnet werden, je nachdem die beiden unendlich fernen Punkte der Geraden reell oder imaginär sind oder zusammenfallen.] In the paragraph that precedes this one, Klein made the remark that in hyperbolic geometry, straight lines have two points at infinity, that in spherical geometry, straight lines have no point at infinity, and that in Euclidean geometry, straight lines have two coincident points at infinity. This is a hint for the meaning of his expression "following the usual terminology" in the above sentence. An explanation was given in a note by Stillwell (Note 33 on the same page), in which Stillwell recalled that the points on a differentiable surface were called hyperbolic, elliptic or parabolic according to whether the principal tangents are real, imaginary or coincident. Stillwell also recalled that Steiner used these names for certain surface involutions, the involutions being called hyperbolic, elliptic or parabolic depending on whether the double points arising under these involutions are respectively real, imaginary or coincident.

[118] It was not uncommon, in the nineteenth century, in transliterating Russian names into French, to replace the letter v by the double letter ff, especially if the letter v occurs at the end of the name.

[119] with some exceptions; for instance, Beltrami, in some of his letters to Houël and in his letters (in French) to Helmholtz, wrote Lobatcheffsky; see e.g. [23], p. 87, p. 295.

skij. Sommerville, in his famous Bibliography, used the form Lobačevskij. Rosenfeld and his translator use the form Lobačevskiĭ, and there are still other possible forms.

In this book, I have chosen the transliteration "Lobachevsky", for no special reason, and I have tried to stick to it. It is the one that has been used by Lobachevsky's celebrated translator into English, G. B. Halsted, in his biographical note [62]. (We note that Halsted himself also used other forms, like Lobachevski, in his report on Engel and Stäckel's *Urkunden zur Geschichte der nichteuklidischen Geometrie* [66], Lobacévski, in his biographical notice on Hoüel [65], and Lobachévski in his report on the Lobachevsky prize [63].)

Likewise, I changed Kasan on the front page of the *Pangéométrie* (also spelled Cazan in Lobachevsky's *Crelle's Journal* article *Géométrie Imaginaire*) into Kazan, as it is usually spelled today, and Nijniy-Novgorod into its modern spelling, Nizhny Novgorod.

There is another point on translation (which is also a point on terminology) that I would like to make right away. This concerns the title of the first work published by Lobachevsky, namely, his 1829–1830 memoir, whose Russian title is: О началах геометрии (O nachalakh geometrii). There are different translations of the title, e.g. "On the principles of geometry" (used by Hoüel, by Kagan's and Rosenfeld's translators, by Katz, and by others), or "On the foundations of geometry" (used by Gray, Sommerville and others). The difference concerns the Russian word nachala. There is a third translation, namely, "On the elements of geometry", which has been used by fewer authors, and which was the one used by G. B. Halsted.[120]. We can also find the title "On the elements of geometry" in some translations of Russian journal articles where this work is referred to. I have adopted the third possibility, after a correspondence with Ernest Vinberg, who told me that he is convinced that Lobachevsky used consciously the word nachala in reference to Euclid's *Elements*, called in Russian Начала Евклида (Nachala Evclida).

The same comment of course also applies to Lobachevsky's memoir published in 1835–1838, whose Russian title is Новые начала геометрии с полной теорией параллельных (Novye nachala geometrii s polnoj teoriej parallel'nykh), which is translated by some as "New foundations of geometry, with a complete theory of parallels" and by others as "New principles of geometry, with a complete theory of parallels", and for which I have adopted, for the same reason, the translation "New elements of geometry, with a complete theory of parallels"

The commentary that follows is divided into five parts.

The first part, entitled *On the content of Lobachevsky's Pangeometry*, contains a mathematical commentary on the *Pangeometry*. It also includes a short review on the part of the theory that is only outlined in Lobachevsky's memoir. As I already said, I think that every student in geometry would benefit from reading Lobachevsky's text, and this part of my commentary is intended to help the reader to find his way in the subject.

[120]It should be noted that Halsted also used at some other places the word "principles".

The second part, entitled *On hyperbolic geometry and its reception*, has a historical flavour. My intention there was not to write a (short) history of the subject. I only wanted to give the reader an idea of the degree to which the reception of hyperbolic geometry by the mathematical community was difficult and painful. In particular, I have quoted and translated several documents which show that until the first decade of the twentieth century, there were still fierce opponents of hyperbolic geometry, whose opinions were published in good mathematical journals. As a general rule, I have tried to provide references and quotations that are not contained in the majority of the books on the history of the subject.

The third part of the commentary is called *On models, and on model-free hyperbolic geometry*. This is a commentary on the theory of hyperbolic geometry, in the way Lobachevsky conceived it, as opposed to the same theory worked out in models (namely, the Poincaré models, the Klein–Beltrami model and the Minkowski model), as it is usually done today. As a matter of fact, working out hyperbolic geometry in models has been a current trend since the end of the nineteenth century. After having personally taught, for several years, hyperbolic geometry using models, as most of my colleagues do, I discovered the beauty of model-free hyperbolic geometry by reading some of the original texts, and by talking with Norbert A'Campo. I have become convinced that this second approach is preferable, for teaching and for writing in this field. Of course, working without models requires a certain degree of maturity and of abstraction, one reason being the lack of obvious coordinate systems. But I find this approach intellectually much more satisfying. Let me note right away that I am not an advocate of teaching or of working out hyperbolic geometry as a purely logical system, but, on the contrary, of doing it in the purely geometric way that we find in Lobachevsky's *Pangeometry*, and in the texts written by some of Lobachevsky's followers, of which I give the bibliographical references in the fourth part of this commentary. In fact, this is the main reason for which I decided to translate and to write a commentary on Lobachevsky's *Pangeometry*.

The fourth part of the commentary, entitled *A short list of references on model-free geometry*, is an immediate sequel to the third part. It contains a short list of references, with a few comments on works written on model-free hyperbolic geometry. About half of the texts that I refer to have a rather cultural and historical value, and the others are more technical, and they can be used for learning the subject.

Finally, the fifth part, entitled *Some milestones for Lobachevsky's works on geometry*, contains a short review of Lobachevsky's works on geometry, together with some comments on their reception, as well as some remarks on translations and later editions. To the best of my knowledge, such an updated account was missing in the mathematical literature.

# 1. On the content of Lobachevsky's *Pangeometry*

The memoir can be divided into five parts.

The first part (pages 279 to 285 [81–87 in this book] of the original French text; pages 3 to 12 of the present translation) is a general introduction to non-Euclidean geometry, in which Lobachevsky discusses the notion of parallelism, which in principle is a classical notion, but which needs to be adapted to the setting of hyperbolic geometry. He then introduces other important notions that are useful in hyperbolic geometry, such as limit circles (horocycles) and the angle of parallelism function.

In the second part (pages 285 to 300 [87–102 in this book] of the original text; pages 12 to 30 of the translation), Lobachevsky established trigonometric formulae for spherical geometry, and then for hyperbolic geometry.

Most of the results presented in these first two parts are contained in Lobachevsky's previous writings. In the *Pangeometry*, Lobachevsky made a survey of these results, referring to his *Geometrische Untersuchungen zur Theorie der Parallellinien* [100]. Although the material presented in these two parts is not new, the summary that Lobachevsky made of it is useful, and it contains interesting new remarks.

In the third part (pages 301 to 315 [103–117 in this book] of the original text; pages 30 to 50 of the translation), Lobachevsky develops the analytic theory of hyperbolic geometry, more precisely, the differential geometry of curves, surfaces and solids.

The fourth part (pages 315 to 338 [117–140 in this book] of the original text; pages 50 to 75 of the translation) concerns the application of the analytic theory to the computation of definite integrals. These integrals are generally obtained as values of lengths, of areas and of volumes of various figures, calculated in different manners. The theory that Lobachevsky developed in this fourth part is comparable to the theory of elliptic integrals, as developed for instance by Euler, in which the definite integrals were computed as lengths of ellipses and other figures in the Euclidean plane.

The last part (pages 338 to 340 [140–142 in this book] of the original text; pages 75 to 77 of the translation) contains some remarks on astronomical observations that might have been helpful for the question of whether the physical world in which we live is hyperbolic or not. Lobachevsky's conclusion is that the question cannot be answered using the astronomical records that were available to him.

Let me describe now in more detail the content of the *Pangeometry*.

Lobachevsky's memoir starts with some general remarks on Euclidean and non-Euclidean geometry, in particular, on the impossibility of proving, by "elementary geometry" (that is, using Euclid's *Elements*), that the angle sum in an arbitrary triangle is equal to two right angles. Lobachevsky then recalls some of the subtleties that are inherent in the definition of parallelism. As is well known, in the Euclidean plane, two lines are said to be parallel if they do not intersect,[121] and, from a given point, one can draw a unique parallel to a given line that does not contain this point. This notion of

[121] In Euclid's *Elements*, Book I, Definition 23 says: "*Parallel* straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction." (Heath's translation [72], p. 190).

parallelism is insufficient in hyperbolic geometry. Indeed, in the hyperbolic plane, it is useful to distinguish between two kinds of non-intersecting pairs of lines, namely, *parallel* lines and *hyperparallel* (also called *ultraparallel*) lines.

Let us be more precise.

Given a line $L$ in the hyperbolic plane and a point $x$ not on $L$, there exist, in the pencil of lines that contain $x$, infinitely many lines that intersect $L$, and infinitely many lines that do not intersect $L$. The line $L'$, in that pencil, that makes a right angle with the perpendicular from $x$ to $L$ does not intersect $L$. The last result holds in neutral geometry (it follows from the fact that the angle sum in an arbitrary triangle is $\leq \pi$). In the Euclidean plane, this line $L'$ is the unique line that does not intersect $L$. But in the hyperbolic plane, there are other non-intersecting lines. A *parallel* line to $L$ through $x$ is defined to be a line that separates, in the pencil of lines containing $x$, the family of lines that intersect $L$ from the family of lines that do not intersect $L$. Such a limiting line is disjoint from $L$. A line is said to be *hyperparallel* to $L$ if it is not parallel to $L$ and if it is disjoint from $L$.

This definition of parallel line in hyperbolic geometry, as a line separating the family of intersecting lines from the family of non-intersecting lines, is the one that was used by each of the three founders of hyperbolic geometry, namely, Lobachevsky, Bolyai and Gauss. We note that in the Euclidean plane, the definition of parallel lines that mimics the one of hyperbolic geometry contains some redundancy, but is equivalent to the usual one. In other words, the notion of parallelism that is used in hyperbolic geometry is the generalisation, to neutral geometry, of the notion of parallelism in Euclidean geometry.

In the hyperbolic plane, there are exactly two distinct lines through $x$ that are parallel to $L$; they are represented in Figure 33. In the Euclidean plane, these two lines coincide, and there is only one line through $x$ that is parallel to $L$.

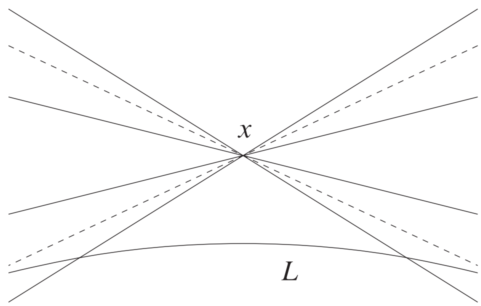


Figure 33. The two dashed lines are the parallels through $x$ to the line $L$.

An equivalent way of defining parallelism in hyperbolic geometry is to say that two lines are parallel if they meet at infinity, with an appropriate definition of the expression "meeting at infinity".[122]

[122] Using a terminology introduced by Hilbert, two parallel lines in the hyperbolic plane have a common *end*.

One must also note that in hyperbolic geometry, there is a notion of "direction of parallelism". Two lines are parallel with respect to some definite direction on each of them, the *direction of parallelism*.

In the memoirs that Lobachevsky wrote before the *Pangeometry*, e.g. in his *Geometrische Untersuchungen zur Theorie der Parallellinien* to which he constantly refers in his *Pangeometry*, he studied in detail the parallelism relation, showing in particular that it is symmetric and transitive.[123] We also note in passing that the definition of parallelism in dimension 3 is based on the planar definition. More precisely, two lines in 3-space are said to be parallel if they are coplanar and if they are parallel in the plane that contains them.[124]

After these remarks on parallelism, Lobachevsky discussed some primary notions of geometry. He noted that it is easier to start with the notion of circle than with that of straight line. This remark could have been motivated by the fact that the notion of line involves in some way or another the use of infinity.[125] Indeed, a line must be infinitely extendable, a fact which is usually expressed in a precise way in terms of the so-called Archimedean axiom, or, in modern terms, using an axiom that says that a line is isometric to the real line. The definition of a circle as a set of points that are equidistant from some fixed point does not involve such a use of infinity. Lobachevsky then defined a straight line in the plane as an intersection locus of a family of pairs of circles, a definition in which a line appears as the equidistant locus from two given points. Likewise, Lobachevsky defined a plane in 3-space as the intersection locus of a family of pairs of spheres.

After this, Lobachevsky introduced the notions of limit circle (an object which today is commonly called horocycle) and limit sphere (horosphere).[126] A limit circle

[123] In proving the transitivity, one must be careful about the direction of parallelism on each of the lines involved. The statement is that if two lines are parallel to a third one *with respect to a common direction on the third line*, then the first two lines are mutually parallel.

[124] The same thing holds, of course, in Euclidean geometry.

[125] We recall that Legendre's famous proof of the fact that the angle sum in a triangle is at most equal to two right angles uses the fact that a line is infinitely extendable. (Of course the result is false on a sphere.)

[126] Limit circles and limit spheres are of paramount importance in hyperbolic geometry. In his previous treatises (e.g. *Geometrische Untersuchungen zur Theorie der Parallellinien*, §31 and 34), Lobachevsky used the expressions *boundary line* and *horicycle* to denote a limit circle, and *boundary sphere* and *horisphere* to denote a limit sphere. We point out this fact, because it is sometimes thought (mistakenly) that the words "horicycle" and "horisphere" have been coined by Poincaré. The word "horicycle" is assembled from the Greek words "hori" (boundary) and "cyclos" (circle). János Bolyai independently discovered limit circles and limit spheres, and he called them "$F$-cycles" and "$F$-surfaces" (§11 of the Appendix, [24]). Gauss, in a letter he wrote on the 6th of March 1832 to Wolfgang Bolyai, after he read János Bolyai's *Appendix*, suggested the use of "paracycle" instead of $F$-cycle and "parasphere" instead of $F$-surface. Gauss wrote in that letter: "Thus, for instance the surface and the line your son calls $F$ and $L$ might be called parasphere and paracycle: they are, in essence, a sphere and a circle of infinite radii. One might call hypercycle the collection of all points in a plane that are at the same distance from a straight line; similarly for hypersphere". [So könnte z.B. die Fläche, die Dein Sohn $F$ nennt, eine Parasphäre, die Linie $L$ ein Paracykel genannt werden: es ist im Grunde Kugelfläche, oder Kreislinie von unendlichem Radius. Hypercykel könnte der Complexus aller Punkte heissen, die von einer Geraden, mit der sie in Einer Ebene liegen, gleiche Distanz haben; eben so Hypersphäre.] Beltrami's paper *Teoria fondamentale degli spazii di curvatura costante* [17] contains a detailed study of horocycles and horospheres. Horocycles can be explicitly drawn on Beltrami's pseudo-sphere. Beltrami further developed the description of these objects in several letters to Hoüel, see

is a limit of a family of circles passing through a given point and whose centres tend to infinity while staying on a given line (Figure 34). The line that is used in this definition is called an *axis* of the limit circle. A limit circle is perpendicular to any

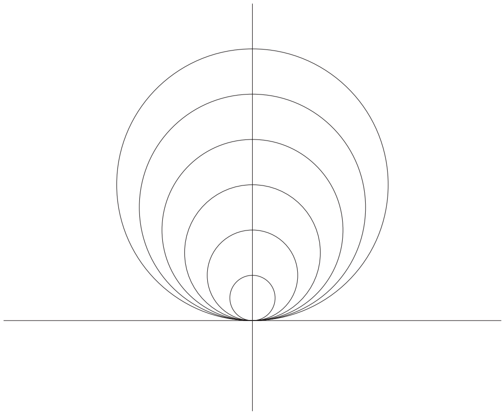

Figure 34. The limit of an increasing family of circles is a *limit circle*, hence the name "limit circle".

of its axes, and the family of axes of a limit circle forms a pencil of parallel lines. In Euclidean geometry, limit circles are straight lines,[127] and the family of perpendiculars to a straight line is again a pencil of parallel lines. As a sequence of circles converges to a limit circle, the pencil of lines that are orthogonal to these circles converges to a pencil of parallel lines that are orthogonal to the limit circle (see Figure 35). One can foresee from here a duality, in hyperbolic geometry, between families of limit circles and families of parallel lines.

A *limit sphere* (or *horosphere*) in 3-space is obtained by rotating a limit circle, contained in a plane embedded in 3-space, around one of its axes. A limit sphere can also be described as the limit of a family of spheres in 3-space passing through a given point and whose centres tend to infinity while staying on a given line. The intersection of a limit sphere with an arbitrary plane is either a circle or a limit circle.

After having introduced these objects, Lobachevsky described some fundamental results contained in his earlier works, in particular, in his memoir *On the elements of geometry* (1829) [95] and in his *Geometrical researches on the theory of parallels* (1840)

e.g. the letters dated 1st of April 1869, 12 October 1869, and 19 December 1869 ([23], pp. 87, 100, 107). Horospheres are at least as important as hyperplanes. They turned out to be one of the fundamental objects in mathematics, even beyond hyperbolic geometry. One can mention in this respect a basic idea of I. Gelfand, which he formulated in the 1950s and which is still in active use, namely, building a harmonic analysis theory based on the horospherical transform, which amounts to integrating functions on homogeneous manifolds along orbits of unipotent subgroups playing the role of horospheres.

[127] Kepler also thought of a straight line as a circle whose centre is at infinity, cf. M. Kline's *Mathematical Thought from Ancient to Modern Times* [86], p. 290. Before Kepler, Nicholas of Cusa, in his *De docta ignorantia* (1440) described how a circle with increasing radius tends to a straight line (the last reference was communicated to me by Jeremy Gray).

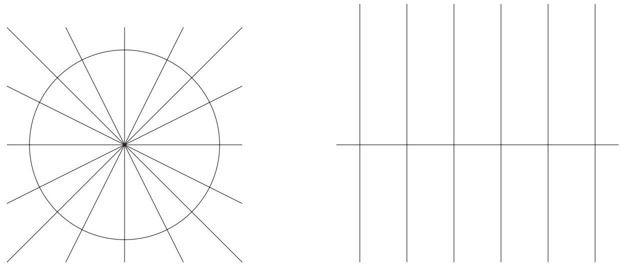

Figure 35. As the circle converges to a limit circle, the pencil of lines orthogonal to that circle (represented in the left-hand side of the figure) converges to the pencil of parallel lines that are orthogonal to the limit circle (represented in the right-hand side of the figure).

[100]. Both of these memoirs contain an exposition of the bases of non-Euclidean geometry. The results recalled by Lobachevsky include the following:

• Under the axioms of Euclidean geometry deprived of the parallel axiom (that is, in a geometry which, as we already noted, is sometimes called "neutral geometry"), there are two possibilities: either the angle sum in any triangle is equal to two right angles, or the angle sum in any triangle is less than two right angles. The first case occurs in Euclidean geometry, and the second case occurs in the *Pangeometry*.

• In spherical geometry, the angle sum function is a multiple of the angular excess function.[128]

After these basic results, Lobachevsky recalled the definition of the angle of parallelism and the construction of limit circles and of limit spheres in hyperbolic space, and he reported on the following important related results:

• In hyperbolic space, the geometry of a limit sphere is Euclidean. More precisely, Lobachevsky made the observation that if on a given limit sphere limit circles are considered as the "straight lines", and if the "angle" between two limit circles is defined as the dihedral angle between the diametral planes[129] that contain these limit circles, then the axioms of plane Euclidean geometry are satisfied by such a geometry.[130]

[128]This result is generally attributed to the Flemish mathematician Albert Girard (1595–1632), who stated it in his *Invention nouvelle en algèbre* (Amsterdam, 1629). Excerpts of this text are reproduced in O. Terquem's *Nouvelles Annales de Mathématiques*, t. 14 (1858). Euler gave a very simple proof of Girard's theorem in his memoir *De mensura angulorum solidorum* [47], see the account in Rosenfeld [131], p. 31.

[129]A *diametral plane* of a limit sphere is, by definition, a plane containing an axis of that limit sphere. It may be useful to note here that in hyperbolic 3-space, the dihedral angle between two planes is defined, as in Euclidean geometry, using the 2-dimensional definition: one takes a point on the line that is common to the two planes, and a perpendicular to this line in each of the two planes; the two perpendiculars so obtained lie in a common plane, and the dihedral angle is the angle that the two lines make in that plane. It follows from the axioms of neutral geometry that this angle does not depend on the choice of the point on the common line.

[130]It is fair to recall here that the fact that the geometry of the horosphere is Euclidean was independently obtained by J. Bolyai. Bolyai's work appeared in print two years after Lobachevsky's first published result on the subject. Let me also note the following interesting remark by L. Gérard in his thesis, [53], p. 3: "One can believe that Lobachevsky and Bolyai would never have thought about horocyles or horospheres if their first goal had not been to find a surface on which they could apply all the reasoning of Euclidean

It is important to note that Lobachevsky, in doing this, described a model of Euclidean geometry that lies inside hyperbolic 3-space. This idea goes far beyond the common classically dominating view that considered Euclidean geometry as the geometry of the real physical world that surrounds us. Lobachevsky's idea provides us with a new model of Euclidean geometry.

The fact that the Euclidean parallel postulate holds in the horosphere geometry is based on the following lemma, which is not proved in the *Pangeometry*, but which is proved in §28 of the *Geometrische Untersuchungen*: It we have three planes in hyperbolic 3-space that pairwise intersect each other in lines that are parallel,[131] then the sum of the three dihedral angles that they pairwise make is equal to two right angles.

Beltrami gave a different proof of the fact that the horospheres in hyperbolic 3-space are naturally equipped with a Euclidean structure. His proof is extracted from differential geometry, and it consists in showing that the metric on horospheres induced by the metric of hyperbolic 3-space has zero curvature.[132]

• The distance function from a variable point on a given line $L$ to a line $L'$ that is parallel to $L$ decreases as the variable point moves on $L$ in the direction of parallelism. A related remark that was made by Lobachevsky in his *Geometrische Untersuchungen* §33 is that parallel lines have the character of asymptotes.[133]

It is sometimes natural to talk about *parallel rays* instead of *parallel lines*. The notion of direction of parallelism is inherent in the definition when one talks about parallel rays.

• The sequence of lengths of pieces of equidistant horocycles bounded by two parallel lines that are axes of these horocycles is a geometric sequence, with multiplicative factor $< 1$, if this sequence of horocycle pieces moves in the direction of parallelism. The same property holds for the areas of the successive regions bounded by these horocyclic arcs.

A consequence of the last result is that the total area of the surface contained between two parallel lines, which is unbounded in the direction of parallelism and bounded on the other side by a piece of horocycle, is finite. This contrasts with the situation in Euclidean geometry, where the distance between two parallel lines is constant,[134]

geometry." [Il est permis de croire que Lobachevsky et Bolyai n'auraient jamais songé à l'oricycle et à l'orisphère s'ils ne s'étaient pas proposé avant tout de trouver une surface à laquelle puissent s'appliquer tous les raisonnements de la Géométrie euclidienne]. We emphasize the fact that this kind of use of Euclidean reasoning in hyperbolic geometry is intrinsic. It is done inside hyperbolic space, and it is not comparable to the use of Euclidean notions that is made when one works in a Euclidean model of hyperbolic geometry.

