The Geometrical Work of Girard Desargues

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J. V. Field
J. J. Gray

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J. V. Field
Science Museum
London SW7 2DD
England

J. J. Gray
The Open University
Walton Hall
Milton Keynes MK7 6AA
England

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Preface

Our main purpose in this book is to present an English translation of Desargues' *Rough Draft of an Essay on the results of taking plane sections of a cone* (1639), the pamphlet with which the modern study of projective geometry began. Despite its acknowledged importance in the history of mathematics, the work has never been translated before in its entirety, although short extracts have appeared in several source books.

The problems of making Desargues' work accessible to modern mathematicians and historians of mathematics have led us to provide a fairly elaborate introduction, and to include translations of other relevant works. The translation of the *Rough Draft on Conics* (as we shall call it) thus appears in Chapter VI, the five preceding chapters forming an introduction and the three following ones giving translations of other works by Desargues.

Chapter I briefly reviews parts of ancient geometrical works available to Desargues which seem to be relevant to his own work, namely theorems in Euclid's *Elements*, the first four books of Apollonius' *Conics* and some remarks by Pappus in his *Collection*. These Hellenistic works belong to the 'high' mathematical tradition whose development has been the main theme of all histories of mathematics. It is from these works that Desargues took the theorems whose theory he was to reformulate in the *Rough Draft on Conics*. However, although he was well read in the works of this 'high' mathematical tradition, all Desargues' works other than the *Rough Draft on Conics* belong to the 'low' mathematical tradition of practical geometry. For instance, he wrote on such matters as drawing in perspective and setting up the gnomon of a sundial. Moreover, he earned his living in professions much concerned with practical mathematics, working as a military engineer and as an architect. It is thus not surprising to find that the concerns of the 'low' mathematical tradition seem to have played a significant part in the thinking behind the *Rough Draft on Conics*. Since this 'low' mathematical tradition has received relatively little attention from historians, our account of it, in Chapter II, is somewhat more detailed than our summary of 'high' mathematics in Chapter I.

In Chapter III we discuss the reception accorded to the *Rough Draft on Conics* by mathematicians of Desargues' own generation and of succeeding generations up to that of Poncelet. Chapter IV provides a detailed outline of the mathematical content of the *Rough Draft on Conics*, using modern notation. Several passages in our translation of the work, in Chapter VI, have been keyed to the summary in Chapter IV, so that the reader can follow Desargues' reasoning without necessarily having to work through the treatise line by line. The final introductory chapter, Chapter V, is the translators' preface, which is mainly concerned with linguistic and textual problems. It includes lists of the technical terms used in the *Rough Draft on Conics* (Desargues' famous botanical vocabulary) and in his *Perspective* of 1636, the terms being given in both French and English. This preface is followed by the translation of the *Rough Draft on Conics*, in Chapter VI. The footnotes labelled by a, b, c, etc, which are intended to provide a running commentary, are our own. So too are the illustrations which accompany the text, Desargues' own illustrations apparently being irretrievably lost. Chapter VII contains a translation of Desargues' treatise on perspective of 1636. This work has never been published in English before. (The corresponding French text is given in Appendix 5.) Chapter VIII contains translations of the three geometrical propositions by Desargues which were printed at the end of Abraham Bosse's expanded account of Desargues' perspective method in 1648. The first of these propositions is Desargues' famous and beautiful theorem on two triangles in perspective. Chapter IX gives a translation of Desargues' newly rediscovered work on sundials, first published in 1640. The diagrams (in the Notes to Chapter IX) are our own, since the single known copy of the work contains only text.

Several shorter items have been added as appendices. They are Descartes' interesting, and apparently very carefully phrased, verdict on Desargues' *Rough Draft on Conics* (Appendix 1), Beaugrand's defence of classical methods (Appendix 2), Blaise Pascal's *Essay on Conics* (Appendix 3), a note on Johannes Kepler's invention of points at infinity in 1604 (Appendix 4), and finally, as already noted, the French text of Desargues' perspective treatise of 1636 (Appendix 5). This treatise has not been reprinted since it appeared in a much emended version in Poudra's edition of Desargues' works in 1864.

Our bibliography is not intended to be exhaustive, but includes all primary and secondary sources cited in our text.

It will probably be clear from the present authors' separate publication records that in several chapters of this book the work of one or other author predominates. We should, however, like to point out that neither of us, working alone, could have produced what either of us would have regarded as an adequate translation of the work with which we are chiefly concerned, the *Rough Draft on Conics*. We hope that together we add up to a Desargues scholar.

April 1986 J. V. F.
J. J. G.

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We are grateful to Dr Jan Hogendijk for his detailed critical reading of an earlier draft of our typescript from the point of view of a historian of mathematics. We believe our work has benefited considerably from his attention. Our study of linear perspective, in Chapter II, led us into the territory of the art historian. Here we are grateful for the professional advice and guidance kindly given us by Professor Martin J. Kemp, whose forthcoming book, *The Science of Art*, sets perspective in a wider artistic context than we have attempted to describe and gives an account that is not only more detailed than our own mathematical sketch but also largely complementary to it.

We are grateful to Dr S. S. Demidov and Professor A. P. Yushkevich for the interest they showed in our work when a summary of it was presented (by J. J. G.) at a seminar in the history of mathematics at Oberwolfach. This interest led to the appearance of a Russian version of the lecture in *Istoriko-matematicheskie issledovaniya*, XXIX (1985).

The text of Desargues' treatise on sundials of 1640 was believed lost until Anthony J. Turner discovered a copy of it in 1983. We are grateful to Mr Turner not only for giving us a photocopy of the work before his discovery was published, but also for allowing us to include a translation of the treatise in the present work (see Chapter IX).

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Contents

Chapter I

The Greek Legacy

When Desargues circulated fifty copies of his *Brouillon project d'une atteinte aux evenmens des rencontres du Cone avec un Plan* (*Rough Draft of an Essay on the results of taking plane sections of a cone*) in 1639, he was contributing to a lively contemporary study of geometry. Descartes' novel algebraic methods had been published two years before, and in 1639 Mydorge published a more classical treatment of the conic sections. The classical authors themselves were increasingly well studied. Desargues had available Commandino's Latin edition of Euclid's *Elements*, published in 1572, as well as his Latin edition of the first four books of Apollonius' *Conics*, published in 1566 with extensive commentaries by Eutocius, Pappus and Commandino himself. The last four books of the *Conics* were unknown in Desargues' time. Two editions of Pappus' *Collection* had also been published by Commandino (posthumously) in 1588 and 1602. In this chapter we sketch what in these ancient works forms the background to Desargues' remarkable text. The mathematical details of his reformulation of those ideas is described in more detail in Chapter IV. Ironically, the modern (Greek-less) English reader is in some ways scarcely more able than Desargues was to approach the originals. There is, of course, Heath's three-volume edition of Euclid's *Elements* (see Bibliography for details). But only the first three books of Apollonius' *Conics* exist in English, translated by R. C. Taliaferro in 1939; the first seven books exist in the French translation of Ver Eecke, 1923. Heath provided an extensive detailed commentary on the *Conics* in 1896. Finally, Book VII of Pappus' *Collection* is only now translated into English, by A. Jones (see the Bibliography); happily Ver Eecke put all of it that survives into French in 1933.

The Classical Geometry of Conic Sections

The ancient geometers presented not only a body of results, but a way of deriving them and hence of formulating the basic concepts of geometry. Since some of these methods and ideas were accepted by Desargues while others were deliberately rejected, it will be necessary to look at them both in a little detail. We start by asking: What was geometry supposed to be about? The answer is that it was about the concept of magnitude in a rather general and elusive sense. Informally, it is clear that magnitudes like line segments, plane figures and angles are the staple diet of geometry; but formally and philosophically it is not easy to state what these concepts mean. What Euclid's *Elements* embodies is one way of systematizing the study of them so that one can reason deductively about them. While the approach he so successfully presented in that book certainly does not resolve the philosophical problems, it is more relevant for us to pursue the mathematical subtleties it contains. The reader is referred to Mueller's book (1981) for a thorough discussion of many aspects of the *Elements*; we shall be highly selective.

Euclid distinguished between the concepts 'line segment' and 'length' in a way one tends to blur today. To Euclid, one line segment is equal to another if they can be made to coincide exactly, and one is shorter than another if it can be made to coincide exactly with a piece of the second segment. Line segments may be added (by juxtaposition) and subtracted (by inserting one in the other). Indeed, any strict modern definition of length for line segments would recognize that length is a function defined on the set of segments and satisfying some obvious intuitive rules (invariant under motion, additive, etc.). The point to grasp is that when, in the *Elements*, one segment is said to be equal to another it means that they can be made to coincide with one another exactly (as his fourth Common Notion says). It follows that they have the same length; one does not first measure the lengths and deduce that the segments are equal. To a Greek geometer the logical absurdity of such a manœuvre would have been obvious.

The same is true of area. The area of a figure is a primitive concept in the *Elements*, not reducible to that of a product of lengths. Euclid did not show two figures were equal in area by computing their areas, but by a dissection and motion argument, as the famous proof of Pythagoras' theorem literally shows. As is well known, the theorem says that if ABC is a triangle right angled at A, then the sum of the squares on AB and AC is equal to the square on BC. In *Elements* (I,47)—by which we mean Book I, Proposition 47—this is proved as shown in Fig. 1.1.

The square on AC ($ACKH$) is equal in area to twice the area of triangle BCK, since they are on the same base, CK, and between the same parallels. Triangles BCK and ECA have the same area because they are congruent, and that area is half the area of the rectangle $CELP$. So the areas of the square $ACKH$ and the rectangle $CELP$ are equal. The square $ABFG$ can likewise be cut up and fitted on to $BLPD$, so the theorem is proved

In the second book of the *Elements* Euclid showed how, given any rectilineal figure, a square can be found equal in area to the given figure. This result establishes that all rectilineal figures are comparable in size or, as we might say, that they are ordered magnitudes, and it establishes a 'simplest' one

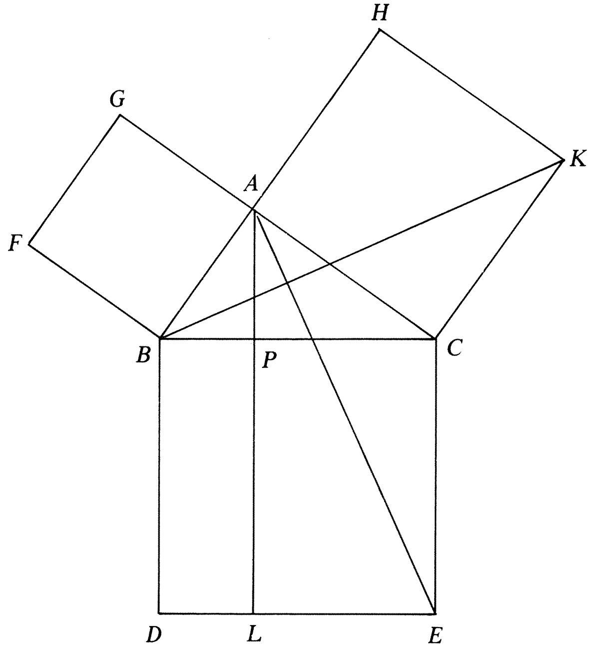

Fig. 1.1

of each magnitude. But it does not reduce the concept of area to the concept of length or product of lengths. We insist upon this point because Apollonius' treatment of conics is in this spirit, and because Desargues' response to it was to be complicated and interesting.

Although area is not regarded as meaning product of lengths, the Greek understanding of product of line segments was that it was a figure. They spoke of a product as the rectangle of the two segments, and treated it as if it was an area. Similarly, this dimensional attitude forbade division of line segments, but it allowed the formation of ratios. The story of how this led to a theory of proportion, the comparison of four magnitudes, in the difficult case when the magnitudes did not have a numerical ratio one to another need not concern us here—it is well told, from a variety of standpoints, in Fowler (1986), Knorr (1975), Mueller (1981, Chapter 3) and Szabó (1978).

But we need to recall briefly the rules Euclid established in Book V for handling ratios, for Desargues took them for granted and today they are almost forgotten. First the rules we need:

Alternando:	$a:b::c:d$	implies	$a:c::b:d$,
Separando and componendo:	$a:b::c:d$	implies	$a-b:b::c-d:d$,
and conversely:	$a:b::c:d$	implies	$a+b:b::c+d:d$.

These rules were established for the general magnitude, and so apply automatically only when a and b, and simultaneously c and d, are magnitudes of the same kind. That is, a and b might be lengths, and c and d be areas, but the two pairs cannot be mixed up. One cannot form the ratio of a length to an area, for example. Nor can one, in Greek geometry, multiply two magnitudes together in general. But one can multiply lengths, of course, and in Book VI Euclid does show that if a, b, c and d are line segments and $a:b::c:d$, then the rectangles ad and bc are equal in area.

The point is, in short, that Greek geometry is about geometric objects, and they are not studied by passing to and from the measure of their sizes. That approach was advocated, more or less, by Descartes, and indeed urged on Desargues by Descartes in his letter to him (see Appendix 1 below). Its success is one reason why Desargues' work has become so difficult to read, and indeed why the word 'area' has become so thoroughly ambiguous.

With this understanding of the concepts behind us, we can now look more briefly at the theorems in the *Elements* which Desargues referred to. There are three:

(1) *The First Secant Theorem* (III,35). If in a circle two straight lines cut one another, the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other (see Fig. 1.2).

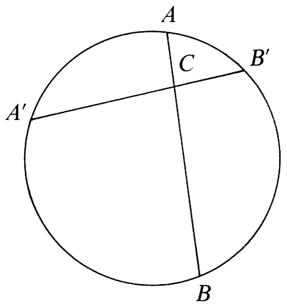

Fig. 1.2

I.e. if AB and $A'B'$ are secants crossing at C then $AC.CB = A'C.CB'$.

(2) *The Second Secant Theorem* (III,36). If a point be taken outside a circle and from it there fall on the circle two straight lines, and if one of them cut the circle and the other touch it, the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point of contact and the convex circumference will be equal to the square on the tangent (see Fig. 1.3).

I.e. if DB is a tangent and DCA is a secant, then $DA.DC = DB^2$.

