Menelaus' Spherics in Greek and Arabic Mathematics

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Abstract

We present the history of Menelaus' *Spherics* (second c. AD), explaining its importance and quoting in detail some of the significant propositions. We include this work in the general context of Greek and Arabic mathematics, highlighting the far-reaching connections with modern mathematical ideas.

Keywords

Menelaus of Alexandria · Spherics · Spherical geometry · Spherical trigonometry · Greek mathematics · Arabic mathematic · Theodosius of Tripoli · Autolycus of Pitane · Pappus of Alexandria

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1 Introduction

Menelaus' *Spherics* (second c. AD) constitute one of the most remarkable achievements of Ancient Greek mathematics, along with Euclid's *Elements*, Apollonius *Conics*, Diophantus' *Arithmetica*, Archimedes' treatises on geometry and statics, Theodosius' *Spherics*, Pappus' *Collection*, to name only some of the most important Greek mathematical works. Among these treatises, Menelaus' *Spherics* is certainly the least understood by historians and the least familiar to mathematicians of the modern period. As a matter of fact, we know that Fermat was nourished by the work of Diophantus, that Christiaan Huygens was deeply immersed in the writings of Archimedes and Apollonius, that Euler was an assiduous reader of Diophantus and Pappus and that he was well acquainted with Theodosius' *Spherics*, not to mention of course Euclid's *Elements*, a work which all mathematicians of the past thoroughly studied. But very few of the modern mathematicians mention Menelaus' *Spherics*,[1] a fact that might seem surprising, especially since a relatively large number of medieval Arab mathematicians were well acquainted with this work.

I can see two reasons for which Menelaus' *Spherics* remains unknown. First, there are problems inherent to the available versions: no Greek manuscript has survived, and the ones we possess, in Arabic or translated from the Arabic, present intricate differences in form and wording. Another reason is that this work is difficult to access from the mathematical point of view. Indeed, the existing manuscripts do not contain proofs of several of the difficult propositions. I will say more on this below.

Menelaus' *Spherics* opened up a new research field. Saying this, we are not talking about the subject of the geometry of the sphere which, as a general topic, already existed (we mentioned Theodosius' *Spherics*, but the sphere was studied before Theodosius, by Autolycus, Euclid, Archimedes, and others), but of the field known today as the "intrinsic geometry of surfaces," which can be described, precisely like Menelaus investigated the geometry of the sphere, as the study of the properties of curvilinear triangles on a surface which is not the Euclidean plane. Indeed, the vast majority of Menelaus' propositions are concerned with such

[1] An exception is Marin Mersenne, who was fond of old mathematical texts and who was aware of this work. Indeed, Mersenne published in (Mersenne 1626) a Latin edition of the third book of the *Spherics*, together with works of Autolycus, Archimedes, Apollonius, Theodosius, Diophantus and Pappus. But one century later, Anders Lexell, who was Euler's closest collaborator, and who was a specialist of spherical geometry, writes in his memoir (Lexell 1784): "From the time where the *Spherical elements* of Theodosius were handed in written form, it is very difficult to find, in the treatises of the geometers, other elements concerning further refinements of the theory of the figures drawn on the surface of the sphere, than those that are usually exposed in the *Elements of Spherical Trigonometry* and that are aimed at solving spherical triangles." It is not completely clear to what work with the title *Elements of Spherical Trigonometry* Lexell refers (this may be Euler's *Principes de la trigonométrie sphérique tirés de la méthode des plus petits et des plus grands* (Euler 1753)), but it is obvious that he was not aware of Menelaus' work, written two centuries after that of Theodosius, nor of the works of Naṣīr al-Dīn al-Ṭūsī and the other Arab mathematicians of the eleventh to thirteenth century who worked extensively on spherical trigonometry.

properties. Menelaus' major achievement is that he highlighted for the first time the notion of spherical triangle, and he developed the bases for its study.

The Arabs included this work into their body of knowledge and they understood its novelty. Let us quote right away Aḥmad ibn Sa'd al-Harawī (tenth century) from the very beginning of his edition of Menelaus' treatise (English translation in (Rashed & Papadopoulos, 2017, p. 500)):

> The way that Menelaus followed in that book is marvelous; nobody preceded him and nothing had been known to his predecessors on that subject. Did he foresee that this kind of geometrical science is unique in itself, that it has laws suitable to it and lemmas that lead to the goal which are different from those that one is willing to demonstrate by using straight lines and the determination of planes by their intersections?

The sentence about "straight lines and the determination of planes by their intersections" refers to the fact that Menelaus avoided as much as possible the use of Euclidean solid geometry which is the ambient geometry of the sphere; the expression "straight lines and planes" refers to Euclidean lines and planes. Let us also quote Menelaus himself, from the oldest fragment known of the *Spherics*, which we also edited and translated into English in (Rashed & Papadopoulos, 2017, p. 504):

> I have invented a kind of proof which is excellent and admirable. Several things relative to surface <properties> on the spherical surfaces appeared to me; I don't think they were offered to anybody else. And I ordered the lemmas and the proofs in an order so that it will be easy for those who are fond of science to rise up and attain universal and honorable sciences. In this way, with the help of particular propositions that occur to them from the experiences they acquire from that art, they will become skillful and outstanding in the universal propositions, given the multitude of universal propositions that can occur to them, and that become for them proofs and indications for the inaccessible and the difficult directions in the principles of Spherics, their role and what ensues from them.

The field that Menelaus opened in his *Spherics* can be especially appreciated in the context of non-Euclidean geometry, a domain of mathematics that became a particularly active research field in the last quarter of the twentieth century.

In the pages that follow, I will review this work in its context, discussing at the same time the mathematics it contains, the history of the texts that we possess, and the impact of this work on the Arab mathematicians of the period ninth to thirteenth century, who are certainly, among the mathematicians who read this work, those who have most understood its importance.

The content of the rest of this chapter is the following:

In Sect. 2, I briefly review works on the geometry of the sphere by two predecessors of Menelaus, namely, Theodosius of Tripoli and Autolycus of Pitane, pointing out relations between these works and Euclidean geometry, making connections with the work of Menelaus, with a later work by al-Ḥasan Ibn al-Haytham and hinting to some relations with the development of modern analytic spherical geometry.

Section 3 contains some preliminary general facts about Menelaus' *Spherics*, including its division and its main characteristics.

Section 4 is the heart of this chapter. I discuss in it some significant propositions of Menelaus' *Spherics* and at the same time I give an overview of the major themes that are addressed in this work. I believe that before commenting on the history of this book, one has to understand the subject it treats, and for this, a knowledge of at least some of the important statements and methods of proof is necessary.

I have divided this book, as in my edition with Rashed of al-Harawī's rectification (Rashed and Papadopoulos 2017), into seven major themes. The reader interested in a more complete discussion of the mathematical content of this work is referred to Part II (pp. 125–395) of our edition.

It may be noticed that some of the statements of Menelaus' propositions are analogues of propositions in Euclid's *Elements* (e.g., the triangle inequality, equality and inequality criteria for triangles, etc.), but one must be aware of the fact that this is only an apparent similarity in the sense that the proofs of the spherical propositions are very different from those of their Euclidean analogues. This is due to the fact that the properties of the underlying surfaces are very different: On the one hand, we have the Euclidean plane, in which two lines may intersect or not, and where any two lines starting at the same point diverge, and on the other hand, we have the sphere, in which any two lines intersect and where two lines starting at a point start by diverging and then converge to the same point. Furthermore, in the Euclidean plane, the angle sum in a triangle is constant and equal to two right angles, whereas on the sphere, the angle sum in a triangle is non-constant and is always greater than two right angles. There are many other substantial differences between the two geometries.

In §5, I discuss the influence of Menelaus' *Spherics* on Pappus. This is a rare occasion where the influence of this work on a later author from Greek Antiquity is visible.

Sections 6 and 7 constitute an important part of this chapter; they concern the influence of Menelaus' *Spherics* on Arabic mathematics of the period that starts at the ninth century and that ends at the fifteenth, where a considerable amount of work was done in translating this treatise and in trying to complete the proofs of the difficult propositions that are contained in it. This effort led to a development of important mathematical theories by the Arabs, especially in spherical geometry.

In §8, I discuss the connection of the *Spherics* with astronomy, a relation which, in my opinion, has been exaggerated by modern authors.

In the following, when I quote mathematical propositions, I have sometimes paraphrased the original text to make it more easily readable by a modern mathematician. The translations close to the original together with the original text of Menelaus' *Spherics* are contained in the volume (Rashed and Papadopoulos 2017).

2 Two Predecessors: Theodosius and Autolycus

Before talking about Menelaus' *Spherics*, I will make a short review of the works of two forerunners, Theodosius of Tripoli (first c. AD) and Autolycus of Pitane (fourth century BC). I will start with Theodosius, the immediate predecessor of Menelaus, mentioning his influence on the latter, and then, I will consider Autolycus, on whose work was based that of Theodosius.

Theodosius' *Spherics* was written in the first century BC, that is, two centuries before Menelaus. The two works, which carry the same title, are concerned with the properties of the sphere. Theodosius' treatise was obviously not foreign to Menelaus. In fact, Menelaus in a more than one sense has placed himself in the continuity of Theodosius: It suffices to recall that at the beginning of Book II al-Harawī's rectification (a version of this work to which we shall often refer), Menelaus declares that his goal is to improve and generalize some propositions of Theodosius (Rashed & Papadopoulos, 2017, p. 684). Furthermore, Menelaus, at some places of his *Spherics*, refers either explicitly or at least implicitly to Theodosius' work, and uses some of his propositions. We have commented at length on this fact in our edition (Rashed and Papadopoulos 2017) of al-Harawī's rectification, and we have also highlighted there the major differences between the two works.

The best modern-language edition of Theodosius' *Spherics* is still Ver Eecke's (Theodosius, Les Sphériques de Théodose de Tripoli 1927). It is very close to the text from which it is translated, it reproduces faithfully the mathematical content of the work, and it is accompanied by a solid mathematical commentary.[2] Let me start with some general remarks on this work.

Theodosius, in his *Spherics*, makes a distinction between a *theorem* and a *construction*. Such a distinction, which has already been made explicit and commented upon by Proclus in his *Commentary to the first book of Euclid's* Elements, is highlighted by Heiberg and by Ver Eecke.[3] The latter, after the number of each proposition, writes whether it is a theorem or a construction. An example of a construction in Theodosius' *Spherics* is Proposition 2 of Book I, whose statement is: *To find the center of a given sphere*. This is an analogue of Proposition I of Book III of Euclid's *Elements* whose statement is: *To find the center of a given circle*. An important remark that fits here is that in Menelaus' *Spherics*, which is the main topic of the present chapter, all the propositions are meant to be constructive.

Theodosius' *Spherics* is divided into three books. None of the statements of the propositions in the first two books is close to any statement of a proposition of Menelaus' *Spherics*. In this respect, the two treatise are very different. If we want to summarize in very few words the difference between them, we can say that the first one is concerned with circles drawn on the surface of the sphere whereas the second one is concerned with triangles drawn on the surface of the sphere. This is especially true for the first two books of Theodosius.

Let us highlight some statements of Theodosius' *Spherics*.

Book I starts with definitions: that of the sphere (by which is meant the solid sphere), its surface, its center, and the notion of diameter. After the definition of the

[2] There are editions and commentaries done by authors with a poor knowledge in mathematics, and I prefer not to dwell upon this.

[3] Ver Eecke's edition is based in part on Heiberg's, see the comments in (Theodosius, Les Sphériques de Théodose de Tripoli 1927) p. LII of the Introduction.

latter, it is said that it is a rotation axis of the sphere, a sentence which Ver Eecke considers as a later addition ([36, p. 1, Footnote 3]). Indeed, the property that a diameter of the sphere is a rotation axis is not used in the treatise. It is used in Autolycus' work which we review below.

Theodosius defines the pole of a circle on the surface of the sphere as a point on this surface such that all the Euclidean segments starting at this point and ending at a point on the circle are equal. In particular, no attempt is made to define the pole intrinsically on the surface of the sphere (for instance, as a center with respect to lengths of arcs of great circles on that surface).

Theodosius then gives the definition of "two planes having the same inclination." His definition is the same as Euclid's. We recall that the definition of the angle made by two planes is contained in Euclid's Book XI (Definition 6), and that the definition of two pairs of planes having the same inclination is the content of the next definition of the same book. We also note that unlike in Euclid's *Elements*, besides this relative notion of inclination, there is no absolute notion of angle made by two planes in Theodosius' *Spherics*, neither in the definitions nor in statements or in the proofs. In his propositions and in his proofs, Theodosius only talks about *perpendicular* great circles and *oblique* great circles. The notion of angle between two lines is only used for the Euclidean lines in the solid sphere, and never for two arcs of circles on the surface of the sphere. These constitute fundamental differences between the *Spherics* of Theodosius and Menelaus. Indeed, Menelaus uses the dihedral angle between two planes to define the angle made by two circles on the sphere, and hence, to define the notion of angle at the vertex of a spherical triangle, and he applies this notion in an essential way in every proposition and in every proof, with very few exceptions.

After the definitions, the first book of Theodosius' *Spherics*, in Ver Eeckes' edition, consists of 23 propositions.[4] These propositions concern basic properties of the sphere and its intersections with planes. For instance, Proposition I.1 says that the intersection of the sphere with a plane is a circle. We already mentioned Proposition I.2, which is a construction problem: *To find the center of a given sphere*. A great circle is the intersection of the sphere with a plane passing through the origin. Proposition I.11 says that on the sphere, two great circles cut each other in two equal parts. Proposition I.12 is a converse and is less trivial: it says that if two circles on the sphere cut each other into halves, then these two circles are great circles. Proposition I.13 says that if, on a sphere, a great circle cuts another circle with right angles, then it cuts it into two equal parts and passes by its poles. Proposition I.16 says that the distance from a great circle to a pole is equal to the side of a regular inscribed quadrilateral (i.e., a square). Proposition I.17 is a converse. Proposition I.18 is a construction problem: *To construct a segment equal to the diameter of a given circle on the sphere*. Proposition I.21 is also a construction problem: *To find the pole of a given circle on the sphere*.