[131] We note that Lobachevsky already proved, in §25 of the *Geometrische Untersuchungen*, that the parallelism relation is transitive, so that in the present context we do not need to specify which pairs of lines are mutually parallel.

[132] This is further discussed in the footnotes accompanying my translation of the *Pangeometry*.

[133] It may be worth mentioning here that in the neutral plane (that is, the plane that satisfies all of Euclid's axioms except perhaps the parallel postulate), declaring the absence of asymptotic straight lines is equivalent to accepting Euclid's parallel postulate.

[134] In the neutral plane, the property that given two disjoint lines, the distance from a point on one of them to the other line is constant (that is, the property that this distance does not depend on the chosen point) is equivalent to Euclid's parallel axiom. The relation (or, rather, the confusion) between parallel lines and equidistant lines was a source of mistakes since Antiquity. The Neoplatonic philosopher and mathemati-

which implies that the area of a region bounded by two parallel lines and a common perpendicular (playing the role of the hyperbolic geometry horocycle) is always infinite.

The previous result also implies that in the hyperbolic plane, the distance from a point on a line $L$ to a parallel line $L'$ tends to 0 exponentially if the point on the line $L$ moves to infinity in the direction of parallelism.

Lobachevsky then studied the *parallelism function* (also called the *angle of parallelism* function) in the hyperbolic plane. This is the function, denoted by $\Pi$, that assigns to each segment of length $p$ the acute angle that this segment makes with a ray that starts at an endpoint of this segment and that is parallel to a second ray that starts perpendicularly at the other endpoint of the segment (see Figure 36). The value of the parallelism function only depends on the length $p$ of the segment and not on the segment itself. Thus, the parallelism angle of a segment of length $p$ is denoted by $\Pi(p)$.[135]

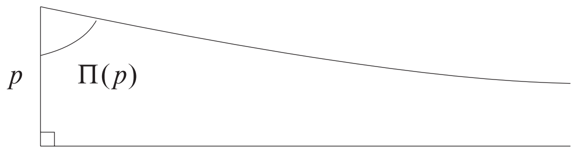


Figure 36. The parallelism angle $\Pi(p)$ corresponding to the segment $p$.

We note that the definition of the angle of parallelism uses the fact that the hyperbolic plane is doubly homogeneous; that is, that any two segments of the same length are congruent (i.e. they are equivalent through a motion of the plane). This fact is a

cian Proclus, in his *Commentary on the first Book of Euclid's Elements* [122], mentioned that Posidonius (135–51 B.C.), a philosopher, historian, physicist, astronomer and mathematician, native of Apamea (in Syria), *defined* parallel lines as lines situated in the same plane and that "come neither near nor apart"; that is, lines that are at a constant distance from each other. (The notion of *distance* was not used, and Posedonius expresses this fact by saying that "all perpendiculars drawn from one line to the other one are equal", see [122], Ver Eecke's edition, p. 153, or Morrow's edition, p. 138.) This definition by Posidonius is considered to be the result of a proof he attempted of Euclid's parallel postulate, using the notion of equidistance; cf. Pont [121], p. 150. Pont argues that it was observed since Greek Antiquity that Posidonius's and Euclid's definitions of parallel lines are not equivalent. The medieval mathematician Ibn al-Haytham (965–1039) again defined a parallel to a line $D$ as the locus of endpoints of a segment of constant length moving on $D$ and perpendicular to $D$ (see Pont [121], p. 169). We note that the notion of "motion" was not used by Euclid, for philosophical reasons and following Aristotle's ideas. (A discussion of this fact is beyond our goal in this commentary; we refer the reader to the discussion in Rosenfeld [131], p. 111–112.) The medieval mathematician Nasir al-Din at-Tusi (1201–1274) and, after him, the Renaissance mathematicians Christopher Clavius (1538–1612), Giovanni Alfonso Borelli (1608–1679) and Giordano Vitale (1633–1711) were aware of the relation of Euclid's postulate to the notion of equidistance, and they tried to prove that Euclid's axioms, without the parallel postulate, imply that the equidistant set to a line is a line, cf. [26], p. 13, and [121], pp. 200 and 368. Finally, we mention that Leonhard Euler (1701–1783) and Joseph Fourier (1768–1830), in their attempts to prove Euclid's parallel postulate, assumed the fact that an equidistant set to a line is a line (cf. [121], pp. 281 and 554).

[135] In Lobachevsky's writings, as in other writings of the same period, and following the tradition of Euclid's *Elements*, the word "line" (to which we have preferred the word "segment" in the present translation) is often identified with the length of that line (i.e. of that segment). Thus, Lobachevsky says that $\Pi$ is a function of the line $p$, meaning that it is a function of the real number $p$.

consequence of the axioms of neutral geometry.

Given a point $x$ in the hyperbolic plane and a line $L$ that does not contain it, if $p$ denotes the length of the perpendicular from $x$ to $L$, then the two parallel lines from $x$ to $L$ make an angle equal to $2\Pi(p)$.

In Euclidean geometry, the parallelism function $p \mapsto \Pi(p)$ is constant, and equal to $\frac{\pi}{2}$. In hyperbolic geometry, this function is strictly decreasing, and it establishes an analytic one-to-one correspondence between the set of length values $p$ that vary between 0 and $\infty$ and the set of angle values that vary between $\frac{\pi}{2}$ and 0.

Let us note a relation between the parallelism function and the angular deficit function defined on triangles.

We recall that given a triangle with angles $\alpha, \beta, \gamma$, its *angular deficit* is defined as $\pi - (\alpha + \beta + \gamma)$. In the hyperbolic plane, the angular deficit of any triangle is positive.

The parallelism function is related to the angular deficit function when this function is extended so that it is also defined for triangles having vertices at infinity.

Indeed, consider a right triangle one of whose edges adjacent to the right angle being infinitely long, and the length of other edge adjacent to the right angle being equal to $p$. (The hypothenuse is then also infinite, and it is parallel to the infinitely-long edge adjacent to the right angle.) Figure 36 represents such a triangle. The angle at the vertex which is at infinity is equal to zero and the angle sum of this triangle is $\pi/2 + \Pi(p)$. Thus, the angular deficit of this triangle is $\pi/2 - \Pi(p)$.

The parallelism function is of paramount importance in hyperbolic geometry, in particular because it establishes a natural correspondence between angle measure and length measure. To see one important consequence of this fact, we first recall that in neutral geometry, there is a canonical measure for angles. Such a measure can be defined as the ratio of the given angle to the total angle at any point, this total angle being equal to four right angles.[136] There is no canonical measure for lengths of segments in Euclidean geometry; that is, one has to choose a unit of length before talking about actual length. In contrast, in hyperbolic geometry, there *is* a canonical measure for length, and precisely, this measure can be deduced from the canonical measure for angles, via Lobachevsky's parallelism function. J. H. Lambert, in his *Theorie der Parallellinien* (1766) [139] already noticed (using arguments different from those of Lobachevsky) that in (the still hypothetical) hyperbolic geometry, there exists a canonical measure for length.[137]

[136]This can be made precise by using congruence. Indeed, using angle congruence, one can first define values of rational fractions of total angles, and then of all fractions by using limiting arguments.

[137]The existence of a canonical measure of length in hyperbolic geometry is a property of paramount importance. As a matter of fact, this property is equivalent to the negation of Euclid's parallel postulate. In his letter to Gerling, dated 11 April 1816 (see [52], Vol. VIII, p. 168), Gauss also talked, in the following terms, about the existence of this canonical measure of length, under the hypothesis of the failure of the parallel postulate: "It would be desirable that Euclidean geometry were not true, for we would then have a universal measure a priori. One could use the side of an equilateral triangle with angle = 59°59'59".9999 ... as a unit of length." [Es wäre sogar wünschenswerth, dass die Geometrie Euklids nicht wahr wäre, weil wir dann ein allgemeines Mass a priori hätten, z. B. könnte man als Raumeinheit die Seite desjenigen gleichseitigen Dreiecks annehmen, dessen Winkel = 59°59'59",9999 ...] We mention (as an anecdotic remark) that Gauss

Besides the conceptual importance of the function $\Pi$ in hyperbolic geometry, this function allows shorter trigonometric formulae (as we shall see below) and concise expressions for several geometric quantities. For instance, in Lobachevsky's notation, the circumference of a circle of radius $r$ in the hyperbolic plane is equal to $2\pi \cot \Pi(r)$.[138]

The parallelism function $\Pi(p)$ was already introduced by Lobachevsky in his first published memoir, the *Elements of geometry* (1829) [95], where it is denoted by $F(p)$. This function was used later on by several authors, with a reference to Lobachevsky. For instance, it is mentioned in Beltrami's *Saggio di interpretazione della geometria non-Euclidea* [16] and in Klein's *Über die sogenannte Nicht-Euklidische Geometrie* [82].

The parallelism function is related to the hyperbolic cosine function by the formula

$$\sin \Pi(p) = \frac{1}{\cosh p},$$

assuming the curvature of the space to be equal to $-1$.[139]

Lobachevsky first defined the parallelism function for positive values of $p$, and then, he defined a continuous extension of that function to all real values of $p$. For a negative $p$, the value $\Pi(p)$ is determined from the value $\Pi(-p)$ by the relation

$$\Pi(p) + \Pi(-p) = \pi.$$

This simple definition of an extension is usefully applied at several points in the *Pangeometry*.

After the discussion on the parallelism function, Lobachevsky established trigonometric formulae for triangles in spherical geometry. He proved that for a right spherical triangle with edges $a$, $b$, $c$ and opposite angles $A$, $B$, $\frac{\pi}{2}$ respectively, we have

$$\left.\begin{aligned} \sin A \sin c &= \sin a, \\ \cos b \sin A &= \cos B, \\ \cos a \cos b &= \cos c. \end{aligned}\right\} \tag{32}$$

was a member of a commission for weights and measures.

[138]Thus, one can *define* (although this is not the most natural definition) the parallelism function $\Pi(r)$ as $\cot^{-1}\left(\frac{1}{2\pi} C(r)\right)$, where $C(r)$ is the circumference of a circle of radius $r$.

[139]For a space of curvature $K = -1/k^2$, the formula becomes $\sin \Pi(p) = 1/\cosh \frac{p}{k}$. The existence of the constant $k$ appears in the works of all three founders of hyperbolic geometry (Lobachevsky, Bolyai and Gauss), although none of them did relate it to curvature. This constant appears in various forms, and we mention in this respect a letter from Gauss to F. A. Taurinus, written on 8 November 1824: "The assumption that in a triangle the sum of three angles is less than 180° leads to a curious geometry, quite different from ours, but thoroughly consistent, which I have developed to my entire satisfaction, so that I can solve every problem in it with the exception of the determination of a constant, which cannot be designated *a priori*. The greater one takes this constant, the nearer one comes to Euclidean geometry, and when it is chosen infinitely large, the two coincide." (Greenberg's translation.) [Die Annahme, dass die Summe der 3 Winkel kleiner sei als 180°, führt auf eine eigene, von der unsrigen (Euklidischen) ganz verschiedene Geometrie, die in sich selbt durchaus consequent ist, und die ich für mich selbst ganz befriedigend ausgebildet habe, so dass ich jede Aufgabe in derselben auflösen kann mit Ausnahme der Bestimmung einer Constante, die sich apriori nicht ausmitteln lässt. Je grösser man diese Constante annimmt, desto mehr nähert man sich der Euklidischen Geometrie und ein unendlich grosser Werth macht beide zusammenfallen.] ([52], p. 187.)

More generally, for an arbitrary spherical triangle with edges $a$, $b$, $c$ and with opposite angles $A$, $B$, $C$ (Figure 37), we have

$$\left.\begin{aligned}&\sin a \sin B = \sin b \sin A,\\ &\cos b - \cos a \cos c = \sin a \sin c \cos B,\\ &\cot a \sin b = \cot A \sin C + \cos b \cos C,\\ &\cos a \sin B \sin C = \cos B \cos C + \cos A.\end{aligned}\right\} \tag{33}$$

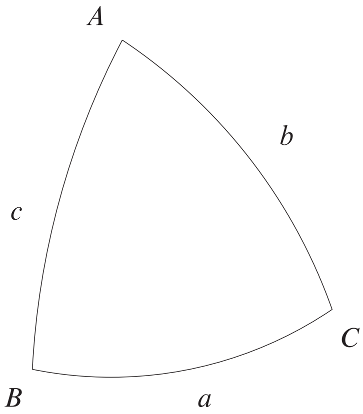

Figure 37

An important remark that we can make here is that in Lobachevsky's discussion, the above formulae concern a geometric sphere embedded in hyperbolic 3-space, and not a sphere in Euclidean 3-space, where spherical geometry is usually worked out.

Lobachevsky observed that the formulae he obtained for the geometry of a sphere in hyperbolic 3-space coincide with those of "ordinary" spherical geometry, that is, of the geometry of a sphere in Euclidean space. He considered this as a "truly remarkable fact" saying that spherical trigonometry is part of neutral geometry. Lobachevsky concluded that in this sense, spherical geometry is independent of Euclid's parallel axiom.[140]

The spherical triangle on which Lobachevsky worked is realised on a sphere centred at a vertex of a right hyperbolic rectilinear triangle that he started with, and these formulae for spherical trigonometry are used in the proofs of the trigonometric formulae

[140]We note that a similar fact was highlighted by Beltrami, several years later, in his paper *Teoria fondamentale degli spazii di curvatura costante*. Beltrami concluded this paper as follows: "The geodesic spheres of radius $\rho$, in an $n$-dimensional space of constant negative curvature $-\frac{1}{R^2}$, are the $(n-1)$-dimensional spaces of constant curvature $\left(\frac{1}{R\sinh\frac{\rho}{R}}\right)^2$. Therefore spherical geometry can be regarded as part of pseudo-spherical geometry. [Le sfere geodetiche di raggio $\rho$ nello spazio ad $n$ dimensioni di curvatura costante negative $-\frac{1}{R^2}$ sono spazi ad $n-1$ dimensioni di curvatura costante positiva $\left(\frac{1}{R\sinh\frac{\rho}{R}}\right)^2$. Quindi la geometria sferica può riguardarsi come contenuta nella pseudosferica.]

for this right hyperbolic triangle. The proofs of the hyperbolic trigonometric formulae also involve the consideration of a horospherical (Euclidean) triangle, drawn on the horosphere passing through a vertex of the rectilinear triangle and tangent to the plane of that triangle.

Thus, Lobachevsky's proof of his hyperbolic geometry trigonometric formulae uses at the same time spherical geometry and Euclidean geometry. We shall discuss this in more detail below. It is because of such reasonings that hyperbolic geometry is considered as a "pangeometry" (in the etymological sense of the word).

It is interesting to see how spherical geometry gradually developed from a subject with strong practical and computational aspects to an abstract independent field. Spherical geometry is considered today at the same level as Euclidean and hyperbolic geometry, being one of the three geometries that describe spaces of constant curvature. The three planar geometrical systems, Euclidean, hyperbolic and spherical, are also usually described as the geometries in which the angle sum of triangles is $2\pi$, $< 2\pi$ and $> 2\pi$ respectively.

It may be useful to say here a few words on the evolution of spherical geometry.

Spherical geometry was studied since antiquity, where it was intimately linked to astronomy. Eudoxus of Cnidus (410–355 B.C. ca.), who was a student of Plato, worked on that subject. A book by Menelaus of Alexandria (70–140 A.D. ca.), entitled the *Spherics*, survived in an Arabic translation, and it is sometimes considered as an attempt by its author to write a treatise on spherical geometry that would be the analogue of Euclid's *Elements*. In this treatise, Menelaus systematically studied spherical triangles whose edges are segments of great circles, in the way Euclid studied triangles in the plane. The treatise contains the first elements of spherical trigonometry. Ptolemy's *Almagest* (written in the second century A.D.) contains a systematic exposition of spherical trigonometry. Spherical geometry has been further developed by the medieval Arabic and Arabic-speaking mathematicians. According to Youschkevitch, the first treatment of spherical trigonometry as a research field in itself, that is, without being subject to astronomy, was done by Nasir al-Din at-Tusi ([157], p. 142). According to Rosenfeld, the modern form of spherical trigonometry, as well as of all trigonometry, is due to Euler; see [131], p. 31 ff., for a short account of Euler's work on spherical geometry and trigonometry. According to Bonola [26], p. 82, the recognition of spherical geometry as the geometric system in which the angle sum of triangles is greater than two right angles is an eighteenth-century invention, mainly due to J. H. Lambert, who is also one of the most important precursors of hyperbolic geometry. Before Lambert, spherical geometry was considered as a chapter in Euclidean geometry, namely, the study of the geometry of a sphere in Euclidean space.[141]

Volume XIII of Gauss's *Collected works* [52] contains notes on spherical trigonometry.

[141] Likewise, since the discovery (made by Lobachevsky) that the geometry of the spheres in hyperbolic 3-space is the same as the geometry of the spheres in Euclidean 3-space, one can say that spherical geometry can be studied as a chapter in hyperbolic geometry; see also Beltrami's concluding remarks of his *Teoria fondamentale degli spazii di curvatura costante* [17] mentioned in the preceding note.

In the nineteenth century, spherical geometry formulae were developed by F. A. Taurinus, who pointed out, in 1825, an interestingly simple passage between hyperbolic and spherical trigonometry. Indeed, in his *Theorie der Parallellinien*, Taurinus obtained the fundamental trigonometric formulae for hyperbolic geometry (which, from his point of view, was purely hypothetical) by working on a sphere of "imaginary radius", and he called the hyperbolic geometry that he so obtained the *logarithmic-spherical geometry* ("Logarithmisch-sphärischen Geometrie"), see the texts by Taurinus in the edition by Stäckel and Engel [139]; cf. also Bonola's *Non-Euclidean geometry*, [26], p. 79.[142] In the same way, Lobachevsky, in his *Elements of geometry* [95] (see the conclusion of that paper, p. 65 of the German translation) and in his *Geometric researches on the theory of parallels* (see p. 45 of the Appendix of Bonola's edition [26]) noted a passage between the equations of hyperbolic trigonometry and those of spherical trigonometry, which consists in replacing, in the trigonometric equations of spherical geometry, the edge lengths $a$, $b$, $c$ of a triangle by the imaginary quantities $a\sqrt{-1}, b\sqrt{-1}, c\sqrt{-1}$, and setting in these equations the terms $\sin\Pi(a)$, $\cos\Pi(a)$ and $\tan\Pi(a)$ to be equal to $1/\cos a$, $\sqrt{-1}\tan a$ and $1/\sqrt{-1}\sin a$ respectively. A similar idea has been repeated by several authors. Beltrami, in his *Saggio di interpretazione della geometria non-Euclidea* [16] and in his *Teoria fondamentale degli spazii di curvatura costante* [17] showed that the trigonometric formulae of the pseudo-sphere can be obtained from those of the usual sphere by considering the objects on the pseudo-sphere as being on a sphere of imaginary radius $\sqrt{-1}$, and he attributed this observation to E. F. A. Minding [112] and to D. Codazzi [32].[143] Klein, in his *Über die sogenannte Nicht-Euklidische*

[142] Gauss corresponded with Taurinus on this subject, see Volume XIII of Gauss's *Collected works*. It is known that Taurinus was driven to despair after Gauss stopped answering his letters, and that he bought all the remaining copies of a booklet he had published on the theory of parallels (reproduced in [139]) and burnt them (see [155], p. 50, and [139], pp. 249–250).

[143] Ernst Ferdinand Adolf Minding (1806–1885) was among the first important mathematicians who worked on the differential geometry of surfaces in the way inaugurated by Gauss in his *Disquisitiones generales circa superficies curvas*. Starting from 1840, Minding taught 40 years at the Russian University of Dorpat (the university to which Bartels, who had been Lobachevsky's teacher in Kazan, moved in 1821). Minding became a member of the St. Petersburg Academy of Sciences. He studied the question of when a surface can be developed onto another one, in terms of curvature. He considered surfaces of constant curvature, and in particular he gave the equation of the famous *pseudo-sphere*, the surface that had to play a major role in the work of Beltrami. One should also note here that this surface was already considered by Gauss, see [52], Vol. VIII, p. 265. (But, of course, without the name pseudo-sphere, and the fact that it was a model of hyperbolic geometry was not yet discovered.) Minding's paper [112] contains trigonometric formulae for geodesic triangles on a surface of constant negative curvature that coincide with Lobachevsky's trigonometric formulae, contained in his paper *Géométrie imaginaire*. Lobachevsky's paper was published in Volume 17 (1837) of *Crelle's Journal*, that is, three years before Minding's paper. An important remark made by Minding in his paper [112] is that the trigonometric formulae on a surface of constant negative curvature have the same form as those on a sphere, up to replacing, in the spherical formulae, the radius $R$ of the sphere by the imaginary quantity $R\sqrt{-1}$. A similar remark had already been made by Lobachevsky in his 1829 paper [95], concerning the passage from the spherical geometry formulae to his hyperbolic geometry formulae. The importance of Minding's paper [112] was pointed out 28 years after that paper appeared, by Beltrami, in his *Saggio di interpretazione della geometria non-Euclidea*. Beltrami made the link between Minding's work on the intrinsic geometry of surfaces of constant negative curvature and Lobachevsky's work on non-Euclidean geometry. It may be interesting to note here that according to records transmitted by Karimullin and Laptev in [80] p. 20, Lobachevsky borrowed from the Kazan University Library all volumes

*Geometrie* [82] made the same observation. We note that it is because of such a "duality" between the trigonometric formulae for spherical and for hyperbolic geometry that Beltrami chose, for hyperbolic geometry, the name "pseudo-spherical geometry". Likewise, W. E. Story, in his paper *On the non-Euclidean trigonometry* [142], working in Klein's setting of projective geometry, showed that some trigonometric formulae of hyperbolic geometry can be deduced from those of spherical geometry by replacing each length by the corresponding length divided by $2ik$ and each angle by the corresponding angle divided by $2ik'$, where $k$ and $k'$ are two constants. (No mention of Lobachevsky is made in that paper, written in 1881, in which the author refers to the works of Cayley and Klein.)