In each case Euclid's proof first takes the case when a secant passes through the centre and then when it does not, and uses Pythagoras' theorem.

(3) *The Intercept Theorem* (VI,2). If a straight line be drawn parallel to one of the sides of a triangle it will cut the sides of the triangle proportionally, and, if the sides of the triangle be cut proportionally, the line joining

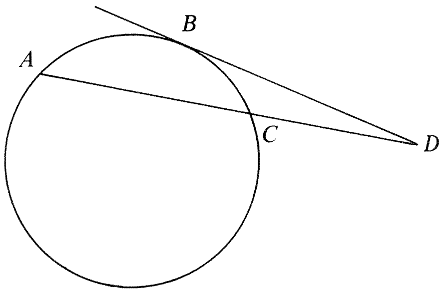

Fig. 1.3

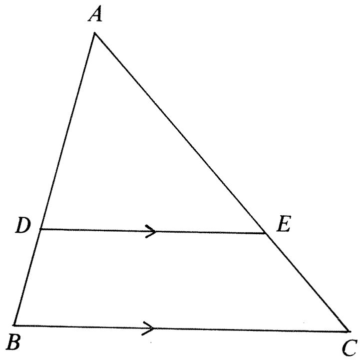

Fig. 1.4

the points of section will be parallel to the remaining side of the triangle (see Fig. 1.4).

I.e. $DE \| BC \rightarrow \dfrac{BD}{DA} = \dfrac{CE}{EA}$ and conversely.

Euclid's proof is in terms of the equal areas BDE and CDE.

The value of this theorem is that it enables ratios along a line to be transferred to equal ratios along other lines by parallel projection.

Apollonius

We now turn to Apollonius, Apollonius described a cone as follows (see Fig. 1.5, we introduce some modern notation). Given a circle and any point, A, not in π, the plane of the circle, a cone is generated by drawing the straight line through the fixed point A to the circle and sweeping the line around the circle. The line is taken to be extended indefinitely in each direction, so a double cone is produced having a circular base, but the vertex of the cone will not in general be vertically above the centre of the base. The line AO joining the vertex to O, the centre of the base, Apollonius called the *axis* of the cone. He defined a conic section as any plane section of the cone missing the vertex, and discussed the line pair very briefly. It seems that Apollonius was the first to regard the double branched hyperbola as a single object, but he was not always able to treat it on a par with the other sections. For his systematic description of conics, Apollonius argued as follows. If π', the plane of section, is parallel to the base, π, it cuts out a circle as the conic section. Otherwise the plane of section meets the base in a line, l, say. Draw the line, δ, perpendicular to l and passing through the centre of the base, suppose it meets the circle at B and C. The triangle ΔABC Apollonius called the *axial triangle*, since it contains the axis. The plane of section and the plane of the axial triangle meet in a line, δ' (see Fig. 1.5).

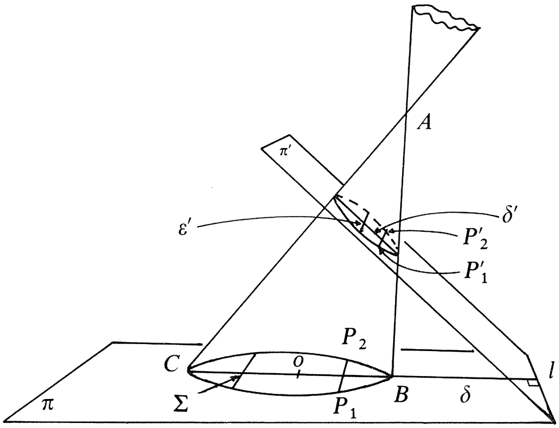

Fig. 1.5

To study the conic section in π' which corresponds to the circle in the base, Apollonius considered corresponding chords in each figure. Chords in π parallel to l, e.g. P_1P_2, correspond to chords of the section, say, P'_1, P'_2, which are also parallel to l. A pair of such form, with A, a triangle AP_1P_2 and an intercept $P'_1P'_2$ to which the intercept theorem applies, and it follows that δ' bisects each such chord to the section. So δ' is a diameter of the conic, since Apollonius defined a *diameter* as a line bisecting all the chords in a family of parallel chords; he called these chords *ordinates* to the diameter δ'. Suppose one such ordinate is also a diameter, let us call it ε'. Since corresponding chords parallel to δ and δ' meet on l the intercept theorem does not apply, and ε' does not correspond to a diameter of the circle. However, it is true (and is proved in I,16,17) that ε' bisects every chord parallel to δ', and so when ε' exists Apollonius called ε' and δ' *conjugate* diameters. Moreover, he showed (*Conics* II,5 and 6) that the tangents to the conic at the extremities of δ' (resp. ε') are parallel to ε' (resp. δ'). ε' and δ' meet at C', where they bisect each other, and C' Apollonius called the *centre* of the conic. Since the vertex, A, is in arbitrary position the diameter δ' is not necessarily an *axis*—Apollonius' term for a diameter which is perpendicular to its ordinates.

To further the discussion Apollonius introduced a quantity called the *parameter*, a line PL at right angles to the plane of section, such that

$$\frac{PL}{PA} = \frac{BC^2}{BA.AC} \quad \text{in the case of the parabola,}$$

and such that

$$\frac{PL}{PP'} = \frac{BF.FC}{AF^2} \quad \text{in the case of the ellipse or hyperbola,}$$

where AF is drawn parallel to δ', meeting the base at F, and the diameter, δ', through P meets AC at P'. Using the parameter Apollonius derived for each curve a relationship between the ordinate ML and the segment of the diameter EM at an arbitrary point L, which was the equation of the conic. By means of these equations Apollonius put an end to the need for the consideration of three dimensions in his study of conics. While these expressions look formidable in the *Conics*, it is not difficult to transcribe them into modern notation. It is established in Heath's *Greek Mathematics* I,139, that if p is the parameter and d the length of δ, then with respect to axes along the diameter and the ordinate tangent to the conic at P, the sections have equations

$$y^2 = px \qquad \text{(parabola),}$$

$$y^2 = px + \frac{px^2}{d} \quad \text{(hyperbola),}$$

$$y^2 = px - \frac{px^2}{d} \quad \text{(ellipse).}$$

In fact the names for the sections derive from the fact that y^2 either equals, exceeds, or falls short of px. (The sense of these words still survives when they describe figures or manners of speech: parable, hyperbole and ellipsis.) Apollonius devoted much of the first book of his *Conics* to freeing his description of a conic from any seeming dependence on the choice of diameter.

The tangent to a conic at a given point was introduced in I,17, as that line which passes only through the given point and otherwise lies outside the conic, whereas any other line through the conic at that point must meet the conic again. Apollonius proved in particular (I,34) that if D is taken on the diameter AB of a central conic, DC the ordinate to the diameter meeting the conic at C, and E is taken on AB such that $BD{:}DA{::}BE{:}EA$, then CE is tangent to the conic at C. He established the converse in I,36, namely that drawing the tangent at C and the ordinate through C to the diameter AB yields four points B, D, A, E in the prescribed proportion. In I,35 he showed that if EC is a tangent to the parabola at C meeting the diameter EB in B, and the ordinate through C meets the diameter at D, then $EB = BD$. One says, following Pappus, that A, B, D and E are separated harmonically. It is an amusing exercise to prove I,36 for the case of the circle using just Pythagoras'

Theorem and the secant theorems. By placing the result about the parabola at I,35 it seems clear that Apollonius regarded it as the case of the phenomenon described in I,34 when one of the points, A, no longer exists. This is typical of a number of instances where Apollonius' geometrical language did not permit him to state explicitly that two results are special cases of a general result, but Desargues' introduction of points at infinity would allow him to be explicit.

Apollonius began Book II with a construction for the asymptotes of a hyperbola. He took PP' as a diameter and drew the tangent to the hyperbola at P. He took L and M on the tangent such that $PL^2 = PM^2 = \frac{1}{4}p \, . \, PP'$. Then, taking C as the centre of the hyperbola, CL and CM are the asymptotes. (In fact, the modern sense of the word asymptote derives from Apollonius' use of it, as in II,12 (below). Its original meaning was simply non-intersecting.) He showed in fact that CL and CM do not meet the curve, but that any line CK, where K lies between L and M, does meet it. Then he showed (II,3) that the converse holds: a tangent meeting the asymptotes at L and M is bisected at its point of contact with the hyperbola. Various properties of the asymptotes were then derived, including (II,14) that the asymptotes approach the hyperbola closer than any given amount. Book II also contains many properties of tangents, amongst other (II,29) that if TQ and TQ' are the tangents to a conic from T and QQ' is bisected at V then TV is a diameter of the conic. Finally he showed how to find a tangent to a conic from a given point not on the conic, or through a given point on it (II,49). To find the tangent at Q on the curve, Apollonius' construction was to drop the perpendicular QN to the axis PP' (which he had shown how to find in II,47) and locate T on the axis such that $P'T{:}PT = P'N{:}NA$. TQ is then the tangent at Q (I,34,36).

The high points of Book III are Apollonius' treatment of the harmonic and focal properties of conics. These are proved by an astute but difficult juggling with the areas of various triangles and quadrilaterals spanned by chords, tangents and asymptotes. This is in keeping with Apollonius' area-based approach to the conics themselves. The following proposition (III,1) is crucial: if TQ and TQ' are tangents to a central conic at Q and Q', and QR' and $Q'R$ are diameters meeting TQ' and TQ at R' and R respectively, then the areas of TQR' and $TQ'R$ are equal. Eventually this led Apollonius to prove (III,37,39) that if TQ and TQ' are tangents, then any line through T meeting the curve at R and S and the chord QQ' at U is divided harmonically ($TR{:}RU{::}TS{:}US$). Moreover (III,38,40) any line through V, the midpoint of QQ', meeting the curve at X and Y and the parallel to QQ' through T at Z, is divided harmonically: $XV{:}VY{::}XZ{:}YZ$.

The first construction breaks down in some cases, for example, when the conic is a hyperbola and the line through T is drawn parallel to an asymptote, because one point, say S, does not exist—in this case Apollonius proved (III,30,31) that $TR = RU$—and also when the line through T is drawn parallel to QQ'—in this case Apollonius proved (III,33) that $TR = TS$. The second construction likewise breaks down when the line through V is parallel to an asymptote, and in this case (III,32) $XV = XZ$.

Apollonius also showed (III,35,36) how to interpret III,39 when one of the tangents TQ is replaced by an asymptote. So we may say that Apollonius viewed the asymptotes as analogous to tangents. That he did not view them exactly as tangents is suggested by the order of the propositions, which indicates that he regarded III,30–36 as the exceptions to the theorems III,37–40 that arise when the necessary constructions cannot be performed. (We recognize for future use that QQ' is the polar of T, that the parallel to QQ' through T is the polar of V, and that T and V are the corresponding poles.)

Apollonius then turned to discuss the foci of central conics. The vital result is III,42: if the tangents at P and P' (the opposite ends of a diameter) meet another tangent at R and R' respectively, then $PR \,.\, P'R' = CD^2$, where C is the centre and D one end of the conjugate diameter to PP'. In particular $PR \,.\, P'R'$ is constant, independent of where the tangent RR' meets the conic.

The foci of a central conic are then introduced (III,45) without being named, as the two points S and S' on the major axis AA' such that

$$AS \,.\, SA' = AS' \,.\, S'A' \,. = \tfrac{1}{4}p \,.\, AA' = CB^2,$$

where p is the parameter corresponding to choosing the major axis as diameter of reference, and BC is the minor axis. Apollonius then showed (III,45–52) that if the tangent to the conic at Q meets the tangents through A and A' at R and R' respectively, then the circle on RR' as diameter passes through S and S'; the angles $RR'S$ and $A'R'S'$ are equal, as are $R'RS'$ and ARS; the angle between the focal radii SQ and QS' is bisected by the tangent; and that the length $SQ + QS'$ is a constant, equal to AA'. There is no discussion of the focus of a parabola, nor indeed of the focus–directrix approach to conic sections (see Fig. 1.6).

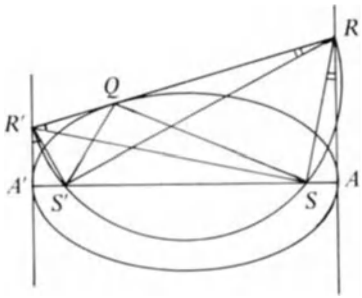

Fig. 1.6

Book IV opens with constructions for tangents which follow from the harmonic properties just described. Apollonius again considered the cases that arise when tangents are to be drawn from points on the asymptotes of a hyperbola, and the construction yields lines parallel to the asymptotes. His constructions break down at this point, but when these exceptional cases do not arise he showed (IV,9) that one obtains the tangents from T by drawing two lines TR_1S_1 and TR_2S_2 meeting the conic at R_1, S_1, R_2 and S_2, locating the fourth harmonic point on each line (say V_1 and V_2) and then joining V_1V_2.

This line meets the conic at Q_1 and Q_2, and TQ_1 and TQ_2 are the sought-for tangents.

We pass over the final sections of Books III and IV only because they did not interest Desargues, noting that in III Apollonius obtained some results with which he could complete Euclid's lost treatment of the locus to three and four lines, and that in IV he showed that distinct conics cannot meet in more than four points.

Throughout the *Conics*, Apollonius endeavoured to treat the different conic sections in a unified way, but he was often forced to discuss the two-branched hyperbola separately, because it meets only one diameter of a conjugate pair, and the parabola because it is not central. The crucial techniques which supplement his use of the equations of the curves are the reference to conjugate diameters, and the use of the concept of area. Area methods, in particular, account not just for the names for the conics, but underlie Apollonius's proofs of the theorems which concern the harmonic properties. They are also at the base of the treatment of the foci of a conic. We shall see that Desargues was able to keep a suitable generalized version of the concept of conjugacy, but that he had to replace the area methods with ones conforming to his novel point of view.