[4] The attribution of the last two propositions to Theodosius, according to Ver Eecke, is doubtful, see [36, Footnotes p. 29 and 30].

From the mathematical point of view, all this is Euclidean geometry; there is no spherical geometry at all.

Let us pass now to Book II.

This book, in Ver Eecke's edition, also consists of 23 propositions. Most of them deal with properties of *small* circles. The book starts with the definition of tangent circles: two circles are tangent to each other if the line that is the intersection of the two planes that contain them is tangent to both of them. Proposition II. 6 says that if a great circle is tangent to a small circle, then it is also tangent to another small circle equal to the first small circle. I have reproduced the figure of this proposition from an Arabic manuscript, Ms. 3464, Topkapi Saray, Ahmet III (Istanbul). This is the drawing on the top in Fig. 1. I would like to take this opportunity to make a comparison with the figure of the same proposition, as redrawn in the plate of the figures of Clavius' English edition of the *Spherics* (Clavius's commentary on the spheriks of Theodosius Tripolitæ 1721), which I reproduced here in Fig. 2. In this plate, this is No. 33.[5] The figure in the ancient Arabic manuscript shows a sense of abstraction which, in my opinion, is absent from the realistic figures in the more recent editions of Clavius, Ver Eecke, etc.

Let me mention two other propositions from the second book of Theodosius' *Spherics*, which are both construction problems. Proposition II.14 says the following:

Given a circle on the sphere which is not a great circle and given a point on it, to describe a great circle tangent to the given circle at the given point.

Proposition II.15 is more involved:

Given on the sphere a circle which is not a great circle and given a point of the sphere situated in the region between this circle and a parallel equal circle, to describe a great circle tangent to the first small circles and passing through the given point. These propositions have a taste of Apollonian constructions of circles tangent to each other in the plane. The figure that comes with the last construction, in the Istanbul 3464 Ms., is reproduced in Fig. 3 below. I have included it here to suggest the complication of the construction. This is also Fig. 42 in Clavius' version (Fig. 2 here).

The content of Book II also belongs to the realm of three-dimensional Euclidean geometry.

Book III of Theodosius' Spherics is concerned with important monotonicity properties that we discuss at length in (Rashed and Papadopoulos 2017). Such properties constitute the culmination of both Spherics, in terms of degree of difficulty. Let me explain by an example what I mean by a monotonicity property. I take as example Proposition III.9 in Theodosius. The corresponding figure, from the manuscript mentioned above, is reproduced here in Fig. 4. The 3 sub-figures correspond to three cases in the proof, and we do no not need to worry about this now. In each case, the action takes place in a hemisphere, bounded by the exterior great circle

[5] In Clavius' edition, the total number of propositions contained in each book is not the same as in Ver Eecke's edition.

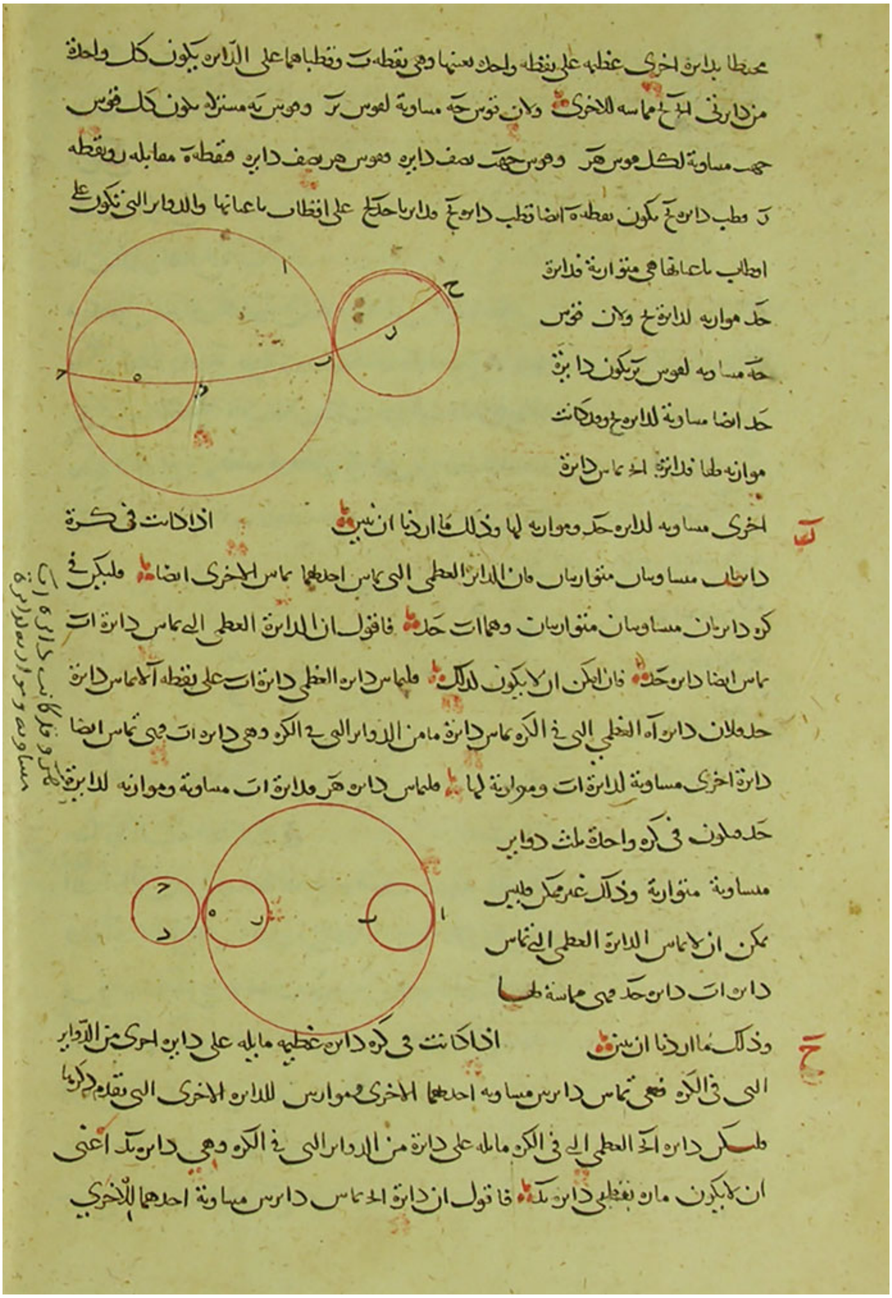


Fig. 1 From Theodosius' *Spherics*. On the top, the figure for Proposition II.6. Ms. 3464, fol. 30, Topkapi Saray, Ahmet III

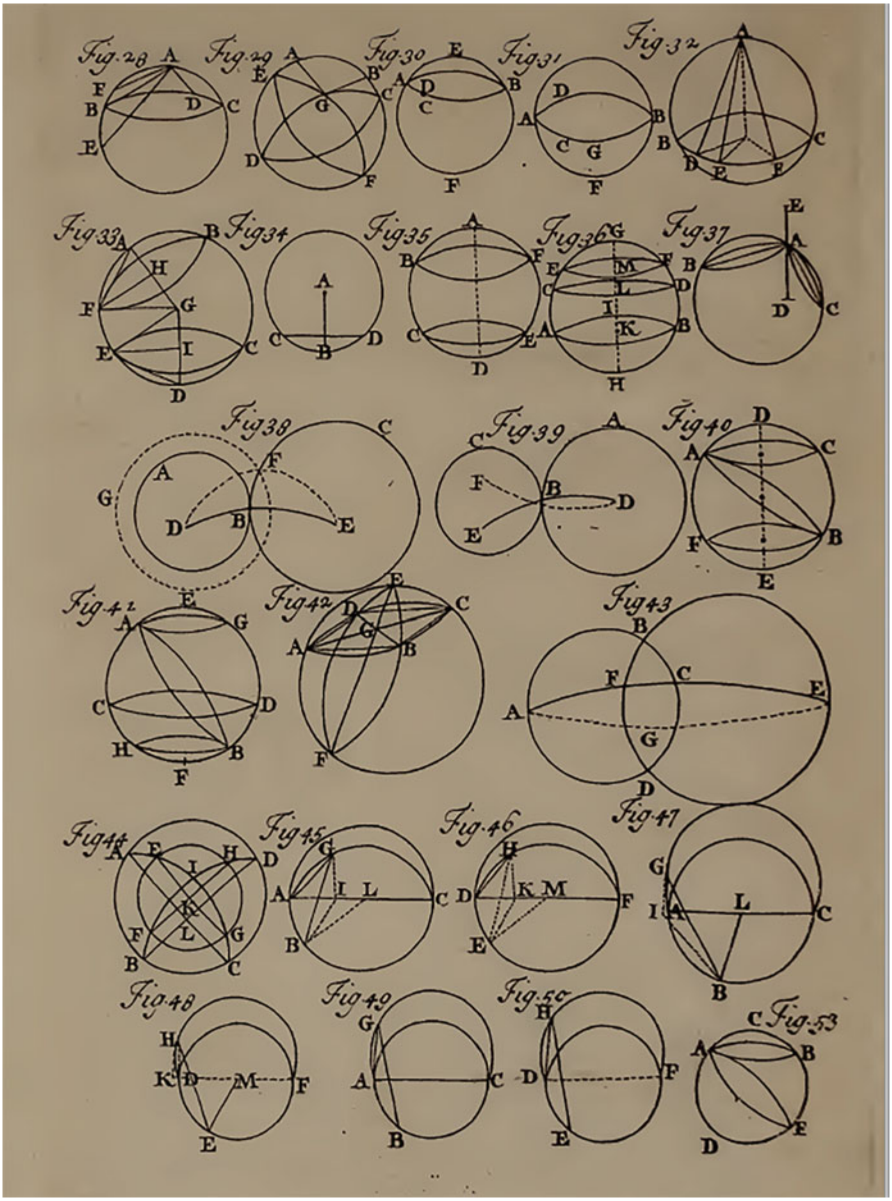


Fig. 2 A plate of figures for Book II of Theodosius' *Spherics*, Clavius' edition (Clavius's commentary on the Spherics of Theodosius Tripolitæ 1721)

represented, and the proposition concerns two arcs of great circles intersecting each other with an acute angle, one which we shall call the equator, and another one, the oblique arc (referring naturally to Fig. 4). In each of the three sub-figures, the point

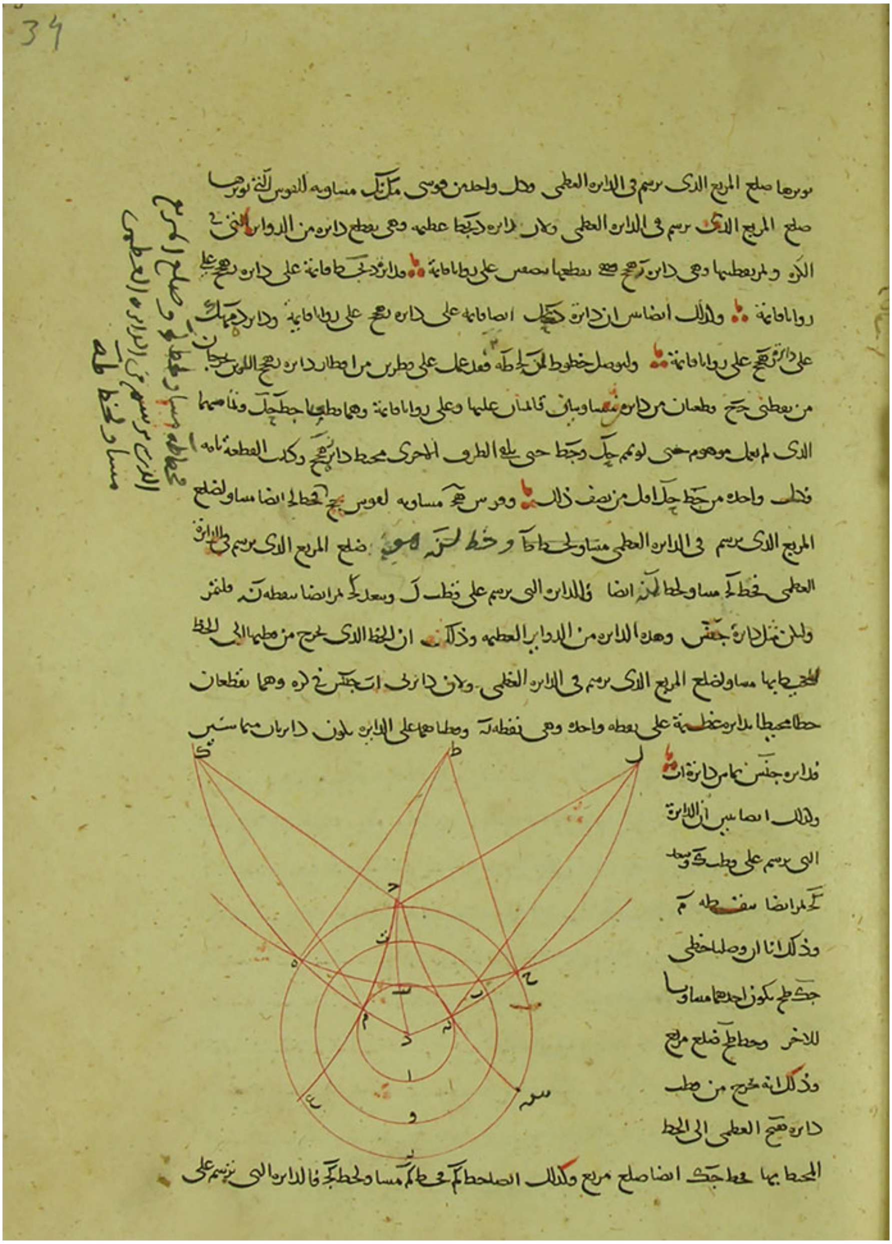


Fig. 3 From Theodosius' *Spherics*, Proposition II.15. Ms. 3464, fol. 30, Topkapi Saray, Ahmet III

on the upper left is a pole of the equator. On the oblique arc we take three points and we consider the two consecutive segments they bound. (There are several segments represented in the figure, and they are all used in the proof, but the statement