It is always instructive to make parallels between results in hyperbolic geometry and results in spherical geometry. There are striking analogies between the trigonometric formulae of these two spaces, and we can see this in Table 1, which contains some formulae that are extracted from the survey paper by Alekseevskij, Vinberg and Solodovnikov in [5]. One can develop the analogies by considering the so-called hyperboloid model of the hyperbolic plane, as the "upper-sheet" ($z > 0$) of the unit sphere

| Hyperbolic plane | Euclidean plane | Sphere |
|---|---|---|
| $\cosh c = \cosh a \cosh b$ | $c^2 = a^2 + b^2$ | $\cos c = \cos a \cos b$ |
| $\sinh b = \sin c \sin \beta$ | $b = c \sin \beta$ | $\sin b = \sin c \sin \beta$ |
| $\tanh a = \tanh c \cos \beta$ | $a = c \cos \beta$ | $\tan a = \tan c \cos \beta$ |
| $\cosh c = \cot \alpha \cot \beta$ | $1 = \cot \alpha \cot \beta$ | $\cos c = \cot \alpha \cot \beta$ |
| $\cos \alpha = \cosh a \sin \beta$ | $\cos \alpha = \sin \beta$ | $\cos \alpha = \cos a \sin \beta$ |
| $\tan a = \sinh b \tan \alpha$ | $a = b \tan \alpha$ | $\tan a = \sin b \tan \alpha$ |

Table 1. The table compares trigonometric formulae for a right triangle with angles $\alpha$, $\beta$, $\gamma$ and opposite edges $a, b, c$, and with right angle at $\gamma$, in the hyperbolic plane (of constant curvature $-1$), in the Euclidean plane and on the sphere (of radius 1). We notice that the hyperbolic formulae are obtained from the spherical formulae by replacing the functions sin and cos of side lengths by the functions sinh and cosh of these side lengths. The formulae in the Euclidean plane are obtained from the corresponding formulae in the other two geometries by taking Taylor expansions, as the edges $a, b, c$ of the triangle tend to 0.

of *Crelle's Journal* published during the period he was there, except Volumes 18 to 21. Minding's paper is contained in Volume 20 (1840). It is thought by some that Lobachevsky was sick during the period where Minding's paper appeared in print and that for that reason he did not notice it. (This was communicated to me by G. M. Polotovsky). More probably, Lobachevsky saw Minding's paper but remained silent about it. The question of why it was so is raised and discussed by Laptev in his papers [90] and [92]. (The last fact was communicated to me by S. S. Demidov.) In any case, these were unfortunate circumstances, because Lobachevsky missed the occasion of using Minding's results to prove rigourously that his geometry is consistent, a fact which Beltrami did several decades later.

of radius $\sqrt{-1}$ in $\mathbb{R}^3$ equipped with a coordinate system $(x, y, z)$ and the quadratic form $x^2 + y^2 - z^2$ and studying it in parallel with the space of spherical geometry being the unit sphere of radius 1 in $\mathbb{R}^3$ equipped with the quadratic form $x^2 + y^2 - z^2$.

Let us also note that this passage from the trigonometric formulae of hyperbolic geometry to those of spherical geometry, made by replacing some real quantities by imaginary ones, has the flavour of a passage, discovered later on, between formulae for volumes of hyperbolic polyhedra and formulae for volumes of spherical polyhedra. This was noticed by Coxeter in [33], who made a relation between computations of Lobachevsky and formulae discovered by Schläfli.

Now we return to Lobachevsky's *Pangeometry*.

After having proved the spherical trigonometry formulae, Lobachevsky established the formula that we already mentioned for the angle of parallelism function:

$$\tan \frac{1}{2}\Pi(x) = e^{-x},$$

from which he deduced

$$\sin \Pi(x) = \frac{2}{e^x + e^{-x}}$$

and

$$\cos \Pi(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

As already noted, these formulae are valid for a special choice of a certain constant that is undetermined. More precisely, there is a constant (the curvature constant) that appears in the formulae, whose value does not follow from the axioms of hyperbolic geometry alone. In the above formulae, the constant is taken to be one, which is equivalent to assuming that the space has constant curvature equal to $-1$.

After that, Lobachevsky derived formulae for hyperbolic trigonometry. Their expression uses the parallelism function.[144] For a right hyperbolic triangle with edges $a$, $b$, $c$, and angles $A$, $B$, $\frac{\pi}{2}$ opposite to $a$, $b$, $c$ respectively, he obtained the following formulae:

$$\left.\begin{array}{l} \sin \Pi(a) \cos A = \sin B, \\ \sin \Pi(c) \cos A = \cos \Pi(b), \\ \cos \Pi(c) \cos B = \cos \Pi(a), \\ \sin \Pi(a) \sin \Pi(b) = \sin \Pi(c), \\ \tan \Pi(c) = \sin A \tan \Pi(a). \end{array}\right\} \tag{34}$$

One can note the formal analogy between some of the equations in (34) and Equations (32) for spherical trigonometry, where some of the arguments, in the passage

[144] In modern textbooks, hyperbolic trigonometry formulae are expressed using the hyperbolic functions sinh and cosh. Lobachevsky did not use the notation of hyperbolic functions $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$ although the notation already existed. According to Cajori's *History of Mathematical Notations*, the hyperbolic functions were introduced in the 18th century by Vincenzo Riccati (1676–1754), and their theory was further developed by J. H. Lambert (1728–1777), who, as we already recalled, was one of the most important precursors of non-Euclidean geometry.

from spherical to hyperbolic geometry, are replaced by the parallelism function of these arguments.

For a general hyperbolic triangle with edges $a$, $b$, $c$ and opposite angles $A$, $B$, $C$ (Figure 37), Lobachevsky obtained

$$\left.\begin{aligned}
&\sin A \tan \Pi(a) = \sin B \tan \Pi(b),\\
&1 - \cos \Pi(b) \cos \Pi(c) \cos A = \frac{\sin \Pi(b) \sin \Pi(c)}{\sin \Pi(a)},\\
&\cos A + \cos B \cos C = \frac{\sin B \sin C}{\sin \Pi(a)},\\
&\cot A \sin C \sin \Pi(b) + \cos C = \frac{\cos \Pi(b)}{\cos \Pi(a)}.
\end{aligned}\right\} \tag{35}$$

Lobachevsky's proof of the trigonometric formulae for right hyperbolic triangles is based on a simple and ingenious construction, which associates to an arbitrary right hyperbolic triangle a spherical and a Euclidean right triangle, whose angles and side lengths are completely determined by those of the right hyperbolic triangle that we started with. The right spherical triangle is constructed on a geometric sphere in hyperbolic 3-space centred at a vertex of the hyperbolic triangle (the plane of the hyperbolic triangle being considered as sitting in hyperbolic 3-space). The right Euclidean triangle is constructed on a horosphere with axis perpendicular to the plane of the hyperbolic right triangle. The construction is explicit, and the formulae involve the parallelism function. In the meanwhile, Lobachevsky associated to the right hyperbolic triangle that we started with, a new right hyperbolic triangle whose angles and edge lengths are also completely determined by those of the first one.

More precisely, to a hyperbolic right triangle with edges $a$, $b$, $c$ and with opposite angles $\Pi(\alpha)$, $\Pi(\beta)$, $\frac{\pi}{2}$ respectively, Lobachevsky associated:

1. A right spherical triangle with edges $\Pi(\beta)$, $\Pi(c)$, $\Pi(a)$ and with opposite angles $\Pi(\alpha')$, $\Pi(b)$, $\frac{\pi}{2}$ respectively.
2. A right hyperbolic triangle with edges $a$, $\alpha'$ and $\beta$, and with opposite angles $\Pi(b')$, $\Pi(c)$ and $\frac{\pi}{2}$ respectively.
3. A right Euclidean triangle with edges $p = r \sin \Pi(\alpha)$, $q = r \cos \Pi(\alpha)$ and $r$, and with opposite angles $\Pi(\alpha)$, $\Pi(\alpha')$ and $\frac{\pi}{2}$ respectively. (The sides of the Euclidean triangle are only determined up to a positive constant $r$; this is related to the fact that homotheties are allowed in Euclidean geometry.)

In these formulae, the following notation is used: a letter $\alpha'$ (with the prime) denotes a length whose angle of parallelism is the complement to a right angle of the angle of parallelism of the length $\alpha$ (without the prime). In other words, we have

$$\Pi(\alpha) + \Pi(\alpha') = \frac{\pi}{2}.$$

The values obtained in 1., 2. and 3. are simple and explicit. One is tempted to use these formulae to deduce, from the trigonometric formulae for right Euclidean and right spherical triangles, trigonometric formulae for right hyperbolic triangles. This is indeed what Lobachevsky does. For me this method of proof is amazing.

We note that the trigonometric formulae (spherical and hyperbolic) are already contained in Lobachevsky's first memoir, the *Elements of geometry* [95], and in his later works, such as the *Geometrische Untersuchungen* (§35 and 36 respectively) [100].

It is interesting to note here the remarks that Lobachevsky made, on the above trigonometric formulae, at the end of his *Elements of geometry*, concerning the non-contradiction issue of his newly discovered geometry (Rosenfeld [131], p. 223, Shenitzer's translation from the Russian; Equations (16) and (17) that are referred to in this text are equivalent to our Equations (33) and (35) respectively):

> After we have found Equations (17) which represent the dependence of the angles and sides of a triangle; when, finally, we have given the general expressions for elements of lines, areas and volumes of solids, all else in the Geometry is a matter of analytics, where calculations must necessarily agree with each other, and we cannot discover anything new that is not included in these first equations from which must be taken all relations of geometric magnitudes, one to another. Thus if one now needs to assume that some contradiction will force us subsequently to refute the principles that we accepted in this geometry, then such contradiction can only hide in the very Equations (17). We note, however, that these equations become Equations (16) of spherical trigonometry as soon as, instead of the sides $a$, $b$, $c$ we put $a\sqrt{-1}$, $b\sqrt{-1}$, $c\sqrt{-1}$; but in ordinary Geometry and in spherical Trigonometry there enter everywhere only ratios of lines; therefore ordinary Geometry, Trigonometry and the new Geometry will always agree among themselves.

This is clearly an indication of Lobachevsky's intuition that if there were a contradiction in hyperbolic geometry, then there would also be one in Euclidean geometry or in spherical geometry, and vice-versa. The reason for this close relationship between the three geometries is that, as we already noted, Lobachevsky's method of proof of the hyperbolic-trigonometric formulae involves transformations that go back and forth between hyperbolic, spherical and Euclidean geometries.

It may be useful to note here that there have been subsequent proofs of the trigonometric formulae, that are also synthetic (that is, that do not use models), and that do not require going into 3-dimensional hyperbolic space. I am thinking here of the important variational method used by L. Gérard in his thesis, *Sur la géométrie non euclidienne* (1892) [53].

After having derived his non-Euclidean trigonometric formulae, Lobachevsky, taking second-order approximations of the functions $\sin\Pi(x)$, $\cos\Pi(x)$ and $\tan\Pi(x)$ for small values of $x$, proved that for infinitesimally small triangles, the trigonometric formulae of hyperbolic geometry become those of Euclidean geometry. He also made the fundamental remark that some of the trigonometric formulae that he obtained for infinitesimal triangles lead to the fact that the angle sum of an infinitesimal triangle is equal to two right angles, thus proving that at the infinitesimal level, hyperbolic geometry is Euclidean.

These trigonometric formulae are at the foundations of the analytic theory of hyperbolic geometry, which is developed in the rest of the memoir.

It is important to note the passage, in the structure of the *Pangeometry*, from an axiomatic and rather abstract approach of hyperbolic geometry (described in the first and the second parts) to an approach based on the distance function, involving concrete computations (in the third and the fourth parts).

The third part of the *Pangeometry* starts with an exposition of the general theory of representation of curves in the hyperbolic plane using coordinates that resemble the $xy$-coordinates of Euclidean analytic geometry. After presenting the general theory, Lobachevsky derived formulae for equations of lines, of circles and of limit circles. In such a representation, a point $P$ in the hyperbolic plane is parametrised with reference to an oriented line, called the $x$-axis, and equipped with an origin $O$. The $x$-coordinate of the point $P$ is taken to be the distance $OM$ from the coordinate origin $O$ to the projection $M$ of $P$ on the $x$-axis (see Figure 38). The $y$-coordinate of $P$ is taken to be

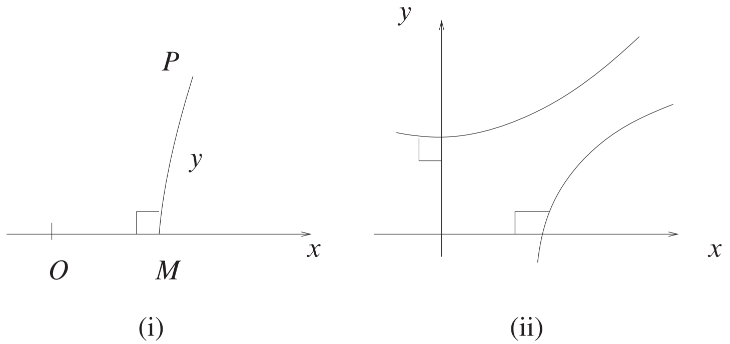


Figure 38. This figure is intended to show one difference between rectangular coordinates in hyperbolic and in Euclidean geometry. On the left-hand side, $x$ and $y$ are the rectangular coordinates of the point $P$. On the right-hand side are represented two perpendiculars to the $x$- and $y$-axes that do not intersect.

the length of the segment $PM$. Coordinates are taken with the usual appropriate signs, depending on which side of the $x$-axis the point $P$ lies, and on which side of the origin the projection $M$ of $P$ on the $x$-axis lies. Lobachevsky calls these $(x, y)$-coordinates "rectangular coordinates". Note that no $y$-axis is needed in that construction, and that the $y$-coordinate, unlike the case in Euclidean geometry, is not equal to the distance from the point $P$ to some $y$-axis. This is related to the fact that there are no "rectangles" (quadrilaterals with four right angles) in the hyperbolic plane. It may also be useful to note that if we take two perpendicular axes in the hyperbolic plane, calling them an $x$- and a $y$-axis, and if we try to mimic the way Euclidean coordinates are usually defined, then one problem arises from the fact that two perpendicular lines erected at points on the $x$- and the $y$-axes do not necessarily intersect (Figure 38 (ii)).

In Lobachevsky's rectangular coordinates, the parallelism function is again a useful tool for writing concise formulae for equations of lines and curves. For instance, the equation of a circle of radius $r > 0$ centred at the origin is

$$\sin \Pi(x) \sin \Pi(y) = \sin \Pi(r).$$

This is also an equation that is used in transforming rectangular coordinates into polar coordinates.

In polar coordinates, the point $P$ in the hyperbolic plane is represented by a pair $(\theta, r)$, with $r > 0$ being the distance from $P$ to the origin $O$, and $\theta$ being the angle between the rays $Ox$ and $OP$. The following equations determine $\theta$ in terms of $x$, $y$ and $r$:

$$\sin \theta = \frac{\tan \Pi(r)}{\tan \Pi(y)}$$

and

$$\cos \theta = \frac{\cos \Pi(x)}{\cos \Pi(r)}.$$

From the equation of a circle given above, by making a small transformation and after making $r$ tend to infinity, Lobachevsky obtained the following equation of a limit circle:

$$\sin \Pi(y) = \tan \frac{1}{2} \Pi(x).$$

Equations of lines in the hyperbolic plane are non-linear, and their expression is more complicated than that of equations of circles.[145] One complication for writing equations of lines stems from the fact that a perpendicular to the $x$-axis dropped from a variable point $x$ on a line $L$ does not make a constant angle with $L$. Another difference with the case of Euclidean geometry is that, in hyperbolic geometry, there are three possibilities for the position of a line with respect to the $x$-axis: the line may intersect the $x$-axis, or it may be parallel to it, or it may be hyperparallel (that is, it may be neither parallel to nor intersecting that axis).

Let us give a concrete example of an equation of line. If we erect a perpendicular to the $x$-axis at the point $O$, then the equation of a line $L$ intersecting this perpendicular at distance $a$ from $O$ and making an angle $\alpha$ with that perpendicular, as in Figure 39, is

$$\cos \Pi(y) = \frac{\cos \Pi(a)}{\sin \Pi(x)} - \sin \Pi(a) \cot \Pi(x) \cot \alpha,$$

$x$ and $y$ being the rectangular coordinates of an arbitrary point on $L$.

The equation of a line $L$ that does not intersect the $x$-axis and that is not parallel to it has a shorter expression. Taking the coordinate origin $O$ to be the foot of the

[145] One may recall here Lobachevsky's remark made in the first pages of the *Pangeometry*, saying that from the synthetic point of view, the notion of circle is more elementary than that of line.

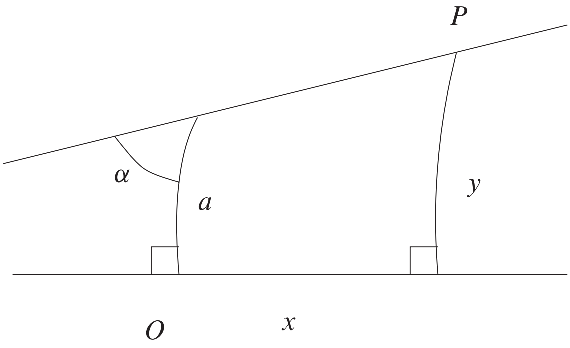


Figure 39

common perpendicular to $L$ and the $x$-axis and denoting by $b$ the length of the common perpendicular segment, the equation of $L$ is

$$\cos\Pi(b) = \cos\Pi(y)\sin\Pi(x).$$

Using the $(x, y)$-representation of lines, Lobachevsky proved that in the hyperbolic plane, if two lines are not parallel and do not intersect, then they have a common perpendicular.

After the section on equations of curves in the hyperbolic plane, the *Pangeometry* contains a section on arc length and on surface area.

Let me make here a remark on area.

The notion of area in hyperbolic geometry is delicate. Of course, it is easy (and pleasant) to start with the definition of the area of a triangle as being the angular deficit of this triangle. But this definition is highly non-intuitive, compared to the definition of area in the Euclidean plane based on the definition of the area of a rectangle as the product of its base and altitude. There are no rectangles in the hyperbolic plane. There is no natural formula for the infinitesimal area element in the hyperbolic plane, because there is no natural choice of coordinates. The work done by Lobachevsky on arclength, area and volume is highly non-trivial. Lobachevsky obtained the following formulae for the infinitesimal arc element, respectively in rectangular and in polar coordinates:

$$ds = \sqrt{dy^2 + \frac{dx^2}{\sin^2\Pi(y)}}$$

and

$$ds = \sqrt{dr^2 + d\varphi^2\cot^2\Pi(r)}.$$

These formulae were obtained by first writing a formula for the distance between two points in the plane, and then neglecting high order terms.

With these formulae, Lobachevsky started a new set of ideas in non-Euclidean geometry, namely, the computation of length, area and volume by integrating infinitesimal elements.

It is important to emphasise the fact that Lobachevsky developed a differential geometric theory that lives in a two and a three-dimensional vector space equipped with a metric that is not the Euclidean metric. This theory may be considered as a first example of an intrinsic Riemannian geometry that is not the Euclidean one. The theory was worked out before Riemann developed the concept of differential geometry on manifolds.

The formulae for the infinitesimal line element, and from there, the formulae for the infinitesimal area and volume elements, were obtained by Lobachevsky using the trigonometric formulae, and the first-order expansions for the (trigonometric functions of) the parallelism function, namely,

$$\cot \Pi(x) = x,$$
$$\sin \Pi(x) = 1 - \frac{1}{2}x^2$$

and

$$\cos \Pi(x) = x.$$

As a first application of his infinitesimal formulae, Lobachevsky made explicit computations of lengths of circles and of limit circle arcs. Again, the expressions obtained use the parallelism function $\Pi$, which permits economical notation.

After establishing the formulae for the circumference and area of a circle, Lobachevsky showed that these formulae lead at the infinitesimal level to the well-known formulae of Euclidean geometry.

Integrating the infinitesimal length in the special cases of limit circle arcs and of circles, Lobachevsky recovered the formulae that he had found earlier, using more global methods. He made the computations in various coordinates: rectangular, polar and limit circle coordinates. He obtained a formula for the infinitesimal area of a plane region by using a subdivision of the plane by a network of infinitesimally close limit circles having a common axis. In the case where this axis is the $x$-axis, the area element takes the particularly simple form

$$\cot \Pi(y) dx.$$

From this formula, Lobachevsky obtained a formula for the area of a region contained between two parallel lines, unbounded in the direction of parallelism and bounded by a segment perpendicular to these parallel lines. He showed how to recover, starting with the infinitesimal area form, the area of a circle. He then used this infinitesimal area element to find an expression of the area of a triangle, and he showed that this area is equal to the angular deficit, that is, to the difference between two right angles and the angle sum of this triangle.

It is interesting to note that the fact that the area of a triangle is equal to its angular deficit is obtained in Lobachevsky's memoir as a theorem. In this respect, we recall that it is possible to *define* area in hyperbolic geometry by starting with the notion of area of a triangle, as angular deficit, and then using additivity and a limiting procedure to define the areas of some more general figures.[146]

Another interesting formula that Lobachevsky gave in the *Pangeometry* is the one for the area of a right rectilinear triangle in terms of the lengths of its edges.

He applied the formula for the area of a triangle to obtain a formula for the area of a circle by a method different from the one he had previously given.

Lobachevsky then studied the notion of area of a curved surface in hyperbolic 3-space. He defined the area of such a surface by taking limits of sums of areas of triangles forming a decomposition of the surface (that is, a "triangulation" of that surface), as the dimensions of all the triangles tend to zero.[147] He obtained the following formula for the infinitesimal area element of a curved surface of equation $z = f(x, y)$, in the rectangular 3-space coordinates:

$$d^2S = \sqrt{\left(\frac{dz}{dx}\right)^2 + \frac{1}{\sin^2 \Pi(y)}\left(\frac{dz}{dy}\right)^2 + \frac{1}{\sin^2 \Pi(y) \sin^2 \Pi(z)}}\frac{dxdy}{2 \sin \Pi(z)}.$$

He then obtained the following formulae for the infinitesimal volume element, in rectangular and in polar coordinates, respectively:

$$d^3P = \frac{dxdydz}{\sin \Pi(y) \sin^2 \Pi(z)},$$

$$d^3P = \frac{1}{4} \cos\theta (e^r - e^{-r})^2 dr d\omega d\theta.$$

After establishing these formulae, Lobachevsky expressed areas and volumes of some special figures by integrals.

Computing areas and volumes in various manners is the main theme of the fourth part of the *Pangeometry*. Using different ways for measuring the area or the volume of a given geometrical figure, Lobachevsky obtained new formulae for definite integrals. One example is the computation of the area of a triangle by integrating the infinitesimal area form, and then equating the result with the formula of the area being equal to the angular deficit. This leads to formulae like

$$\sin A \int_0^{\frac{\pi}{2}-A} \frac{\tan \omega d\omega}{\sqrt{\cos^2 A - \sin^2 \omega}} + A = \frac{\pi}{2}.$$

[146]This was for instance Lambert's approach, when he worked out the (hypothetical) consequences of the failure of Euclid's parallel postulate. We also recall that Gauss, in a letter to Wolfgang Bolyai, dated 6 March 1832 (this was after Gauss read János Bolyai's work on non-Euclidean geometry), wrote a proof of the fact that the area of a triangle in the hyperbolic plane is equal to its angular deficit. Gauss's proof is contained in Volume VIII of his collected works [52], p. 220, and it is different from the one given by Lobachevsky in the *Pangeometry*, although both proofs involve the inclusion of an arbitrary triangle in a triangle having infinite-length edges. Gauss's proof is presented in detail by Coxeter in [35], p. 296–299,

[147]This definition of the area of a surface is a natural generalisation of the definition of the length of a curve. The Schwarz paradox that shows the shortcomings of such a definition in the case of surfaces appeared several years after Lobachevsky's writings, cf. Schwarz [135].

Let us recall in this respect that Lobachevsky was interested from his early works onwards in obtaining new formulae for definite integrals. His first published memoir, the *Elements of geometry* (1829) [95] already contains integrals and computations of volumes. In that memoir, Lobachevsky recovered some of the formulae that were obtained by Legendre in his *Exercices de calcul intégral*. We also recall that Lobachevsky's *Géométrie imaginaire* (1837) [99] is almost exclusively devoted to the computation of integrals. In that memoir, he obtained formulae for volumes of pyramids and of simplices in 3-space. The last section of the *Application of imaginary geometry to certain integrals* (1836) [98] contains a list of 50 integral formulae.