Pappus

The last Greek geometer we need to discuss is Pappus. In his *Collection* Book VII, he described various lemmas to the lost *Determinate Section* of Apollonius which are typically the kind of manipulations with ratios that Desargues liked to use. For example, Pappus showed that if $AD . DC = BD . DE$ then

$$\frac{BD}{DE} = \frac{AB . BC}{AE . EC} .$$

But of much more interest is Pappus' commentary on the lost porisms of Euclid. Pappus described the concept of porism this way: 'The various kinds of porism, however, belong neither to the class of theorems nor to that of problems but in some sense have a form intermediate between the two (in such a way that their statements can have the appearance of theorems or problems).' Here, as an example is his fourth porism. Let $PQRS$ be a quadrilateral—see Fig. 1.7. PQ and RS meet in T, PR and QS in U, and PS and QR in V. The figure so obtained is nowadays called a complete quadrilateral. Pappus considered its intersection with an arbitrary line, obtaining thereby six points A, A', B, B', C and C' (A on PQ, A' on RS, B on PS, B' on QR, C on PR, C' on SQ) and showed that

$$\frac{AA' . BC'}{AB . C'A'} = \frac{AA' . CB'}{AC . VA'} .$$

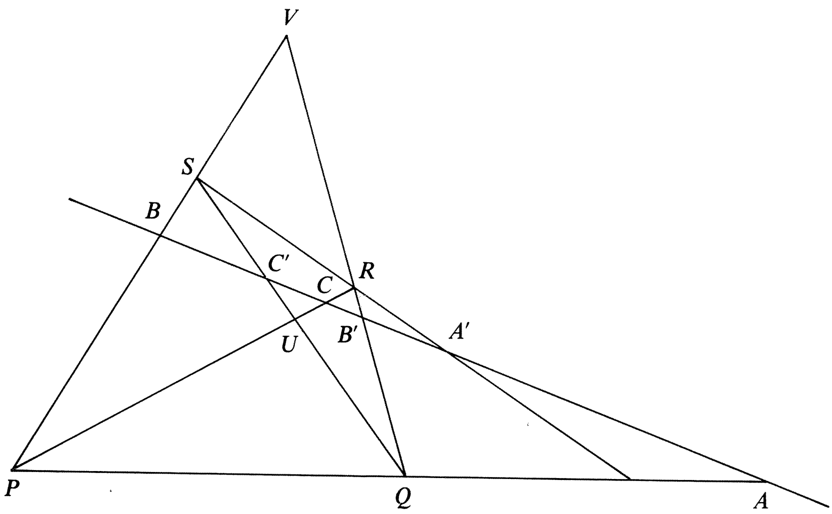

Fig. 1.7

Or rather, he showed, what is equivalent, that if A, A', B, B', C and C' satisfy that relationship, then R, S and A' lie on a line. The proof proceeded by drawing QNM parallel to AA' and chasing ratios much in the fashion Desargues was later to use. In this case collinearity could have been established by appealing to the converse of Menelaus' theorem, but when Pappus reached that point he missed that trick and continued to chase ratios until the conclusion was established—in effect, proving the converse of Menelaus' theorem without saying so.

Porism 5 is interesting, because it amounts to a construction for the fourth harmonic point. Pappus proved that if, in Fig. 1.8, $AB/BC = AD/DC$ then A, H and T are collinear. He next established what we would consider was the special case when DZ is parallel to BC and the porism asserts $AB = BC$. He observed that the converse is also true.

He then considered four lines AH, AB, AC, AD which meet two lines through H in H, B, C, D and H, E, F, G, respectively (see Fig. 1.9), and showed

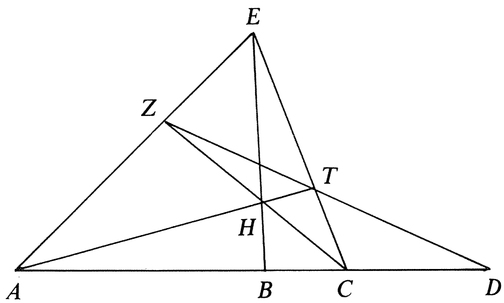

Fig. 1.8

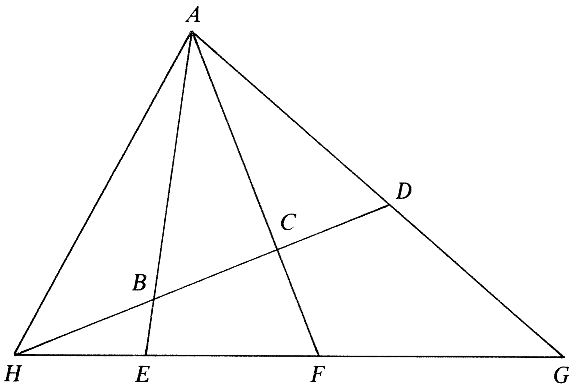

Fig. 1.9

that

$$\frac{HB.CD}{HD.CB} = \frac{HE.FG}{HG.FE}.$$

The proof involved drawing *HK* parallel to *FA*, and *LM* parallel to *DA* and using the intercept theorem. Although this lemma only proves the invariance of cross-ratio under a perspectivity keeping one point fixed (*H*), the general projectivity is obtained by applying it to two such perspectivities, a step not, however, taken by Pappus. Indeed, we do not know of any occasion on which the classical geometers used the cross-ratio of four points in general position. All that was used by them, or Desargues later, was the case of the harmonic division of four points. This is the case of Porism 5, and is the only configuration of four points on a line studied by Apollonius.

Lemmas 12, 13, 15 and 17 are now known as Pappus' theorem, which asserts that if *A*, *B* and *C* lie on one line *A'*, *B'* and *C'* lie on a second, then the points of intersection of *AB'* and *A'B*, of *AC'* and *A'C*, and of *BC'* and *B'C* lie on a line (see Fig. 1.10).

It is worth pointing out, as for example van der Waerden did in *Science Awakening* (1961, p. 287), that Pappus' opening remarks to Book VII of the

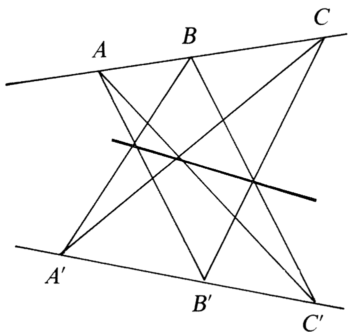

Fig. 1.10

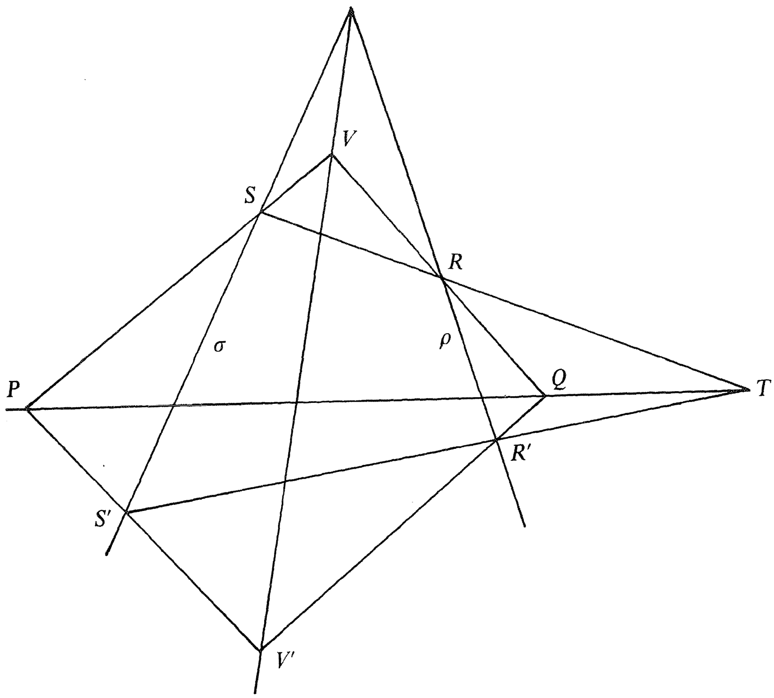

Fig. 1.11

Collection can be made to say something very interesting. Pappus showed that given a quadrilateral $PQRS$ in which PQ meets RS at T and PS meets QR at V, if P, Q and T are held fixed and S and R lie on fixed lines σ and ρ, then V also lies on a fixed line (see Fig. 1.11). If two positions for S and R are taken as shown (S, R and S', R') yielding a second position for $V(V')$ then one obtains the diagram for Desargues' theorem (and indeed a proof of its converse). However, Pappus did not make this observation himself, and in fact what he wrote was generally regarded as very obscure. He did not even say that the three lines σ, ρ, and the one traced out by V are concurrent—a triviality to be sure. Even Halley, when he included his Latin translation of these remarks of Pappus on Euclid's porisms in his edition of Apollonius's *De sectione rationis* in 1706, confessed that he found this passage impenetrable. It was first understood by Simson in 1723, so we can be certain that Desargues had no knowledge of it, nor has it ever been suggested that his discovery of his famous theorems is anything other than original with him. Rather, Pappus' fecund *Collection* is an example of how differently and, in this case, less successfully Greek mathematicians viewed essentially the same material.

Chapter II

Applied Geometry

Thus far we have considered the academic tradition of geometry. It is clear that Desargues' *Rough Draft on Conics* must be seen as related to (indeed derived from) this tradition. None the less, in attempting to relate the *Rough Draft on Conics* to other works by Desargues we shall be obliged to examine a different tradition, the practical tradition of applied geometry, to which all Desargues' other works belong—with the exception of a few pages about teaching the reading and writing of music, which were published in Mersenne's *Harmonie Universelle* (1636).[1] Since we are particularly interested in the ideas we find in Desargues' work on conics, we shall concentrate chiefly upon studies of linear perspective, for linear perspective was the subject of the only geometrical work Desargues seems to have published before he wrote his *Rough Draft on Conics*. However, at the very end of the main body of the text of his short work on perspective, published in 1636,[2] Desargues refers briefly to solutions of two other groups of geometrical problems: those associated with the cutting of stones for use in architecture and those associated with setting up the gnomon of a sundial. He in fact went on to publish treatises on each of these two groups of problems. The one concerned with the cutting of stones, published in 1640, is entitled *Brouillon project d'exemple d'une maniere universelle du S.G.D.L. touchant la practique du trait a preuves pour la coupe des pierres en l'Architecture*... (*A rough draft of an example showing a universal method of Monsieur Girard Desargues of Lyon for employing guide lines in cutting stones for Architecture*...). It makes a sophisticated use of three-dimensional geometry but seems to be unrelated to Desargues' work on conics. Matters stand quite otherwise, however, with Desargues' extremely brief treatise on setting up a sundial, also published in 1640. This treatise, entitled *Brouillon project du S.G.D.L. touchant une maniere universelle de poser le style & tracer les lignes d'un Quadran aux rayons du Soleil*..., is translated in Chapter IX below. Its interest in the present context is that it is concerned with the cone of which the sundial's lines are sections. Indeed, the treatise mainly consists of instructions for the actual physical construction of elements of this cone—the mathematics of the matter apparently being left as an exercise for the reader. The degree to which Desargues concentrates upon practical matters seems to be unusual in a sundial treatise (though we should note that we cannot know how many equally practical treatises may perhaps have circulated in manuscript in craftsmen's workshops and subsequently have vanished without trace—never having found their way into print). Desargues' heavy stress upon the cone arises from his practical treatment, but it was not uncommon for writers of sundial treatises to point out that the lines to be drawn on the dial would be conic sections. However, they usually took the matter no further than that. Jonas Moore's elegant description of the cone swept out by the line joining the Sun to the tip of the gnomon, and cut by the plane of the dial, a description we quote in Chapter IX below, comes from his preface to a translation of Bosse's expanded version of Desargues' treatise and was presumably supplied because the work in question gives considerable emphasis to the cone. As we have seen, this emphasis has its origin in Desargues' own contribution to the treatise, not that of Bosse. It links Desargues' treatise on sundials with his work on conics and suggests that the work on conics in earlier sundial treatises may perhaps have made a contribution to his treatment of conics in the *Rough Draft on Conics* of 1639. We shall, in fact, discuss some sundial treatises which seem to present significant antecedents to Desargues' work, but our main concern will be with treatises on linear perspective, as presenting possible antecedents for the fundamental concept of central projection.

In the Renaissance, works concerning sundials and those concerning linear perspective were equally seen as belonging to the practical tradition associated with the theory of vision. In the sixteenth century, this theory, formerly known by the Latin name *perspectiva*, gradually took on the Greek name *optics*. The name is appropriate since the geometrical parts of the theory can in fact be traced back to the *Optica* and *Catoptrica* of Euclid.[3] The theory of perspective construction, originally distinguished from the theory of natural vision, *perspectiva communis*, by names such as *perspectiva artificialis* ('artificial perspective') or *perspectiva pingendi* ('perspective for painting'), gradually adopted the older Latin name that had formerly applied to the whole theory of vision and thus eventually became known simply as 'perspective'.

In the fifteenth century, the study of linear perspective assumed considerable importance on account of prevailing fashions in the fine arts. From this time onwards there is evidence for the evolution of techniques of using perspective constructions to give an illusion of a third dimension, but even as late as the early seventeenth century, texts on perspective still bear the marks of their Euclidean origin in repeated references to the 'cone' or 'pyramid' of rays issuing from the eye and terminating in the object seen by the eye.[4] This 'cone of vision' is not usually a true cone in the mathematical sense, since its imagined base, the outline of the object, may be of any shape. The plane of the picture, upon which a perspective image of the object is to be drawn, is imagined as intersecting this cone of vision, so that, in modern terms, the perspective picture is a projection of the object on the picture plane.

The juxtaposition of a 'cone' and a plane of section may seem irresistibly suggestive to a modern mathematician, but it appears that the idea of considering the conic sections together in relation to the apex of the cone suggested itself to no-one who thought it worth while to commit the idea to writing. Perspective was seen as the 'degrading' of one shape into another, not as a reciprocal relationship.

We may suggest several reasons why this should have been so. The first is that conic sections (other than the circle) were not included in treatises on practical geometry. Or, being considered difficult, they received a very brief mention, sometimes supplemented with illustrations (showing each conic in a separate cone), as in Albrecht Dürer's *Underweysung der Messung mit Zirkel und Richtscheit* (1525, Latin translation 1532). Only the parabola receives a slightly more extended treatment, as being the shape of a burning mirror.