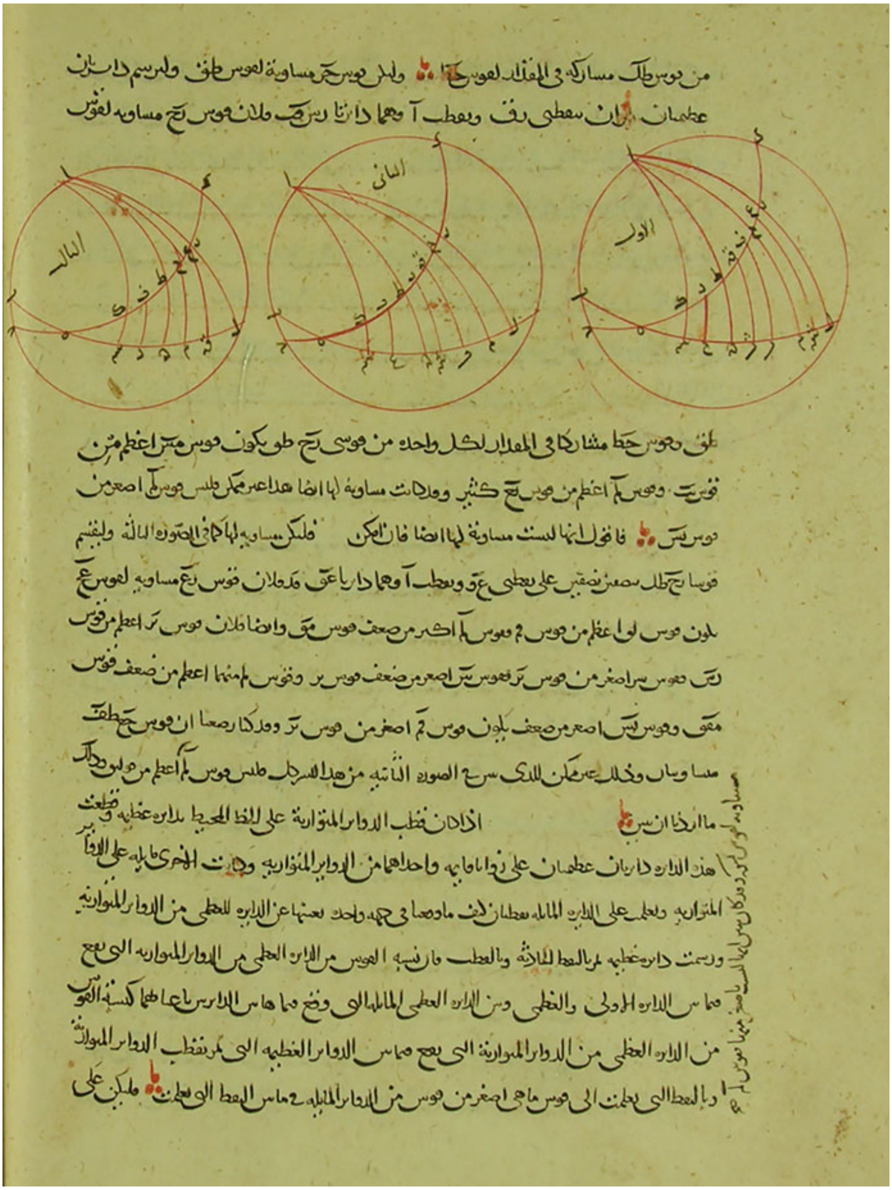


Fig. 4 From Theodosius' *Spherics*, Proposition III.10. Ms. 3464, fol. 50, Topkapi Saray, Ahmet III

involves only two of them.) We project these two segments perpendicularly (that is, using lines emanating from the pole), onto the equator. The proposition says that the ratio of the segment which is closer to the acute angle to its projection is greater than the ratio of the other segment to its projection. The word "monotonicity" that we used to describe the theme of this proposition refers to such an inequality. More precisely, this word refers to the fact that if we start, instead of two consecutive segments on the oblique line, with a sequence of more than two consecutive segments on this line, then the sequence of ratios of one segment to its projection are all different and they vary monotonically: the more a segment is close to the acute angle of the triangle, the more the corresponding ratio is greater. We note that in the analogous Euclidean situation where we project perpendicularly segments on a Euclidean straight line onto another straight line, the conclusion of the statement is that the ratios are all equal.

Despite the fact that his methods were criticized by Menelaus who obtained more general results and who proved them using intrinsic methods, Theodosius remains, to the best of our knowledge, the first mathematician who highlighted such monotonicity properties which are proper to spherical geometry. We mentioned above that from the purely mathematical point of view, in Books I and II of Theodosius' *Spherics* all the statements are Euclidean. On the contrary, this proposition and other similar ones in Book III brings us to the heart of spherical geometry.

Between the Ancient Greek period and the modern one, monotonicity properties in Theodosius' Book II have been featured by Ibn al-Haytham, the preeminent tenth-century Arab mathematician, see (Rashed 2006–2014), in particular pp. 49–105 of the French edition. Roshdi Rashed, the author of this edition of Ibn al-Haytham, considers (rightly) that the developments by Ibn al-Haytham that were based on this proposition of Theodosius and similar ones constitute an early chapter in infinitesimal geometry. Note by the way the related multiplicity of lines in Fig. 4, which suggest a passage to the limit.

Let me mention here that several modern treatises on hyperbolic geometry start with propositions similar to Theodosius' III.9 that we just mentioned, and that such propositions are used in the foundations of the infinitesimal theory of non-Euclidean theory. For instance, in the doctoral dissertation of Louis Gérard, *Sur la géométrie noneuclidienne* (Gérard 1892), defended in 1892 and written under the direction of Poincaré, a monotonicity property similar to III.9 is used for establishing a functional equation which gives directly the trigonometric formulae of non-Euclidean geometry. In fact, the development in Gérard's thesis is done in the context of Lobachevsky geometry, but the same arguments, with inequalities in the reverse direction, give the trigonometric formulae for spherical geometry. I mention this here in order to suggest the power of these inequalities. Later authors have also used the same approach in the analytic development of non-Euclidean geometry, see, e.g., Barbarin's book (Barbarin 1902), one of the early French treatises on non-Euclidean geometry, in which the author makes at the same time a comparison between the spherical and the hyperbolic cases, starting precisely with these monotonicity properties.

From Proposition III.9 and its figure, it is also clear that spherical triangles without the name are present in Theodosius' Book III. Ver Eecke, on p. X-XI of the Introduction to his edition of Theodosius' *Spherics*, rightly states that the latter was surely aware of some of the properties of spherical triangles, but that he did not include this object in his work because he did not succeed in identifying their important properties.

Book III of Theodosius' *Spherics* contains 14 propositions, some of which are much more involved than the one we mentioned. Menelaus, in his *Spherics* gave improved and stronger versions of some of these propositions. We shall review this in the next section, but let me mention here, in this connection, the beginning of Book II of Menelaus' *Spherics*, in al-Harawī's edition. We read the following on the critique of Menelaus toward Theodosius, regarding the method used to prove these propositions (Rashed & Papadopoulos, 2017, p. 684):

> Aḥmad ibn Abī Saʿd al-Harawī said that Menelaus overcame the difficulties of this science, something which was not within the reach of somebody else. Despite the fact that he mastered it, and albeit the majesty of what he was undertaking, he bypassed numerous lemmas which are needed by someone who examines this book and does not have the rank of Menelaus.
>
> We can see him criticizing Theodosius in his book on the Spherics; he sees that the method that he followed is not satisfactory, since it comprises complications and the drawing of numerous <straight> lines, and that Theodosius did not take into account, by this method, the properties of the figures which take place in the sphere, that is, the states of the angles generated by the intersection of circles.

> [I swear] on my life that everything that Theodosius proved in that book, Menelaus proved it in an easier way, and he intended that the proof is by the direct method.[6]

Theodosius' work was influenced by that of Autolycus, in particular, his two treatises *On the moving sphere* and *On risings and settings* written about 300 hundred years before.

Autolycus lived in the fourth century BC, about the same time as Euclid. His treatise *On the moving sphere* contains 12 propositions and is considered as the oldest complete Greek treatise that reached us. The expression "moving sphere" in the title denotes the Celestial sphere which, according to the theory that was in use at that time, rotates uniformly around an axis that crosses the Earth which sits at the center of the universe and which passes through the poles.

From the mathematical point of view, the twelve proposition of Autolycus are easy. I will just limit myself to three propositions from Autolycus' work, my aim being just to give an idea of some of the background of the two *Spherics*, especially the one of Theodosius. The reference I use is (Autolycus de Pitane 1979).

Proposition 1 says the following:

> *If a sphere rotates with a uniform velocity around an axis, all the points situated on the surface of the sphere and that are not on the axis describe parallel circles that have the same poles as the sphere and that are perpendicular to the axis.*

[6]"By the direct method" means without using a proof by contradiction.

Proposition 6 says:

> *If on the (moving) sphere the fixed great circle (that is, the horizon) is oblique to the axis, it is tangent to two equal parallel circles: the one, situated on the side of the visible pole, will always be visible, and the other one, situated on the side of the invisible pole, will always be invisible.*

One may compare this proposition to Theodosius' Proposition II.8 which we quoted above.

Proposition 12 of Autolycus says:

> *If on the (moving) sphere a fixed circle always cuts into two equal parts an arbitrary moving circle of the sphere and if none of the two is neither perpendicular to the axis nor passes by the pole of the sphere, then the two circles are great circles.*

The reader may notice the relation between the last proposition and Proposition I.12 of Theodosius' *Spherics* which we quoted above.

Some propositions of Autolycus' *Moving sphere* are used in Theodosius' astronomical treatises, *On habitations* and *On the days and nights* which we mention in §8.

We pass now to Menelaus' work.

3 Menelaus' *Spherics*: Preliminary Remarks

Talking about Menelaus' *Spherics*, one must necessarily mention the Arab commentators of this work, since this work reached us only through a certain number of Arabic versions, all of them containing more or less substantial differences among themselves.

We start by talking about the division of this work.

There is no general agreement on the division of Menelaus' *Spherics*. Naṣīr al-Dīn al-Ṭūsī, in the preface of his edition which we quote in (Rashed & Papadopoulos, 2015, p. 13), writes the following:

> This treatise consists of three books in some copies, and of two in some others. Concerning the three books, in most cases, the first one contains thirty-nine propositions, the last one twenty-five propositions, and the middle one, in many copies, contains twenty-four propositions, and in the copy of Ibn ʻIrāq, twenty-one propositions. And in some of them, the first one contains sixty-one propositions, the second one nineteen propositions, and the third one twelve propositions. Regarding the copies with two books, the first one contains sixty-one propositions and the last one thirty propositions. And there are differences in some propositions, because some made two propositions a single one and conversely. But in general, the total number of propositions in the treatise is between eighty-five and ninety-one. I referred in the margin to some of the books and the number of propositions they contain, in red and in black, and sometimes I did it in the text.

Ibn Abī Jarrāda, in a manuscript we quote in (Rashed & Papadopoulos, 2017, p. 17) (see also §6 below), mentions a division into four books.

We have discussed at length the divisions of the various versions of the *Spherics* that survive in Chapter I of our edition of al-Harawī's rectification (Rashed and Papadopoulos 2017), which is divided into two books. At the beginning of Book II, a sentence explains this last division (Rashed & Papadopoulos, 2017, p. 684):

> As we have shown the lemmas which we needed (i.e., in the first 61 propositions), let us return now to what Theodosius wanted to show; we prove it using a universal assertion, without admitting in it the impossible.[7] We shall then show his error and rectify what is spoiled.

We conclude from this that Menelaus suggests that his goal in Book II is to improve some propositions of Theodosius. A rapid inspection of the two treatises shows that the propositions in question are those of Book III of Theodosius' *Spherics*. Indeed, Book II of Menelaus' Spherics contains a certain number of propositions that are concerned with the comparison of projections of segments of a great circle onto another such segment, in the spirit of Proposition III.9 of Theodosius' *Spherics* which we recalled above.

Although al-Harawī's rectification of the *Spherics* is divided into two books, the numbering of the proposition is continuous, from 1 to 91.[8] In what follows, it will be practical for us to use this numbering.

Now about the content.

The most important object of study in Menelanus' *Spherics* is the spherical triangle.[9] This is a figure made by three arcs of great circles called its *sides*, each two sides intersecting at a point called a *vertex* of the triangle. A side of a triangle is always assumed to be less than a semicircle. In this way, a side is the unique shortest arc between its vertices. This justifies the fact that in what follows, we shall sometimes call *segment* an arc of great circle which is smaller than a semi-circle, and *angle* an angle made by two segments at an intersection point.

Menelaus avoided as much as possible proofs that use solid Euclidean geometry. This is the point of view of intrinsic geometry of surfaces which we mentioned in the introduction. Furthermore, Menelaus always sought for new proofs. In the few cases where a proof similar to that of an analogous proposition in Euclid's *Elements* is possible in the spherical setting, he either does not give any proof at all (like in Proposition 6 which says that if we draw from the two vertices of the base of a

[7] This means by a direct method, without using proofs by contradiction.

[8] Regarding the number of propositions, one should note that there are statements for which Menelaus gives two different proofs, and each proof is counted as a new proposition (this is the case, for example, of Propositions 8 and 9). Thus, in some sense, there are less than 91 propositions.

[9] Two Arabic expressions مُثَلَّث (threefold) and شَكل ذو ثَلَاثَة أضلَاع ("trilateral figure") are used by the Arabic translators of Menelaus for the Greek term *τρίπλευρον* ("trilateral"). Let us recall that Euclid, in the *Elements* (Definition I.3), defines trilateral figures as those contained by three straight lines.

triangle two arcs of great circles that meet within the triangle, then the sum of these arcs is smaller than the sum of the two legs of the triangle), or he gives a different proof (like in Proposition 8 which says that if two triangles have two pairs of equal sides such that the angle contained by one pair belonging to one of these triangles is greater than the angle contained by the corresponding pair in the other triangle, then the base of the triangle that has a greater angle is greater than the base of the other triangle).