In addition to his interest in integral calculus, Lobachevsky's aim was to show that his new geometry is useful in a domain other than geometry, namely, analysis. Furthermore, although Lobachevsky did not have a proof in the sense we intend it today that hyperbolic geometry is consistent,[148] drawing as many consequences as possible of the new theory without meeting any contradiction was of course very reassuring.

Some of the integrals that Lobachevsky found are recorded in integral tables, see for instance the tables by Bierens de Haan, *Tables d'intégrales définies* [22], and Gradshteyn and Ryzhik's *Table of integrals, series, and products* [54].

Computations of areas and of volumes of figures in hyperbolic space were carried out, after Lobachevsky's work on the subject, by several authors. In 1852, Schläfli worked on volumes of spherical polyhedra on the $n$-dimensional sphere,[149] establishing a formula for the differential of the volume. There is a relation between Schläfli's formula and the formulae of Lobachevsky, of which Schläfli was not aware. (It seems that Schläfli was not altogether aware of Lobachevsky's work.)

We also note that W. E. Story, in his paper *On the non-Euclidean geometry* (1881) [142], computed areas of conics in the hyperbolic plane, and his results are expressed, as in Lobachevsky's work, as elliptic integrals.

An account of Lobachevsky's integrals obtained as volumes of bodies in hyperbolic 3-space is given in Coxeter's paper *The functions of Schläfli and Lobatschefsky* [33]. In that paper, Coxeter made the relation between the formulae of Schläfli and those of Lobachevsky. He also reported on an integral formula giving the volume of a birectangular non-Euclidean tetrahedron in terms of its dihedral angles. This formula is considered today as being of paramount importance. It was discovered by Lobachevsky about 100 years before Coxeter's paper was written, and it had remained almost unnoticed during that whole period. One may also add to this that Coxeter's revival of Lobachevsky's work did not get full attention until the work of Thurston in the 1970s. Lobachevsky's formulae for volume are treated in Chapter 7 of Thurston's Princeton Notes [143], and they are also reviewed in the appendix of Milnor's paper, *Hyperbolic geometry: The first 150 years* [110]. In that appendix, Milnor showed how one can deduce from Lobachevsky's work a formula for the volume of an ideal tetrahedron. Milnor wrote an expanded version of volume computation in hyperbolic

[148] and in fact, there also was no proof at that time that Euclidean geometry was consistent!

[149] Schläfli was among the first mathematicians who worked in dimensions $> 3$.

geometry, in a paper published in Milnor's *Collected papers* [111]. (The paper is only published there.) It is also worth noting that before Milnor, Laptev wrote, in 1954, a paper on volumes of pyramids, in which he described in modern terms ideas of Lobachevsky [89]. Laptev's paper was not mentioned by Milnor. The paper [151] by Vinberg contains a survey of Lobachevsky's computations of volumes of hyperbolic tetrahedra, mentioning the works of Schläfli, Coxeter, Laptev and Milnor. Vinberg's paper is written in the pure model-free tradition of Lobachevsky.

In connection with the computation of volumes of tetrahedra, it may also be interesting to recall that there is a letter dating from 1832 from Gauss to Wolfgang Bolyai, in which Gauss proposed, as a problem for Wolfgang's son János, to determine the volume of a tetrahedron,[150] cf. [52], p. 224. Supposedly, Gauss was unaware of Lobachevsky's calculations of volume, which were published two years before.[151]

Computing volumes of hyperbolic polyhedra is nowadays a very active field of research, especially in dimension three. This subject, inaugurated by Lobachevsky, is so vast that it would be unreasonable to try to report on its development in the present commentary.

To conclude this circle of ideas, we note that in the nineteenth century, computing elliptic integrals was fashionable in Russian mathematical circles, partly because of Euler's influence. Lobachevsky was the first mathematician to compute such integrals using the differential calculus in hyperbolic space.

Lobachevsky concluded the *Pangeometry* with some considerations that have a different character. They concern the fact that the world in which we live might be hyperbolic. Lobachevsky tried to check this fact by using data that were available to him on extremely large distances, namely, distances between stars.[152]

Using astronomical observations in order to test the physical validity of hyperbolic geometry (namely, that the angle sum in an interstellar triangle is strictly less than two right angles) has been a concern for Lobachevsky for a long period of time.

Already in his first written treatise (cf. the *Elements of geometry* (1829), Engel's translation [45], p. 23–24), Lobachevsky tried to use existing angle measurements of a triangle with two vertices at opposite positions of the earth with respect to the sun (that is, two positions that are six months apart) and with the third one at the star Sirius, to determine whether the angle sum in interstellar triangles is equal to two right angles. It

[150] Since in hyperbolic geometry the area of a triangle can be defined as the angular deficit of that triangle, it is a natural question to try to express the volume of a tetrahedron in terms of its dihedral angles. It turns out there is no straightforward formula for this, but there are formulae that make use of an integral function which is now called the *Lobachevsky function*. This function is closely connected to the Euler dilogarithm function. The known formulae for volumes of 3-dimensional tetrahedra involve the value the Lobachevsky function at the dihedral angles.

[151] Gauss's correspondence with W. Bolyai, published in Gauss's *Collected works* [52], Vol. VIII, is also reproduced in a paper by P. Stäckel and F. Engel [140].

[152] Lobachevsky was also the head of Kazan's University Observatory, a position he was appointed to in 1819, cf. [79]. We note by the way that Gauss, who was a co-discoverer of hyperbolic geometry, was also the director of the Göttingen Observatory, a position he held from 1807 until his death in 1855. Let us also mention that likewise, Nasir al-Din at-Tusi, who did valuable work on spherical trigonometry and on the parallel postulate, and who, for that reason, is considered as an important forerunner of non-Euclidean geometry, was the founder and the director of the observatory of Maragha (a city in contemporary Iran).

is known that Lobachevsky consulted for that matter the observations made at Kazan observatory, of which he was the director during several years. In the introduction to his *New elements of geometry with a complete theory of parallel lines*, Lobachevsky wrote the following (Halsted's translation):

> The futility of the efforts which have been made since Euclid's time during the lapse of two thousand years to perfect it awoke in me the suspicion that the ideas employed might not contain the truth sought to be demonstrated, and for whose verification, as with other natural laws, only experiments could serve, as, for example astronomic observations.

Lobachevsky's use of astronomical measurements is also mentioned in Milnor's paper [110].

After he made his computations, Lobachevsky's conclusion was that with a high degree of exactness, one cannot deduce the failure of Euclidean geometry, and thus, either the world in which we live is Euclidean, or the distances that we are capable of measuring are infinitesimal compared to the size of the world. Lobachevsky's measurements, in connection with his views on physics, are also reported on in an interesting paper by N. Daniels, *N. Lobachevsky: Some anticipation of later views on the relations between geometry and physics* [37]. Let us also note that it is told that Gauss also tried to measure angular deficit of large physical triangles, though he considered triangles that are smaller in size than those of Lobachevsky, namely, triangles whose vertices are at the summit of three mountains.[153] It may also be interesting to recall in this respect that Gauss, at some point, following Schweikart, used the term "astral geometry" to designate hyperbolic geometry.

Finally, one may remark that such measurements might have shown in principle that the sum of the values of a certain triple of angles is different from 180°, which

[153] Gauss's biographers do not all have the same attitude towards the story of these measurements. K. Reich, in [127] (p. 80 of the French translation) gives names of three mountains that Gauss used for such measurements: Hoher Hagen, Inselsberg and Brocken. J. Gray (personal communication) believes that Gauss saw very quickly that such measurements would be inconclusive. P. Dombrowski considers the fact that Gauss made terrestrial or astronomical measurements as a test for the validity of Euclidean geometry in the space that surrounds us as a "conjecture which cannot be proved with certainty", see [39], p. 131. W. K. Bühler considers the whole story about Gauss checking whether the three-dimensional space in which we live is Euclidean or non-Euclidean by measuring distances between mountains is a myth, see [28], p. 100. It may be useful to recall here that in any case, geodesy constituted a significant part of Gauss's activity. This is reported on by all his the biographers, see e.g. [43], [28] and [127]. Gauss was interested in measuring the earth's surface irregularities, and understanding their relation with the density of matter beneath this surface. He was also interested in measuring latitudes, and their relation with the gravitational field strength. We also mention that Gauss, who was a member of a commission for weights and measures, was personally involved in a huge project (1818–1832) of measuring precisely the area of the Kingdom of Hanover, and the method used was to cover this surface by a grid of triangles and to add the areas of these triangles. He invented a machine (called "heliotrope"), based on optical signals and movable mirrors, for measuring distances between points on the earth's surface, and he also invented a method for determining the angles of the triangles so obtained. Gauss worked personally on these measurements, and he published theoretical articles on the subject. It is not surprising that several theoretical scientists that were Gauss's contemporaries considered that he was losing valuable time in doing such measurements, see e.g. excerpts from his correspondence with F. Bessel quoted in [43], p. 120, and from the one with W. Olbers quoted in [39], p. 129.

would have implied that the physical space in which we live is not Euclidean, but that measuring the actual angle sum of the (hypothetical) non-Euclidean triangle that has the three given points as vertices requires the knowledge of the surface on which this triangle lies in three-space. It seems that this issue was not addressed neither by Gauss nor by Lobachevsky. This remark was made by Beltrami already in 1870 (see his letter to Hoüel dated 4 January 1870 [23], p. 123).

One may question *a posteriori* the seriousness of these computations made by Gauss and Lobachevsky. But the fact that these two mathematicians considered that the world in which we live might not be Euclidean is important, and the idea was new.

# 2. On hyperbolic geometry and its reception

The discovery of hyperbolic geometry may be considered as the most important geometrical discovery of the nineteenth century.[154] In this section, I recall a few facts about this event and its consequences, with a particular stress on the way hyperbolic geometry was received in the few years that followed Lobachevsky's death.

As is well known, the road to hyperbolic geometry was tortuous, and it involved a gigantic amount of energy spent on trying to prove an erroneous statement, namely, that Euclid's parallel postulate follows from the other axioms. Proving this false statement was the main concern of many first-rank geometers, for nearly twenty centuries. Some of these geometers published their "proofs" and then retracted them, and others did not retract their proofs. There are many known examples.

A case which is not so well known is that of Euler, who attempted two proofs of the parallel postulate, reported on by his devoted student and relative Nicolaus Fuss, who also was Euler's assistant and the permanent secretary at St. Petersburg's Academy of Sciences during many years.[155] The first of Euler's attempt is based on the assumption that an equidistant locus to a line is a union of two lines, and the second one is based on the existence of similar triangles. It is known that both assumptions are equivalent to Euclid's parallel postulate.

Let me also cite the case of Joseph-Louis Lagrange, who, by the end of his life, presented to the French Academy of Sciences a memoir in which he "proved" the parallel postulate, but who, at the last moment, interrupted the reading and withdrew the manuscript, saying: "Il faut que j'y songe encore" [I have to think further]. This story is related in Barbarin's *Géométrie non euclidienne* [11], p. 15, and in Bonola's *Non-Euclidean geometry* [26], p. 52, who quotes de Morgan [115]. The same event is told in more detail in Pont's *Aventure des parallèles* [121], pp. 231–234, where Lagrange's manuscript is also analysed.

Finally, let me mention an unpublished 250 page manuscript by Joseph Fourier, written between 1820 and 1827. This manuscript contains about twenty attempts to prove the parallel postulate. Pont made a thorough analysis of this manuscript in [121], pp. 531–586. Even though the attempts by Fourier were unsuccessful, it appears that Fourier defined a line $l$ to be parallel to another line $l'$ if $l$ separates the lines that intersect $B$ from those that do not intersect $l'$. Thus, Fourier used (at about the same period, and independently) the same definition of parallelism as Lobachevsky, Bolyai and Gauss. Furthermore, Fourier's manuscript contains the notion of horocycle; cf. [121], p. 541.

[154] Recall David Hilbert's sentence: "Let us look at the principles of analysis and geometry. The most suggestive and notable achievements of the last century in this field are, as it seems to me, the arithmetical formulation of the concept of the continuum in the works of Cauchy, Bolzano and Cantor, and the discovery of non-Euclidean geometry by Gauss, Bolyai, and Lobachevsky." [74]

[155] The manuscripts, handwritten by Fuss, are kept in the St. Petersburg Academy of Sciences, and they were disclosed in 1961. The titles of the two manuscripts are *Euleri doctrina parallelismi* and *Euleri elementa geometriae ex principio similitudinis deducta*. They are analysed in Pont's *Aventure des parallèles* [121], pp. 281–282. Pont quotes a paper by Y. A. Belyi, *On an elementary geometry handbook by L. Euler* [15].

It is also known that each of the three founders of non-Euclidean geometry, namely, Lobachevsky, Bolyai and Gauss, spent several years trying to prove the parallel postulate.

P. Stäckel and F. Engel, in their paper on Gauss and the Bolyais, [140], quoted a passage from Janós Bolyai's autobiography in which he stated that until the year 1820, he was seeking a proof of the parallel postulate. A. Châtelet, in [31], p. 136, claimed that J. Bolyai, a few years after his famous *Science of absolute space*[156] appeared in print, came back to working on a proof of Euclid's parallel postulate, and that for some time he even thought he had succeeded in finding one.

Regarding Gauss, we can quote a letter he sent to Wolfgang Bolyai in 1799:

> It is true that I have come upon much which by most people would be held to constitute a proof [of the parallel postulate]: but in my eyes it proves as good as *nothing*.[157]

G. B. Halsted, who published an English translation of Bolyai's *Science of absolute space*, wrote on page ix of the Preface to his translation (see [26]), and after citing excerpts from Gauss's correspondence:

> "From this letter we clearly see that in 1799 Gauss was still trying to prove that Euclid's is the only non-contradictory system of geometry, and that it is the system regnant in the external space of our physical experience. The first is false, the second can never be proven."

Rosenfeld concluded from Gauss's correspondence that Gauss, up to the year 1816, was still trying to prove the parallel postulate, and that it is only in 1817 that he started being convinced of the non-provability of this axiom (see [131], p. 215).

Now we come to Lobachevsky.

It seems that during his studies, and at least until the year 1820, Lobachevsky was trying to prove the parallel postulate (see Engel [45], p. 381).

A. V. Vasiliev published in 1909 a set of class-notes that were taken by a student at Kazan University, Mikhail Temnikov, at geometry lessons that Lobachevsky gave during the academic years 1815–1816 and 1816–1817. These notes contain "proofs" of the parallel postulate by Lobachevsky. In one of these proofs, Lobachevsky used a lemma which is true in Euclidean geometry but not in hyperbolic geometry, and which he thought he proved using the axioms of neutral geometry. The lemma says the following (see Figure 40): Given a line $AB$, if we take successive lines $AC$, $CD$, $DE$, $EF$ such that $AC \perp AB$, $CD \perp AC$, $ED \perp CD$ and $EF \perp ED$, and all of the successive lines turn in the same direction (say to the right, as in Figure 40), then the

[156] J. Bolyai's work was published in 1832 as a 28-page Appendix to his father's *Tentamen juventutem studiosam in elementa matheseos purae elementaris ac sublimioris methodo intuitiva evidentiaque huic propria introducendi, cum appendice triplici*, cf. [24]. Note that there is no general agreement on whether to use as short title for Bolyai's Appendix "Science of absolute space" or "Absolute science of space"; see the various translations in item [24] of the bibliography.

[157] Gauss's *Collected works*, Vol. VIII [52], pp. 159–160 [... aber was in meinen Augen so gut wie NICHTS beweist ...]. The English translation is borrowed from [131], p. 214.

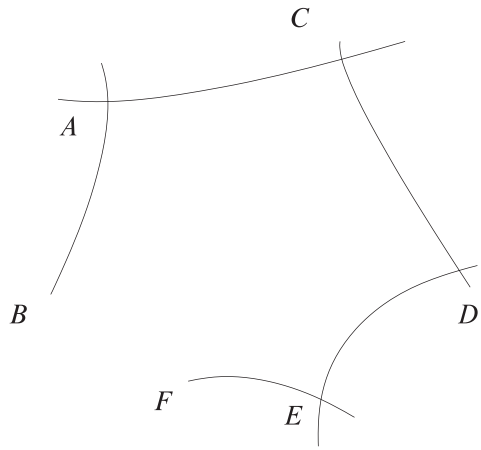


Figure 40. All the intersections in this picture are perpendicular. In the Euclidean plane, the line $EF$ necessarily cuts the line $AC$. In his early class-notes, Lobachevsky wrote (mistakenly) that in neutral geometry, the line $EF$ necessarily cuts either $AB$ or $AC$.

line $EF$ necessarily intersects either $AB$ or $AC$. Another lemma that Lobachevsky thought (mistakenly) was valid in neutral geometry is the following: Any line that cuts perpendicularly one of the sides of a 45° angle cuts the other side. Lobachevsky's proofs are reported on by Pont in [121], p. 252 ff. Lobachevsky realised in the year 1823 or before that all these proofs were unsuccessful, and he made in his book *Geometry* (written in 1823, but published posthumously) the remark that all the existing proofs of the parallel postulate are false.[158]

The date at which Lobachevsky stopped his attempts to prove the parallel postulate and started working on a new geometry in which this postulate is not satisfied, is an important date, but unfortunately it is only approximately known.[159] On p. 415 of [152] and on p. 472 of [153], Vucinich wrote the following: "B. L. Laptev, on the basis of new documentary material, shows that as early as 1822, Lobachevsky was convinced of the futility of all efforts to adduce proofs for the parallel postulate and that most probably by 1824 he was preoccupied with constructing a new geometry". Houzel, in [77], quoted Laptev [88] for saying that Lobachevsky gave in 1823 a course in geometry in which he declared (Houzel's translation from the Russian):

> Up to now, one has not succeeded to find a rigorous proof of that truth. Those which were given may be named only explanations, but do not deserve to be considered, in the full sense, mathematical proofs.

Houzel considers this as an indication of the fact that at that date Lobachevsky was

[158] quoted by Pont [121], p. 251.

[159] It is also conceivable that Lobachevsky went back and forth several times between the two convictions, namely, that the parallel postulate is a consequence of the other axioms of Euclid, and that it is not. Developing a theory where this postulate is denied was useful in both cases, either to look for a contradiction (in the first case) or to obtain a coherent theory where this axiom does not hold (in the second case).

still working on a proof of Euclid's parallel postulate. It is true that the text shows that Lobachevsky does not exclude such a proof. Laptev, in his 1976 book [91] affirms however that Lobachevsky already in 1817 abandoned all efforts to deduce the parallel postulate from the other postulates.

Stories like the above, about first-rank mathematicians, including the founders of hyperbolic geometry, writing "proofs" of the parallel postulate, can be amusing, but as always, there is an unfortunate effect on mathematics. Each time a mathematician of the stature of Legendre, whose *Geometry* was considered as *the* modern version of Euclid's *Elements*, announces that he had settled the parallel problem (Legendre included a proof of this fact in the third edition of his *Geometry* (1800)), then the natural short-term consequence is to freeze further research on the subject. Lobachevsky, in his *New elements*, before starting the exposition of his new theory, spent several pages explaining where the arguments of his predecessors (namely Legendre and Bertrand) failed. There is also a sad aspect in the story of non-Euclidean geometry. The sad aspect is that the first results, which laid the basis of the theory, took an incredibly long time to be acknowledged, and caused much more depression than joy for their authors. The three founders of the theory, namely, Gauss, Bolyai and Lobachevsky, never got any significant credit for the work they did on the subject until after their death (and they lived a relatively long period of time after they made their discoveries). Gauss, of course, was a formidable mathematician, and during his lifetime, he was generally considered as the greatest living mathematician. But non-Euclidean geometry did not contribute in any sense to Gauss's fame, and in his early biographies, his work on hyperbolic geometry is hardly mentioned. As a matter of fact, Gauss never sought credit for his work on hyperbolic geometry, and the reason (or, at least, the reason he gave in his correspondence with some of his friends) seems to be that he simply wanted to avoid controversy.[160]

Janós Bolyai practically stopped publishing mathematics just after his work on non-Euclidean geometry appeared in print, with the bitter feeling that Gauss had plagiarised his work. When he came across Lobachevsky's memoir published in German, Bolyai was convinced that Gauss, having stolen his ideas, published them under the false name of Lobachevsky. Bolyai left several thousands of pages of handwritten notes, which were disclosed and studied after his death. They contain results (which were new at the time he wrote them) on various subjects, including number theory, the axiomatisation of mathematics, and the use of complex numbers in geometry. Bolyai is now considered as a genius. But this recognition was posthumous.

Likewise, Lobachevsky had a difficult lifetime, and the full acknowledgement of his work came only several years after his death. His memoir *On the elements of geometry*, which he submitted for review to the St. Petersburg Academy of Sciences in

[160] J. Gray noted that P. Dombrowski has suggested that it might be that Gauss never published his ideas on hyperbolic geometry because he did not get the foundations of the subject to his satisfaction. In this respect, Gray remarks that Gauss barely published his work on elliptic functions, which was in much better shape. We refer to Gray's article [57] for a discussion of Gauss's role in the discovery of hyperbolic geometry. It is also conceivable that Gauss, after reading Bolyai's *Appendix*, considered that the problem was completely settled and therefore he did not really pursue research in the field.

1832, received a negative report, written by M. V. Ostrogradsky, an influential member of the Academy. When he read Lobachevsky's memoir, Ostrogradsky, who was a specialist in integral calculus, ignored its geometric content, and he based his opinion on a computation made by Lobachevsky of two definite integrals. In Rosenfeld's *History of non-Euclidean geometry* [131], p. 208 (quoting Kagan), we find the following translation of Ostrogradsky's review, that is extracted from the St. Petersburg Academy Archives:

> "Having pointed out that of the two definite integrals Mr Lobachevsky claims to have computed by means of his new method one is already known and the other is false, Mr Ostrogradsky notes that, in addition, the work has been carried out with so little care that most of it is incomprehensible. He therefore is of the opinion that the paper of Mr Lobachevsky does not merit the attention of the Academy."

Vucinich, in [152], p. 314, wrote that beyond that report, Ostrogradsky continued his harsh criticism of Lobachevsky's work until his death.

After the negative report written by Ostrogradsy, Lobachevsky's work was published in a local journal, the *Kazan Messenger* (Kazansky Vestnik) [95], and it remained unnoticed for years.

E. B. Vinberg wrote in his Note on Lobachevsky [150] that the geometric works of Lobachevsky were derided and were regarded as an eccentricity of a respected professor and university rector.

Rosenfeld reported, in [131], p. 219, on one single Russian mathematician, P. I. Kotelnikov, professor at Kazan University, who recognised the value of Lobachevsky's work, and who praised him during his lifetime. Rosenfeld and his translator cited as follows an excerpt from an address, entitled "On bias against mathematics", delivered by Kotelnikov at Kazan University in 1842:

> In this connection I cannot pass over in silence that the futile millennial attempts to prove with all mathematical rigour one of the fundamental theorems of geometry, to the effect that the angle sum in a rectilinear triangle is equal to two right angles, inspired Mr Lobachevsky, a revered and meritorious professor of our university, to undertake the prestigious task of building a whole science, a geometry based on the new assumption that the angle sum in a triangle is less than two right angles – a task that is bound to gain recognition sooner or later.