Second, although treatises on perspective usually purported to deal generally with the 'cone' of vision, they were actually mainly concerned with what would be useful to their assumed readership, that is, artists. The style of the section on perspective in Dürer's *Underweysung* is typical: there is a brief sketch of generalities (to enable one to construct a convincing architectural setting for the scene in the picture), followed by a series of recipes for foreshortening such stock elements as cubes, polygonal well-heads, column capitals and human figures. For the more complicated examples, it was usual to suggest, as Dürer did, that a mechanical contrivance be used to construct the points where particular visual rays intersected the picture plane.

Third, since perspective constructions were intended as an aid to the making of pictures, treatises confined their attention as far as possible to constructions which could be carried out on a piece of paper, or on the panel, canvas or wall upon which a sketch was to be made as a preliminary to painting. That is, they reduced the problem to a two-dimensional one, as far as possible. Some of the more elaborate treatises do include three-dimensional figures, but only after the author has exhausted the usefulness of two-dimensional ones. Reducing the problem to two dimensions is achieved by effectively eliminating consideration of the apex of the cone of vision (that is, in modern terms, the centre of projection) and replacing it by an 'eye point' or 'distance point' which lies in the plane of the picture.

Our Fig. 2.4 shows the construction described by Leone Battista Alberti (1404–1472) in his *De pictura* (1435). Alberti's description is important historically because it is the earliest surviving account of a perspective construction, but it is not clear how widely this particular construction was in fact used.[5] In our figure, the plane of the picture $QRST$ is imagined as vertical. P is the 'principal point', which is 'opposite the eye' (i.e. it is the foot of the perpendicular from the centre of projection to the plane of the picture). All

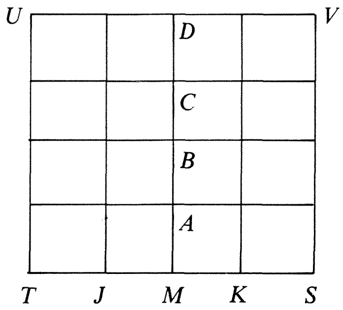

Fig. 2.1. Ground plane, with grid of squares, $STUV$.

lines in the object that are perpendicular to the plane of the picture ('orthogonals') meet at P when they are represented in perspective in the picture, so P is also called the 'vanishing point' (by later writers). The point D is the 'eye point', lying on the horizontal line through P in the plane of the picture. The length ND is the distance between P and the eye (i.e. the distance of the picture plane from the centre of projection).

The mathematics, or geometrical optics, underlying this construction can be expressed in fairly simple terms, as follows: Suppose the ground plane $STUV$ to be marked out in a square grid, as shown in Fig. 2.1. (In many paintings of the time, this grid appears in the guise of a tiled floor or pavement.) In the picture plane, $QRST$, the grid will appear as shown in Fig. 2.2. Figure 2.3 shows the plane LME, that is the vertical plane through E, the eye of the observer, and the point P. In more modern terms, we might think of it as a side elevation of the scheme. Since $STUV$ is a square grid, we can

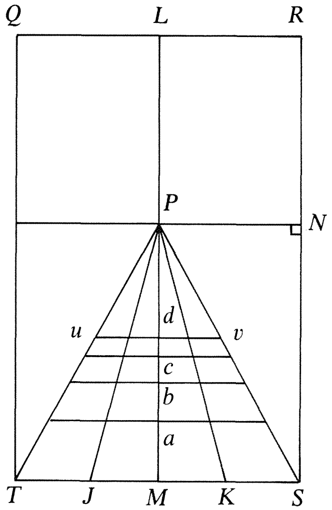

Fig. 2.2 Picture plane, $QRST$. a, b, c, d, u, v are the projections of the points A, B, C, D, U, V.

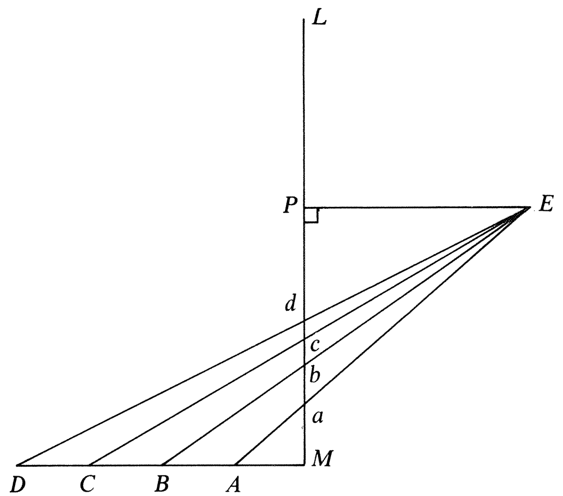

Fig. 2.3. Plane *LME*. Position of eye is *E*. Distance of eye from picture $= EP$.

superpose Fig. 2.3 onto Fig. 2.2 so that the points *D C B A M* in Fig. 2.3 lie on the points *T J M K S* in Fig. 2.2, respectively.

When this is done, the lines $(PN)_2$ and $(PN)_3$ will lie in a straight line, so that in the composite diagram the point *E* will lie on *PN* produced. Also, we have

$$(dM)_3 = (dM)_2,$$

since they are in fact the same line segment. Therefore, in the composite diagram $(d)_3$ will lie on *uv* produced. Similarly, $(a)_3$, $(b)_3$ and $(c)_3$ will lie on lines through *a*, *b* and *c* parallel to *TS*.

The composite diagram is therefore as shown in Fig. 2.4 (from which some lines have been omitted for the sake of clarity). The figure is that for Alberti's *costruzione legittima*, where the lines that will be omitted from the final picture are shown dashed. It is clear from the previous diagrams that in the *costruzione legittima*, as shown in Fig. 2.4,

$$\begin{aligned}\text{distance of eye from picture} &= (EP)_3 \\ &= (EN)_4.\end{aligned}$$

The constructions given in manuals written for artists all resemble that given by Alberti. However, his *De pictura* seems to be addressed at least equally to patrons. It is largely concerned with establishing the credentials of painting as one of the liberal arts, and thus a suitable concern for gentlemen. The very brief account of linear perspective seems to be part of a claim that painting partakes of the dignity of mathematics and therefore should be seen as making a fifth with the four mathematical arts taught in universities: arithmetic, geometry, astronomy and music.

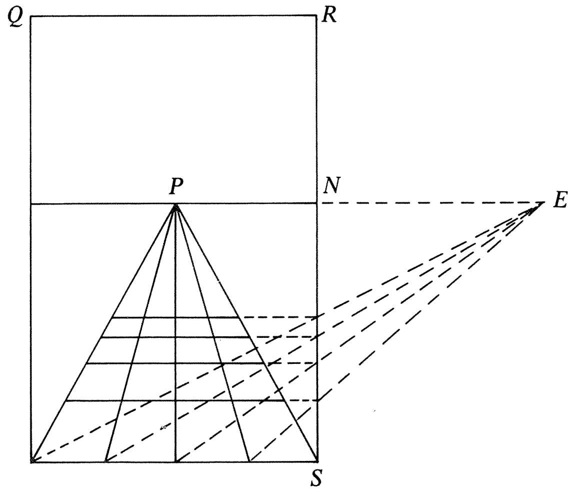

Fig. 2.4. The result of superimposing Fig. 2.3 and Fig. 2.2, so that $(LM)_3 = (RS)_2 = (RS)_4$. This is the *costruzione legittima* described by Alberti (1435).

Although Alberti's account of perspective scarcely goes beyond generalities, he gives the impression that artists calculated foreshortenings for every object in their pictures. However, art historians have pointed out that the numerous surviving works of art of this period (at the height of the fashion for linear perspective) show that artists did *not* carry out perspective constructions for every element.[6] It is clear that Renaissance artists, like those who wrote for them (many of them also practising artists), were concerned almost exclusively with accurate perspective constructions of those elements which could be used to give an adequate sense of depth as part of the design of the picture.

To give a sense of depth, the foreshortened images had to be easy to read, so it was usual to use architectural elements or highly symmetrical geometrical shapes. This is, for instance, the technique employed by Piero della Francesca (c. 1416–1492) in his *Story of the True Cross*, painted during the 1450s in the church of San Francesco, Arezzo. Here the sense of depth is almost entirely conveyed by quasi-classical architecture and foreshortened crosses. A crucifix hung above the main altar of the church, close to the frescoes, providing the image Piero omitted from his cycle, as well as a three-dimensional reference for the crosses in the pictures.

Striking examples of pictures consisting almost entirely of simple elements, all shown in apparently accurate perspective, are provided by small-scale pictures in inlaid work ('intarsia' in Italian). Such panels seem to have been fashionable in the fifteenth century, for example, as the backs of seats in choir stalls or as the doors of cupboards (which might be shown as if open, displaying imaginary contents).[7] Many such panels were made to decorate the palace of Federico da Montefeltro (1422–1482), Count (later Duke) of

Urbino. Piero della Francesca had connections with Federico's court and it seems to have been for someone in Urbino that he painted his small panel *The Flagellation of Christ* (59 cm × 81 cm), now in the Galleria delle Marche (Urbino). This picture has something of an air of being a perspective exercise (in which it resembles intarsia panels) and is one of the very few fifteenth-century paintings whose perspective scheme is sufficiently visible to permit realistic attempts at a reconstruction of the painter's mathematics.[8]

Piero della Francesca was known not only as a painter but also as a very able mathematician, and his treatise on perspective, *De prospectiva pingendi*, dedicated to Federico da Montefeltro and probably written in the 1460s, is commended as 'the best in method and form' by a mathematician of the following century, Egnazio Danti (1536–1586), in his preface to his edition of the treatise on perspective written by the architect Giacomo Barozzi da Vignola (1507–1573).[9] However, Piero's treatise remained in manuscript (it was first printed in 1899), presumably because it was too difficult for most practising artists, who preferred the 'common rules' to be found in the humbler works which Danti proceeds to list.[10]

The first book of Piero's treatise is concerned with flat shapes that may be rendered in perspective simply by using constructions like that described by Alberti (see above). However, in the remaining two books Piero moves on to consider elaborate three-dimensional bodies which are transformed into their perspective images by making use of their plans and elevations. This technique of 'perspective by intersection' bears a close resemblance to the Descriptive Geometry developed in the eighteenth century by Gaspard Monge (1746–1818).

Although Piero's treatise is largely concerned with constructions that most artists would probably have regarded as far too difficult to apply in practice, it does also contain what is probably—if it is not a later interpolation in the manuscript—the earliest description of a perspective construction that later became very popular in elementary treatises. This is what was later to be known as the 'distance point construction', whose origin was probably in a fourteenth-century workshop rule. It is mentioned briefly by Alberti in *De pictura*, merely as a method of checking the accuracy of the rendering of the 'pavement' by drawing a diagonal of the complete square, which should also be a diagonal of the individual 'tiles' through which it passes. The first printed account of the distance point method (as a technique for construction) is in *De artificiali perspectiva* (1505) by Johannes Viator (Jean Pélerin, ?1445–?1522). In his introduction to Vignola's treatise, Danti dismisses this work as 'more abundant in pictures than in words', but it was extremely popular, going through numerous printed editions.[11]

The diagonal line that Alberti proposed as a check on the perspective construction is the line shown as Tv in Fig. 2.5. Its use will clearly permit one to construct the pavement by drawing the lines PT, PJ and so on, through the 'centric point' P, and then drawing parallels to TS through the points in which these lines cut Tv. In fact, this construction is equivalent to using the

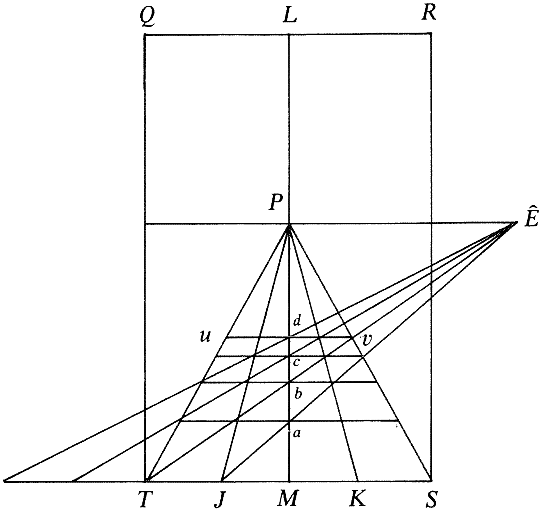

Fig. 2.5. The result of superimposing Fig. 2.3 on Fig. 2.2 so that $(LPM)_3 = (LPM)_2$; E then coincides with $\hat{E}$.

costruzione legittima, as can be seen by superposing our Figs 2.2 and 2.3 once more, this time in such a way that the points $L\,P\,M$ in each diagram are coincident with one another. It is clear that the points a, b, c, d will also coincide. The composite diagram is shown in Fig. 2.5.

The proof that all the lines through $\hat{E}$ (the 'distance point') really are diagonals of $n \times n$ squares of tiles with one side parallel to the base of the picture, TS, is very simple in modern terms. Just as the lines TU, MD, SV in Fig. 2.1 appear to converge in Fig. 2.2 (as lines Tu, Md, Sv) to a 'point at infinity', depicted as P, so all the diagonals parallel to TBV also appear to converge to a point, labelled $\hat{E}$ in Fig. 2.5. To see that the point $\hat{E}$ satisfies $P\hat{E} = PE$, where E is the eye, as in Fig. 2.3, one simply looks at the whole configuration from above. The line of sight joining E to the 'point at infinity' on TBV, meets the picture plane at $\hat{E}$, and is parallel to TBV. So angle $PE\hat{E}$ = angle $BTP = 45°$, and therefore triangle $EP\hat{E}$ is isosceles, and $PE = P\hat{E}$.

A similar argument could be applied to the other lines through $\hat{E}$ in Fig. 2.5. We leave it as an exercise to the reader to extend the proof to all diagonals of the tiles of the pavement.