It is conceivable that Menelaus did not give a proof of the famous *Sector Figure* (Proposition 66) because he did not have a proof which does not use the ambient Euclidean geometry. It is also possible that Ptolemy, who uses extensively this proposition in his *Almagest*, gave a proof of it (see (Halma, 1813, pp. 50–55)) because of the absence of proof in Menelaus' treatise. We shall mention several times this proposition in what follows.

Furthermore, a large number of arguments of Menelaus rely heavily on polarity properties on the sphere that do not hold in the Euclidean place.

Thus, in most of the cases, the proofs of Menelaus' propositions of spherical geometry cannot be used for their Euclidean analogues. We shall give explicit examples in the next section.

4 The Propositions

There is more than one way of grouping the propositions of Menelaus' *Spherics* into topics: such a grouping depends on the importance that we attach to each particular theme and on the relations we can make between the various topics. I have collected the propositions of Menelaus' *Spherics* into seven groups, corresponding to seven themes, which I review now.

Group 1: Construction of Angles; Propositions 1, 41–44

I start with Proposition 1: *Given an arbitrary angle, and given an arbitrary point on a given line, we can construct an angle equal to the given angle having its vertex at the given point and a side on the given line.*

This proposition is of paramount importance, and it is abundantly used in the proofs of the rest of the propositions of the *Spherics*. Its proof, unlike the vast majority of the other propositions, uses the solid geometry of the ambient space. More precisely, it relies on constructions from Theodosius' *Spherics* and a criterion from Euclid's *Elements* for the equality of two angles seen from the center of a circle.

A remark is necessary here. We mentioned that most of Menelaus' proofs are intrinsic, that is, they use notions defined on the surface of the sphere (lines, angles, circles, etc.) and not from the ambient space. However, this cannot be true for *all* propositions. The reason is that Menelaus did not start with axioms for spherical geometry, since he thought of the sphere as a surface embedded in Euclidean 3-space, therefore the axioms of solid Euclidean geometry are sufficient for the study of such an object. Thus, some of the propositions, and in particular the first one, necessarily rely on Euclidean geometry. Establishing an autonomous set of axioms for spherical geometry is a modern idea.[10]

Let us now pass to another proposition from the first group, namely, Proposition 41. The general idea is to construct, in a spherical triangle, a segment (that is, an arc of great circle) from a vertex to its opposite side, making with this side an angle equal to one of the angles of the triangle at the extremity of this side. The construction is not possible in the general case, and I will explain why, but let me first state the proposition under Menelaus' hypotheses:

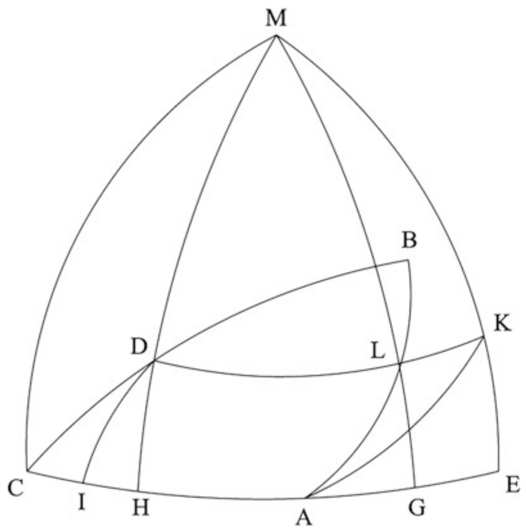
> *Given a triangle ABC such that AB + BC is smaller than a semi-circle and such that the angle A is obtuse, and given a point D on BC, to construct a line from D that cuts the side AB with an angle* $\widehat{DIC}$ *equal to the angle A of this is triangle.*

The proof is contained in (Rashed & Papadopoulos, 2017, pp. 628–630). I have reproduced here Menelaus' figure (Fig. 5), just to suggest the fact that the construction is involved. This proposition is used in later propositions of the *Spherics*, and in particular in the last propositions.

Let me make a few comments on Proposition 41.

The Euclidien analogue of this proposition does not need any particular hypothesis: one takes the parallel to *AB* passing through *D*; it makes with *AC* an angle equal to the angle at *A* of the triangle *ABC*. But in the spherical case, parallel lines do not exist, and the problem is much more complicated. Without specific assumptions, the

Fig. 5 Figure for Proposition 41



[10] An early formulation of a set of axioms for spherical geometry is in (Halsted 1885).

desired line does not always exist: See Fig. 6 for the particular case where each of AB and BC is a quarter of a circle. In this case, a great circle from D cannot make with AC a right angle, because D is not a pole of AC (since the pole is at B). In the case where we allow $AB + BC$ to be greater than a semi-cercle, the situation is not better: Take for example the case where the angle A is acute and C right, with AB and BC both greater than a quarter of a circle and let D be a pole of AC (Fig. 7). Then every line from D to the side AC makes a right angle with it, therefore this angle cannot be equal to the angle at A.

In Proposition 42, the same construction is done under the hypothesis that the angles at B and C are both acute. This proposition is used later (e.g., in the proof of Proposition 45). I have reproduced Menelaus' figure in Fig. 8 here. I have also reproduced (Fig. 9) the page of an Arabic manuscript containing the three propositions, 42, 43, 44 on the same subject. Notice the concision of the writing (the three propositions fit in one page).

Group 2: Comparison Between Triangles; Propositions 4, 8, 9, 13–19 and 20–23 This group consists of congruence and inequality criteria for spherical triangles. More precisely, there are two sorts of theorems in this group: (1) congruence theorems, i.e., theorems saying that if we have two triangles such that some elements (side lengths or angles) of one are equal to the corresponding elements of the second, then the two triangles are congruent; (2) non-congruence theorems, i.e., theorems

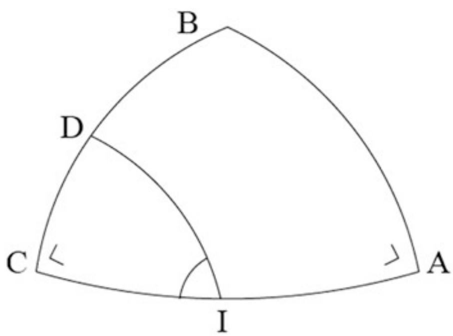


Fig. 6 Proposition 41 fails with the assumption that AB and BC are equal to a quarter of a circle

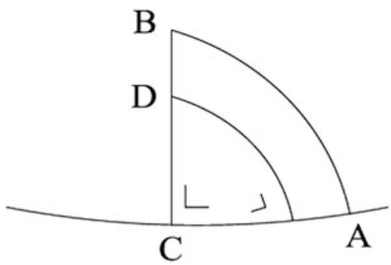


Fig. 7 Proposition 41 fails when AB and AC are greater than a quarter of a circle

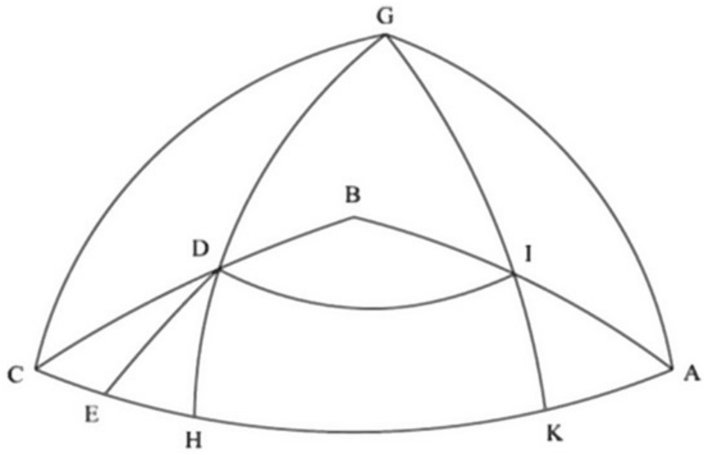


Fig. 8 Proposition 42

saying that if we have two triangles such that some elements (side lengths or angles) of one are greater than the corresponding elements of the second, then some other element of the first is greater (or smaller) than the corresponding element of the second.

An example of a congruence theorem is Proposition 19, which has no analogue in Euclidean geometry. It says that if two spherical triangles have equal angles, then they are congruent.

We note incidentally that Menelaus never makes statements such as "the triangles are congruent", or "the triangles are equal", or equivalent expressions. Instead, he always gives precise statements, saying what side in the first triangle is equal to what side in the second triangle and what angle in the first one is equal to what angle in the second one.

Examples of non-congruence theorems are Propositions 8 and 9, which are in fact a single theorem concerning two triangles having two pairwise equal pairs of sides. They say that if in one of these triangles, the angle contained by two equal sides (the legs) is greater than the angle in the second triangle contained by a pair of legs equal to the pairs of legs in the first triangle, then the first triangle has a greater base than the second. This proposition also holds in Euclidean geometry (Euclid's Proposition I.24).

Group 3: Geometry of Triangles; Propositions 2, 3, 5–7, 10–12 and 24–26
These propositions concern the geometry of individual triangles. For instance, Propositions 2 and 3 concern isosceles triangles. They say that if a triangle has two equal sides (the legs), then these legs subtend with the base equal angles, and conversely, if a triangle has two equal angles at the base, then the two legs subtending them are equal. These are analogues of Propositions I.5 et I.6 of Euclid's

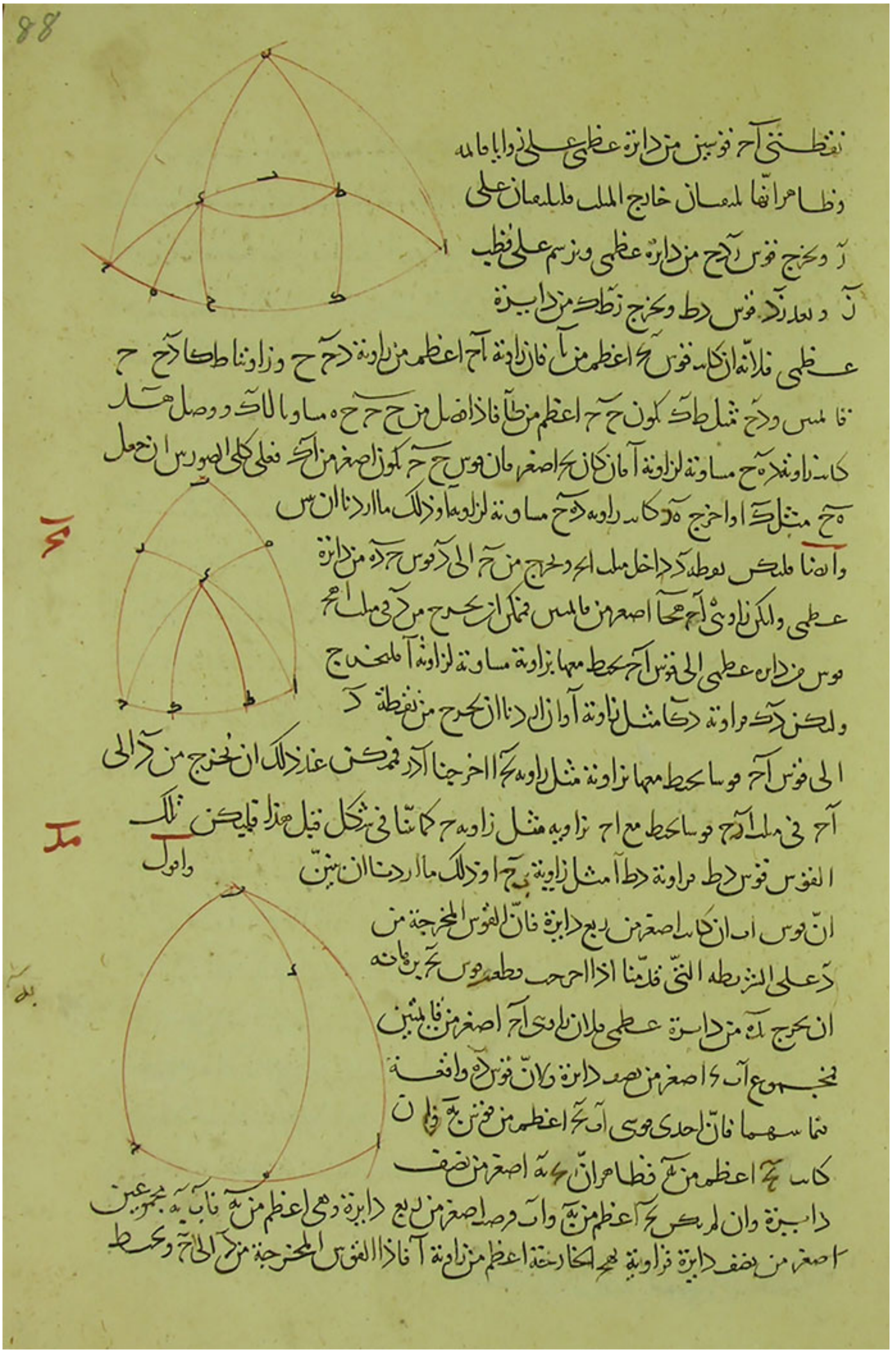


Fig. 9 Propositions 42, 43 and 44. Ms. 3464, fol. 88, Topkapi Saray, Ahmet III

Elements. The proofs though are very different in the spherical case; they are based on the duality between lines and their poles, and they also use some propositions in solid geometry from Theodosius' *Spherics*. Proposition 5 is the triangle inequality for spherical triangles. Let us dwell on Proposition 11, which is used in a crucial way at several places in the *Spherics*:

Let ABC be a spherical triangle. Consider the exterior angle $\widehat{BCD}$. Then we have the following three equivalences:

1. $\widehat{BCD} = \widehat{A}$ *if and only if* $AB + BC$ *is a semi-circle;*
2. $\widehat{BCD} > \widehat{A}$ *if and only if* $AB + BC$ *is smaller than a semi-cercle;*
3. $\widehat{BCD} < \widehat{A}$ *if and only if* $AB + BC$ *is greater than a semi-cercle.*

Let us prove the first statement. We produce the sides AB and AC until we obtain a lune. One of the vertices of this lune is A, and we may set the point D to be second vertex (Fig. 10). If $\widehat{BCD} = \widehat{A}$, then $\widehat{BCD} = \widehat{D}$. In the triangle CBD, since $\widehat{C} = \widehat{D}$, we have $BC = BD$ (this is the congruence criterion of Proposition 3). Since $AB + BD$ is a semi-cercle, $AB + BC$ is also a semi-cercle.