Kotelnikov's case of support for Lobachevsky is really the "exception that proves the rule".

Lobachevsky continued nevertheless working and publishing on geometry. According to Kagan, it is probable that a very heavy teaching load and his complete involvement in administrative tasks prevented Lobachevsky from falling into profound mental depression, as did J. Bolyai and F. A. Taurinus for the reason that their work on non-Euclidean geometry was not acknowledged.

After the hostility, disdain, and mockery with which his geometrical writings were received in Russia, Lobachevsky decided to write memoirs in French and in German, and he sent his manuscripts to Western European journals, seeking for recognition in the West. Strangely enough, these works remained unnoticed for several years following their publication. There was however one exception: Gauss came across Lobachevsky's paper published in German, and he was so much impressed that he decided to read his works written in Russian.[161] Of course, this exception is a major one, because of Gauss's fame, but Lobachevsky was not aware of Gauss's interest, and Gauss, in his writings, never mentioned the name of Lobachevsky, except in private correspondence. Lobachevsky died with the feeling that his geometrical works were completely unacknowledged.

Let us now review some of the developments that occurred after the decade that followed Lobachevsky's death.

The publication, in the period 1860–1865, of Gauss's correspondence with Schumacher, was certainly the first major factor that drew the attention of the mathematical world to the work of Lobachevsky, or at least to his name.

In 1865, A. Cayley published a *Note on Lobatchewsky's imaginary geometry* [30], in which he reported on Lobachevsky's trigonometric formulae for hyperbolic geometry, making a comparison between these formulae and the trigonometric formulae of spherical geometry. Although Cayley failed to understand Lobachevsky's main ideas (see Rosenfeld [131], p. 220, and Yaglom [155], p. 168), the note helped to make some of Lobachevsky's work known to the mathematical community.

In 1872, V. V. Bunyakovsky (1804–1889), a member of the St. Petersburg Academy of Sciences, wrote a paper in French, in the Memoirs of that Academy [29], in which he claimed to establish a contradiction between Lobachevsky geometry and the visual notion of space. As Rosenfeld puts it in [131], p. 210, although Bunyakovsky's paper contains a critique of Lobachevsky's work, the difference between his report and the previous reports, written by the academician Ostrogradsky and his students, is that Bunyakovsky spoke respectfully of Lobachevsky's talents.

[161] Cf. Gauss's letter to J. F. Encke, February 1841, published in his *Collected works*, [52], Vol. VIII, p. 232: "I am making reasonable progress in learning to read Russian and this gives me a great deal of pleasure. Mr Knorre sent me a small memoir of Lobachevsky (in Kazan), written in Russian, and this memoir, as well as his small German book on parallel lines (an absurd note about it has appeared in Gersdorff's *Repertorium*) have awakened in me the desire to find out more about this clever mathematician." The translation is from the English version of Rosenfeld's *History of non-Euclidean geometry*. The original text says: [Ich fange an, das Russische mit einiger Fertigkeit zu lesen, und finde dabei viel Vergnügen. Hr. Knorre hat mir eine kleine in russischer Sprache geschriebene Abhandlung von Lobatschewsky (in Kasan) geschickt und dadurch so wie durch eine kleine Schrift in deutscher Sprache über Parallellinien (wovon eine höchst alberne Anzeige in Gersdorfs Repertotium steht) bin ich recht begierig geworden, mehr von diesem scharfsinnigen Mathematiker zu lesen]. Vucinich, in his *Science in Russian culture* ([152], p. 309) mentioned that Gauss learned Russian by reading Bunyakovsky's *Foundations of mathematical probability*, "an extremely well written work". G. Waldo Dunnington [42] claimed that the reason why Gauss learned Russian was his desire to read Lobachevsky's work on non-Euclidean geometry in the original. Bühler [28], p. 155, and Rosenfeld [131], p. 219, made the same assertion. Polotovsky wrote in [120] that this is a myth. In any case, Yaglom noted that "Gauss had outstanding linguistic abilities and the learning of a foreign language often served as a pastime for him" ([155], p. 167).

Full recognition of Lobachevsky's work and credit for his discovery of non-Euclidean geometry were given in G. Battaglini's paper *Sulla geometria immaginaria di Lobatschewsky* [14] (1867), in E. Beltrami's famous paper *Saggio di interpretazione della geometria non-Euclidea* [16] (1868), and in F. Klein's *Über die sogenannte Nicht-Euklidische Geometrie* [82] (1871). The paper by Beltrami is usually referred to for establishing the relative consistency of non-Euclidean geometry with respect to the Euclidean one. It was translated into French by Houël, and published in the same year in the *Annales Scientifiques de l'École Normale Supérieure*. Klein had learned about Lobachevsky's work at a seminar conducted by Weierstrass, at the University of Berlin, in 1870.

Beltrami, in the introduction of his *Saggio di interpretazione della geometria non-Euclidea*, explaining the goal of his paper, wrote the following:

> We tried, to the limit of our capabilities, to make for ourselves an idea of the results to which the doctrine of Lobachevsky leads; and following a method which seems to us completely conformal to the good traditions of scientific investigation, we tried to provide that doctrine with a firm basis, before we admit for that theory the necessity of a new order of entities and of concepts. We believe that we succeeded in our goal with respect to the planar part of this doctrine, but we believe that this goal is impossible to attain for what concerns the rest.[162]

The French translation of Beltrami's paper by Houël (a translation that was approved by Beltrami)[163] is more precise, for what concerns the last sentence quoted. Translated into English, the sentence says the following:

> We believe that we succeeded in our goal for what concerns the planar part, but it seems to us that it is impossible to attain this goal in the case of dimension three.[164]

The fact that Beltrami was dubious about Lobachevsky's three-dimensional geometry posed in particular the question of the validity of Lobachevsky's proofs of his trigonometric formulae (which Beltrami knew were the same as the trigonometric

[162] [Abbiamo cercato, per quanto le nostre forze lo consentivano, di dar ragione a noi stessi dei risultati a cui conduce la dottrina di Lobatschewsky; e, seguendo un processo che ci sembra in tutto conforme alle buoni tradizioni della ricerca scientifica, abbiamo tentato di trovare un substrato reale a quella dottrina, prima di ammettere per essa la necessità di un nuovo ordine di enti e di concetti. Crediamo d'aver raggiunto questo intento per la parte planimetrica di quella dottrina, ma crediamo impossibile di raggiungerlo in quanto al resto.] Stillwell's translation of the first sentence reads (cf. [141], p. 7): "In this spirit we have sought, to the extent of our ability, to convince ourselves of the results of Lobachevsky's doctrine; then, following the tradition of scientific research, we have tried to find a real substrate for this doctrine, rather than admit the necessity for a new order of entities and concepts." Stillwell's translation of the second part of the sentence seems more logical, but I did not follow it because it is not faithful to the original in this particular case.

[163] Beltrami and Houël exchanged several letters concerning this translation, see [23].

[164] [Nous croyons y avoir réussi pour la partie planimétrique ; mais il nous semble impossible d'y parvenir dans le cas de trois dimensions.]

formulae of the pseudo-sphere), since these proofs involved constructions in the three-dimensional hyperbolic space. In this respect, Beltrami made the following remark, in his letter to Hoüel dated 18 November 1868 (pp. 65–66 of [23]):

> … and even more the reflection (which never left me) that the pseudo-spherical trigonometry was deduced by Lobachevsky from his stereometric theorems and was nevertheless really correct, led me to postpone the publication of my researches until all my difficulties were resolved.[165]

In the following year, Beltrami published another paper on the same question, the *Teoria fondamentale degli spazii di curvatura costante* [17], in which he settled the question regarding the validity of Lobachevsky's three-dimensional geometry.

One can follow the fascinating evolution of Beltrami's ideas during that period in reading his letters to Hoüel, and the invaluable notes accompanying them in [23].[166]

Beltrami wrote in the *Teoria fondamentale degli spazii di curvatura costante*:

> … One can check that Lobachevsky's theory coincides, except for nomenclature, with the geometry of 3-dimensional space of negative constant curvature.[167]

Beltrami declared at several occasions that his pseudo-sphere model gave him such a satisfying and convincing picture of the existence of Lobachevsky's planar geometry that (curiously enough) it made him doubt the existence of Lobachevsky's three-dimensional geometry. For instance, Beltrami wrote, in the conclusion of his *Saggio di interpretazione della geometria non-Euclidea* [16] (p. 26 of Stillwell's translation):

> The preceding seems to confirm at every point the interpretation of noneuclidean planimetry by means of surfaces of constant negative curvature. The nature of this interpretation is such that there cannot be an analogous, equally real, interpretation of noneuclidean stereometry.[168]

We also quote in this respect a passage from a letter written by Beltrami to Hoüel, dated 18 November 1868:

> That paper (the one called *Saggio etc.*) has been written during last year's fall, after reflections that made life difficult for me at the time of the publication of your translation of Lobachevsky.[169]

[165] [... et encore plus la réflexion (qui ne m'avait jamais quitté) que la trigonométrie pseudosphérique était déduite par Lobatschewsky des théorèmes de sa stéréométrie, et se trouvait cependant bien exacte me portèrent à différer la publication de mes recherches jusqu'à la solution de toutes mes difficultés.]

[166] The book [23] contains 65 letters sent by Beltrami to Hoüel, between the years 1868 and 1881. Among these letters, 17 were written in the important year 1869.

[167] [Si puo verificare che la teoria di Lobatschewsky coincide, salvo nei nomi, colla geometria dello spazio a tre dimensioni di curvatura costante negativa.]

[168] [Da quanto precede ci sembra confermata in ogni parte l'annunciata interpretazione della planimetria non-euclidea per mezzo della superficie di curvatura costante negativa. La natura stessa di questa interpretazione lascia facilmente prevedere che non ne può esistere una analoga, egualmente reale, per la stereometria non-euclidea.]

[169] Beltrami refers here to Hoüel's translation of the *Geometrische Untersuchungen zur Theorie der Parallellinien*, cf. [100].

> I think that at that time, the idea of constructing non-Euclidean geometry on a perfectly real surface was very new. However, I will confess that, dragged somehow too far by the complete explanation that my surface gave of non-Euclidean planimetry, I was first tempted to question the stereometric researches of Lobatschewsky, and to see in them only a kind of "geometric hallucination" (it was the way I qualified them; of course, in my mind) which I was hoping to reduce, after some more mature thoughts, to their true meaning. In other words, I thought at that time that the *construction of non-Euclidean planimetry on a surface I call pseudo-spherical would exhaust the range of this transcendental planimetry*, in almost the same way as the ordinary construction of complex variables on a plane entirely exhausts all their arithmetic significance.[170]

We remark incidentally that at the time of this discussion led by Beltrami, only 2- and 3-dimensional (Euclidean or non-Euclidean) geometries were conceived. Higher-dimensional spaces were still not considered.

B. Riemann, in his 1854 *Habilitationsvortrag*, *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen* (On the hypotheses that lie on the foundations of geometry), in which he dwell upon the three kinds of constant curvature geometries on surfaces, never mentioned the name of Lobachevsky, despite the fact that his mentor, Gauss, knew about Lobachevsky's work. This can appear as a surprise, especially if one notices that Riemann mentioned Legendre and his unsuccessful attempts to prove the Euclidean parallel postulate, as an example of what Riemann considered as the "darkness" by which geometry was covered since the times of Euclid. But the situation becomes somehow clearer if one recalls that the identification of Lobachevsky's non-Euclidean plane with a surface of negative constant curvature was done only later on, namely, in the works of Beltrami [16], [17] and of Klein [82]. One must also note in this respect that Riemann was not interested in deriving a geometry in the fashion of Lobachevsky (that is, in an axiomatic way, following Euclid).[171] Riemann's *Habilita-*

[170][Le dit écrit (celui qui a pour titre "Interprétation etc.") a été rédigé en l'automne de l'année passée, d'après des réflexions qui avaient pollué en moi à l'époque de la publication de votre traduction de Lobatschewsky. Je crois qu'alors l'idée de construire la géométrie non-euclidienne sur une surface parfaitement réelle était tout-à-fait nouvelle. Je vous avouerai cependant que, entraîné un peu trop loin par l'explication complète en tous points que ma surface donnait de la planimétrie non-euclidienne, je fus tenté d'abord de ne pas croire à la partie stéréométrique des recherches de Lobatschewsky et de n'y voir qu'une espèce de "hallucination géométrique" (c'est ainsi que je la qualifierais; dans ma pensée, bien entendu) que j'espérais pouvoir, après des réflexions plus mûres, réduire à sa vraie signification. En d'autres termes, je croyais alors que la *construction de la planimétrie non-euclidienne sur la surface que j'appelle pseudo-sphérique épuisait entièrement la portée de cette planimétrie transcendentale*, à peu près comme la construction ordinaire des variables complexes sur un plan épuise toute entière leur signification arithmétique.] (cf. [23], p. 65–66.)

[171]According to J. Gray (personal communication), the fact that Riemann was so surprised by Gauss's choice of the topic for his Habilitation examination indicates that it is likely Gauss and Riemann never talked about the foundations of geometry. As a matter of fact, it seems that Gauss and Riemann almost never talked at all. In this respect, Yaglom reported in [155], p. 168 that "Gauss was not a good teacher and rarely appreciated the merits of his pupils. [...] Riemann took Gauss's course on the method of least squares, but apparently they had no personal contacts." Of course, this does not contradict the fact that Riemann was highly influenced by Gauss's work on the differential geometry of surfaces.

*tionsvortrag* was published in 1867, the year following Riemann's death [129], which was one year before Beltrami published his *Saggio* and his *Teoria fondamentale* [16], [17].[172]

As part of the story, we also mention Beltrami's paper *Sulla superficie di rotazione che serve di tipo alle superficie pseudosferiche* (On the rotation surface that serves as a model for pseudo-spherical surfaces) [18] in which the author studied the pseudo-sphere, the surface that served as a model for Lobachevsky's geometry. The pseudo-sphere is a surface of revolution in Euclidean 3-space, whose meridian is a tractrix. The word "interpretation" in the title of Beltrami's paper *Saggio di interpretazione della geometria non-Euclidea* (Essay on the interpretation of non-Euclidean geometry) refers to the pseudo-sphere as being a model of Lobachevsky's geometry in the sense that the geodesics on the pseudo-sphere, in the sense of differential geometry, have the same behaviour as that of the straight lines in the hyperbolic plane. This discovery was a very important step in the history of non-Euclidean geometry, since it provided the first concrete example of a surface embedded in Euclidean three-space that is locally isometric to the hyperbolic plane. But some problems remained of course, because the pseudo-sphere is not a faithful representation of the hyperbolic plane, since this surface is not simply connected (it is homeomorphic to a cylinder). Moreover, the universal cover of the pseudo-sphere is a horodisk (a surface bounded by a horocycle) in the hyperbolic plane, and a horodisk is not isometric to the hyperbolic plane, since it is not complete. Beltrami was aware of these facts. The question of the existence of a surface in Euclidean space whose intrinsic geometry is that of the complete hyperbolic plane remained open for several years. It was eventually solved in 1901, that is, after Beltrami's death, by David Hilbert, who showed that there does not exist any analytic non-singular surface embedded in Euclidean three-space whose intrinsic geometry is that of the hyperbolic plane.[173]

Regarding the later development of non-Euclidean geometry, it is worth mentioning another work that probably has not received the attention it deserves. This is the work of Joseph-Marie de Tilly (1837–1906), a member of the Royal Belgian Academy

[172]Hoüel translated Riemann's *Habilitationsvortrag* into French, but no French journal agreed to publish it. Concerning this question, Beltrami wrote in a letter to Hoüel dated 4 December 1868 (cf. [23], p. 68–69): "I regret the bad mood of French journals towards the translation of Riemann's posthumous Memoir, and I cannot resign myself to consider that this decision is final. If I am not mistaken, there is, in that work, amid many obscure passages, some true light shafts, and I do not know whether some of the most important concepts that we encounter there have ever been presented in such a clear way". [Quant à la traduction du Mémoire posthume de Riemann, je regrette la mauvaise disposition des journaux français, et je ne puis me résigner à la considérer comme définitive. Si je ne me trompe, il y a dans ce travail, au milieu de nombreuses obscurités, des véritables traits de lumière, et je ne sais si quelques unes des conceptions les plus importantes qu'on y rencontre, ont été jamais présentées d'une façon aussi lucide.] Hoüel's translation was eventually published in the Italian Journal *Annali di Matematica* cf. [129], at the insistance of L. Cremona, who was the journal's editor (see Beltrami's letter to Hoüel dated 8 January 1869, [23], p. 72).

[173]The analyticity condition was soon after shown to be unnecessary by G. Lütkemeyer, in his Göttingen thesis *Über den analytischen Charakter der Integrale von partiellen Differentialgleichungen* (1902). As a matter of fact, there is no $C^2$ differentiable embedding of the hyperbolic plane in $\mathbb{R}^3$. N. Kuiper showed in 1955 that there exists a $C^1$-embedding of the hyperbolic plane in $\mathbb{R}^3$. We also mention that there exist $C^\infty$ embeddings of the hyperbolic plane in $\mathbb{R}^5$, cf. D. Blanuša (1955). Jeremy Gray communicated to me that the existence of such an embedding is much older, cf. [114].

of Science, who was an officer in the Belgian army and who taught mathematics at the Military School. In the 1860s, de Tilly, who was not aware of the work of Lobachevsky, developed independently a geometry in which Euclid's parallel postulate does not hold. He introduced the notion of distance as a primary notion in the three geometries: hyperbolic, Euclidean and spherical. He developed an axiomatic approach to these geometries based on metric notions, and he highlighted some metric relations between finite sets of points; see for instance his *Essai sur les principes fondamentaux de la géométrie et de la mécanique* [144] and his *Essai de géométrie analytique générale* [145].

It is also important to note that after the first period of fancy for hyperbolic geometry, several of the earlier attempts to prove Euclid's postulate (e.g. those of Wallis, Saccheri, Lambert and Legendre) began to be regarded as containing valuable mathematical results. Several of these approaches were by contradiction, that is, their authors searched for a contradiction in the consequences of the negation of Euclid's parallel postulate, and it turned out later on that some of the consequences that were obtained were among the important results proved by Lobachevsky, Bolyai and Gauss. It is in this sense that Wallis, Saccheri, Lambert, Legendre and others are considered as the predecessors of hyperbolic geometry. In this respect, we recall that Beltrami wrote in 1889 a paper, *Un precursore italiano di Legendre e di Lobatschewski* [19], that rehabilitated Saccheri's *Euclides ab omni naevo vindicatus*, giving excerpts from that book that contained various theorems which had been previously attributed to Legendre, Lobachevsky and Bolyai.[174] Beltrami's paper was soon followed by translations of Saccheri's works into English by Halsted (published in instalments in the *American Mathematical Monthly*, starting from 1894), into German by Stäckel and Engel (1895) [139], and into Italian by Boccardini (1904) [134]. In 1887, Gino Loria published a paper entitled *Il passato e il presente delle principali teorie geometriche* (The past and present of the main geometric theories) [106], which was translated into German by F. Schutte and published in 1888. Loria eventually expanded his article into a 476 page book with the same title, in which he reviewed the development of geometry from antiquity until the first years of the twentieth century. Of major importance are also the compilations by Stäckel and Engel, *Die Theorie der Parallellinien von Euklid bis auf Gauss* [139] and by Bonola, *Geometria non-euclidea, esposizione storico-critica del suo sviluppo* [26].

Despite all these developments, non-Euclidean geometry continued to be a controversial subject until the end of the nineteenth century. Its main ideas were not understood by a large number of mathematicians, and the bases of the theory were put into doubt by several others. For instance, it was difficult, even for many geometers, to conceive the fact that the area of a triangle is uniformly bounded, that there exist triangles whose angle sum is arbitrarily small, that there are disjoint straight lines with no common perpendicular, that there exist triples of points that are not on the same circle, and so on. The common mathematician regarded hyperbolic geometry as a mere

[174] Saccheri's book was published in 1773, the year of Saccheri's death, and it was soon forgotten, until Angelo Manganotti, who, like Saccheri, was a Jesuit priest, found by accident a copy of that book and communicated it to Beltrami.

logical curiosity, but there were also harsh opponents of the subject. Let me give a few examples.

J. Barbarin, in his paper *La correspondance entre Hoüel et de Tilly* [13], pp. 50–61, reported that around the year 1870, the French Academy of Sciences was truly flooded by attempts to prove the Euclidean postulate, and that these attempts were so numerous that the Academy appointed a special commission, called "Commission des parallèles", to deal with this matter. In 1869, a public conflict took place at the Academy, between supporters and detractors of non-Euclidean geometry. After that conflict started, all the notes on the subject that were submitted to the Academy were systematically returned to their authors [154].[175]

In that conflict, the detractors of non-Euclidean geometry included Joseph Bertrand, who was a professor of Physics and of Mathematics at Collège de France and who was a member of the Academy of Sciences since 1856,[176] and Charles Dupin, who was a member of the Academy since 1818.[177] The conflict became an open one when Bertrand and Dupin claimed that a proof of the parallel postulate by Jules Carton, presented at the Academy, was correct, cf. [3].[178] Joseph Liouville and Irénée-Jules Bienaymé participated in the battle, supporting the fact that it is not possible to give a proof of the parallel postulate.[179] Some episodes of that conflict are related by Pont, [121], pp. 652–660.

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[175]It seems that the suspension of discussions on the parallel postulate at the Academy of Sciences was first asked by the supporters of a proof of the parallel postulate, see Beltrami's letter to Hoüel dated 23 December 1869 [23], p. 110. We recall by the way that there have been other instances of subjects whose discussion was banned from the French Academy. For instance, discussions of papers related to squaring the circle or on the possibility of perpetual motion were prohibited, being considered as useless and time-consuming. As a matter of fact, discussions about squaring the circle were already banned in 1775, that is, long before the proof of impossibility of such a construction, due to the large amount of noise that was made around this problem. We recall that the solution of the problem was obtained in 1882 by Ferdinand von Lindemann, as a corollary of his proof that $\pi$ is transcendental. It is also worth noting in this respect that the first major step in the proof that $\pi$ is transcendental was obtained by Lambert, who is mentioned several times in this book, and who is often considered as the precursor of hyperbolic geometry who came closest to its discovery. Lambert gave the first proof, in 1761, of the fact that $\pi$ is irrational, and he conjectured that $\pi$ is transcendental.

[176]Joseph Bertrand (1822–1900) became the permanent secretary of the Academy of Sciences for mathematics in 1874, and he was elected at the *Académie Française* in 1884.

[177]Charles Dupin (1784–1873) was also a famous politician.

[178]Besides his note to the Academy of Sciences, Carton wrote a book on the subject, entitled *Vrais principes de la géométrie euclidienne et preuves de l'impossibilité de la géométrie non euclidienne*, cf. [4]. Carton's proof was refuted by Hoüel in his *Note sur l'impossibilité de démontrer par une construction plane le principe de la théorie des parallèles dit Postulatum d'Euclide*, cf. [75]. Eventually, it turned out that this proof by Carton of the parallel postulate was a forgery. In a letter to Hoüel, dated 6 January 1870, Beltrami reported that Carton's proof was copied from a "proof" by C. Minarelli, that had appeared 20 years before, in the *Nouvelles Annales de Mathématiques*, No. 8, pp. 312–314, and that Minarelli's "proof" had already been refuted several times (see [23], p. 121).