We do not wish to present our proofs of either the *costruzione legittima* or the distance point construction as reconstructions of the thought-processes of their inventors. The second proof is entirely anachronistic. However, although details of the first proof are almost certainly also anachronistic, it should be noted that the method of superposition is not. A similar superposition of planes that are actually at an angle to one another is found in Vitruvius' method of constructing the lines on a sundial (the 'analemma'), described in his treatise on architecture (*De architectura*, Book IX, Chapter

VII)—a text newly rediscovered in the fifteenth century, and held in high esteem. Moreover, similar conventions of superposition seem to have been followed in technical drawings, where they allowed corresponding parts to be seen in relation to one another (as well as making economical use of expensive materials such as parchment and paper).[12] If we use the method of superposition, the validity of the *costruzione legittima* (long considered the only valid construction—hence its name) is more easily seen than that of the second (distance point) construction. This might explain why the apparently more complicated *costruzione legittima* was, it seems, the first to be accepted as correct. The popularity of the distance point construction was no doubt due to its being easier to use. Mathematical validity was not the prime concern of the artist and the distance point method permitted the whole construction to be carried out within the picture field—provided, of course, that the (ideal) viewing distance was no greater than half the picture width. Although this may seem an awkward condition, it is in fact met in a surprisingly large number of perspective pictures. Artists were presumably well aware that the illusion of space produced by perspective construction was effective even when the eye was far from the ideal viewing position that was built into the construction.[13]

The first general account of the theory of linear perspective occurs not in a work addressed to artists, but in one intended for professional mathematicians, namely the commentary on the *Planisphaerium* of Claudius Ptolemy (fl. 129–141) written by Federico Commandino (1507–1575) and published in Venice in 1558.

Ptolemy's *Planisphaerium*, which survives only in a fragmentary state, is concerned with stereographic projection, actually the projection of points and circles on the celestial sphere onto the plane through its equator, using the south celestial pole as the centre of projection. This is the projection that is used on most planispheric astrolabes, so it would undoubtedly have been familiar to all mathematical astronomers throughout the Middle Ages as well as the Renaissance.[14] In its surviving form, Ptolemy's work is practical in its approach, not proving general results but instead giving detailed calculations relating to particular examples. Commandino's introduction and commentary appear to be designed to repair this deficiency. He seems to have recognized the mathematical connection between Apollonius' work on conics (to which he refers) and Ptolemy's use of a cone of projection. He also (implicitly) connects Ptolemy's cone with the Euclidean cone of vision, and begins his *Commentary* by finding the perspective rendering of a general point. He then turns to the projection of circles. Ptolemy had assumed the general result that all the sections he took of his cone were circles. Commandino first deals with circles then sets out to find the conditions *on the cone* for the projected section to be elliptical, parabolic or hyperbolic. In effect, he is imagining each cone as containing a circle plus one other conic section. He does not consider any more general projective relationship.

Vignola's treatise on linear perspective, *Le Due regole della prospettiva* *practica* ... (1583) edited by Egnazio Danti (to which we have already referred) seems to have become one of the standard texts on the subject. It therefore seems certain that Desargues would have been acquainted with the work, and he may have noted Danti's reference (in his preface) to Commandino's contribution to the theory of perspective. Danti's reference is brief and vague, but Desargues might be expected to wish to follow it up, since we know he was familiar with Commandino's translations of Apollonius. It is also possible that Desargues may have come across Commandino's *Commentary* independently, merely as another treatise by an author he already knew. In any case, it is tempting to suppose that Desargues may have been familiar with a work which, to the modern eye, so notably stops short just where Desargues' own works were to begin.

Commandino's *Commentary* is noteworthy for plunging at once into three dimensions. Another work which does the same, but which seems to have been entirely without influence, is the brief treatment of perspective written by another professional mathematician, Giovanni Battista Benedetti (1530–1590). The work, which is entitled *De rationibus operationum perspectivae* (*On the reasons for the operations of perspective*), was first published in Benedetti's *Diversarum speculationum ... liber* (1585) but much of it may have been written many years earlier. It is a most elegant piece of mathematics. Benedetti uses the correct three-dimensional constructions to show why artists can produce convincing results by using the two-dimensional constructions found in manuals. There are repeated appeals to three-dimensional theorems from the *Elements*. However, Benedetti contents himself with merely exposing the principles involved, and gives no detailed treatment of particular applications. Nor does he mention curved lines of any kind.[15]

Benedetti had, however, given a brief three-dimensional discussion of conic sections in relation to their cone as an appendix to his treatise on sundials, *De gnomonum umbrarumque solarium usu liber* (1574). The appendix is concerned with an instrument for drawing conic sections whose functioning involves setting up the vertical angle of the cone and the angle between the axis of the cone and the plane of section. Benedetti makes it clear that his instrument can be used to draw any conic section (though his diagrams show only ellipses) and by considering the three-dimensional relationship of section to cone, using triangles through the apex of the cone (as Desargues was to do), he proves several theorems that are not to be found in the *Conics* of Apollonius and which are general theorems applying to all types of conic. For instance, he shows that if a plane section is taken through two cones with a common apex and common axis, but different vertical angles, then the two conic sections so produced will in general be of different eccentricity.[16]

Benedetti's new theorems are true of all types of conic section precisely because the section is seen as related to the cone, but he does not attempt to relate the different types of conic one to another through the cone, and he does not mention either perspective or projection. Highly competent mathematician though he is, Benedetti's work on conics must be seen as part of an essentially practical tradition. This concern with a practical problem, namely how to set up the new instrument to draw a given conic section, presumably accounts for the fact that once this problem had been solved Benedetti did not, apparently, go on to prove any further theorems.[17]

A somewhat different attitude was taken by Commandino in his treatise on sundials, published together with his edition of Ptolemy's *Analemma* (1562). Commandino provides diagrams of every type of sundial that he believes was used in the Ancient world, and is concerned to relate as many of his results as possible to the propositions found in Apollonius' *Conics*. In fact, Commandino seems to have one eye on the descriptions of ancient sundials given by Vitruvius (in *De architectura*, Book IX, Chapter VII) and the publication of his treatise caused Daniele Barbaro (1513–1570)[18] to make very considerable modifications to his commentary on this part of *De architectura* when the commentary appeared in a second edition in 1567.

Barbaro also wrote a fairly substantial treatise on perspective, *La Practica della Perspettiva* (1569). The mathematical content of this work is, as Barbaro acknowledges, very largely drawn from Piero della Francesca's *De prospectiva pingendi* (probably written about a century earlier) but its publication must be seen as marking a stage in the *social* history of mathematics. It shows that the general level of mathematical education had risen far enough for Piero's work to seem to be worth publishing, if in a rather simplified form and directed to a readership of gentlemen rather than craftsmen. One of Barbaro's additions is a quite lengthy discussion of perspective stage sets, which is explicitly related to Vitruvius' references to stage scenery (e.g. in *De architectura*, Book VII). Here again we find mathematical studies being related to a tradition of humanist classical scholarship as well as to a practical craft tradition.

As, during the later sixteenth century, correct naturalistic perspective came to be taken for granted in works of art (and indeed in stage sets) we find an increasing number of treatises on perspective which seem to be addressed to those concerned with the practical problems that were the traditional province of the mathematician: problems of surveying, navigation and fortification. For example, it was recognized that an understanding of the laws of perspective made it possible to convert a scout's sketch of a fortress into a ground-plan which would be of use to prospective attackers.

Military applications of this kind almost certainly lay behind the treatise on perspective written by the engineer Simon Stevin (1548–1620) for the Stadtholder of the Netherlands Prince Maurice of Nassau (1567–1625). Stevin's work was first published in Flemish at Leiden in 1605, and a Latin translation, by Willebrord Snel (1580–1626), appeared in the same year.[19] Like the *Perspectivae libri sex* (1600) written by Commandino's pupil Guidobaldo del Monte (1545–1607), Stevin's work is clearly addressed to his fellow mathematical practitioners. However, neither work attempts to unify the theory in any way. Both proceed by treating example after example, though Guidobaldo's work is notable for proving that the fact that orthogonals meet at the 'centric point' is only a special case of the general theorem that lines parallel to one another become concurrent lines when they are shown in perspective.[20]

A French version of Stevin's work appeared in the *Œuvres mathématiques de Simon Stevin* (1634) edited by Albert Girard (1595–1632). Girard—now best remembered as an algebraist—also edited the rather less sophisticated mathematical works of Samuel Marolois (d. before 1651), published in Amsterdam in 1628. The title-page of this edition, shown in Fig. 2.6, speaks for itself. The study of perspective is put together with the practical requirements of the surveyor, navigator (note the ship and astronomical instruments accompanying the artist at the top right), masons and building workers, and a classical soldier holding a plan of one of the star-shaped forts that were designed to withstand the bombardment of cannon (which had only in the last century become markedly more dangerous to the enemy than to the user). Marolois' mathematical interests, and those of his presumed readership, are clearly much wider than those of artists. As we have seen, Desargues shared most of these interests and, although he did not write about specifically military applications of mathematics, we know he worked for a time as a military engineer. It is possible that he was present in this capacity at the siege of La Rochelle in 1628. Later, he was to practise as an architect.

As we have already noted, the only geometrical work Desargues published before the *Rough Draft on Conics* was his *Perspective* of 1636. The work is only twelve pages long, but is otherwise fairly conventional in its style of presentation, using one worked example as a means of describing a new method of making an accurate perspective drawing of an object whose dimensions and distance from the observer and the picture plane are known. The chosen example is also conventional, and is, in fact the same as that in the fifth problem in Stevin's treatise. Desargues' style of exposition is that of the series of recipes for foreshortening which, as we have seen, made up the main part of treatises on perspective addressed to painters. In his own work on perspective of 1648 Abraham Bosse (1602–1676), who employed Desargues' perspective construction and was a successful practising artist as well as a teacher, explicitly mentions that Desargues' method of exposition is conventional. Desargues' vocabulary, however, is not.

In the *Perspective* as in the *Rough Draft on Conics* Desargues gave new names to things which already had accepted names. The new terms in the *Perspective* are not so strange as those in the later work, but it is tempting to see their introduction as an indication that Desargues not only regarded his work on perspective as original but also believed it to have implications beyond drawing problems similar to that described in the text. However, Desargues merely lists the new vocabulary at the beginning of the work, without any explanation. Abraham Bosse remarks upon the new vocabulary in his introduction to his own *Perspective* of 1648 (which included a reprint of Desargues' work) but he does not offer any explanation of it.

Desargues' diagram shows a building, but the problem posed in his

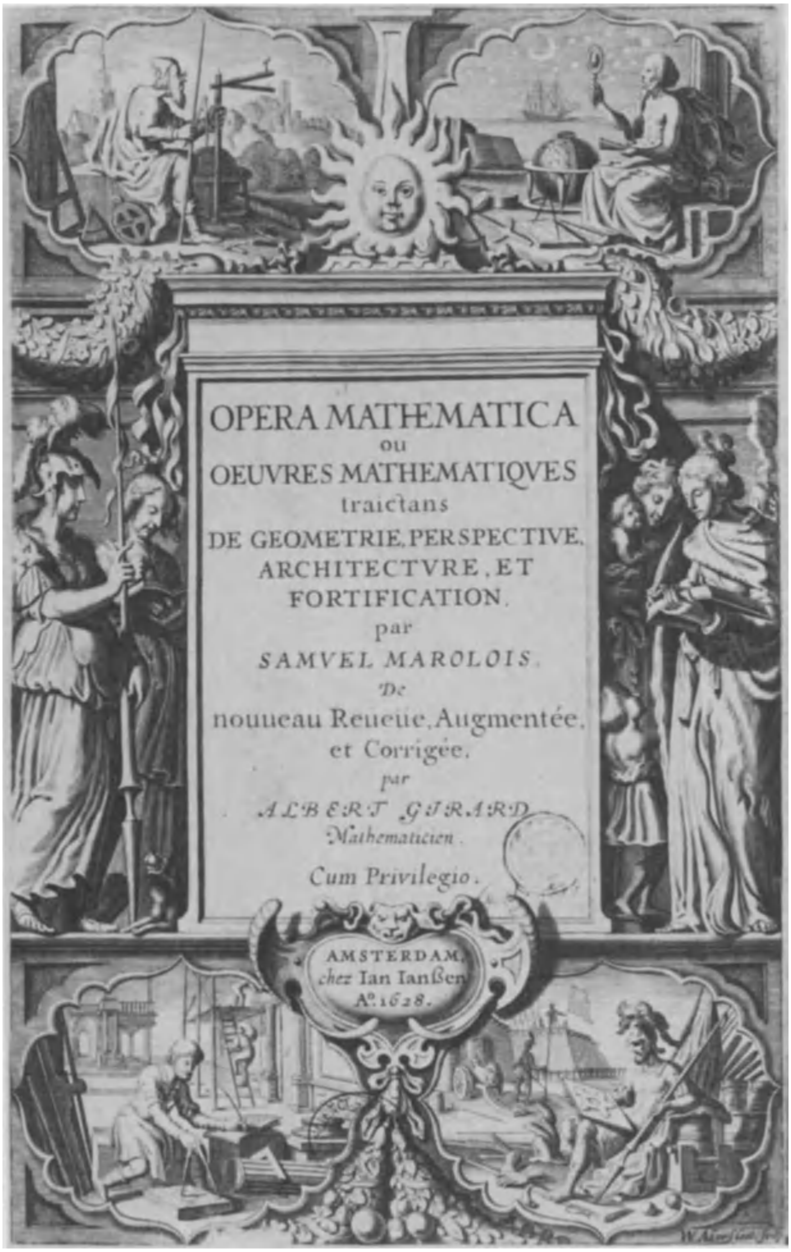

Fig. 2.6. Titlepage of the mathematical works of Samuel Marolois (Amsterdam 1628) showing the practical pursuits associated with the study of perspective. The artist (top right) is accompanied by astronomical instruments and the other figures represent the crafts of the surveyor (top left), the mason or architect (bottom left) and the soldier or military engineer (bottom right). Courtesy of the Trustees of the Science Museum, London.

Perspective is, in fact, the traditional one that is shown in our diagram of Alberti's construction (Fig. 2.4), namely to construct the squared 'pavement' upon which the figures stand in many Renaissance pictures. The 'pavement' is like a chess board lying with one side parallel to the lower edge of the (vertical) plane of the picture. For practical purposes one also needs to be able to construct arbitrary subdivisions of each square of the chess board.

The advantage of Desargues' perspective construction lies in its not requiring the use of any point in the picture plane beyond the edge of the actual picture. It was one of the recognized disadvantages of Alberti's construction that one had to use an 'eye point' which lay beyond the edge of the picture, at a distance equal to the distance between the picture and the eye of the observer. As we have seen above, this objection also applies to the distance point construction if the viewing distance of the picture is to be greater than half the picture's width. In Desargues' construction, the function of the eye or distance point in constructing divisions along the orthogonals is carried out by a point which always lies within the picture, whatever the viewing distance is to be. Several such improved versions of Alberti's construction were to be invented in the course of the seventeenth century. Desargues' seems to have been one of the first.