The converse is obtained by reversing the arguments and using the congruence criterion of Proposition 2. The proofs of (2) and (3) are of the same type, based on the notion of lune and using Proposition 7 (inequality criteria for triangles).

Let me remark that if one proves (1) and (2), then (3) follows immediately by a *reductio ad absurdum*, but Menelaus does not allow this kind of reasoning.[11] This is why a proof of (3) is needed.

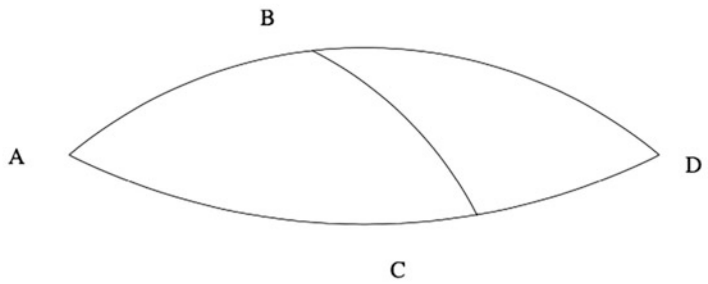


Fig. 10 Proposition 11

[11] Regarding this fact, let us quote Ibn Abī Jarrāda: "I said that Menelaus presented numerous lemmas for the proof of this proposition; but al-Harawī had shown some of them by a *reductio ad absurdum*; and we said that this does not belong to Menelaus' method. I showed all what was presented in the course of the proof without a *reductio ad absurdum*." (see the exact reference in (Rashed and Papadopoulos 2017) p. 41).

Proposition 12 of the *Spherics* is of paramount importance in spherical geometry; it says that the angle sum in any spherical triangle is greater than two right angles. It is a direct consequence of Proposition 11. We reproduce this proof, because it is also a noteworthy illustration of the intrinsic methods of Menelaus.

To prove Proposition 12, let ABC be a spherical triangle and let us consider the exterior angle $\widehat{BCD}$. It suffices to prove that $\widehat{BCD} < \widehat{A} + \widehat{B}$.

We suppose without loss of generality that $\widehat{BCD} > \widehat{A}$ (otherwise the conclusion we seek is obviously true). Then, we construct an angle $\widehat{DCE} = \widehat{A} < \widehat{DCB}$ with E on the extension of the segment AB, as in Fig. 11. Now Proposition 11 applied to the exterior angle $\widehat{DCE}$ of the triangle AEC implies that $AE + EC$ is a semicircle. Thus, $BE + EC$ is smaller than a semicircle. Applying again Proposition 11, this time to the triangle BCE, gives $\widehat{CBA} > \widehat{BCE}$. Adding $\widehat{A}$ to both sides gives $\widehat{BCD} < \widehat{A} + \widehat{B}$, which is the desired result.

Group 4: Division of Triangles; Propositions 27–40

This group consists of a set of propositions that concern inequalities obtained by dividing a spherical triangle by a segment (that is, an arc of great circle) connecting a vertex to the opposite side. The arc may be a side bisector, or an angle bisector, and there are also other kinds of division of a triangle, for instance, by an arc of great circle joining the midpoints of two sides. I will report more precisely on some of these propositions.

I start with a proposition which I find particularly interesting, namely, Proposition 27. It says the following:

> *In a spherical triangle ABC, let D and E be the midpoints of AB and BC, and DE an arc of great circle joining them. Then DE > AC/2.*

The importance of this kind of proposition in modern metric geometry cannot be overemphasized. It is equivalent to a concavity property for distances in spherical

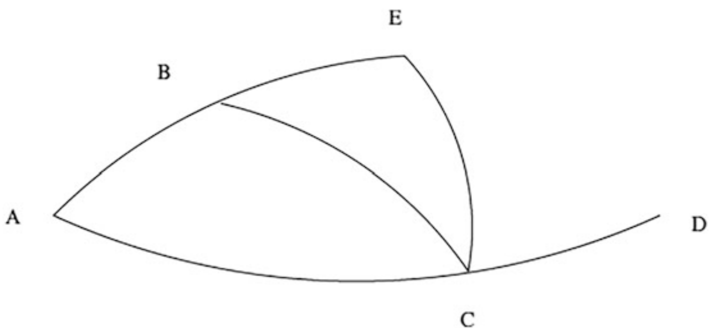


Fig. 11 Proposition 12

geometry, and more generally, in a space of positive curvature. Herbert Busemann took the conclusion of Proposition 27 as a definition of a metric of positive curvature, see (Busemann 1948; Busemann 1955; Busemann 1961).[12]

The proof of this proposition is a beautiful example of the intrinsic methods of Menelaus:

We produce ED until a point G such that $DG = DE$ and we produce AG and CB until they meet at a point H (Fig. 12).

Since $DG = DE$, $AD = DB$ and $\widehat{EDB} = \widehat{ADG}$, the two triangles EDB and ADG are congruent (This is one of the congruence theorems, Proposition 10 of the *Spherics*). Thus, we have $EB = AG$ and $\widehat{EBA} = \widehat{DAG}$.

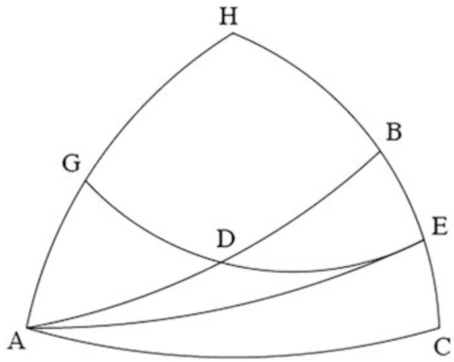
From Proposition 11 on exterior angles which we recalled above, applied the triangle HBA and its exterior angle $\widehat{B}$, $AH + HB$ is a semi-circle. Thus, $AH + HE$ is greater than a semi-circle. Now we join A and E by a segment of great circle. Applying again Proposition 11 to the triangle HEA, we get $\widehat{CEA} < \widehat{EAG}$.

Now the two triangles CEA and EAG have a common edge EA, and we have $CE = AG$, and $\widehat{CEA} < \widehat{EAG}$. Therefore, $AC < EG$ (this is Proposition 8 of the *Spherics*). Since $DE = \frac{1}{2}GE$, then $DE > \frac{1}{2}AC$ and the proposition is proved.

Let us note that to perform the construction in the proof of Proposition 27, the sides of the triangle ABC must be small enough (for instance, we do not want the point H to fall inside the triangle ABC, which could happen if the segment AH is very long). We also note that in the definition of positive curvature one can assume the triangles small (the definition is local).

It is also important to know that there is an analogue of this proposition in the setting of hyperbolic geometry (the other classical non-Euclidean geometry), with the inequality in the conclusion reversed.

Fig. 12 Proposition 27



[12] Busemann was not aware of the fact that the property was stated explicitly in a theorem of Menelaus. I highlighted this fact in (Papadopoulos, 2019).

A more precise form of Proposition 27 can be proved using spherical trigonometry and a formula of Euler giving the area of a spherical triangle. I have included this proof in the first appendix to this chapter.

Among the other propositions in the same group, Propositions 28 and 29 concern again a triangle ABC with a segment DE joining the midpoints of AB and CB, but here, instead of comparing the segment DE to the side AC, one compares angles. More precisely, the propositions give the inequality $\widehat{BDE} < \widehat{BAC}$ under various conditions on the angles of the triangle ABC. Note that in the Euclidean case, we have the equality $\widehat{BDE} = \widehat{BAC}$.

A recent paper provides analogues of propositions 28 and 29 of Menelaus' *Spherics* in the setting of hyperbolic geometry, see (Senapati 2019). I am mentioning this to point out that such statements are still actual.

Group 5: Monotonicity; Propositions 45–62, 64, 65, 81–91

This group contains 32 propositions that express monotonicity properties. We already explained, in §2, what we mean by such properties and we talked about their importance. At the basis of such a property, we generally have a spherical triangle and projections of segments contained in one side onto another side. Such projections may use arcs of great circles making equal angles with the base side on which we project, like in Proposition III.9 we quoted from Theodosius *Spherics*, but there are other kinds of projections: using arcs of great circles perpendicular to the base, using small circles parallel to one side, etc. All these monotonicity results provide inequalities that show fundamental differences between spherical geometry and the two other classical geometries (Euclidean and hyperbolic). Furthermore, like for Proposition 41 which we recalled above, various kinds of hypotheses on the shapes of the triangles (angles or side lengths) are necessary in the statements of these propositions.

There is a progression in the results of Group 5 in terms of the sophistication of the statements and difficulty of the proofs. Some of these propositions are improvements of propositions of Theodosius on the same topic. I will mention two of these propositions, limiting myself to the statements.

Proposition 82 says the following:

Let ABC be a triangle with a right angle at A and acute angle at B, and with BC at most equal to a quarter of a circle. Consider on BC two arcs BD and EG and take the perpendiculars DH, EI, GK to the base AC (Fig. 13). *Then,*

if $BD = EG$ then $AH > IK$ and $AB - DH < EI - GK$;

if $AB - DH = EI - GK$, then $BD > EG$;

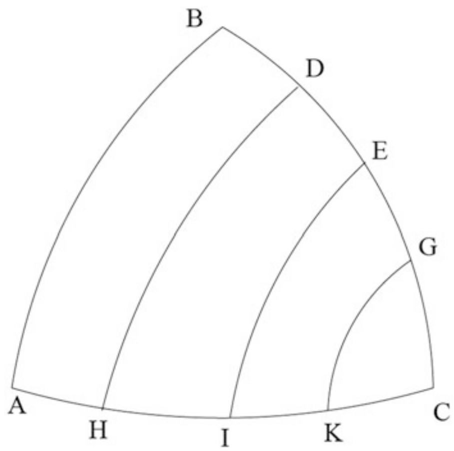
if $BD + (AB - DH) = EG + (EI - GK)$, then $BD < EG$;

if $BD - (AB - DH) = EG - (EI - GK)$, then $BD < EG$.

Furthermore, in every case, we have

$$\frac{\operatorname{crd} 2BD}{\operatorname{crd} 2EG} > \frac{\operatorname{crd} 2(AB - DH)}{\operatorname{crd} 2(EI - GK)}.$$

Fig. 13 Proposition 82



Proposition 89, in the same group, says the following:

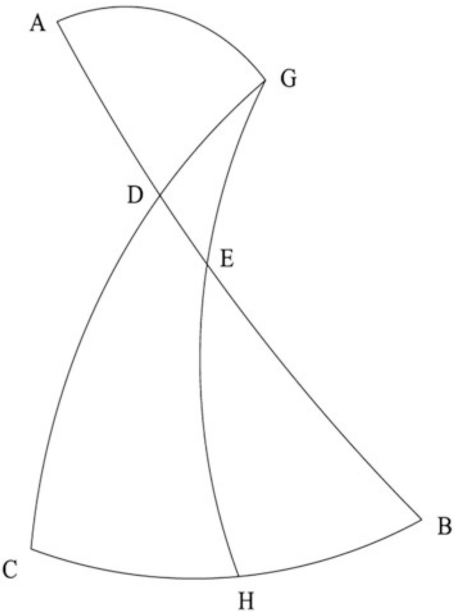
Let AB and BC be two arcs of great circles. Consider two points D and E on AB and take the perpendiculars DC, EH on BC (Fig. 14). *Then,* $\frac{\mathrm{crd}2CH}{\mathrm{crd}2DE}$ *is equal to the ratio of the product of the diameter of the sphere by the diameter of the (small) circle parallel to BC and tangent to AB to the product of the diameters of the two circles parallel to BC and passing by D and E.*

In Fig. 15, we have reproduced the page containing the last three propositions (89 to 91) from Ms. 3464; the reader can notice again the conciseness of the writing. Let me mention in this regard that I know of no written proofs of these three propositions, or of any of the last eleven propositions of the treatise. Furthermore, all the texts that reached us contain some ambiguities in the statements of these propositions. We have discussed the problems raised by these propositions in our edition (Rashed and Papadopoulos 2017).

Group 6: The Sector Figure and Its Applications; Propositions 66–70 and 72–80
Proposition 66 is the so-called Sector Figure. establishes a relation between six sides of a figure called "complete quadrilateral". This is the figure obtained by producing two pairs of opposite edges of a convex spherical quadrilateral until they pairwise meet at a point.[13] A picture of a complete quadrilateral is given in Fig. 16, where we start with the quadrilateral *ADEG* and we produce *AG* and *DE* until they meet at the point *B*, and *AD* and *GE* until they meet at the point *C*. Notice that each convex spherical quadrilateral gives rise to four complete quadrilaterals, depending on the choice of two adjacent sides. Using the notation of Fig. 16, the proposition says that we have:

[13] Convexity is understood here with respect to the intrinsic geometry of the sphere: A figure is said to be convex if (1) it is contained in an open hemisphere; (2) for any two points in this figure, the arc of great circle joining them is contained in the figure.