[179]Regarding Liouville's implication in the conflict, let me note an amusing remark by Beltrami, in his letter to Hoüel dated 30 December 1869 ([23], p. 111): "I am very happy that a geometer with the force and the (very rightly deserved) authority of Liouville shares the ideas that we consider as correct. Nevertheless, from my point of view, a little bit egoistic for what concerns truth, I would have preferred somebody else instead; because I know that Mr Liouville has a very determined antipathy for Italy and the Italians, and this will probably prevent him from considering with care anything that comes from that side." (and Beltrami developed that remark in the rest of the letter). [Je suis bien content qu'un géomètre de la force et de l'autorité (bien justement acquise) de M. Liouville partage les idées que nous croyons exactes. Seulement, à mon point

During that conflict, Beltrami was hesitant about the opportunity of publishing in France the translations made by Hoüel of his two fundamental papers *Saggio di interpretazione della geometria non-Euclidea* [16] and *Teoria fondamentale degli spazii di curvatura costante* [17]. In a letter to Hoüel dated on 23 December 1869 ([23], p. 109–11), Beltrami wrote, commenting an article that appeared in the same year in the magazine *Cosmos: Revue encyclopédique hebdomadaire des progrès des sciences et de leurs applications*, relating the events that took place at the Academy of Sciences that accompanied the publication of J. Carton's note:

> Nevertheless, I am happy that this fact highlighted again that subject in France. At the same time, I must make for you a further declaration: that is, if you think that the insertion of the translation of the two Memoirs in the École Normale Supérieure Collection (a collection which is somehow official) might hurt the beliefs of some high-ranking persons in science and in teaching, or be a source of annoyance, or alter your good relations with eminent scientists that were your old masters, I will completely free you from your promise, no matter how much it is dear to me, and I will advise you to postpone the thing to a later moment, when the debate will be pacified and until we get right to the bottom of the matter. Or else I would ask you to confide the translations that you kindly prepared to an independent Science journal.[180]

While the Academicians were discussing proofs of the parallel postulate, most of them had not heard about Lobachevsky's work. In January 1875, in a letter to Hoüel, mentioned in [121], p. 636, Darboux wrote:

> I know nothing about Pangeometry, and nobody in Paris knows about it. I am perfectly sure, at least among the members of the Institute, nobody has ever heard about it.[181]

Several years passed before the conflict ended. In his book *La science et l'hypothèse* Poincaré mentioned that by the time in which the book was written (1902), the French Academy received "only" one or two proofs of the parallel postulate every year.

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de vue, un peu égoïste de la vérite, j'aurais préféré que ce fût un autre; car je sais que M. Liouville a une antipathie très-décidée pour l'Italie et pour les Italiens, et cela l'empêchera probablement de faire un peu attention à ce qui vient de ce côté.]

[180]Quoi qu'il en soit, je suis content que ce fait ait rappelé en France l'attention sur le sujet en question. En même temps je dois vous faire une ample déclaration : c'est que s'il vous paraissait que l'insertion de la traduction de mes deux Mémoires dans le Recueil de l'École Normale Supérieure (Recueil officiel en quelque sorte) pût froisser les convictions de quelques personnes haut-placées dans la science et dans l'enseignement, donner lieu à des désapprobations, ou altérer vos bons rapports avec des savants éminents qui ont été vos maîtres d'autrefois, je vous dégagerais complètement de votre promesse, quelque chère qu'elle fût pour moi, et je vous conseillerais de remettre la chose à plus tard, quand le débat serait calmé et lumière faite. Ou bien je vous prierais de confier les traductions que vous avez bien voulu préparer à quelque journal indépendant de Science.]

[181][Je ne connais pas la Pangéométrie et personne ne la connaît à Paris. J'en suis parfaitement sûr, au moins parmi les membres de l'Institut on n'en a jamais entendu parler.]

Of course, the fact that there exists a geometry which is not the Euclidean one opened the way to the assumption that the geometry of the space that surrounds us might not be Euclidean. This dispute between supporters and detractors of non-Euclidean geometry had also ramifications in the physical sciences. Ethical and philosophical arguments were also used in this debate. Morals were shaken by the appearance of non-Euclidean geometry, because some rules that had been considered as irrefutable for centuries (they were mathematical rules!), were said to be obsolete.

From the philosophical point of view, the opponents of non-Euclidean geometry included Rudolf Hermann Lotze and Gottlob Frege. We refer to the book by Gray [55] (in particular p. 190 and 328) for a short account of this subject and in reference to these two philosophers.

The mathematician Jules Andrade recounted in a paper published in 1900 in l'*Enseignement Mathématique* [6] that at an important meeting (on which he does not give further details), he heard the following affirmation:

> Ethics is concerned about the proof of Euclid's postulate, because if certainty deserts even mathematics, then, alas, what could happen with moral values ![182]

Let us also quote a passage from Dostoyevsky's novel *The brothers Karamazov* (written in 1880) in which the existence of non-Euclidean geometry raises theological problems:

> (Ivan talking to his brother Alyosha, [40], p. 218) I tell you that I accept God simply. But you must note this: If God exists and if He really did create the world, then, as we all know, He created it according to the geometry of Euclid and the human mind with the conception of only three dimensions in space. Yet there have been and still are geometricians and philosophers, and even some of the most distinguished, who doubt whether the whole universe, or to speak more widely the whole of being, was only created on Euclid's geometry; they even dare to dream that two parallel lines, which according to Euclid can never meet on earth, can meet somewhere at infinity. I have come to the conclusion that, since I can't understand even that, I can't expect to understand about God. I acknowledge humbly that I have no faculty for settling such questions, I have a Euclidean, earthly mind, and how could I solve problems that are not of this world?[183]

When did the conflict between supporters and opponents of non-Euclidean geometry end? The answer is not completely clear. In his 1895 biographical note on Lobachevsky [62], G. B. Halsted wrote the following:

[182][Il y a quelques années, dans une réunion que je ne préciserai pas davantage, j'ai entendu, formulée gravement, cette opinion "que la morale elle-même est intéressée à la démonstration du postulatum d'Euclide; car, disait-on, si la certitude déserte même les mathématiques, que deviendront, hélas, les vérités morales!]

[183]Jeremy Gray quotes this too in his *Worlds out of nothing* [55], and he thinks that the reference to parallel lines meeting at infinity surely suggests that projective geometry is at issue (personal communication). Of course, Dostoyevsky's doubts also concern the existence of a geometry in dimensions higher than three.

> [non-Euclidean geometry's] day of probation is safely passed, and one might better square the circle and invent perpetual motion than make the slightest objection on non-Euclidean geometry.

This claim has to be taken with care, because at the time where Halsted wrote it, there were still fierce opponents to hyperbolic geometry. Let me give a few examples.

In the first five volumes of the *American Mathematical Monthly* (starting in 1894), Halsted published in instalments a paper entitled *Non-Euclidean geometry: Historical and expository*. The paper consists essentially of his translation of Saccheri's *Euclides ab omni naevo vindicatus* [134]. The same issues contain several articles whose aim was to point out contradictions in non-Euclidean geometry. For instance, in Volume 1 of that journal (1894), in a short note entitled *Easy corollaries in non-Euclidean geometry* [61], Halsted wrote that "in Lobatschewsky's geometry, there may be a triangle whose angle-sum differs from a straight angle by less than any small given angle however small". In a later issue of the same year (pp. 247–248), J. N. Lyle wrote a note which starts with Halsted's sentence as a hypothesis, and then he argued for a dozen of lines, and concluded with the following words: "Since the conclusion is absurd, the hypothesis from which it is deduced must be absurd." The same author, in another article of the same volume (pp. 426–427) wrote that "Lobatschewsky in his *Theory of parallels* argues from false premises with such plausibility and subtle sophistry as to allure many into accepting both premises and conclusions as sound geometry."

It would be easy to give other excerpts of this kind, that is, excerpts from papers written by opponents of non-Euclidean geometry that were published in serious mathematical journals. Such a situation lasted until the end of the first decade of the twentieth century.

Michel Frolov, a mathematician who was also a general in the French army, and the author of a book entitled *La théorie des parallèles démontrée rigoureusement, Essai sur le livre I des éléments d'Euclide* (Basel, 1893; a second edition was published in 1899), wrote several articles against non-Euclidean geometers in *L'Enseignement Mathématique*. I will quote now a long excerpt from one of these articles, because it gives a quite good idea of how some mathematicians regarded hyperbolic geometry, at the beginning of the twentieth century.

> Non-Euclidean geometry, created by Gauss and his collaborators, and which, according to the famous Mr J. Bertrand, had no seriously convinced disciple thirty years ago, is very fashionable today. Among its supporters, we can find members of Sciences Academies, university professors and school teachers. It is used for the integration of differential equations; it is even hoped that it will be used in the solution of the three-body problem. Hundreds of writings are published by its followers, with the aim of propagating this doctrine, whose most characteristic feature is a most complete confusion between straight and curved lines.[184] As a consequence of this

[184]Frolov's remark is justified in some sense, but this is the fate of hyperbolic geometry; when we draw pictures of disjoint hyperbolic lines on a (Euclidean) piece of paper, we draw them as curved lines, for otherwise, these lines would appear as either equidistant or intersecting.

confusion, the notion of straight line is completely lost, and without such a firm notion, the study of Geometry becomes unrealistic. It is like studying music while having no ear. That is not all: Some people invoke the Geometry of beings without width and of inhabitants of empty spheres.[185] They present doctoral theses on that doctrine. They distribute special prizes and honourable mentions for books written on that doctrine.[186] Finally, some people attribute to it a *large philosophical significance, because according to its followers, by showing the inanity of Kant's ideas on space, this doctrine has basically ruined the metaphysics of criticism* (a sentence quoted from P. Mansion's, *Métagéométrie*, Mathesis, October 1896, p. 41).

There are very worrying symptoms that make us fear that this doctrine will soon acquire a place in the teaching curricula, all the more so since it has already penetrated a few treatises on geometry. It is not sufficient to affirm that the neo-geometers have no other goal than to train themselves on the analysis of various hypotheses. It seems that hypotheses should never be admitted in mathematics, and that it is urgent that specialised journals devoted to the teaching of mathematics address the important issue that can be summarised in the following question: Is non-Euclidean geometry true or false?

[...] Non-Euclidean geometry is a doctrine that is only hypothetical. It is based on the negation of Euclid's Axiom XI, or postulatum, which was rightly considered, for more than twenty centuries, as an evident truth, confirmed by all the physical facts.

This hypothesis, which nothing justifies, immediately led to a chain of paradoxes, which appeared to its inventors as containing no contradiction. They thought they had discovered a marvellous doctrine, which would change the face of mathematics and which would throw upon it abundant light.

Its partisans multiplied the proofs that establish that Euclid's postulatum is unprovable, and it seems that on the contrary, the professional mathematicians did nothing to defend the Geometry of Euclid and Archimedes against the invasion of this subversive doctrine that reverses all the geometrical facts of the physical world.[187]

[185]Reference is made here to a famous passage by Poincaré in his article *Les géométries non euclidiennes* [117], repeated in his *La science et l'hypothèse* [118].

[186]Reference is probably made to the Lobachevsky Prize and to its three accompanying "honourable mentions" that were awarded in 1897.

[187][La géométrie non euclidienne, créée par Gauss et ses collaborateurs Lobatschevsky et Bolyaï, et qui, selon l'illustre M. J. Bertrand n'a eu, il y a trente ans, aucun disciple sérieusement convaincu, est très en vogue de nos jours. Elle compte, parmi ses partisans, des membres des académies des sciences et des professeurs des universités et des collèges. On s'en sert pour intégrer des équations différentielles ; on espère même résoudre le problème des trois corps, en l'envisageant dans son domaine. Des centaines d'écrits sont publiés par ses adeptes, dans le but de développer et de propager cette doctrine, dont le trait le plus caractéristique est la plus grande confusion des lignes droites et des lignes courbes, de sorte qu'on y perd facilement toute

The article from which the quotes are extracted and translated was published in Volume 2 (1900) of *L'Enseignement Mathématique* [51]. We note that the newly founded journal (1899) had a prestigious list of sponsors that included Appell, Cremona, Klein, Mittag-Leffler, Picard and Poincaré. There are other papers by the same author, in the same journal, in which he presented what he considered as contradictions in non-Euclidean geometry. Responses to these writings by other mathematicians were published during the first decade of the twentieth century, in *L'Enseignement Mathématique* and in other journals. In connection with these articles, in Volume 4 (1902) of *L'Enseignement Mathématique*, p. 330, there is the following Note from the Editorial board of the journal:

> We recall to our readers that l'*Enseignement Mathématique* is at the disposal of the Euclideans and the non-Euclideans. Without taking sides in one sense or another, this Journal is a platform which is open to all mathematicians.[188]

Frolov gave the following interesting example in his support to the claim he made, that confusion reigns in non-Euclidean geometry. The example is based on a letter from Gauss to Schumacher, dated 12 July 1831,[189] in which Gauss claims that if we start with an equilateral triangle of centre $O$ in the hyperbolic plane, and if we make the three vertices of that triangle tend simultaneously to points $a$, $b$ and $c$ at infinity while staying on the angle bisectors, we obtain, as a limiting figure, an infinite tripod, formed by the rays $Oa$, $Ob$, $Oc$ (see Figure 41). Frolov wrote on that example:

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notion de ligne droite, sans laquelle l'étude de la Géométrie devient illusoire. Autant vaudrait étudier la musique sans avoir d'oreille. Ce n'est pas tout : on invoque la Géométrie des êtres sans épaisseur et de ceux qui habitent des sphères creuses. On présente des thèses de doctorat sur cette doctrine. On distribue des prix d'encouragement et des mentions honorables pour des ouvrages qui contribuent à son perfectionnement. Enfin, on lui attribue une *grande portée philosophique, car selon ses adeptes, en montrant l'inanité des idées de Kant sur l'espace, elle a ruiné, par la base, la métaphysique du criticisme* (phrase citée de M. Mansion, *Métagéométrie*, Mathesis, Octobre 1896, p. 41).

Ce sont des symptômes très inquiétants, qui font craindre que cette doctrine ne tardera pas à conquérir une place dans l'enseignement, d'autant plus qu'elle a déjà pénétré dans quelques traités de géométrie. On a beau affirmer que les néogéomètres n'ont d'autre objet que de s'exercer à des analyses mathématiques sur des hypothèses variées. Il semble que les hypothèses ne devraient jamais être admises en mathématiques et qu'il est urgent que les revues spéciales, consacrées à l'enseignement mathématique, s'occupent de la question qui se résume en une évidente alternative : la Géométrie non euclidienne est-elle vraie ou fausse ?

La Géométrie non euclidienne est une doctrine toute hypothétique, reposant sur la négation de l'axiome XI ou du postulatum d'Euclide, qui fut, avec raison, considéré pendant plus de vingt siècles comme une vérité évidente, confirmée par tous les faits physiques.

Cette hypothèse, que rien ne justifie, conduisit immédiatement à un enchainement continu de paradoxes qui a paru à ses inventeurs ne contenir en lui aucune contradiction. On a cru avoir découvert une doctrine merveilleuse, appelée à changer la face de la Mathématique et à verser sur elle des flots abondants de lumière. Ses partisans s'empressèrent aussitôt de la rendre inattaquable en multipliant les preuves pour établir que le postulatum d'Euclide est indémontrable. Au contraire, les mathématiciens de profession paraissent n'avoir rien fait pour défendre la Géométrie d'Euclide et d'Archimède de l'invasion de cette doctrine subversive renversant tous les faits géométriques du monde physique.]

[188] [Nous rappelons à nos lecteurs que l'*Enseignement Mathématique* est à la disposition des euclidiens et des non-euclidiens. Sans prendre de parti dans un sens ou dans un autre cette Revue est une tribune ouverte à tous les mathématiciens.]

[189] *Collected works* [52], Vol. VIII, p. 216.

> Gauss's adepts, who were more daring, if not more clever than him, explained that the master was mistaken, and that the equilateral triangle does not, at the limit, become equal to its angle bisectors, but to the union of straight lines $QR$, $ST$, $UV$, which are pairwise asymptotic. This correction was necessary, because it would have been strange that the edges of the triangle break by themselves. But such an explanation assumes an occult force which suddenly separates the edges of the triangle from each other, because even if one can easily conceive lines that do not intersect, it is difficult to understand what could prohibit us from connecting the three points $a$, $b$, $c$, taken on the angle bisectors of the triangle as far as wished from the centre $O$. Gauss's conception appears to be clearer than the one of his disciples. But instead of trying to making clear this useless confusion, one would better acknowledge that enlarging the triangle does not change the value of its angles, and that the angle sum remains constant, which amounts to rejecting the hypothesis and accepting Euclid's postulatum. [190]

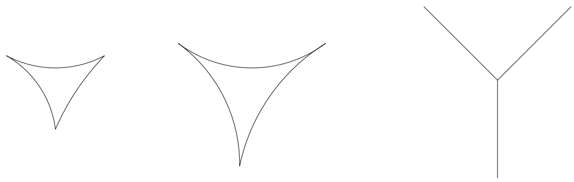

Figure 41. The figure is drawn after a figure contained in Gauss's letter to Schumacher. It shows a tripod which, according to Gauss, appears as the limit of hyperbolic triangles.

We must admit that strictly speaking, Frolov is right in pointing out what can be considered as a mistake by Gauss, since the limit of the triangle, in the most natural sense that can be given to the word limit, is not the tripod, but a figure that is called today an "ideal triangle", viz. a triangle whose three vertices are at infinity and whose angles are all equal to zero. By the way, a modern geometer will notice that Gauss's intuition fits well in the context of *coarse geometry*, that was developed during the last two decades of the twentieth-century by M. Gromov. We also note that Lobachevsky, in the *Pangeometry*, made use of triangles with zero angles.

Frolov then expounded what he considered to be contradictions in Lobachevsky's writings, and after these and other similar attacks on non-Euclidean geometry, Frolov

[190] [Ce sont ses adeptes, plus hardis sinon plus judicieux que lui, qui ont expliqué que le maître s'était trompé et que le triangle équilatéral se transforme, à la limite, non pas en ses bissectrices, mais en trois droites $QR$, $ST$, $UV$, asymptotes réciproques deux à deux. Cette correction devenait nécessaire, car il serait étrange que les côtés du triangle rectiligne se brisent d'eux-mêmes. Mais une telle explication suppose une force occulte qui détache brusquement les côtés du triangle l'un de l'autre, car s'il est facile de concevoir des droites qui ne se rencontrent pas, il est difficile de comprendre ce qui empêcherait de joindre entre eux les points $a$, $b$, $c$ pris sur les bissectrices du triangle équilatéral, aussi loin que l'on veut du centre $O$. La conception de Gauss semble plus claire que celle de ses disciples. Mais, au lieu de débrouiller cette confusion inutile, il vaut mieux reconnaître qu'avec l'agrandissement du triangle ses angles ne varient pas et que leur somme reste constante, ce qui revient à rejeter l'hypothèse et à accepter le postulatum d'Euclide.]

concluded his paper with the following words:

> We have the right to wish that this paradoxical and contradictory doctrine will not be introduced in teaching, where it can distort the students' intelligence.[191]

The mathematician C. Vidal concluded as follows a paper that also appeared in *L'Enseignement Mathématique* (1902) [147]:

> From all the preceding discussions, it seems to us that it should follow at least the impression that the non-Euclidean doctrine is not as firm as people tend to say. And it is possible that soon will come the day in which the non-Euclideans will no more dare feigning, regarding their adversaries, a self-confidence and a spirit of conciliation which are always accompanied by some irony. Besides, it is understandable that some geometers, having been cheated by ingenious sophisms, and being thoroughly engaged in the non-Euclidean adventure, find it difficult to put aside their errors. But whether they wish it or not, we think that they will be defeated by logic.[192]

In 1908, Klein wrote in his *Elementary mathematics from an advanced viewpoint*[193] ([85], p. 177 of the English translation of Part II (Geometry)):

> Today, every person of general culture has heard of the existence of a non-Euclidean geometry, even though a clear understanding of it can be attained only by experts.[194]

On page 185 of the same book, Klein wrote:

> *Every teacher certainly should know something of non-Euclidean geometry*. Thus, it forms one of the few parts of mathematics which, at least in scattered catch-words, is talked about in wide circles, so that any teacher may be asked about it at any moment.[195]

At the same time, there were still papers against non-Euclidean geometry, published in mathematical journals. In a survey on the parallel axiom, written in the 1913 issue of the *Mathematical Gazette* [50], W. B. Frankland wrote:

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[191] [On a le droit de souhaiter que cette doctine paradoxale et contradictoire ne soit pas introduite dans l'enseignement, où elle pourrait fausser l'intelligence des élèves.]

[192] [De toutes les discussions qui précèdent doit se dégager, ce nous semble, au moins cette impression que la doctrine non-euclidienne n'est pas aussi ferme qu'on veut bien le dire. Et peut-être le jour n'est-il pas éloigné où les non-euclidiens n'oseront plus affecter, vis-à-vis de leurs adversaires, une assurance et un esprit de conciliation qui ne vont point sans quelque ironie. On comprend, d'ailleurs, que certains géomètres, trompés par d'ingénieux sophismes et engagés à fond dans l'aventure non-euclidienne, aient quelque peine à revenir de leur erreur. Mais, qu'ils le veillent ou non, ils seront, croyons-nous, vaincus par la logique.]

[193] Klein wrote that book for school teachers.

[194] [heute hat wohl sogar jeder allgemein Gebildete schon einmal etwas von der Existenz einer nichteuklidischen Geometrie gehört, wenn auch ein klares Verständnis für sie doch nur der Fachmann erreichen kann.]

[195] [*Jeder Lehrer muß notwendig etwas von der nichteuklidischen Geometrie kennen;* denn sie gehört nun einmal zu den wenigen Teilen der Mathematik, die zum mindesten in einzelnen Schlagworten in weiteren Kreisen bekannt geworden sind; nach ihr kann daher jeder Lehrer jeden Moment gefragt werden.]

> I want to come now to the latter half of the eighteenth-century, when Bertrand of Geneva proved the parallel axiom finally and completely.[196]

and, further, in the same paper,

> This real demonstration, as it seems to me, after more than ten years of reflection [...] excludes Lobachevsky's hypothesis.

All this shows that even though hyperbolic geometry is for us today not only a geometry which is as natural as Euclidean geometry, but also a geometry that offers the most active and the most rewarding subjects for research, its birth was probably the most painful one of any mathematical subject, and its acceptance was incredibly prolonged.

Let me add that there is something paradoxical in this story. On the one hand, the absolute reign of Euclidean geometry ended with the discoveries of Lobachevsky, Bolyai and Gauss. But on the other hand, these three authors showed that Euclid was right in treating the parallel postulate as a postulate and not as a theorem, and in this sense their works led to a rehabilitation of a statement made by Euclid, that had been controversial for 2000 years.

[196]The author refers to the Swiss mathematician Louis Bertrand (1731–1812), a disciple of Euler, who also published a "proof" of the parallel postulate. Bertrand's proof has been analysed by several mathematicians, including Lobachevsky, Kagan, Buniakovski, and Gérard in his thesis. Gauss also discussed this proof in his correspondence with Schumacher. (See the detailed account on Bertrand's work by Pont in [121], pp. 303–330). Louis Bertrand is not to be confused with the French mathematician Joseph Bertrand (1822–1900), who is mentioned elsewhere in this book, and who was a fierce opponent of non-Euclidean geometry.