In Desargues' diagram (see Fig. 7.1, p. 146) pencils of lines radiate from the substitute 'eye point' and the vanishing point, marked F and G. Each pencil defines a series of ranges of points on transversals. However, Desargues' chosen conventional method of exposition effectively precludes any mathematical discussion of the properties of pencils or ranges. The representation of distances in directions parallel to the picture plane is solved by the simple array of similar triangles contained in triangle GCB. (Vertical distances diminish in the same way as horizontal ones.) The representation of horizontal distances perpendicular to the plane of the picture is more complicated, and is carried out by means of a set of rules which Desargues does not prove to be correct. These rules divide the line AG in equal parts, as shown by the successive parallels HT, QO, VP. They are more easily seen to be correct on redrawing the plan view of the imaginary chess board.

After the worked example, Desargues adds about a page which he says is intended particularly for mathematicians. In this page he states a series of simple theorems 'which might be enunciated differently in other connections but are stated here in a manner relating to perspective . . .'. He also notes that this subject is a fertile source of important propositions.

The propositions Desargues states in this passage are mainly designed to bring out the analogy between lines that are parallel to one another and lines that meet at a point. This analogy may be seen to arise quite naturally in the context of linear perspective, since most families of lines which are parallel in the subject to be portrayed are transformed into convergent lines in the picture, while some remain parallel. The difference in behaviour is determined, as Desargues points out, by whether or not the lines are also parallel to the picture plane. Desargues does not point out this analogy explicitly, and does not suggest the existence of a 'point at infinity', but the propositions themselves, and their arrangement as a series, do seem very suggestive. The idea of a cross-ratio is entirely lacking in this work.

The final paragraph of the *Perspective*, announced as a proposition which cannot be expressed as briefly as the preceding ones, contains a tantalizing mention of the wholly original notion of the perspective rendering of a general conic: 'Given to portray [i.e. render in perspective] a flat section of a cone, draw two lines in it whose appearances [in the picture] will be the axes of the figure which will represent it' ('Aiant à pourtraire une coupe de cône plate, y mener deux lignes, dont les aparences soient les essieux de la figure qui la representera').

When taken in conjunction with the rest of the work, this tantalizing final proposition of the *Perspective* suggests almost irresistibly that at least some of the ideas we find in the *Rough Draft on Conics* of 1639 were present in Desargues' mind when he wrote the earlier work. This is confirmed by Mersenne's comment, in his *Harmonie Universelle*, that in 1636 Desargues was already preparing a work on conic sections.[21] It thus appears that the ideas we find in the *Rough Draft on Conics* may have arisen in the context of a mathematical investigation of linear perspective. In the course of the fifteenth and sixteenth centuries many crafts had become more mathematical, among them the craft of drawing in perspective (for drawing in perspective, according to rules, must surely be accounted a craft even if it has sometimes contributed to the making of beautiful works of art). In fact perspective had attracted the attention of competent professional mathematicians, such as Benedetti and Egnazio Danti, as well as that of men with a background in the (increasingly mathematical) crafts associated with the practical tradition of optics, such as surveying. Both perspective construction and the construction of sundials had become associated with wider mathematical concerns. The theory behind them was thus ready to give rise to mathematical generalizations.

When we come to consider the *Rough Draft on Conics* in detail, we shall see that the crucial elements Desargues brought from the practical tradition to the academic tradition of geometry were the concern with problems as being three-dimensional (the practical tradition dealt with the real world, not with diagrams on paper) and the concept of projection from object to image. In Desargues' work, however, the concept of projection is not exactly that we find in earlier texts. Commandino's *Commentary* on Ptolemy's *Planisphaerium* and all the texts on linear perspective are concerned with a projection which renders a three-dimensional object as a two-dimensional image. It is thus not very surprising that they do not (apparently) recognize the essentially symmetrical relationship between object and image. Indeed, in Renaissance usage, objects rendered in perspective are usually said to be 'degraded' ('digradato' in Italian). The emphasis is thus upon what has been *changed* by the projection: dimensionality and shape. Desargues was to concentrate instead upon the elements that remained *unchanged*. His important original contribution seems to have been the concept of invariance.

Indeed, one might say that his *Rough Draft on Conics* is concerned with the fact that the property of being a conic is invariant under central projection. Since the *Rough Draft on Conics* introduces so many new terms, it seems rather ironic that Desargues does not even mention this new concept, let alone propose to name it. His new terms are all for geometrical entities that can be shown in a diagram.

Although hindsight leads one to see revolutionary concepts taking shape in Desargues' *Perspective* of 1636, the main body of the work does not appear revolutionary, and it was at first welcomed as a highly practical contribution to the technique of drawing in perspective. Fairly soon, however, the work became the subject of charges of plagiarism. Desargues was accused, unjustly it would seem, of publishing as his own, ideas he had taken from the unpublished papers of the late Jacques Aleaume (or Alleaume, 1562–1627). The charges, countercharges and ensuing controversy have been discussed in some detail by Taton (1951). The controversy seems to have been fairly acrimonious, and to have gradually died away rather than actually coming to an end. In particular, it is not clear what contribution it may have made to the series of quarrels between the engraver Abraham Bosse, who roundly proclaimed himself a pupil of Desargues, and the newly founded Académie Royale de Peinture et de Sculpture, whose president was the painter Charles Le Brun (1619–1690).

Initially, relations seem to have been harmonious, though since Bosse was merely an engraver, he was not admitted to full membership of the academy. His relations with its members were probably also affected by the fact that he was a protestant (this was a time of considerable religious tension in France) and came from the provinces (Touraine). It should also perhaps be remembered that Blaise Pascal (1623–1662), whose opinions Bosse cited in defence of Desargues' work and reputation, was by this time (the 1650s) clearly committed to the Jansenist side in the long quarrel that split the ranks of French catholics. Overtly, however, the first dispute between Bosse and the academy broke out over the opinions Bosse expressed concerning the *Trattato della Pittura* of Leonardo da Vinci (1452–1519).

A French translation of Leonardo's unfinished treatise was published in 1651 (the Italian text which had appeared in the same year was the first published edition of the work), and the academy considered using it as the basis for a course of lectures on painting. In this treatise, as printed in 1651, Leonardo does not concern himself with linear perspective but with *aerial* perspective (that is the change in colours with distance), and with such matters as the rendering of relief by means of light and shade. There are, however, occasional approaches to mathematics: as when it is said, in Chapter XXV, that an object is best painted from a distance which is three times its height. Bosse commented unfavourably on the disorganized nature of the work (modern scholars agree it is not a finished treatise) and objected to many of its recipes—such as the one about the best distance mentioned above. It appears that some academicians agreed with him, but by 1660, after further disputes, Bosse no longer taught perspective at the academy.[22] He had claimed, however, in his *Traité des Practiques Geometrales* of 1665, that everyone used Desargues' method of constructing scales for drawing in perspective, since they found it the easiest and the most effective. There never seems to have been any suggestion that Desargues' construction was mathematically invalid, but there were apparently charges that it was not useful because its assumptions—that a picture would be seen with only one eye from a fixed position—were unrealistic.[23] These were, and are, the assumptions always made in using linear perspective constructions,[24] and their limitations do indeed show up quite strikingly in certain circumstances, though for the most part they pass unnoticed.[25] Thus the mathematical apparatus may have seemed a mere encumbrance when final adjustments had in any case to be made by eye. Moreover, Bosse records that it was said in anger, and afterwards retracted, that Desargues, not being an artist, had no right to tell artists their business.[26] The times had indeed changed since the 1430s when Alberti had tried to claim for painters some of the dignity of mathematicians.

Chapter III

Mathematical Responses to Desargues' *Rough Draft on Conics*

As we have already mentioned, the earliest responses to Desargues' work on perspective (1636) included accusations of plagiarism. The response to his *Rough Draft on Conics* was more cautious, and eventually comparatively enthusiastic. The earliest recorded reactions include that of J. Beaugrand (?–1640) who pointed out, correctly, that various construction problems in geometry could be solved on strictly Apollonian lines and not, as Desargues had seemed to imply, only by his new methods. One is reminded of the fact that at the beginning of his *Geometrie* (1637) Descartes had been at pains to show that *his* new method could be used to solve problems which had defeated the Ancients. Not all the responses were so negative. In the *Rough Draft on Conics* Desargues himself mentions two mathematicians who apparently took a sympathetic attitude to his work: Jean Pujos and Jean-Baptiste Chauveau.

Pujos is a somewhat shadowy figure, though, as Taton has remarked, the fact that Desargues credits him with having given a proof of one of the theorems in the *Rough Draft on Conics* indicates that he must have been a competent geometer (Taton, 1951, p. 173 and note 89). Moreover, in an earlier passage of the *Rough Draft on Conics* Desargues indicates that Pujos gave three-dimensional proofs, which probably means that he too, like Desargues, used perspective methods (Taton, 1951, p. 156 and note 68). However, most of the works Pujos is known to have published are merely polemical pamphlets in defence of the mathematics of his eccentric patron Paul Yvon, Sieur de Laleu (see Mersenne, *Correspondance*, II, 550, 551; III, 230(n), 357(n), 358).

Chauveau's mathematical activity is somewhat better recorded than that of Pujos. It forms the subject of a brief article by P. Tannery, in which Chauveau is identified as having taught mathematics in Paris and as being the probable author of two manuscript treatises preserved in the Bibliothèque Nationale in Paris: one treatise on the geometry of indivisibles (length nine folios) and one on the elements of conics (also of length nine folios) (see Tannery, *Mémoires* *scientifiques*, VI, pp. 282–286, article originally published in 1895). The subject of the first of these treatises indicates that Chauveau was in touch with the advances being made in the mathematics of the day. The subject of the second may perhaps be due to the influence of Desargues.

We also know that Carcavy, a former colleague of Fermat in Toulouse, approved of Desargues' work and spoke up for it in Mersenne's informal academy. Many years later, in 1656, he was to express enthusiasm for Desargues' achievement in two letters addressed to Christian Huygens (see Huygens, *Oeuvres*, Vol. I, pp. 418–419, pp. 431–432 and Taton, 1951, pp. 194–196).

At the time the *Rough Draft on Conics* was published, Desargues seems to have been a member of Mersenne's circle, which possibly accounts for the discussion of his work in which Carcavy participated. Indeed, since the *Rough Draft on Conics* was apparently printed only in 50 copies, which were circulated to Desargues' friends and acquaintances, the printing may have been undertaken mainly with a view to provoking such discussion—after which the work might be modified for wider publication. This was apparently what happened with other works of Desargues, such as his book on gnomonics, which later appeared in an extended version edited by Bosse. This, too, was the fate of the brief *Perspective* of 1636—though in this case Bosse also reprinted Desargues' original version (see our previous chapter).

The most famous and most mathematically gifted of Desargues' followers was also a member of Mersenne's circle: the young Blaise Pascal, who sometimes accompanied his father Etienne to meetings of the informal academy. Blaise would have heard of Desargues' work from his father, who was a close friend of Desargues, and he wrote his *Essay on Conics* in late 1639, when he was only 16. This essay mainly consists of statements of theorems and gives only brief indications of how they might be proved; the longer *Traité* which Pascal wrote later is now lost, and can only be partially reconstructed from Leibniz's comments on it, written in 1676—see Taton (1955, pp. 85–101).

Pascal's brief *Essay* is chiefly remarkable for the theorem which still bears his name, and for which he gave no proof: If six points A, B, C, D, E and F are taken on a conic, AB and DE produced to meet at L, BC and EF produced to meet at M, and CD and FA to meet at N, then L, M and N lie on a line. As is clear from the way he stated the theorem first for six points on a circle and then for the general conic, it is likely Pascal's proof depended on an argument about transversals to circles, but it is impossible to be sure. The converse of the theorem is also true, but Pascal did not state it. A translation of the *Essay* is given in Appendix 3.

For his part, Desargues seems to have been delighted by his young follower's response. In 1642 he wrote that he would only reveal his methods when 'the demonstration of the great proposition named after Pascal sees the light of day. And what Pascal can say is that the first four books of Apollonius are but a case, and indeed an immediate consequence, of that great proposition.' (Text quoted in Taton, 1955, p. 53.) There is an element of exaggeration here, as with the claim made by Mersenne in 1664 that Pascal could draw 400 corollaries from a universal proposition which between them covered almost the whole of Apollonius' *Conics*. It might be more reasonable to allow that the insight of taking a projective point of view allows one to recapture most of the *Conics*.

Pascal worked on a *Traité des coniques* intermittently until at least 1654 and even till 1659, but it remained unpublished at his death in 1662. In 1670 Oldenburg's attention was drawn to its existence by Huret's attack on the method of Desargues and Pascal, and in 1673 he asked Leibniz to investigate. Leibniz wrote a letter to Pascal's nephew Etienne Perier, who was keen to see his uncle's papers find their way into print, and told him about his findings; this, together with a few notes by Leibniz and Tschirnhaus, are all that remain of the *Traité*. It seems that the work was in four sections: one on the projective theory of conics; one on the 'mystical and conical hexagram' (which Taton has argued is not the same as the hexagram in Pascal's theorem—see Taton (1955, pp. 60–64)); a third on tangents to the circle; and a fourth on centres and diameters. Whether it was not published because of Pascal's theological problems with mathematics or for other reasons, by not appearing it plays only a negative role in the story of responses to Desargues' work.

To go back to 1640, in that year Beaugrand returned to the attack with an unpleasantly polemical letter which revealed how little he understood Desargues' originality. This early demonstration of the Parisian hothouse effect succeeded in casting doubt on Desargues' work, and although Beaugrand died at the end of 1640 the controversy revived in 1642 to further exaggerate these criticisms. Mathematicians cautiously supported Desargues, as did eminent practitioners like Abraham Bosse, but the nearly impenetrable style of the *Rough Draft on Conics* cannot have helped. So it is tantalizing in the extreme to discover that Desargues did indeed write a second book on the same subject, his *Leçons de ténèbres* of 1640, which unhappily is now completely lost. Indeed it has vanished so utterly that Taton devotes several pages in the introduction to his edition of the *Rough Draft on Conics* to establishing that the work ever existed, and as to its contents all that can be said is that it also contained a projective treatment of conic sections.