Fig. 14 Proposition 89



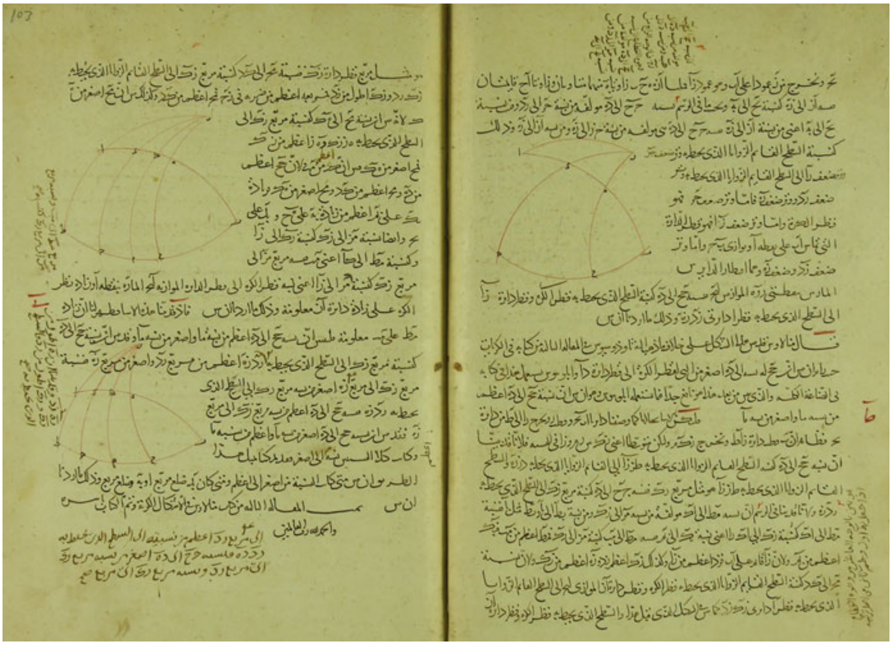


Fig. 15 Propositions 89 to 91 of the *Spherics*, Ms. 3464, fol. 99, Topkapi Saray, Ahmet III

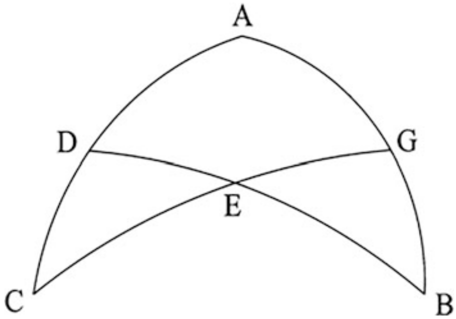


Fig. 16 The Sector Figure (Proposition 66)

$$\frac{\text{crd } 2AG}{\text{crd } 2BG} = \frac{\text{crd } 2AC}{\cancel{\sin}\ \text{crd } 2CD} \cdot \frac{\text{crd } 2DE}{\text{crd } 2EB}.$$

Expressed in the language of sines—which Menelaus did not have, but which was used by the Arab commentators of the *Spherics*—the relation is:

$$\frac{\sin\ 2AG}{\sin\ 2BG} = \frac{\sin\ 2AC}{\sin\ 2CD} \cdot \frac{\sin\ 2DE}{\sin\ 2EB}.$$

In some sense, this spherical proposition is close to the spherical Sine Rule for triangles; indeed the Sine Rule can be deduced easily from the Sector Figure, and conversely, there is a very short proof of the Sector Figure using the spherical Sine Rule. One should note however that the discovery of the spherical Sine Rule occurred only in the eleventh century. In the meanwhile, the Sector Figure has been the only trigonometrical tool[14] used in astronomy. In particular, the proposition was heavily used by Ptolemy in the *Almagest* as the main ingredient for his astronomical calculations. Furthermore, it is the only result of the *Spherics* which is quoted under the name "Menelaus' Theorem".[15]

Proposition 66 occupies a particular place in the *Spherics*. It is situated among a series of propositions whose proofs are completely intrinsic to the surface of the sphere, but the proof given to this proposition in al-Harawī's redaction and in all the other Arabic manuscripts that survive is, like the one given by Ptolemy in his Almagest, purely Euclidean. (In fact, the proofs given by the Arabs are either Ptolemy's or a small variation it.) At the same time, such a proof was considered as faulty by the various Arab commentators and editors of the *Spherics* because it is

[14] We are using here the the word "trigonometry" for a formula that gives relations between the elements of a figure which is not necessarily a triangle.

[15] There are other names for this proposition. Ptolemy, in the *Almagest*, used the expression "secant figure", which is also the name that was generally adopted by the Arabs الشَّكل القَطَّاع in their commentaries on Menelaus' and Ptolemy's works and in the treatises they wrote on spherical geometry.

not in the spirit of Menelaus' methods. Whether Menelaus had a proof of this proposition that does not rely on solid Euclidean geometry remains a mystery.

The other propositions in this group are proved using the Sector Figure. They involve divisions of spherical triangles by arcs of great circles joining vertices to the sides that subtends them. These results concern angle and external angle bisection (Propositions 73 and 79), angle division into three equal parts (Propositions 74 and 75), division by altitudes (Propositions 78, 80), and various other kinds of divisions. These propositions may all be proved using the spherical Sine Rule, and the Sector Figure Theorem is used in the absence of this theorem. We have discussed in detail all these propositions in our commentary in (Rashed and Papadopoulos 2017). In particular, Proposition 80 needs a careful discussion, and we comment on it here. In al-Harawī's version, the proposition says the following:

The altitudes that are produced from the angles of a triangle to their opposite sides intersect in one point.

The statement as it stands is not correct and needs a hypothesis. Let me explain this. Consider the example of a spherical triangle whose three angles are right. In this case, any arc of great circle drawn from a vertex to the opposite side that subtends it is an altitude. Obviously, three arbitrary such arcs drawn from the three vertices to the opposite sides do not necessarily intersect in a common point. Thus, the theorem fails in this case. There are much more general cases where the theorem fails.

The statement becomes correct if we add to it a further hypothesis. One possibility is the following:

Suppose that in a given triangle two altitudes intersect in a point within the triangle. Then there exists a third altitude passing through this point.

With such a hypothesis, the proof in al-Harawī's edition of Menelaus, of the fact that the three altitudes intersect in one point, becomes correct. This proof involves several applications of the Sector Figure Theorem.

Group 7: Proposition 71

I have dedicated a special section to Proposition 71 because I consider it as very special, both mathematically and historically, and I will explain why.

First, the statement is particularly interesting. It expresses a non-trivial property of spherical triangles, and I have not seen anywhere else, in the old and modern literature, a statement that bears any resemblance to it. Secondly, the proof of this proposition has given rise to an abundant literature by the Arab mathematicians who have revised and commented Menelaus' text. It seems that it was precisely in trying to prove this proposition that these mathematicians discovered the Sine Rule and the other formulae of spherical trigonometry, see the comments in (Rashed and Papadopoulos 2017) and Sect. 7 below. Moreover, the proof uses an important property that was obviously known to Menelaus, namely, the invariance of the spherical cross ratio, although he used it without proof. The invariance of the cross ratio is one of the main tools of projective geometry.

I have given a detailed proof of this proposition and reviewed several points in the history of the important mathematical activity that took place around it, between the ninth and the thirteenth centuries, in my two papers with Rashed (Rashed and Papadopoulos 2014; Rashed and Papadopoulos 2015) .

Let me recall the statement of this proposition:

Consider two triangles ABE and LMO with right angles at B and M, and where the angles A and L are acute and equal. Then,

$$\frac{\mathrm{crd}\,(AE+AB)}{\mathrm{crd}\,(AE-AB)}=\frac{\mathrm{crd}\,(LO+LM)}{\mathrm{crd}\,(LO-LM)}.$$

I have reproduced the figure that accompanies this proposition in Fig. 17 here.

As a matter of fact, inspired by this kind of result in spherical geometry, several theorems in hyperbolic geometry are proved in the recent paper (Papadopoulos and Su 2017). In particular, a more precise analogue of Proposition 71, in the setting of hyperbolic geometry, is given there (Theorem 2.1), based on hyperbolic trigonometry. Going backwards, a more precise form of the same proposition in the setting of spherical geometry can be given using the same outline, and we have reproduced it in the first appendix to the present paper. Let me also mention here that Menelaus' theorem as well as the invariance of the cross ratio which is used in the proof of Proposition 71 have been adapted and used in the case of hyperbolic geometry, see (Papadopoulos and Su 2017; Papadopoulos and Yamada 2013). I am saying all this to give a hint of the freshness of the ideas behind this and other propositions of the *Spherics*.

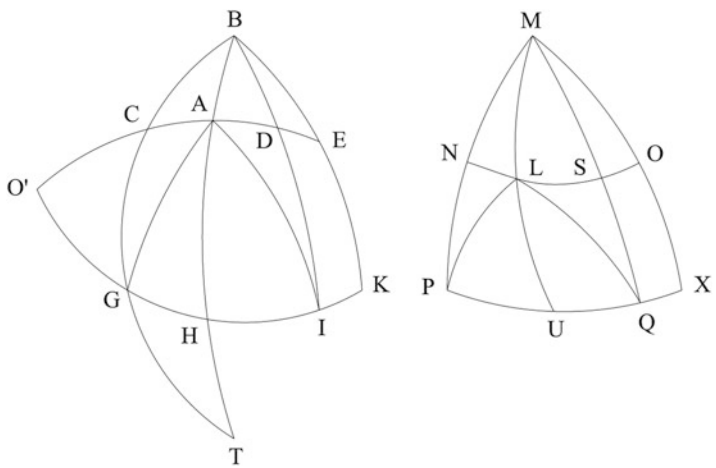


Fig. 17 Proposition 71

5 Pappus

Pappus of Alexandria is among the rare mathematicians from Greek Antiquity who have embraced Menelaus' intrinsic methods. Book VI of his *Mathematical collection* contains a sequence of theorems on spherical geometry, some of which are proven using the intrinsic geometry of surfaces, in the pure style of Menelaus. As a matter of fact, the first proposition in this book concerns the triangle inequality, for a spherical triangle. Pappus' statement is the following:

> *If, on a spherical surface, three arcs of great circles which are all smaller than the semicircle mutually cut each other, then two of these arcs, taken in an any way, are greater than the remaining arc.*

The reader may notice that there is no mention of spherical triangle in this statement, but right after the proof of this proposition, Pappus notes: "Menelaus, in his *Spherics*, calls such a figure a trilateral (τρίπλευρον)". Pappus uses this word in the subsequent propositions. Another remark that fits here is that whereas Menelaus' proof of the triangle inequality is purely intrinsic, Pappus' proof uses the ambient Euclidean solid geometry (it is based on Proposition XI.20 of the *Elements*, which is a triangle inequality that concerns the three dihedral angles that form a solid angle.) The reason for which Pappus' proof must necessarily use Euclidean geometry is that this is the first proposition he gives on spherical geometry. As we already said, regarding Menelaus, spherical geometry, for both authors, based on the axioms of Euclidean geometry, and the first proposition, in any case, must use the Euclidean setting.

But the proofs of the next propositions of Pappus are intrinsic to the sphere. The statements are not always equivalent to those of Menelaus, and some of Pappus' propositions complement some propositions of his predecessor.

Let us now mention a few propositions of Pappus.

Proposition 2 says the following (Ver Eecke, 1982, p. 371):

> *If two arcs of great circles are drawn internally, above one side of a trilateral, then they are smaller than the remaining two sides of the trilateral.*

In other words, the proposition says that if we take an interior point in a triangle and if we join by two arcs of great circles that point to the vertices of the base, then the sum of these two arcs is smaller than the sum of the two legs of the triangle. This proposition is equivalent to Proposition 6 of the Spherics whose statement is (Rashed & Papadopoulos, 2017, p. 518):

> *If we produce from two extremities of a side among the sides of a triangle two arcs of great circles that meet within the triangle, then they are smaller than the remaining sides.*

Then, follow 9 propositions which are proved using Menelaus' methods. Some of the statements are close to propositions in Menelaus' Spherics, with slightly different hypotheses regarding the bounds on the lengths of the sides, and some other statements are contained in Theodosius *Spherics* (with proofs using Menelaus' intrinsic methods). As an example of Pappus' intrinsic methods, let us consider his proposition VI.4, which is equivalent to Proposition 38 of Menelaus (with a slightly different hypothesis concerning the lengths of the sides). The latter belongs to Group 4 of the Spherics, concerning the division of a triangle.

Menelaus' statement is the following (Rashed & Papadopoulos, 2017, p. 618):

If the sum of two unequal sides in a triangle is smaller than a semicircle, if we separate from the two extremities of the remaining side two equal segments, and if we produce from the two resulting points two arcs till the angle that the two sides contain, then they contain with the two sides two unequal angles, the greater at the smaller and the smaller at the greater, and the two produced arcs are smaller than the two sides of the triangle.

Pappus' statement is the following (Ver Eecke, 1982, p. 373):

Let four arcs of great circles AB, AC, AD, AE cut an arc of great circle BE, let the arc BC be equal to the arc DE, and let each of the arcs AB, AC, AD, AE be smaller than a quadrant. Then, the sum of the arcs BA, AE is greater than the sum of the arcs CA, AD.

We reproduce Pappus' proof in order to show the method (see Fig. 18).

Let us cut the arc *CD* into two equal parts at the point *Z*. Let us draw through the points *A*, *Z* the arc of great circle *AZH*, and let us set the arc *ZH* to be equal to the arc *AZ*. Let us draw through the points *H*, *E* the great circle *HEK* and through the points *H*, *D* the great circle *HD*Θ. Since the arc *HZ* is equal to the arc *ZA* and the arc *DZ* is equal to the arc *ZC*, it follows that the arc *DH* is also equal to the arc CA. For the same reason, the arc *EH* is also equal to the arc *BA*. But since the two arcs *AD*, *DH* are established internally on one of the sides *HA* of the trilateral *HEA*, it follows that the arcs *AD*, *DH* are smaller than the arcs *AE*, *EH*. Thus, the arcs *AE*, *EH* are greater than the arcs *AD*, *DH*. But the arc *EH* is equal to the arc *AB* and the arc *HD* is equal to the arc *AC*. Therefore, the sum of the arcs *BA*, *AE* is greater than the sum of the arcs *CA*, *AD*. This is what we had to prove.

I know of no other mathematician than Pappus before the modern times (that is, before the nineteenth century) who integrated in his writings Menelaus' methods of intrinsic geometry of the sphere.

6 Translations, Rectifications and Difficulties

Everything we know about the history of the translations and the various versions of Menelaus' *Spherics* is reported on in the first part of the edition (Rashed and Papadopoulos 2017), pp. 3–123 and in the two articles (Rashed and Papadopoulos 2014; Rashed and Papadopoulos 2015). I will summarize here some important points on that history.