# 3. On models, and on model-free hyperbolic geometry

Lobachevsky's *Pangeometry* is a foundational essay on abstract, postulational hyperbolic geometry, that is, hyperbolic geometry without models. Nowadays, students and researchers in hyperbolic geometry work in models, and it is safe to say that a large proportion of these students never realise that hyperbolic geometry can be studied without models, simply because they never did a computation or worked out an exercise without using a model. In the following lines I would like to say a few words on models versus model-free hyperbolic geometry whose study, in my opinion, deserves more attention than it actually has received.

We already recalled that the first model of hyperbolic geometry was produced by Beltrami in 1868, in his *Saggio di interpretazione della geometria non-Euclidea* [16], that is, thirteen years after Lobachevsky's death. In this paper, Beltrami described the "pseudo-spherical model" of the hyperbolic plane, a non-simply connected surface of constant negative curvature embedded in Euclidean 3-space. Furthermore, by making a special choice of coordinates, Beltrami found a representation of the hyperbolic plane as the open unit ball in the Euclidean plane, in which the Euclidean chords play the role of hyperbolic lines, and where the projective transformations of the plane that preserve the ball play the role of the hyperbolic isometries. This model was later on called the Klein–Beltrami model, because Klein added to it some extra structure, that gave an interpretation within non-Euclidean geometry to the points in the Euclidean plane that lie outside the unit ball.[197] This model made hyperbolic geometry, at a very early stage, subject to projective geometry: non-Euclidean geometry happens on a subset of projective space and its isometry group is a subgroup of the projective group.[198] Perpendicularity in the hyperbolic plane can be expressed in terms of polarity in projective geometry, and so on. We can quote here F. Klein, who already in 1871,

[197] It is true that Beltrami's pseudo-sphere had already been described by Gauss and Minding, as a surface of negative constant curvature (cf. [52], Vol. VIII, p. 265, and [112]) and that a surface of constant negative curvature which is identical to the Poincaré disk model is implicitly contained in Riemann's *Habilitationsvortrag* (mentioned by J. Gray in [55], p. 192). But neither Gauss nor Minding did state a relation between these surfaces and the parallel postulate or the non-Euclidean geometry discovered by Lobachevsky. Beltrami was the first to do that, although Minding came close to that conclusion.

[198] Klein's *Lectures on non-Euclidean geometry* [84] despite their name, are almost entirely dedicated to projective geometry. Klein included the study of the three geometries (Euclidean, hyperbolic and spherical) in the context of projective geometry in much the same way as Beltrami included them in the context of differential geometry (in the sense inaugurated by Gauss). As a matter of fact, Klein, after Cayley, included *all* geometry in the context of projective geometry. We quote Klein, from his *Elementary mathematics from an advanced viewpoint* ([85], p. 134 of the English translation of Part II (Geometry)): "This whole manner of viewing the subject was given an important turn by the great English geometer A. Cayley in 1859. Whereas, up to this time, it had seemed that affine and projective geometry were pooper sections of projective geometry, Cayley made it possible, on the contrary, *to look upon affine geometry as well as metric geometry as special cases of projective geometry. "Projective geometry is all geometry"*." [Diese ganze Art der Betrachtung hat nun durch den großen englischen Geometer A. Cayley 1859 eine wichtige Wendung erfahren: während es bisher schien, als ob die affine und projektive Geometrie ärmere Ausschnitte der metrischen seien, macht es Cayley möglich, umgekehrt *die affine sowohl als die metrische Geometrie der projektiven als besondere Fälle einzuordnen: "projektive geometry is all geometry"*.]

considered, following a similar idea of Cayley concerning Euclidean geometry, that hyperbolic geometry should be taught as a chapter in projective geometry. In his *Über die sogenannte Nicht-Euklidische Geometrie* [82], Klein wrote the following:

> We shall not further pursue the philosophical speculations [of Gauss, Lobachevsky and Bolyai] which led to the works in question; rather, it is our purpose to *present the mathematical results of these works, insofar as they relate to the theory of parallels, in a new and intuitive way, and to provide a clear general understanding.*
>
> The route to this goal is through projective geometry. By the results of Cayley, one may construct a projective measure on ordinary space using an arbitrary second degree surface as the so-called fundamental surface. Depending on the second degree surface used, this measure will be a model for the various theories of parallels in the above-mentioned works. But it is not just a model for them; as we shall see, it precisely captures their inner nature. (Stillwell's translation).[199]

Since the appearance of Thurston's Notes on hyperbolic geometry in 1976 [143], which put the subject at the forefront of all research in the geometry and topology of low-dimensional manifolds, several good textbooks have been written on that subject. Most of these books treat hyperbolic geometry in models, mainly in the Poincaré ball and upper half-space models, in the Klein–Beltrami projective model, and sometimes in the Minkowski model.

The three founders of hyperbolic geometry, namely, Bolyai, Lobachevsky and Gauss, made their computations without models. This means that the proofs of the results they obtained use qualitative arguments, in purely geometric terms, rather than arguments from linear algebra, as it is done in most of the modern textbooks on hyperbolic geometry, where the subject is worked out in models.

Models are useful, in several ways. First of all, as we already noted, the existence of models was used to prove the consistency of hyperbolic geometry. Indeed, the Beltrami model (as do the Poincaré and the other models), shows the consistency of hyperbolic geometry relative to Euclidean geometry. Roughly speaking, having a model of hyperbolic space sitting inside Euclidean space, in which the primary notions of hyperbolic geometry are Euclidean notions, shows that if the axioms of Euclidean geometry are consistent, then those of hyperbolic geometry are also consistent.[200] Klein proved that projective geometry is independent of the parallel axiom. In considering the

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[199] [Sie sollen indes nicht etwa die philosophischen Spekulationen weiterverfolgen, welche zu den genannten Arbeiten hingeleitet haben, vielmehr ist ihr Zweck, *die mathematischen Resultate dieser Arbeiten, soweit sie sich auf Parallelentheorie beziehen, in einer neuen anschaulichen Weise darzulegen und einem allgemeinen deutlichen Verständnisse zugänglich zu machen.* Der Weg hierzu führt durch die projectivische Geometrie. Man kann nämlich, nach dem Vorgange von Cayley, eine projektivische Maßbestimmung im Raume konstruieren, welche eine beliebig anzunehmende Fläche zweiten Grades als sogenannte fundamentale Fläche benutzt. Je nach der Art der von ihr benutzten Fläche zweiten Grades ist nun diese Maßbestimmung ein Bild für die verschiedenen in den vorgenannten Arbeiten aufgestellten Parallellentheorien. Aber sie ist nicht nur ein Bild für dieselben, sie deckt geradezu, wie sich zeigen wird, deren inneres Wesen auf.]

[200] We also recall that Lobachevsky knew that the consistency of plane Euclidean geometry would follow from the consistency of hyperbolic geometry, when he realised the Euclidean plane as a horosphere in hyperbolic 3-space.

three geometries (Euclidean, hyperbolic and spherical) as part of projective geometry, Klein reduced the problem of the consistency of these three geometries to that of projective geometry. Hilbert was the first to raise the consistency problem in precise terms, and he reduced the consistency of the axiom set of Euclidean geometry (and of hyperbolic and spherical geometries) to that of the real number system.[201] To conclude this circle of ideas, we recall that Gödel, in 1931, showed that the consistency of the real number system cannot be proved. To be a little more precise, we also recall that Gödel showed that arithmetic with multiplication is not provably consistent. Without entering into the precise meanings of these statements, I am mentioning them here in order to highlight the fact that the consistency question was a long and difficult project, and therefore it is not surprising that Lobachevsky, who constantly worried about the consistency of his geometry, did not succeed in proving it.

Another aspect of models is that working in them turns out to be efficient for doing computations. Indeed, it is relatively easy to write formulae for distances, areas and volumes of some simple objects using the Poincaré models, and it is rather straightforward to write proofs of the trigonometric formulae of hyperbolic geometry in these models (especially if the formulae are known in advance).

Models of hyperbolic geometry also have applications in other fields. For instance, the Poincaré models have applications in number theory, in complex analysis and in the theory of differential equations; the Beltrami–Klein model served as a basis of the definition of the Hilbert metric on an arbitrary convex body; the Minkowski model has applications in physics, and so on.

However, I personally find that from an aesthetic and from a more conceptual viewpoint, working with models can be frustrating. In the Poincaré models, in which the hyperbolic lines are the Euclidean circles orthogonal to the boundary, and where angles coincide with Euclidean angles, the results on hyperbolic geometry are usually proved using the underlying Euclidean geometry. This may be reassuring in some sense, but it can be intellectually unsatisfactory, once we know that all the computations can be done without reference to any model. Furthermore, all Euclidean models of hyperbolic space suffer a lack of symmetry; they either distort angles or distances. On the other hand, models also usually have some superfluous symmetry. For instance, the Poincaré disk model has a centre, and this centre plays no role from the hyperbolic viewpoint. In conclusion, all the usual models fail to give a satisfying vision of the homogeneity and the isotropy of hyperbolic space. Finally, some effort is needed to realise that a result obtained in a model follows from the axioms of hyperbolic geometry alone, and not from the extra structure contained in the model. For instance, it is safe to say that

[201] To get an idea of the relation between the Euclidean plane and the real number system, recall that in the Euclidean plane, a point is described by a pair $(x, y)$ of real numbers, and that a line is described by a class of equivalent equations of the form $ax + by + c = 0$ where $a, b, c$ are real numbers. The notions of segment, angle, congruence, alignment, betweenness, and so on, can be defined using these analytic forms. From this, one can see that there is also a three- (and a multi-) dimensional analogue. The definitions and the results of Euclidean geometry can be formulated in terms of definitions and statements concerning real numbers, and it is not difficult to imagine that the consistency of Euclidean geometry can be deduced from that of the real number system. Note also that the real number system can be defined using the integer system.

most students of hyperbolic geometry are unaware of the fact that the fine structure of the boundary at infinity, which is usually treated in the Poincaré model, equipped with actions of isometry groups of hyperbolic space (I am thinking here about notions of tangential convergence, etc.) is independent from the choice of the model, and that there is no need of any model in order to formulate and study these questions.

Let me also mention that one advantage of the model-free point of view is that several arguments in hyperbolic geometry that are done in this setting can be transported with very little changes to the realm of spherical geometry. For instance, there are proofs of the hyperbolic trigonometric formulae that are based on monotonicity properties of edge lengths of trirectangular quadrilaterals, properties that follow from the fact that the angle sum in a hyperbolic triangle is less than $2\pi$. (See e.g. [1], [2] and [53].) These arguments can easily be transported to the setting of spherical geometry, again using monotonicity properties in trirectangular quadrilaterals, based this time on the fact that the angle sum in a spherical triangle is greater than $2\pi$. The proofs of the hyperbolic formulae that are done in the Poincaré models cannot be transported to the realm of spherical geometry.

It is also important to recall in this respect that some of Lobachevsky's arguments make use of Euclidean geometry (and also of spherical geometry), but in an intrinsic manner (as opposed to the extrinsic manner, that is involved in the model point of view). This holds in particular in the *Pangeometry* and in Lobachevsky's previous memoirs, where Lobachevsky, in his proof of the hyperbolic trigonometric formulae, associated to a triangle in the hyperbolic plane a spherical triangle sitting on a geometric sphere centred at a vertex, and a Euclidean triangle sitting on a horosphere. (We recall that for this, it is necessary to consider the hyperbolic plane as embedded in hyperbolic 3-space.) In the same memoirs, Lobachevsky proved that hyperbolic geometry becomes Euclidean at the infinitesimal level, and he recalled this fact whenever he found an occasion for doing so, by showing that the first-order approximations of the formulae that he obtained give the usual formulae of Euclidean geometry. Lobachevsky did not need any Euclidean model for doing so. It is important to make this point clear, because it may seem, to a reader of a modern textbook on hyperbolic geometry, that the proof of the fact that hyperbolic geometry becomes Euclidean at the infinitesimal level uses some Euclidean space underlying a model.

For several years, I taught hyperbolic geometry using models, and like every other teacher of the subject, I noted, insisted on, and abundantly used the fact that in the Poincaré models, hyperbolic angles coincide with Euclidean angles, that hyperbolic lines are Euclidean circles, that horocycles are also Euclidean circles, that hypercycles are again Euclidean circles, and so on. All this is irrelevant for hyperbolic geometry. After having read some of the original texts, I became convinced that the treatment and the teaching of hyperbolic geometry from a synthetic and model-free point of view is more interesting and intellectually more satisfying, despite the fact that it is not always the quickest way to obtain results. Of course, one should not fall in the opposite trap of giving a course based on axioms, where geometry becomes an abstract exercise in logic. Lobachevsky, in his *Pangeometry*, worked in coordinates, and he developed a differential and integral calculus, in a model-free way. The reference to axioms is only made at the beginning of the memoir. Geometry is in large part a process of visual thinking, and Lobachevsky's *Pangeometry* is a wonderful example of how to do it, without the help of any Euclidean model.

# 4. A short list of references

In this section, I give a list of references on non-Euclidean geometry and on its history, with a few comments on the references.

The list is obviously incomplete in every sense of the word, and a complete list would be unreasonably long. Already in 1878, G. B. Halsted compiled a bibliography of non-Euclidean geometry that includes 174 titles [60]. This bibliography was updated and inserted in Lobachevsky's 1883 *Collected geometric works* edition [103], and it included there 272 titles. In 1887, P. Riccardi published a bibliography of more than 1000 entries of works related to Euclid's parallel postulate, written between 1482 and 1890 [128]. The bibliography of the volume *Die Theorie der Parallellinien von Euklid bis auf Gauss* (The theory of parallel lines, from Euclid to Gauss), edited by P. Stäckel and F. Engel (1895), includes about 300 items on the theory of parallel lines published between 1482 and 1837. D. M. Y. Sommerville published in 1911 a list of more than 4000 items, with an author and a subject index, the items being chronologically ordered and covering the period from the beginning of the work on the theory of parallels (IVth century B.C.) until the year 1911.

In any case, a long bibliography would probably be redundant, given the electronic tools that are available today. Thus, I restricted myself to a short list of references that I personally like. As a matter of fact, there are two lists, one on the history of the subject, and the other one on the techniques. For the latter, I chose only references on model-free hyperbolic geometry, in order to be consistent with the rest of the book.

**A. Works on the history**

For the history, and the historical documents, I recommend the following:

(a) R. Bonola, *Non-Euclidean geometry: A critical and historical study of its development*, 1912, reprinted by Dover, 1955 [26]. This book contains excerpts from foundational texts, comments on their developments and several historical notes and technical remarks. The book was originally written in Italian (1906), and the Italian edition reproduces, with some additions, a chapter that appeared as part of a book published in Bologna in 1900 and edited by F. Enriques, *Questioni riguardandi la geometria elementare*. The 1906 book edition was published after its author's death, with an introduction written by Enriques, who was a friend of Bonola. This book, together with the one of Pont [121] referenced below, constitute in my opinion the best references on the history of the work done in relation with the parallel postulate.

(b) P. Stäckel and F. Engel, *Die Theorie der Parallellinien von Euklid bis auf Gauss, eine Urkundensammlung zur Vorgeschichte der Nicht-Euklidischen Geometrie* (Theory of parallel lines, from Euclid to Gauss, a collection of sources for the prehistory of non-Euclidean geometry) (1895) [139]. This book contains important excerpts from works done during the period 1482–1837 by Wallis, Saccheri, Lambert, Gauss, Schweikart and Taurinus. The editors, Stäckel and Engel, also published (separately) other historico-critical editions of works on non-Euclidean geometry. In particular, Engel published a translation of Lobachevsky's *Elements of geometry* (1829) and of his *New elements of geometry with a complete theory of parallel lines* [45]. Stäckel published a critical edition of the correspondence between Gauss and Wolfgang and János Bolyai [140] as well as works on the lives and works of the two Bolyais, see [25].

(c) Vol. VIII of C. F. Gauss's *Werke* [52] contains Gauss's notes and correspondence on non-Euclidean geometry. We recall that Gauss did not publish anything on the subject, and that these notes and letters were published posthumously.

(d) B. A. Rosenfeld, *A history of non-Euclidean geometry*, translated from the Russian by A. Shenitzer [131]. This is an extremely interesting book.

(e) M. J. Greenberg, *Euclidean and non-Euclidean geometries* [58]. This is a geometry textbook that follows a historical approach.

(f) J.-C. Pont, *L'aventure des parallèles / Histoire de la géométrie non euclidienne: Précurseurs et attardés* (1986) [121]. This is an invaluable reference, for the documents it contains and the analysis that is made there of most of the known attempts to prove the parallel postulate, from Greek antiquity until the early twentieth century.

(g) R. Rashed's translations and editions of works of the Arabic predecessors of non-Euclidean geometry. Of particular interest are Rashed's works on Nasir al-Din at-Tusi, Thāibn Qurra, Ibn al-Haytham and Omar Khayyam, cf. for instance [123] and the impressive 5 volume edition *Les mathématiques infinitésimales du IXème au XIème siècle* [124]. We note by the way that a Latin translation of Nasir al-Din at-Tusi's commentary on Euclid, as well as his "proof" of the parallel postulate, are contained in John Wallis's *Collected works* (Volume 2), Oxford 1693. Saccheri knew this proof, and he quoted it in his *Euclides ab omni naevo vindicatus* [134]. At-Tusi's proof might have been written by one of his students. See the comments by Rosenfeld and Youschkevitch in [132], p. 469. The paper by R. Rashed and C. Houzel, *Thābit ibn Qurra et la théorie des parallèles* [125] is easily available, and it contains a translation and a commentary on Thābit ibn Qurra's work on the parallel postulate. Some of the Arabs' work on the parallel postulate is also reviewed in Heath's edition of Euclid's *Elements*. Russian translations and commentaries of several Arabic texts on the parallel postulate were published in Russia by B. A. Rosenfeld and A. P. Youschkevitch, in the second half of the twentieth century. This work was further highlighted in the last quarter of the twentieth century, due in large part to the work of Rashed in France.

(h) K. Jaouiche, *La théorie des parallèles en pays d'Islam. Contribution à la préhistoire des géométries non euclidiennes* [78]. This book contains translations and critical editions of several attempts of proofs and of equivalent forms of the parallel postulate, done by Arabic and medieval Islamic authors.

(i) C. Houzel, *The birth of non-Euclidean geometry* [77]. This article is an excellent concise introduction to the subject. In particular, Houzel analyses the reasons that led Gauss, Bolyai and Lobachevsky (and certainly others), at about the same time (the first third of the nineteenth century), to switch from the effort to prove Euclid's parallel axiom, to the one of developing a new geometry in which this axiom is not satisfied.

(j) Jeremy Gray's books and papers on Gauss, on Bolyai and more generally on the history of geometry. His book *Worlds out of nothing. A course in the history of geometry in the 19th century* [55] is most suitable for a course on the history of non-Euclidean geometry; see also [56].

(k) Finally, I would like to mention the important edition of the *Letters from Beltrami to Hoüel* [23]. Reading these letters, with the introduction, comments and notes written by L. Boi, L. Giacardi and R. Tazzioli, gives a very good idea of the development of non-Euclidean geometry during the period that started about ten years after Lobachesvky's death. One of the interesting aspects of these letters is that at several occasions, Beltrami answered there some mathematical questions that Hoüel asked him, concerning his two epoch-making papers *Saggio di interpretazione della geometria non-Euclidea* [16] and *Teoria fondamentale degli spazii di curvatura costante* [17], which Hoüel translated into French. The editors of the book [23] made a remarkable work in commenting these letters and in highlighting at that occasion important relations between the mathematical works of Lobachevsky, Gauss, Riemann, Beltrami and several of their contemporaries.

**B. Technique**

The following is a selection of texts on model-free non-Euclidean geometry, which I find particularly interesting for learning this point of view.

(a) L. Gérard, *Sur la géométrie non euclidienne* [53]. This is the text of the author's thesis defended in Paris in 1892, and it is a profound and important text on hyperbolic geometry without models. It is written in the pure tradition of Euclid, and the sequence of propositions that are proven there are accompanied by ruler and compass constructions. The author wrote in the introduction: "I stuck to the rule of reasoning only on constructible figures. This condition, which Euclid imposed on himself as an inflexible rule, is no longer followed today; it has even been claimed that this rule is unworthy of Geometry",[202] and for the last statement, he quotes a paper published in the *Memoirs* series of the London Mathematical Society. Part of Gérard's thesis is on the analytic aspect of non-Euclidean geometry. It contains a synthetic proof of the so-called "Pythagorean theorem" for hyperbolic triangles, and of other trigonometric formulae. The text of Gérard's thesis is not easy to

[202][Je me suis astreint à la condition de ne raisonner que sur des figures que l'on sache construire. Cette condition, qu'Euclide s'est imposée comme une règle inflexible, n'est plus guère observée aujourd'hui ; on a été jusqu'à dire que c'est une pure règle de jeu indigne de la Géométrie].

obtain, but some important arguments are repeated in the books by Turc and by Barbarin cited below. Gérard's thesis obtained an "honourable mention" at the occasion of the 1897 Lobachevsky Prize.[203]

(b) P. Barbarin, *La géométrie non euclidienne* [11]. This is an excellent introductory textbook on hyperbolic geometry, although it presents some of the results without complete proofs. The book also contains interesting historical remarks. The third edition of the book (1928) contains supplementary chapters by A. Buhl on the relation between non-Euclidean geometry and physics. It is interesting to note that Barbarin, one of the best French geometers of the last quarter of the nineteenth century, admits in the introduction to his paper, *Sur un quadrilatère trirectangle* [12] that he first thought he had a proof of the parallel postulate. It is not unusual to find attempts to prove Euclid's postulate made by outstanding mathematicians, until the end of the nineteenth century. Barbarin was a high-school teacher in Bordeaux. We owe him several results on hyperbolic geometry, in particular, the first complete classification of conics and quadrics in the non-Euclidean plane, and new formulae for volumes of tetrahedra. Barbarin was second to Hilbert for the third award of the Lobachevsky Prize. G. B. Halsted wrote an interesting biographical note on Barbarin in the *American Mathematical Monthly* [68].

(c) A. Turc, *Introduction élémentaire à la géométrie lobatschewskienne* [146]. Albert Turc (1847–1913) was a linguist besides being a mathematician. This book is an excellent introduction to model-free hyperbolic geometry. It was published posthumously in 1914, and reprinted in 1967.