It seems that Desargues continued both his theoretical and practical work throughout most of the 1640s. His treatise on gnomonics—mentioned in his *Perspective* of 1636 and presumably at least sketched out by then (see note 31 on our translation of the *Perspective*)—was published apparently in a small edition in 1640. In this period Desargues also proposed and solved the problem of showing that a cone on any base whatever admits a circular section, and exhibited the section. His own description of his method does not survive, but can be reconstructed from Mersenne's account—see Taton (1955, pp. 42, 43).

In Abraham Bosse's *Perspective of M. Desargues* (1648), which is a vastly expanded version of Desargues' work of 1636, written in the diffuse mathematical style that seems to be characteristic of Bosse, there appears at the end of the book, after the reprint of Desargues' own *Perspective*, Desargues' beautiful theorem on two triangles in perspective. In modern terms we might state this as: if two triangles are in perspective, then the meets of corresponding sides are collinear. Desargues' statement, however, involves postulating the existence of a number of lines (actually the sides of the triangles and the lines joining corresponding vertices to the centre of perspective) and then stating as a theorem the existence of one more line, that passes through the meets of corresponding sides. The impenetrability of this formulation is attested by the fact that the statement as printed in 1648 is actually incorrect (the existence of one line has been omitted from the conditions).

Presumably Desargues had not discovered this theorem by 1636, since otherwise it could easily have found a place in the collection of interesting new results which forms a tailpiece to the *Perspective* published in that year. In fact, the famous perspective theorem is only one of three geometrical propositions tacked onto the end of Bosse's book. The ascription of their discovery to Desargues is based on Bosse's not giving any other authorship for them, while the ascription of their wording to Desargues rather than Bosse is stylistic: the style, though obscure, is laconic. Unfortunately, we have been unable to find among Bosse's introductory material any indication of the date at which these propositions were discovered, though Bosse is careful to mention that the reprint of the *Perspective* faithfully reproduces the text of 1636, the accusations of plagiarism no doubt still echoing in his ears (Bosse, *Perspective*, 1648, p. 311).

It is, of course, possible that the three geometrical propositions were recent work, receiving their first printing in 1648: the case of Desargues' older contemporary Galileo Galilei (1546–1642) serves to remind us that beautiful discoveries are not the exclusive prerogative of the young. However, it also seems possible that the perspective theorem may have formed part of Desargues' lost *Leçons de ténèbres*, for as Sinisgalli (1978) has pointed out, it is in connection with the perspective rendering of shadows that Simon Stevin shows a diagram which corresponds to the two-dimensional case of Desargues' perspective theorem (Stevin *De Sciagraphia* (1605), see Sinisgalli (1978, p. 134 et seq)). The point is that one triangle can only be the shadow of another cast by a lamp if the two triangles are in perspective. As we have shown in the preceding chapter, there is every reason to place Desargues' mathematical work in the practical tradition to which this work of Stevin's also belongs, so the lost *Leçons de ténèbres* may well have contained not only the material of the *Rough Draft on Conics* of 1639 but also the same sort of material as we find in Stevin's treatise, and thus perhaps also the perspective theorem.

The three geometrical propositions published by Bosse in 1648 seem to represent Desargues' last contribution to mathematics. After this he seems gradually to have withdrawn from intellectual circles, turning from mathematics to architecture, and writing no more.

Most of his architectural achievements date from after 1645. He was apparently at his best designing and building complicated staircases which called upon the full range of his architectural skills. In his native Lyon he executed many delicate constructions, and at Beaulieu constructed a remarkable system of epicycloidal wheels for raising water which was later described by Christiaan Huygens (*Oeuvres*, VII, 112) and repaired by Philippe de la Hire. After several years in Lyon he returned to Paris in 1657, where Huygens recalled him discussing in a lively fashion in 1660. From the reading of his will we know he died in early October 1661; the exact date, place, and cause of death are unknown.

It is often argued, or at least stated, that after a disappointing reception Desargues' works on projective geometry were then buried under a torrent of discoveries associated with the calculus of Newton and Leibniz, which were made possible by the adoption of Cartesian algebraic methods. Taton (1971, p. 49) speaks of '... the sudden vogue of analytic geometry and infinitesimal calculus [preventing] the seventeenth century from witnessing the revival of geometry for which Desargues had laid the foundations.' Morris Kline (1972, p. 301) sees algebraic methods as having been more effective especially in providing quantitative knowledge: '... Hence projective geometry was abandoned in favour of algebra, analytic geometry, and the calculus, ...'. While this view is broadly true, it oversimplifies somewhat. Desargues' work was championed in the later seventeenth century by Philippe de la Hire (1640–1718) who wrote a number of highly regarded books, and projective methods were advocated by no less a mathematician than Isaac Newton. The sad result even so was that projective geometry was not taken up.

La Hire certainly read the *Rough Draft on Conics* thoroughly; for a long time the only known copy of the work was one made by la Hire himself in 1679. Possibly he made his own handwritten copy of the *Rough Draft on Conics* from a printed copy belonging to his father, the painter Laurent de la Hire (1606–1656), a pupil of Desargues and a friend and colleague of Abraham Bosse at the Académie. Philippe de la Hire had by then written his first book on geometry, his *Nouvelle Méthode en Géométrie,...* It too has become extremely rare, and he was later to write that his new method had been found difficult because it involved planes and solids. Whiteside has pointed out (*Mathematical Papers of Isaac Newton*, IV, 271, n. 70) that it received a favourable review, probably from Collins, in 1676, and that Newton may have read it, for Hooke wrote to him mentioning it in 1679. There are certainly similarities between ingenious projective transformations described by both men.

La Hire's point of view in his *Nouvelle Méthode* was entirely projective. He regarded all conics as projections of circles, and used the harmonic division of four points, which he showed was projectively invariant, to obtain theorems about poles and polars. His independence from Desargues' work is apparent from the lack of references to it in the text, which is also in a markedly different—and simpler—style from Desargues'. He does acknowledge him briefly in the preface. He made a very interesting comment on his relationship to Desargues' work at the end of his copy of the *Rough Draft on Conics*, which it is worth quoting in full.

> In the month of July of the year 1679 I first read this little book by M. Desargues, and copied it out so as to get to know it better. It was more than six years since I had published my first work on conic sections. And I do not doubt that, if I had known anything of this treatise, I should not have discovered the method I employed, for I should never have believed it possible to find any simpler procedure which was also general in its application. All the proofs in this work [i.e. Desargues' treatise] are so full of compounding of ratios and take such long and indirect routes that if you compare them with my own proofs of the same results, which do not involve any such compounding, and which include in the first book a much more general version of all that is to be found here; there will be no difficulty in assessing the advantages my method has over the other. They both have the common purpose of showing from the cone the main features of the conic sections by considering the properties of the division of a certain straight line, which Apollonius knew very well, since he made use of all its points of intersection with the section of the Cone, a line which M. Desargues makes an example of his Involution. I have followed Pappus in saying that this line is cut harmonically. This persuades me that Apollonius had, in fact, discovered the property of this line by considering the solid, but, since he could not apply it in a sufficiently simple manner, he employed proofs in the plane in preference to what had led him to discover all these properties; and it was by carefully considering all the properties of this line and all the cases in Apollonius, and comparing each of them with all the others, that I found the means of constructing the general description which I gave in the procedure which I have published.

Disappointed, as it seems, by the reception of his *Nouvelle Méthode*, la Hire next treated conic sections entirely by plane methods. In his *Nouveaux Elémens des sections coniques* he gave a focus–directrix approach to the parabola, and string construction approaches to the ellipse and hyperbola. The book is altogether simple and elegant, if not particularly novel or profound. He published it simultaneously with two other books, one on Cartesian methods in geometry and the other, also Cartesian in the true sense, on the use of conics to solve equations. (See H. J. M. Bos, 1981.) Then in 1685 he published his masterpiece, the *Sectiones Conicae*. This is a beautiful volume, in which, as Chasles said (*Aperçu*, 123): 'Except for the locus to three and four lines and the beautiful basic theorems of Desargues and Pascal, all other known properties of conics are presented here for the first time systematically, uniformly and elegantly.' Its omissions, which are surprising, may suggest this book is not in the projective spirit, but that is not so; it is entirely projective. It begins with a discussion of the harmonic division of a line and its invariance under projection, including the case when a point is sent to infinity, and goes on to discuss harmonic properties of lines and circles. It contains the elegant theorem (not proved in Desargues' *Rough Draft on Conics* but easy to deduce from Theorem 5 (in our numbering)) that as a line, l, rotates about a fixed point P the tangents at its extremities T_1 and T_2 always meet on a fixed line, m (the polar of P) (see Fig. 3.1). An English translation of this is given in Fauvel and Gray (1987).

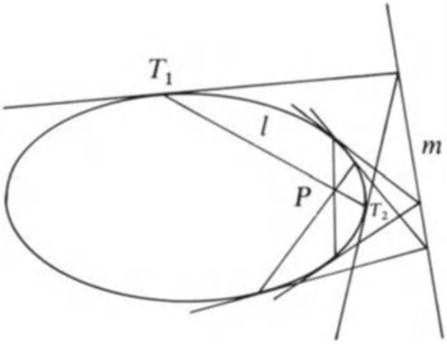

Fig. 3.1

Because all these harmonic properties are invariant under projections, la Hire was then able to transfer them to the general conic, thus treating them all systematically and in a unified way as Desargues had done. The book ends with a lengthy appendix in which la Hire shows how thoroughly he has matched Apollonius' *Conics*, then recently, if poorly, published by Borelli in an edition embracing the first four books, which had survived in Greek, and the next three, translated from the Arabic. This book by la Hire enshrined the projective study of conic sections, so it is worth noting that not only is Desargues' name never mentioned in it, nor is there any discussion of six points in involution. At the end of the book (in the Appendix Omnium Generum) la Hire came as close as Pappus had to showing that the cross-ratio of any four points is invariant under a perspectivity and hence is projectively invariant.

In 1687 this study was further advanced by Isaac Newton, who showed, in Book I of his *Philosophiae Naturalis Principia Mathematica*, how to solve all of the six different problems of the form: find the conic through k points and tangent to m lines, $k + m = 5$. In the course of accomplishing this feat (the first time it had been done) he introduced a projective transformation capable, as he remarked, of transforming any conic to a circle. It is not presented as a projection from one plane to another along rays from a vertex, but a continuous, one-to-one, linear map, and it is strikingly similar to the one given by la Hire in his *Planiconiques*, which was printed in the same volume as his *Nouvelle Méthode*.

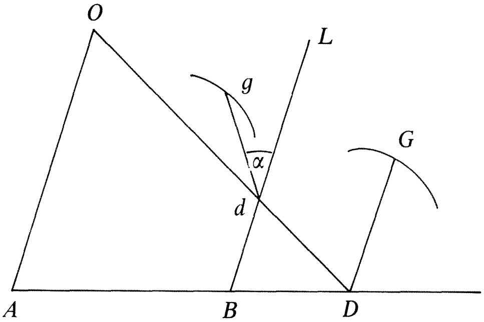

Fig. 3.2

Newton's so-called organic transformation proceeds as follows. Given a curve *HGI*, draw two parallel lines *AO* and *BL* meeting a third line in *A* and *B* (Fig. 3.2). Draw *GD* parallel to *OA* and meeting *AB* at *D*. Join *OD*, meeting *BL* at *d*. Draw *dg* at a given fixed angle, α, to *BL*, and choose *g* so that $dg/Od = DG/OD$. Then as *G* traces *HGI*, *g* traces out the transformed curve *hgi* of the same degree. To see this, without loss of generality one chooses rectangular co-ordinates with axes *AD* and *AO*, when it is soon clear that the transformation sends $G = (x, y)$ to

$$g = \left(\frac{x - ys}{x}, \frac{x + yx - 1}{x}\right), \qquad \text{where } s = \sin\alpha \text{ and } c = \cos\alpha,$$

so it is indeed linear in the sense of sending lines to lines. The line *AO* is sent to infinity, and so, as Newton observed, lines converging to a point on *AO* are transformed into parallel lines. Since the construction sends lines to lines and intersections of curves and lines to intersections of curves and lines, it must send curves of degree n to curves of degree n, as claimed. Newton's proof is essentially the same as ours, if more verbal. As Whiteside says (1960, 288), the hypothesis that Newton took his inspiration from la Hire is interesting and plausible, if difficult to prove. It is well known that Newton looked at at least one of la Hire's works, for he refers in the *Principia* (Book I, Prop. 21, Problem 13, Scholium) to the fact that 'that excellent geometer, M. de la Hire, has solved this problem much after the same way, in his *Conics*, Prop. XXV, Book VIII'. It would be rash to suppose Newton gave the book a detailed study, immersed as he was in creating his own masterpiece.

But perhaps Newton's finest insight into projective transformations comes in his treatment of cubic curves, based on research carried out in the 1660s and published as an appendix to his *Opticks* in 1704. In the published work he conducted a mostly algebraic (and rather obscure) analysis of the general cubic equation and showed that by suitable (affine) co-ordinate transformations any equation could be reduced to one of 72 types. In fact the number is 78, and later writers were happy to find the missing 6, but Whiteside has shown that they were known to Newton from the 1660s. Then in a quite unexpected paragraph Newton remarked:

> If onto an infinite plane lit by a point source of light there should be projected the shadows of figures, the shadows of conics will always be conics, those of curves of the second kind [i.e. cubics] will always be curves of the second kind, those of curves of the third kind [i.e. quartics] always curves of the third kind, and so on without end. And just as the circle by projecting its shadow generates all conics, so the five divergent parabolas by their shadows generate and exhibit all other curves of [the] second kind; while in this manner certain simpler curves of other kinds can be found which by their shadows cast by a point-source of light onto a plane shall delineate all other curves of the same kind. (*The Mathematical Papers of Isaac Newton*, VII, 635.)