Several Arabic versions of the *Spherics* were written starting in the ninth century. Al-Harawī's version, which relies on a version by al-Māhānī and which is a

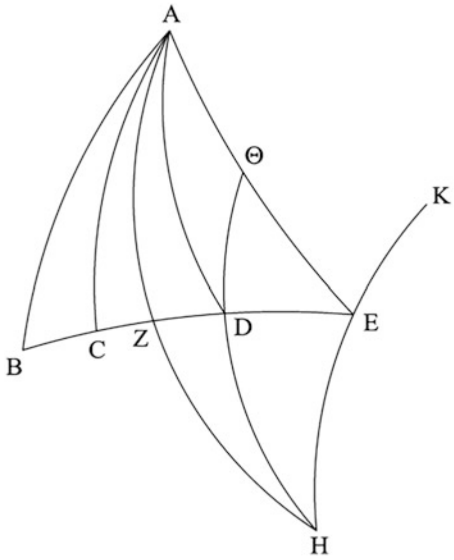


Fig. 18 The figure for Pappus' Proposition VI.4

rectification[16] rather than a translation, is published for the first time, together with an English translation, in the volume (Rashed and Papadopoulos 2017). Besides al-Harawī's version, this volume contains the first critical edition of a fragment containing the 36 first propositions of a translation done in the same century, which we think is the oldest translation that survives. We know that another translation was also done in the same century, by the Arab mathematician and Hellenist Abū ʿUthmān al-Dimashqī. A Latin translation by Gerard of Cremona (1114–1175) and a Hebrew one by Jacob ben Makhir Ibn Tibbon (ca. 1273) were done from Arabic texts. According to the latter, there existed another Arabic translation by Ishāq ibn Ḥunayn, or by his father Ḥunayn ibn Ishāq. Besides these translations, there are also rectifications by Abū Naṣr Manṣūr Ibn ʿAlī Ibn ʿIrāq (born around the year 950), Naṣīr al-Dīn al-Ṭūsī (thirteenth century), Ibn Abī Jarrāda (thirteenth century), Muḥyī al-Dīn al-Maghribī (thirteenth century), al-Yazdī (fifteenth century), and certainly others. A German translation of the rectification by Irāq was published in 1936 by M. Krause, (Krause 1936). These versions differ on several points, with additions of lemmas and intermediate propositions, etc., but the overall mathematical content is the same. There are also writings and developments based on Menelaus' *Spherics* by Ibn Hūd, Thābet ibn Qurra, al-Bīrūnī, al Qiftī, Ibn al-Haytham, and others. We have discussed all this in the volume (Rashed and

[16] the Arabs used the words Islāḥ إِصلَاح, which means "reform", or "revision" (an example is the treatise by Abū Naṣr Manṣūr ibn ʿAlī ibn ʿIrāq, *Islāḥ Kitāb Mānālāwus fī al-ashkāl al-kuriyya*, Leiden, Or. 930), and Taḥrīr تَحرير, which means "redaction" (an example is Nasīr al-Dīn al-Ṭūsī's Taḥrīr Kitāb Mānālāwus fī al-ashkāl al-kuriyya [Hyderabad, 1359 H.]).

Papadopoulos 2017) where we have also made comparisons between significant passages of the existing Arabic versions that flourished from the ninth century on and continued till the thirteenth, and the Latin and Hebrew versions.

We have quoted, in the introduction of the present paper, an excerpt of the Preamble of al-Harawī's rectification, in which the latter expresses his wonder and admiration for Menelaus' geometric reasoning, which he considers as completely new. In the same Preamble, al-Harawī mentions the difficulties inherent to the various translations, due to the fact that the geometers of his time were not capable of understanding several propositions and proofs of this treatise. He writes (Rashed & Papadopoulos, 2017, p. 24):

> When, from that book, the geometers received what they were not used to, and to nothing similar from all other kinds of geometries they knew, they found it difficult. The bad translation and the fact that it was far from being understandable added to their apprehension, so they discarded it and they did not work on it. Nevertheless, no one among them denied the eminence of that book, but they acknowledged that what he showed in this way is easier and more valuable that anything which anybody else invested in.
>
> It was said that a group of geometers had the intention to rectify it, and since they were unable to do it, they asked the help of al-Māhānī, who rectified the first book and some propositions of the second, and he stopped at a proposition of which they said that its aim was difficult, and whose explanation was also hard.
>
> [...] I meditated on what al-Māhānī rectified and I saw during this short period of time that there were defects, and I carefully reviewed what needed to be corrected in the expression, in the meaning, and in the proof. And I persisted until the end of the book, until the proposition which the geometers imagined that al-Māhānī left aside and was not able to rectify. I also found a rectification that was far from being correct due to some modern geometer who claimed he rectified part of it, and left aside the rest. In the thing he said that he rectified there are some corrupt things, from which it was clear to anybody who examined them that he did not understand the purpose of this man [i.e., Menelaus].

According to the testimony of Naṣīr al-Dīn al-Ṭūsī, the situation of the existing versions of the *Spherics* in the thirteenth century was not better, except for the one by Ibn 'Irāq. He writes:[17]

> When I reached the book of Menelaus on the spherical figures, I found it in many different copies, with undetermined problems, together with disappointing rectifications, like the rectifications of al-Māhānī, of Abī al-Faḍl Aḥmad ibn abī Sa'ad al-Harawī and others, some of which are incomplete and others wrong. I was puzzled for what concerns the proof of some problems, until I obtained the rectification of Abū Naṣr Ibn 'Irāq.

It is likely that Ibn 'Irāq's is a rectification of the translation due to the Hellenist and mathematician Abū 'Uthmān al-Dimashqī. We refer again the reader to our discussion in (Rashed and Papadopoulos 2017).

In addition to his redaction of the *Spherics*, Ibn 'Irāq wrote a short treatise which concerns Proposition 71 which we discussed above (in our classification on §3, we

[17] Reference in (Rashed & Papadopoulos, 2017, p. 16)

included it in Group 8), titled *The Rectification of a Proposition of the Book of Menelaus on the Spherics*. A translation of this treatise is provided in the article (Rashed and Papadopoulos 2014).

Ibn Abī Jarrāda, a contemporary of Tūsī, considered that the most popular rectification that existed at his time was still that of al-Māhānī/al-Harawī, which, despite the fact that, according to him, it has many short-comings. Let us quote him (see (Rashed & Papadopoulos, 2017, p. 18)):

> I found that book (Menelaus' *Spherics*) so altered as none of the other geometry books, for reasons that were specific to it, among which are the difficulties in the proofs, for someone who was supposed to clarify them, because of the necessity of these conditions. And among these difficulties is the fact that Menelaus over-summarized it, as he mentioned in the Preamble. And also, among these difficulties, the multiplicity of the problems hidden in the course of the proofs, and of the abandoned ones to which he refers. Among these, that he meant by the ratio of the arcs the ratio of the double of their chords in the entire third book and in part of the fourth, and he meant by the ratio of the arcs their own ratio in other parts of the fourth book, thinking, because of his extreme power in that field, that this would not be hidden to anyone who will undertake the clarification of his book. That was his basis, and it became difficult to anyone who came after him to recognize this. Part of <the difficulties> is also the fact that he refers in the proofs of the propositions of the fourth book to what he has proved in one of his books, titled *The Measurable Figures*, a book which is unknown, or else it is foreign and did not reach us.
>
> A group <of people> proposed to rectify this book; all of them had failed the aim, they stopped before reaching their goal and they did not comply with Menelaus' method. Among them are those who were not able to penetrate it. There are also those who rectified one part of the book, in the given order, and were incapable of going through it, like al-Māhānī, who reached a proposition which he was incapable of solving, and thus he did not complete it. Among them are also those who pretended they have rectified all the book, deluding and disappointing anyone who examines it. They relied on the fact that when one encounters a difficulty in understanding it, he attributes this lack either to himself or to the corruption of the transcription.
>
> Even though [al-Harawī's] rectification is the most famous one among those which the book was subject to, the best that he brings is where he limits himself to repeat Menelaus' expression. But as soon as he proposes himself to prove a neglected <problem> or to clarify a difficulty, or something similar, then he leaves the conditions of Menelaus, like in the first lemma by which he proved the proposition where al-Māhānī stopped, proving it with common sections whereas what he brought does not suffice for the completion of that proposition. Or as he did in the thirty-seventh proposition of the first book: he resorted to a *reductio ad absurdum* for what is hidden in the proof of Menelaus. Regarding the fourth book, which is the third in his rectification, he did not understand anything of it except the ninth proposition and he attributed everything which is in that book concerning the ratios of arcs to the ratios of chords of the double. Everything that I mentioned should be clear to anyone who examines the rectification of al-Harawī.

This translation and rectification work on Menelaus' *Spherics* generated an important mathematical activity that changed the course of Arabic mathematics. This is the subject of the next section.

7 On the Impact on Arabic Mathematics

Among Menelaus' propositions that had a major impact on Arabic mathematics is the Sector Figure (Proposition 66). Several treatises were written on the applications of this proposition. One such oldest known complete treatise is of Thābit ibn Qurra (d. 901), titled *A Treatise on the Sector Figure*, is reviewed and commented on in detail by H. Bellosta in her article (Bellosta 2009). We quote from the Introduction to this treatise (Bellosta, 2009, p. 365):

> I know that people never discussed any geometrical proposition used in astronomy as much as they did for this proposition. They did not overrate its reputation and the reason of their interest for it; this is what we know from the number of times it was used and the extreme need that we have for it in the spherical science. This is certainly the main direction on which many works in astronomy are based. Even though others than Ptolemy obtained this proposition and talked about it, nobody, to our knowledge, improved its statement or < gave> any arrangement or revision that matches its proof.

Other developments concerning the Sector Figure were reported on by Abū al-ʻAbbās al-Nayrīzī and Abū Jaʻfar Muḥammad ibn al-Ḥusayn al-Khāzin, in their commentaries of the Almagest. There is a large literature on the impact of this proposition on Arabic mathematics. The reader may refer to the work of M.-Th. Debarnot (Debarnot 1996; Debarnot 2002), and the latter's edition of the tenth century work of al-Bīrūnī (Debarnot and Al-Bīrūnī 1985). We have commented on this proposition and its applications in the commentary contained in our edition of the *Spherics*; see (Rashed & Papadopoulos, 2017, pp. 292–337).

In the thieteenth century, Naṣīr al-Dīn al-Ṭūsī, wrote a treatise on the Sector Figure (Naṣīr al-Dīn al-Ṭūsī 1891). in which he showed how a complete trigonometric system of spherical geometry can be derived from this proposition. Al-Ṭūsī's treatise was translated into French and edited by Alexandre Carathéodory[18] in 1891 (Naṣīr al-Dīn al-Ṭūsī 1891). See the detailed review in (Papadopoulos 2018). It is only in the eighteenth century that Leonhard Euler published a treatise containing such a complete system, obviously, without being aware of al-Ṭūsī's work, see (Euler 1753). Naṣīr al-Dīn uses in his work the notion of polar triangle. This notion and the spherical Sine Rule were already introduced in the tenth century by Ibn ʻIrāq and Abū al-Wafāʼ al-Būzjānī (940–998). We have recalled, in (Rashed & Papadopoulos, 2017, pp. 795–801), the story of these discoveries and we have pointed out the confusion that reigned in the eighteenth and nineteenth century literature concerning the history of spherical trigonometry, in particular concerning the discovery of the polar triangle.

Naṣīr al-Dīn's treatise is a remarkable historical document that gives a very good idea of the state of spherical geometry, as developed by the Arabs between the ninth and the thirteenth centuries. At the end of Book IV, Naṣīr al-Dīn says that the

[18] Alexandre Carathéodory was a great-uncle of the famous mathematician Constantin Carathéodory.

Ancients have used the quadrilateral figure with confidence, without treating the large number of cases, and he mentions as examples Menelaus' *Spherics* and Ptolemy's *Almagest*. But the moderns, he says, in order to avoid engaging themselves in the examination of the multitude of ratios and the lengthy considerations that the use of compounded ratios requires, imagined and studied other figures, which replace the quadrilateral, which are as much useful and which do not lead to a multitude of cases. The presentation of these methods is the subject of Book V of his treatise.

We already mentioned another theorem of the *Spherics* that had a large impact on Arabic mathematics, especially on the development of spherical geometry with works by al-Būzjānī, al-Khujandī, Ibn Labbān, Ibn ʻIrāq, and al-Bīrūnī. This is Proposition 71. As we already recalled, Menelaus did not give a proof of this proposition, but only an outline of a proof. Writing a complete proof became a big challenge that was taken up by several mathematicians, in Baghdad and elsewhere, between the ninth and the thirteenth centuries. We already mentioned that the proof of this proposition uses an invariance property of the spherical cross ratio, and to prove this invariance, the mathematicians of the tenth century used the spherical Sine Rule which they had established. It is conceivable that the development of this rule was motivated by the research in spherical geometry due to the effort to prove this proposition of Menelaus. In fact, the search for a proof of this proposition acted as a motivation for reshaping the whole field of spherical trigonometry. Likewise, it is possible that the discovery by Ibn ʻIrāq of the notion of polar triangle was also motivated by his work on the proof of Menelaus' Proposition 71. In his *Rectification of a Proposition of the Book of Menelaus on the Spherics*, which is edited and translated in the paper (Rashed and Papadopoulos 2014), Ibn ʻIrāq presents a set of propositions on spherical geometry that are useful in the proof of this proposition, but that are also of independent interest. At the same time, he gives a proof of the proposition under assumptions that are weaker than the ones that Menelaus made. While he worked on the proofs, Ibn ʻIrāq declared that not only Proposition 71, but the whole treatise of Menelaus' *Spherics* needs to be corrected. He reported on the attempts by al-Māhānī and al-Harawī on correcting Menelaus' treatise and he discussed several points in their proofs. We have discussed these points in our article (Rashed and Papadopoulos 2014).

Understanding the proof of Proposition 71 became an important matter, not only for mathematicians, but also for astronomers and other scientists. The last section of our paper concerns astronomy.