(d) N. A'Campo, *Géométrie hyperbolique* [1]. This is a set of notes from a course that A'Campo gave at Orsay in 1976. The interest in that subject arose that year

[203] The Lobachevsky prize for 1897 was awarded to S. Lie, for the third volume of his work "Theorie der Transformationsgruppen," written in 1893. In the same year, three other works were discerned an "honourable mention": L. Gérard's thesis, E. Cesàro's *Lezioni di geometria intrinseca*, 1896, and G. Fontené's *L'hyperespace à* $(n-1)$ *dimensions*, written in 1892. Both Gérard and Fontené were high-school teachers. The three memoirs are reviewed by E. O. Lovett in [107]. It is appropriate here to say a few words on the Lobachevsky prize, which is an interesting element of this story. The prize was created in 1896, by the Physico-Mathematical Society of Kazan, on the suggestion of A.V. Vasiliev, who at that time was a professor at Kazan University. The prize was established as a reward for a work on geometry, preferably non-Euclidean, done within the six years preceding the award. Winners of this famous prize include S. Lie, W. Killing, D. Hilbert, L. Schlesinger, F. Schur, H. Weyl, E. Cartan, V. V. Wagner, N. V. Efimov, A. D. Alexandrov, A. V. Pogorelov, L. A. Pontryagin, H. Hopf, P. S. Alexandrov, B. N. Delaunay, S. P. Novikov, H. Busemann, A. N. Kolmogorov, F. Hirzebruch, V. I Arnol'd, G. Margulis, Y. Reshetnyak, A. P. Norden, B. P. Komrakov, M. Gromov and S. S. Chern. It is always interesting to learn some of the details about establishment and awarding of prizes. For instance, we learn the following from G. B. Halsted's paper *Supplementary report on non-Euclidean geometry* [67]: "On October 22 (Old style) 1900, the Commission of the Physico-Mathematical Society of Kazan found the scientific merit of the works of A. N. Whitehead and W. Killing equal for the great Lobachevsky prize, and had to decide between them by drawing of lots." More details on this event are recounted in T. Hawkins' *Emergence of the theory of Lie groups* [71], p. 179–180. After the 1937 prize (awarded that year to V. Wagner), there was an interruption in further awards by the Society. In that year, the Soviet authorities deprived Kazan University of the privilege of this tradition that this university had so famously established. The prize was replaced, starting in the year 1951, by a similar one under the auspices of the Soviet Union Academy of Sciences. In 1992, Kazan University was able to restore its Lobachevsky prize, and as a result there now co-exist two Lobachevsky prizes.

in Orsay with the appearance of Thurston's Princeton Notes [143]. An extended version of A'Campo's note is contained in [2].

(e) N. V. Efimov, *Higher geometry* [44]. Efimov was also a winner of the Lobachevsky prize.

(f) H. P. Manning, *Non-Euclidean geometry* [108]. This is a concise and clear introduction to the subject, intended for school-teachers. It is probably the first such book written in English.

(g) The books and papers by H. S. M. Coxeter, cf. e.g. [34], [35] and [33].

(h) A. Ramsay and R. D. Richtmyer, *Introduction to hyperbolic geometry* [126]. This is an excellent textbook on synthetic hyperbolic geometry, and, compared to the older ones that are in this list, it is easy to obtain.

# 5. Some milestones for Lobachevsky's works on geometry

In this last section, I have collected some of the important elements of information that are available on Lobachevsky's works on geometry.

Before talking about the written works, let me say a few words about Lobachevsky's teaching of the subject.

The date on which Lobachevsky gave his first course on geometry is not known with certainty, but we can reasonably assume that it was in 1812. Indeed, Lobachevsky received his first degree in 1810 and his "master's" degree in 1812. It was not unusual for graduated students to start to teach before obtaining an official position as an "assistant professor" (which Lobachevsky obtained in 1814). One can add here a remark by Duffy [41], saying that in the early years of Kazan University, there was a lack of teachers, and that in some classes, students were taught by students.

### Unpublished and posthumously published works

We already mentioned the existence of a set of class-notes that have reached us from Lobachevsky's geometry teaching in the years 1815–1816 and 1816–1817. The notes were published in 1909 by A. V. Vasiliev and are kept in the Archives of Kazan University. In 1951, B. L. Laptev published an edition of these class-notes, with his own comments. Laptev also reported on these notes in his *Theory of parallel lines in the early works of Lobachevsky* [88]. A short report on them is also contained in Duffy's paper [41]. The notes contain, among other things, "proofs", by Lobachevsky, of the parallel postulate. These proofs are also discussed by Pont in his book [121].

In 1898, a manuscript on geometry, written by Lobachevsky in 1823, was discovered in the archives of the University of Kazan. The manuscript was published in its original form in Kazan, in 1909, under the title Геометрия (Geometry), with a foreword by A. V. Vasiliev. It is a book on elementary geometry, of high-school level. The book is included in Lobachevsky's *Complete works*, [104], Volume 2, pp. 43–107, with an Introduction and commentary by V. F. Kagan. Although the subject of this book is not hyperbolic geometry, the importance of the research on the parallel postulate is highlighted there. The book contains three "proofs" of Euclid's parallel postulate. Vucinich, in his paper [152], p. 318, following Vasiliev, wrote that in 1823 Lobachevsky sent the manuscript of that book to the Academician N. Fuss, asking his opinion for possible publication. The book was not published, first because Fuss recommended rejection, but also because Lobachevsky himself was not satisfied with it, considering that the proofs of the parallel postulate that it contained were non-conclusive. Engel, in [45], p. 368, reported on the same fact. According to Pont, the book contains errors, and its structure is unusual and differs substantially from that of Euclid's *Elements* (see [121], p. 251). In this respect, Yaglom wrote the following ([155], p. 54):

> Fuss the traditionalist disliked the book because its author tried to represent geometry not in Euclid's spirit, i.e., not as a series of scholastically interpreted deductions from stated axioms, but as a science concerning (real) space, in which measurements of geometric magnitudes play a leading role and in which transformations ("motions") are freely used in proofs of theorems.

In the same book, Yaglom wrote ([155], p. 165):

> Fuss's harsh verdict on Lobachevsky's *Geometry* (the name of the author was unknown to Fuss) was partly due to a basic mistake Lobachevsky made in assessing the purpose of the manuscript, which unfortunately had no author's introduction. Fuss regarded the *Geometry* as a textbook for beginners, while Lobachevsky had obviously intended it as a refresher course in school mathematics.[204]

G. D. Crowe [36], based on [113], has the following comments on the history of this book:

> ... Magnitskii [a controversial and reactionary figure, who was the academic inspector of the Kazan district] helped Lobachevskii towards his discovery in a second, more direct way as well. In September 1822, Magnitskii wrote to rector Nicol'ski that it would be a good thing if each of the university's professors were required to produce a work for publication each year. Lobachevskii responded in the summer of the following year with his textbook *Geometria*. It was in the course of preparing this textbook and evaluating the criticism it received from the Imperial Academy of Sciences that Lobachevskii became firmly convinced in the validity of a non-Euclidean approach to the theory of parallel lines.

N. Parfeniev discovered in 1930 a manuscript by Lobachevsky, written in 1822–1823 and known by the name of "Konspect" (which means "outline"). This manuscript constitutes a companion guide to the 1823 geometry book (see [113]).

Both the 1815–1816 notes and the book on geometry were reported on in a note written by Vasiliev in the 1894–1895 Jahresbericht der Deutschen Mathematiker-Vereinigung, cf. [148].[205] In this note, Vasiliev wrote that it was not possible for him to obtain a copy of a textbook on geometry that was "written by Lobachevsky in 1823 for high schools and which was harshly criticised by Academician Fuss".[206] Regarding the 1815–1816 notes, Vasiliev wrote the following: "I was very pleased that I was brought an old notebook that contained the lectures Lobachevsky held in 1815–1816 at the university. In these lectures there are three accounts of a systematic

[204] Fuss's report is quoted in Engel's biography of Lobachevsky contained in [45] (p. 368).

[205] I thank Manfred Karbe who brought this report to my attention.

[206] Es war auch nicht möglich, eine Copie eines Lehrbuches der Geometrie, das von Lobatschefskij im Jahre 1823 für die Gymnasien geschrieben wurde und von dem Akademiker Fuss einer schroffen Kritik unterzogen worden war, aufzufinden.

treatment of elementary geometry, and in each of them there are three completely different approaches to the theory of parallels."[207] And then Vasiliev continues explaining the three approaches.

It is possible that around the year 1823, Lobachevsky had already shifted his efforts from the attempt to prove Euclid's parallel postulate to the opposite direction, that is, he started developing a geometry that satisfies Euclid's axioms except the parallel postulate, and that does not satisfy that postulate.

We already recalled that on the 12th (Old style; 23th New style) of February 1826, Lobachevsky read to the Physical and Mathematical Section of Kazan University a paper in French, entitled *Exposition succinte des principes de la géométrie avec une démonstration rigoureuse du théorème des parallèles* (A brief exposition of the principles of geometry with a rigorous proof of the theorem on parallels). Lobachevsky prepared a manuscript of that lecture for publication. The manuscript was sent to three university professors to be reviewed, and it was never published, because Lobachevsky's colleagues failed to understand this work. The manuscript did not reach us, and the reviewers, who did not want to write a negative report, claimed that they had lost it (see Yaglom [155], p. 54). Even though the fact that this paper constitutes the opening era of hyperbolic geometry cannot be asserted with certainty, one can safely say that the paper contains the germs of Lobachevsky's work on geometry. Indeed, according to Lobachevsky himself, the main ideas of that paper are repeated in his 1829 memoir, the *Elements of geometry*. A footnote on the first page of the *Elements of geometry* [95] indicates that this work is

> Extracted by the author himself from a paper which he read on the 12th of February, 1826, at a meeting of the Section for Physico-Mathematical Sciences, with the title *Exposition succinte des principes de la Géométrie, avec une démonstration rigoureuse du théorème des parallèles*.

Likewise, in the Introduction to the *New elements of geometry* (1836) [96], Lobachevsky wrote the following:

> Believing myself to have completely solved the difficult question, I wrote a paper on it in the year 1826: *Exposition succinte des principes de la Géométrie, avec une démonstration rigoureuse du théorème des parallèles*, read on the 12th of February, 1826, in the séance of the Physico-Mathematics faculty of the University of Kazan but nowhere printed.

[207] Deshalb war ich sehr erfreut, als man mir ein altes Heft brachte, das die Vorlesungen, welche Lobatschefskij in den Jahren 1815–1816 an der Universität gehalten hatte, enthält. In diesen Vorlesungen finden sich drei Abrisse einer systematischen Bearbeitung der elementaren Geometrie, und in jedem von diesen Abrissen finden sich drei ganz verschiedene Auffassungen der Parallelentheorie."

## Elements of geometry

Lobachevsky's first published work on hyperbolic geometry that reached us is his memoir entitled *On the elements of geometry* (О началах геометрии, O nachalakh geometrii). It was printed in instalments, in the Казанский Вестник (*Kazan Messenger*) in the years 1829 and 1830 [95].

This memoir contains, as we already noted, part of Lobachevsky's 1826 unpublished lecture, and a section on the applications of the new geometric theory to the computation of definite integrals. The memoir already contains, at least in germ, all the ideas on geometry that Lobachevsky worked on and published in more detail and in more precise forms during the rest of his life: the theory of parallels and the bases of axiomatic non-Euclidean geometry, the trigonometric formulae, the differential geometry of curves and surfaces, and the computations of areas and volumes.

The *Elements of geometry* was translated into German and published in 1898 by F. Engel in [45], under the title *Ueber die Anfangsgründe der Geometrie*, and it is also contained in Lobachevsky's 1883–1886 *Collected geometric works* edition [103], Vol. I, pp. 1–67.

In 1832, Lobachevsky submitted his *Elements of geometry* to the St. Petersburg Academy of Sciences. The memoir was examined by the academician M. V. Ostrogradsky who recommended rejection. One may consider the failure of Ostrogradsky to acknowledge the importance of Lobachevsky's work as a major factor in the enormous delay that occurred before this work was noticed by the mathematical community.

In 1834, the literary magazine[208] Сын отечества и Северный архин (*Syn otečestva i Severnyĭ arhiv*; Son of the fatherland and Northern archive) published a critic of Lobachevsky's *Elements of geometry*, which looked more like a mockery. Extracts are quoted in Rosenfeld [131]:

> Glory to Mr. Lobachevsky who took upon himself the labor of revealing, on one hand, the insolence and shamelessness of false new inventions, and on the other the simple-minded ignorance of those who worship their new inventions.

These articles are considered as having been written by some of Ostrogradsky's students, cf. Rosenfeld [131], p. 209, see also the discussion by Polotovsky in [120]. It is interesting to note that Gauss was aware of the critic that appeared in the *Son of the fatherland*.[209]

[208]Rosenfeld ([131], p. 209) mentions two magazines, *Syn otečestva* and *Severnyĭ arhiv*; as a matter of fact, in the year 1829 these two magazines merged into one, which, until the year 1835, appeared under the name *Syn otečestva i Severnyĭ arhiv*.

[209]In his letter to Gerling dated 8 February 1844, Gauss wrote: "At the same time, one must emphasise that a very offensive critic of these ideas appeared in No. 41 of the journal *Son of the fatherland* Сын отечества in 1834, which led Lobachevsky to send a counter-critic that, however, was not accepted until the beginning of 1835." [Zugleich wird dabei bemerkt, dass eine sehr kränkende Kritik dieser Abhandlung in No. 41 eines andern russischen, wie ich vermuthe in Petersburg erscheinenden, Journals "Sohn des Vaterlandes" Сын отечества von 1834 stehe, wogegen Lobatschewsky eine Antikritik eingeschickt habe, die aber bis Anfang 1835 nicht aufgenommen sei.]

The *Elements of geometry* was translated in Esperanto by C. E. Sjöstedt, who published in 1968 a collection of important works on the theory of parallels, translated into this language, that includes writings by Wallis, Saccheri, Lambert, Gauss, Schweikart, Taurinus, Beltrami, Höuel, von Helmholtz, Klein, Clifford, Poincaré, Hilbert and others.

A Greek translation by K. Philippides and K. Philippidou of extracts of the *Elements of geometry* is contained in the Greek–Russian *Selected works* edition [105] published in 2007 by the State University of Nizhny Novgorod.

### The *New elements of geometry, with a complete theory of parallels*

In 1835, Lobachevsky revised part of his *Elements of geometry* and wrote a new version which is 470 pages long, entitled *New elements of geometry, with a complete theory of parallels* (Новые начала геометрии с полной теорией параллельных). This memoir was printed in instalments, in the newly founded periodical Ученые записки Казанского Императорского Университета (Uchenye zapiski Kazanskogo Imperatorskogo Universiteta; Scientific memoirs of Kazan Imperial University). This journal was founded in 1834 by Lobachevsky himself, and it was purely dedicated to science. It replaced the Казанский Вестник (*Kazan Messenger*), in which he had published his *Elements of geometry*. The *Kazan Messenger* was not specifically devoted to science and was rather a cultural publication.

The *New elements of geometry*, which has thirteen chapters, contains a detailed exposition of Lobachevsky's geometrical system. It is written in the spirit of Euclid's *Elements*, where the use of the individual axioms is highlighted in each proof, and with an attempt to use the smallest possible number of axioms. In the introduction to that work, Lobachevsky analysed previous "proofs" of the parallel postulate (namely, those by A.-M. Legendre and by L. Bertrand), exhibiting the gaps in their arguments.

In a note in the first page of his later work, *Geometrical researches on the theory of parallels* (1840), Lobachevsky wrote the following, referring to his *New elements of geometry*:

> I published my first essay on the foundation of geometry in the *Kazan Messenger* in the year 1829. In the hope of having satisfied all the requirements, I undertook hereupon a treatment of the whole of this science, and published my work in separate parts in the *Scientific Memoirs of Kazan University* in the years 1836, 1837, 1838, under the title *New elements of geometry, with a complete theory of parallels*.

An English translation of Lobachevsky's Introduction of the *New elements of geometry*, with a preface, was published in 1897 by G. B. Halsted [64].

The 1898 volume by Engel [45] includes a translation of the *New elements* (with the exception of the last two chapters), under the title *Neue Anfangsgründe der Geometrie mit einer vollständigen Theorie der Parallelinien*.

A French translation, by F. Mailleux, was published in 1899 under the title *Nouveaux principes de le géométrie avec une théorie complète des parallèles*, in the *Mémoires de la Société Royale des Sciences de Liège*.[210]

The *New elements of geometry* is also contained in Lobachevsky's *Collected geometric works*, [103], Vol. 1, pp. 219–486, and in the 1946–1951 *Complete works* edition, [104], Vol. 2, pp. 147–457.

Finally, a Greek translation by K. Philippides and K. Philippidou of extracts of the *New elements of geometry* is contained in the Greek–Russian *Selected works* edition [105] published in 2007 by the State University of Nizhny Novgorod.

### The *Imaginary geometry*

In 1835, the Russian version of Lobachevsky's *Imaginary geometry* (Воображаемая геометрия, Voobražaemaya geometriya) [97] appeared in the *Scientific Memoirs of the Imperial University of Kazan*. This memoir is a more detailed version of the memoir *Geométrie imaginaire* [99], published in *Crelle's Journal* in 1837 (and sent to that journal in 1835). The Russian version was written after the French one. It is reproduced in the 1883 edition of Lobachevsky's *Geometric works* [103].

A German translation with notes, entitled *Imaginäre Geometrie*, was published by H. Liebmann, in 1904, see [97].

The French version was published in the *Geometric works* edition [103], Vol. 2, pp. 581–613 and in the 1946–1951 *Complete works* edition [104], Vol. 3, pp. 16–70.

A Greek translation by K. Philippides and K. Philippidou of the *Imaginary geometry* is contained in the Greek–Russian *Selected works* edition [105] published in 2007 by the State University of Nizhny Novgorod.

### The *Application of imaginary geometry to certain integrals*

In 1836, Lobachevsky published, again in the *Scientific Memoirs*, a memoir entitled *Application of imaginary geometry to certain integrals* (Применение воображаемой геометрии к некоторым интегралам). A German translation was published in 1904 by H. Liebmann, in his *Anwendung der Imaginären Geometrie auf einige Integrale*, see [98].

The memoir was printed in Lobachevsky's 1883 *Geometric works* [103], Vol. 1, pp. 121–218, and in his 1946–1951 *Complete works* [104], Vol. 3, pp. 181–294.

### The *Geometrical researches on the theory of parallels*

In 1840, Lobachevsky published a 61-page memoir in German, whose complete title is *Geometrische Untersuchungen zur Theorie der Parallellinien*, Von Nicolaus Lobatschewsky, Kaiserl. russ. wirkl. Staatsrathe und ord. Prof. der Mathematik bei der Universität Kasan. Berlin 1840. In der Finckeschen Buchhandlung, see [100].

[210] Mailleux was a lawyer, and he became interested in Lobachevsky's works while he was preparing a doctoral thesis in philosophy.

The memoir received in the same year a very negative review in Gersdorff's *Repertorium der gesamten Deutschen Litteratur*, a review that Gauss described in his letter to J. F. Encke on February 1841 as being "absurd".

This memoir may be considered as the first clear account that Lobachevsky gave of his geometrical system. According to Hoüel [76], this memoir reproduces a manuscript entitled *Beiträge zu der Theorie der Parallellinien*, that was submitted in 1840 to *Crelle's Journal*, and that has not been published there.

In 1866, J. Hoüel published a French translation of the *Geometrische Untersuchungen*, with the following title: *Etudes géométriques sur la théorie des parallèles, par Lobatschewsky, conseiller d'Etat de l'Empire de Russie et Professeur à l'Université de Kasan ; traduit de l'allemand par J. Hoüel, professeur de mathématiques pures à la Faculté des Sciences de Bordeaux ; suivi d'un extrait de la correspondance entre Gauss et Schumacher*. Paris, Gauthier-Villars. The same translation was also published, in the same year, in the *Mémoires de la Société de sciences physiques et naturelles de Bordeaux*, t. IV, pp. 83–182, and it was reprinted by Hermann (Paris, 1900).

A Russian translation by A. V. Letnikov was published in 1868, in the *Matematicheskiĭ Sbornik*, Vol. 3, pp. 78–120.

A facsimile reprint of the German original edition appeared in 1887 (Berlin, Mayer and Müller).

The *Geometrische Untersuchungen* was translated into English, by G. B. Halsted, under the title *Geometrical researches on the theory of parallels, by Nicholas Lobatschewsky* (Austin, Texas, 1891). A second edition of the English translation was published in 1914 by the Open Court Publishing Company, Chicago. A facsimile of this edition is contained in Bonola [26].

The *Geometrische Untersuchungen* was also published in Lobachevsky's *Collected geometric works* [103], Vol. II, pp. 553–578.

Finally, a Greek translation by K. Philippides and K. Philippidou of the *Geometrical researches on the theory of parallels* is contained in the Greek–Russian *Selected works* edition [105] published in 2007 by the State University of Nizhny Novgorod.

### The *Pangeometry*

In 1855, Lobachevsky wrote the two versions of his memoir *Pangeometry*, one in Russian and one in French. The Russian version, Пангеометрия appeared in the *Scientific Memoirs of the Imperial University of Kazan*. The French version carries the title *Pangéométrie ou précis de géométrie fondée sur une théorie générale et rigoureuse des parallèles*, and it appeared as part of a collection of memoirs written by professors of Kazan University which were ordered at the occasion of the fiftieth anniversary of that university [102]. The volume title is *Recueil d'articles écrits par les professeurs de l'Université de Kazan à l'occasion du cinquantenaire de sa fondation* (Collection of memoirs written by professors of the University of Kazan on the 50th anniversary of its foundation). This collection appeared in print in 1856, after the death of Lobachevsky (although the year indicated on the cover page of the publication is 1855).

Although the *Pangeometry* was, strictly speaking, a journal publication rather than a book, reprints of that publication were sold. The French version was also sold as a brochure published in Paris by Hermann, in 1895.

The two versions of the *Pangeometry* are reproduced in the first edition of the *Collected geometric works* (1883–1886), [103], and only the Russian version is reproduced in the second edition (1946–1951) [104].

A German translation (translator unknown) entitled *Pangeometrie oder die auf einer allgemeinen und strengen Theorie der Parallelen gegründeten Hauptsätze der Geometrie* was published in 1858, as part of Georg Reimer's *Archiv für wissenschaftliche Kunde von Russland*.

An Italian translation appeared in 1867, in the *Giornale di Matematiche*. The translator's name is not indicated in that article, but it is known that the translation was done by Battaglini himself, the editor of the journal (see the letters from Battaglini to Hoüel, quoted in [23], p. 62).

A second German edition by H. Liebmann, appeared in 1902. See [102] for exact references.

An English version of a small part of this work, translated by H. P. Manning, appeared in D. E. Smith's *Source book in mathematics* [137].

## Conclusion

Going through Lobachevsky's works in chronological order, one realises that most of his ideas are already contained in his first two published memoirs. He repeated them several times, in general with increasing clarity, concision and force. Had his first ideas been accepted by the mathematical community, one can reasonably assume that Lobachevsky would have moved to other development of the theory.

This also leads me to a question (or, rather, a comment) on Milnor's introduction of his paper *Hyperbolic geometry: The first 150 years* [110]. Milnor wrote:

> The mathematical literature on non-Euclidean geometry begins in 1829 with publications by N. Lobachevsky in an obscure Russian journal. The infant subject grew very rapidly. Lobachevsky was a fanatically hard worker, who progressed quickly from student to professor to rector at his university of Kazan, on the Volga.

The first and the third sentences are correct. I have doubts about the second one. It seems to me that it does not give a fair description of the childhood of a theory on which one person worked in isolation, repeating the same ideas during more than 25 years because nobody heard them. The theory was rather stagnant for decades after Lobachevsky's death. At Lobachevsky's death, his ideas were buried with him for several years, followed by several decades of work on the foundations (which was largely motivated by the discovery of hyperbolic geometry but which had little to do with the development of that theory). During that period, many of the good geometers were still asking the question of whether Lobachevsky's theory exists or not.

During and after this work on the foundations, important connections were made between hyperbolic geometry and other fields of mathematics (projective geometry, physics, group theory, representation theory, etc.)

The real substantial progress that was made on the subject itself is due to Thurston, and it came about 150 years after Lobachevsky's first works appeared in print.

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