8 On the Impact on Astronomy

In his introduction to his rectification, al-Harawī writes (Rashed & Papadopoulos, 2017, p. 501): "Ptolemy relies on this book, especially in the second book of the *Almagest* concerning angles and triangles produced by intersections of great circles." In the same introduction (Rashed & Papadopoulos, 2017, p. 504), he writes: "This book has an immense value in astronomy and the properties of the figures that occur when the horizons intersect the ecliptic, the orbs of the meridian, the equator and the circles called the ephemerides that the sun draws at the beginning of the zodiac." Concerning Menelaus' Proposition 66 (the Sector Figure), al-Harawī; writes (Rashed & Papadopoulos, 2017, p. 684):

> And he worked out in that book the figure which Ptolemy calls the "sector", and he constructed upon it several figures. But Ptolemy uses, from that book, numerous propositions in the second book of his treatise the *Almagest*, without attributing them to anybody and without proving anything. Indeed, all what he uses, for what concerns angles, arising from the intersection of the ecliptic and the horizons, and other things, can be proved by this book. The lemmas we need are among those which Ptolemy introduced, on the intersection of two lines and on the composition of the ratios which are made up thereof. And we find that Menelaus moved in that book to the Sector Figure in a way which does not fit his method, since he did not set for this passage any lemma or any discourse and he also did not make it the starting of a book. So either the lemmas of that figure were known by them, widespread, or they disappeared from that book.

The astronomer Ibn al-Haytham, who was familiar with the Sine Rule and who used it, also used the Sector Figure Theorem in his astronomical works.[19]

With all this, we must say that the astronomical side of Menelaus' *Spherics* has been so much exaggerated to the point that some authors have considered that the work (or at least part of it) was written for its use in astronomy. The confusion started with authors like Bjørnbo and Heath who (mistakenly) declared that a substantial part of the *Spherics* is astronomical, or that it has been written for the purpose of astronomy.[20] These statements have been repeated tirelessly and superficially by several modern authors. The first historians to put forward these ideas were probably convinced that among the Greeks, the only sphere worth studying was the celestial sphere, an idea that we, mathematicians, find hard to believe, knowing that the sphere and its intersections by planes is as natural a field of mathematics to study, as are the cones and their intersections by planes, which are the subject of study of Apollonius' *Conics*, and, in fact, as the ordinary plane with its straight lines. It is probable that Heath was influenced in his opinion by A. A. Bjørnbo (Bjørnbo 1902). In (Rashed & Papadopoulos, 2017, p. 3) we have given a list of authors from before the 1950s who repeated the statement that Menelaus' *Spherics* (or at least a substantial part of it) is a work on astronomy. The names of many modern authors may be added to this list, we prefer not to dwell on this.

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[19] See e.g. Proposition 5 of the treatise of Ibn al-Haytham titled *The Configuration of the Motion of Each of the Seven Wondering Stars*, p. 294 sqq., in: Roshdi Rashed, *Les mathématiques infinitésimales du IXe au XIe siècle*, vol. V: *Ibn al-Haytham. Astronomie, géométrie sphérique et trigonométrie*, London, 2006.

[20] Heath's statement, concerning Menelaus' *Spherics*, in (Heath, 1921, p. 265), is bewildering: "Book II has practically no interest for us. The object of it is to establish certain propositions, of astronomical interest only, which are nothing more than generalizations or extensions of propositions in Theodosius *Sphaerica*, Book III."

Another statement of Heath which led to a misinterpretation is that he declared that Book III of Menelaus' *Spherics* belongs to "trigonometry",[21] a statement which is also absurd. Heath writes in (Heath, 1921, p. 265): "It will have been noticed that, while Book I of Menelaus gives the geometry of the spherical triangle, neither Book I nor Book II contains any trigonometry. This is reserved for Book III." In fact, spherical trigonometry has been developed only several centuries later. The only reasonable meaning that can be given to Heath's statement is that Book III (like Books I and II) constitute a study of spherical triangles, but if such a study is termed as trigonometry, then several books of Euclid's *Elements* should also be termed as *trigonometry*. In fact, Heath says (p. 260) that "Greek trigonometry in its highest development [is] in the *Spherica* of Menelaus," which is correct, but it is only a relative statement. It is correct if we mean by this "trigonometry" in its etymological sense, the study of triangles, but then we have to include also Books I and II of the work.

On the other hand, taking about Greek astronomy, it is fair to recall that Theodosius wrote (at least) two astronomical treatises in which the celestial phenomena are studied from a geometric point of view, namely, *On habitations* and *On the days and nights*. The treatise *On habitations* contains 12 propositions which are all concerned with how the stars in the celestial sphere are seen from different places of the Earth. A brief paraphrase of the propositions is contained in Delambre's *Histoire de l'astronomie ancienne*, (Delambre 1817), vol. 1, pp. 235–7. I have reproduced in Fig. 19 a page containing Proposition 9 of the Habitations, which concerns a comparison between the location of the stars that are visible and the duration of the visibility of these stars, at two places that are not situated on the same meridian. For a recent edition of the Arabic translation of the *On habitations*, due to Qusṭa ibn Lūqā appeared in 2011, together with the Latin translation (done from the Arabic) Gerard of Cremona and an English translation, see (Kunitzsch and Lorch 2011).

This being said, I would like to stress, first, on the fact, that Heath remains one of the most valuable specialists of Greek mathematics in modern history, and secondly, that, by an ironic twist of history, it is true that Ptolemy, in his great astronomical treatise, relied heavily on a proposition of Menelaus' *Spherics*, which is not an astronomical work, rather than on the astronomically oriented *Habitations* or *Days and nights* of Theodosius, or on the *Moving sphere* or the *Risings and settings* of Autolycus, the reason being of course that this proposition of Menelaus could be used in an effective way in astronomical calculations.

9 As a Word of Conclusion

In the preceding pages, I have talked about the history of Menelaus' *Spherics*, discussing its relation with ancient and modern research on geometry.

[21] In Heath's treatise (Heath 1921), the study of Menelaus' book is part of the chapter called *Trigonometry*.

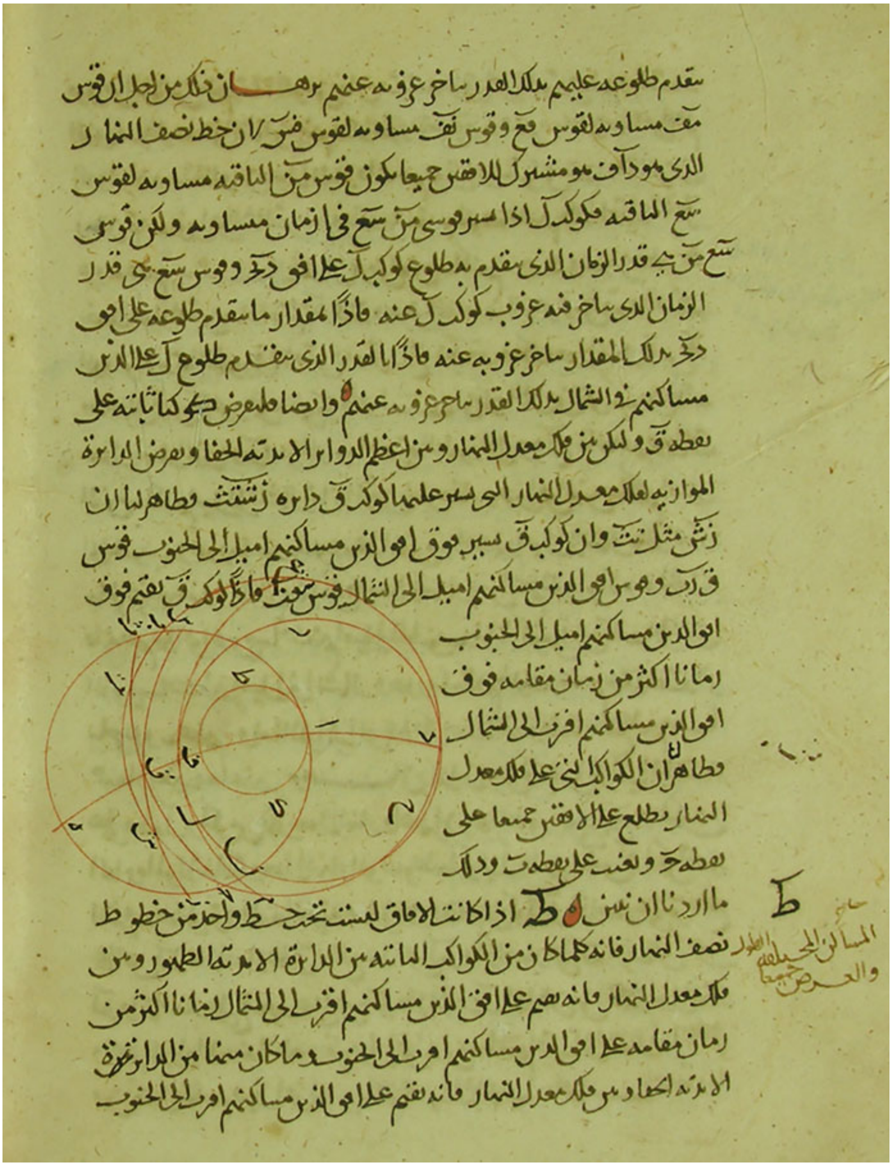

Fig. 19 From Theodosius' *Habitations*; at the bottom of the page, the statement of Proposition 9 and the corresponding figure. Version of Qusṭa ibn Lūqā, Ms. 3464, fol. 121, Topkapi Saray, Ahmet III

In the next two appendices, I have included modern proofs of two propositions of Menelaus, using spherical trigonometric formulae. Such modern proofs are straightforward and they sometimes give more precise statements than the synthetic arguments. However, there is an enormous chasm between these automatic proofs and the depth of Menelaus' geometric proofs, in which the etymology of the word *Theorem*[22] as a contemplation finds it full force. It is in such instances which after all are rare in the history of mathematics that one can recognize the seal of a real mathematician.

Acknowledgments This work obviously owes a lot to Roshdi Rashed with whom I collaborated for the edition (Rashed and Papadopoulos 2017), and I take this opportunity to express my admiration for all the work he has done for the history of Greek and Arabic mathematics.

Appendix I: A Proof of Proposition 27 by Spherical Trigonometry

We start with a formula of Euler from his memoir (Euler 1797) which says that the area of a spherical triangle ABC in terms of the lengths of sides a, b, c is equal to:

$$\frac{1+\cos b+\cos a+\cos c}{4\ \cos\frac{b}{2}\cos\frac{a}{2}\cos\frac{c}{2}}.$$

Consider now a triangle ABC, let D and E be the midpoints of AB and BC, and DE the arc of great circle joining them.

The cosine formulae in the triangles ABC and ADE give:

$$\cos B=\frac{\cos a\cos c-\cos b}{\sin a\sin c}$$

and

$$\cos B=\frac{\cos\frac{a}{2}\cos\frac{c}{2}-\cos DE}{\sin\frac{a}{2}\sin\frac{c}{2}}.$$

Equating the two values of $\cos B$, we obtain

$$\cos DE=\frac{(\cos b-\cos a\cos c)\left(\sin\frac{a}{2}\sin\frac{c}{2}\right)+\cos\frac{a}{2}\cos\frac{c}{2}\sin a\sin c}{\sin a\sin c}.$$

Replacing in the right hand side $\cos a$ and $\cos c$ by their values in terms of $\sin\frac{a}{2}$, $\sin\frac{c}{2}$, $\cos\frac{a}{2}$ and $\cos\frac{c}{2}$, and after some cancellations, we get:

[22] From the Greek, ϑεωρέω, "I watch", the same origin than the word *theatre*.

$$\begin{aligned}\cos DE &= \frac{\cos b + 2\cos^2\frac{a}{2} + 2\cos^2\frac{c}{2} - 1}{4\cos\frac{a}{2}\cos\frac{c}{2}}\\ &= \frac{\cos b + (1+\cos a) + (1+\cos c) - 1}{4\cos\frac{a}{2}\cos\frac{c}{2}}\\ &= \frac{1+\cos b+\cos a+\cos c}{4\cos\frac{a}{2}\cos\frac{c}{2}}\\ &= \frac{1+\cos b+\cos a+\cos c}{4\cos\frac{b}{2}\cos\frac{a}{2}\cos\frac{c}{2}}.\cos\frac{b}{2}\end{aligned}$$

Using Euler's formula we get

$$\cos DE = \cos\frac{1}{2}\mathrm{Area}(ABC)\cos\frac{b}{2},$$

which implies $DE < AC/2$.

Appendix II: A Proof of Proposition 71 by spherical Trigonometry

The following formula gives a more precise version of Menelaus' proposition 71:

Theorem Given a spherical triangle ABC with right angle at A and acute angle at C, and such that AC is smaller than a quarter of a circle, then we have:

$$\frac{\sin(BC+AC)}{\sin(BC-AC)} = \frac{1+\cos C}{1-\cos C}.$$

In particular, the ratio of sines in the conclusion depends only on the angle C, which implies the result of Proposition 71.

The proof is given in the setting of hyperbolic geometry in the paper (Papadopoulos and Su 2017). That proof is based on some non-Euclidean trigonometric transformations. It is easily adapted the spherical case, replacing the hyperbolic functions sinh and cosh by the functions sin and cos. We use the following lemma, which follows from the spherical cosine and Sine Rules.

Lemma We have

$$\tan BC = \cos C \cdot \tan AC.$$

Then we expand $\sin(BC + AC)$ and $\sin(BC - AC)$:

$$\frac{\sin(BC+AC)}{\sin(BC-AC)} = \frac{\sin BC \cos AC + \sin AC \cos BC}{\sin BC \cos AC - \sin AC \cos BC}$$
$$= \frac{\dfrac{\sin AC}{\cos AC} + \dfrac{\sin BC}{\cos BC}}{\dfrac{\sin BC}{\cos BC} - \dfrac{\sin AC}{\cos AC}}.$$

Using the lemma, the last term becomes

$$\frac{1 + \cos C}{1 - \cos C},$$

which is what we wanted to prove.

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