Newton gave no explanation of this remark, but at once moved on to a description of the construction of cubic curves by simple mechanical means. Later writers such as Stirling and MacLaurin gradually tackled this obscure work, but they left this profound paragraph alone. The man who took up the problem and explained it was Jean-Paul de Gua de Malves, in his rather attractive book *Usage de l'analyse* (1740).

The different divergent parabolas are defined by these equations (see Fig. 3.3):

(1) $y^2 = (x - a_1)(x - a_2)(x - a_3), \qquad a_1 < a_2 < a_3,$

(2) $y^2 = (x - a_1)(x - a_2)^2, \qquad a_1 < a_2,$

(3) $y^2 = (x - a_1)^2(x - a_2), \qquad a_1 < a_2,$

(4) $y^2 = (x - a_1)^3,$ or

(5) $y = (x - a_1)(x^2 - ax - b), \qquad x^2 - ax - b$ having no real zeros.

As de Gua explained clearly and patiently, it follows from Newton's analysis that every cubic has a real inflexional tangent. On projecting this to infinity, the cubic acquires the equation

$$y^2 = \text{cubic in } x$$

from which Newton's observation immediately follows. But other writers, like the more influential Euler (*Introductio in analysin infinitorum*, Vol. II, 1748) and Cramer, altogether avoided this projective argument. Although Cramer generally followed de Gua, he did not, for example, compare anomalous behaviour at the origin with corresponding behaviour at infinity in strictly algebraic terms as de Gua had. De Gua's book seems to have been the only one to take up Newton's provocative remark in any generality, although Patrick Murdoch's book of 1746 gives a case-by-case analysis of every cubic and shows how each is derived by a suitable projection.

There were also a number of books written in the eighteenth century on

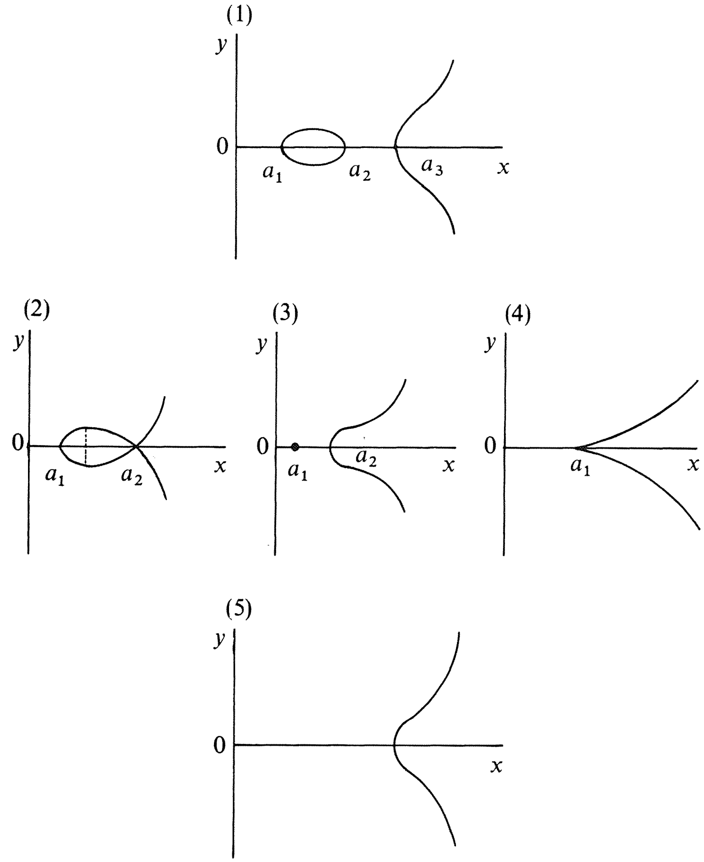

Fig. 3.3

perspective with particular reference to accurate drawing which should be mentioned, although they are in no profound way original (the work of five authors, including Taylor and Lambert, is analysed in Andersen's paper of 1984). Rather, these books represent clear expositions of the mathematical basis of a particular technique, simplified by their authors and made available to all. It is not clear whether gentlemen or artisans were meant to learn from them, but it is clear that their authors did not connect their studies with any of the deeper questions in geometry raised by Desargues and Newton. Since at no point elsewhere in their work did Brook Taylor (1685–1731) or Lambert (1728–1777) grapple with those problems, this is eloquent testimony to the growing neglect into which such problems were falling.

Brook Taylor's *Linear Perspective* (1715) and his more extensive *New Principles of Linear Perspective* (1719) are admirable mathematical monographs, although it seems practitioners could only use simplified versions prepared by others. In them Taylor showed, for example, how to represent any family of parallel lines whether or not they were parallel to the ground plane, and how to represent solid objects seen in perspective. He also dealt with the inverse problem of obtaining true distances given a picture, and of finding the point from which the eye should correctly see a picture. In conclusion he described how the rendering of colours can be dealt with on the basis of Newton's theory of optics.

J. H. Lambert (1728–1777) is another interesting figure who considered the theory of perspective from a practical and a mathematical stand-point. In an early work (*Anlage zur Perspektive*, written in 1752 but not published until 1943) he carried his analysis of the geometric problems as far as the design of an instrument which would reproduce a given horizontal figure as it would appear when seen in perspective. It required only a knowledge of the distance of the eye to the vertical screen. He then published a discussion of perspective in 1759 (the *Freye Perspektive*). This book contains a lengthy and practical discussion of how to draw accurately in perspective, with a careful discussion of the perspective rendering of images already in perspective, such as shadows or pictures in a room. The mathematics, while it has Lambert's customary clarity and careful attention to first principles, is unremarkable. In 1774 he published a second edition of this book, with extensive notes and commentaries, from which it is apparent that he had heard of Brook Taylor and possibly even read him. He added historical comments, drawn exclusively from Montucla, but never referred to Desargues. Of most interest are Lambert's final notes on 'line geometry', in which he posed and solved a number of geometrical construction problems requiring a ruler and a fixed compass only. These problems include:

(1) Given a parallelogram, a line, l, and a point P not on l, draw a line through P parallel to l.
(2) Given a pair of parallel lines l and l', and a point P not on them, draw a line through P parallel to the given lines.
(3) Given a pair of lines l and l' meeting off the page at Q, and a point P not on them, draw PQ. (Taylor had solved this problem in 1715, but gave no proof.)

Lambert's solutions make no use of the theory of pole and polar, which he never described, nor of the equivalent theory of the complete quadrilateral, but rely on his ability to interpret a construction as forming a perspective view of a figure. Lambert's solution to (2), for example, is to draw lines $EPDB$ and ECA (see Fig. 3.4) meeting l at C and D and l' at A and B, then join BC and AD, letting them meet at F, let AP meet EF at G and BG meet AE at R. Then RP is parallel to l and l'.

He showed that this is true by regarding E as a point on the horizon, $ABCD$ as a rectangle, and F and G as midpoints of rectangles. His solution of (3) is very similar, we take the case when P lies outside $l = AC$ and $l' = BD$. Draw PAH and PGB, let GH and AB meet at K and draw KCD. HC meets DG in F and PF passes through Q. Lambert did not remark, but would surely have

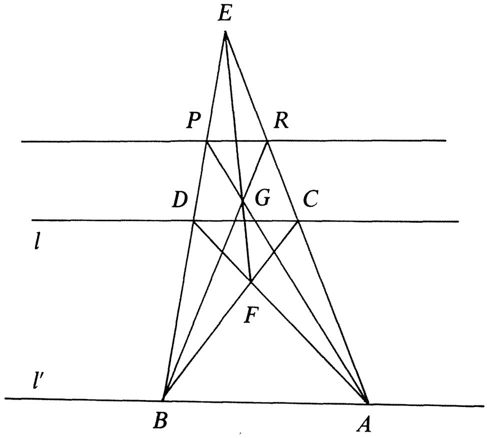

Fig. 3.4

known, that one construction does for both cases (2) and (3), indeed his solution to (2) can be carried out exactly in case (3) as well, see Fig. 3.5. In this figure we can recognize $\triangle PRE$ is in perspective through G with $\triangle ABF$, and CDQ is Desargues' line. Had Lambert known Desargues' work, he would surely have said so at this point. As it is, Lambert's construction is certainly elegant. His interest in 'line geometry' was not widely shared by his contemporaries; but Mascheroni discussed compass only constructions in 1792 and Poncelet in his *Traité* (1822) showed how all ruler and compass constructions can be done with ruler and a fixed (or rusty) compass.

The mathematical tide turned around 1800. At the Ecole Polytechnique, Monge (1746–1818) built a school of French mathematicians around his passion for descriptive geometry, and out of this large institutional milieu with its many career opportunities there came Poncelet (1788–1867) and, a few

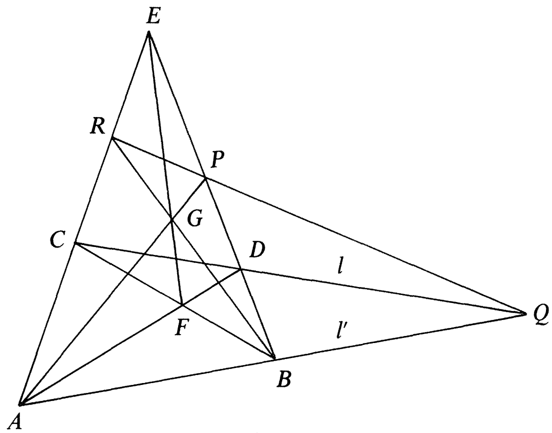

Fig. 3.5

years later, Chasles (1793–1880). In France at least, for both pedagogic and practical reasons, the study of geometry flourished for a generation—and then, quite mysteriously, died a second time.

Monge elaborated a theory, based on orthogonal projection, which was plainly useful for designing forts, interpreting plans, placing guns, and more generally for analysing any three-dimensional problem in two-dimensional terms. These were the standard tasks for the military engineer in the early nineteenth century, as they had been, in only slightly different terms, in the early years of the seventeenth. Thus the social origins of the projective geometry of Monge are very similar to those of the projective geometry of Desargues. And the same is true of their immediate mathematical origins.

Descriptive geometry never quite became the central mathematical discipline its creator had hoped it would be, and its fortunes declined with those of Monge himself. But while it lasted it was a stimulating environment for anyone interested in geometry, stimulating enough indeed for Poncelet to be able to research it as a prisoner of war in Russia in 1813 as a way of keeping himself alive. Incidentally, although Poncelet scrupulously looked for predecessors, by 1822, he could only find Desargues' theorem on perspective triangles as published by Bosse, and a few references to him by Beaugrand and Descartes. Even so, on this basis alone Poncelet praised Desargues highly, calling him the Monge of his century.

If the story of projective geometry in France between 1796 and 1839 is heavily involved with French institutional politics, the German story shows the importance of advances in other branches of mathematics. Möbius (1790–1868), in his (1827), pioneered the method of barycentric or homogeneous co-ordinates, which are excellent for considering the line at infinity and projective transformations. In this way he re-described the theory of conic sections, in particular the theory of pole and polar, and re-created an algebraic theory of duality. (It seems that Möbius, while aware that French mathematicians had developed a general theory of duality, was unaware of the details. In fact, there was quite a row brewing between Poncelet and Gergonne on the subject; but this is a story that still awaits a modern historical analysis. Chasles' comments on it in his *Aperçu* are, as always, interesting—see also Gray (1987).) In Möbius's work the geometric idea of projection is confined to one very clear paragraph, and all transformations are described algebraically. Apparently independently, Plücker (1801–1868) in the 1830s found a rich vein of projective theorems about cubic and quartic curves, and in the 1840s Hesse showed how to put his work onto a clear systematic footing by means of homogeneous co-ordinates. The idea of co-ordinatizing what happens at infinity, and of freely admitting complex numbers, proved to be the way to applying many algebraic techniques to projective geometric questions. The active theory that finally emerged was complex, projective, algebraic geometry. Put that way, it suggests what a long journey it might have to be from Desargues' original ideas to an autonomous discipline within mathematics. But we would rather say it suggests how profound and precious were those same ideas that we now, with some pleasure, publish in English for the first time.

It remains for us to consider, however tentatively, how to account for the difference in reception accorded to projective geometry in the seventeenth and nineteenth centuries. Why was it that the projective viewpoint only seemed fertile second time around? Unfortunately, the most recent discussion of Desargues' work, that by Wilder (1981), concentrates far too closely upon Desargues himself and pays little attention to his contemporaries—who are, after all, the people with whose reactions we are concerned. For example, it seems to us that there is no substance in Wilder's argument that Desargues' interests were unusually varied. They seem rather to have been those one would expect of a mathematically-minded engineer of the time. Naturally, most such men did not write books and have left little trace of their interests, but we may readily compare Desargues with, say, Simon Stevin or Albert Girard.

In fact, the initial response to Desargues' work was quite enthusiastic. As we have seen, further work along the same lines was done by such notable figures as Blaise Pascal and, later, Philippe de la Hire. Desargues may well have had better luck with his contemporaries and immediate successors than is usually argued. However, there seem to have been a number of factors which limited the later development of the subject.

First, although we know, with 20/20 hindsight, that projective methods were potentially very powerful, it may well have seemed at the time that la Hire's elegant *Sectiones Conicae* (1685) was the final word on the projective geometry of conics. The *Conics* of Apollonius had apparently been re-written, and the work showed no need of being extended.

Second, projective theorems can seem isolated one from another, each appearing to be the result of some particular insight. For example, Pascal's theorem on the hexagon or Desargues' on triangles in perspective do not readily suggest that there are more like them waiting to be discovered. Indeed, neither Desargues nor Pascal actually went on to discover such theorems. Results in analytic geometry and the calculus show more interconnections and thus offer more immediate hopes of progress.

Third, Desargues' time saw a turning away from Greek synthetic geometry. The very vastness of the Greek achievement gave it an apparent completeness. Desargues is not the only mathematician whose geometrical work was neglected. For example, Johannes Kepler's invention of the idea of a non-convex polyhedron, published in 1619 (see Field, 1979), was so thoroughly forgotten that when Poinsot re-introduced it in 1810 his paper even contained a reference to the proposition next but one after that in which Kepler had described (and illustrated) two of the very polyhedra whose discovery Poinsot believed himself to be announcing! Mathematical discoveries may indeed be made in a timeless way, as the result of insights, but their acceptance into the mathematical tradition depends upon their becoming part of a system, as Desargues' work did not. Even the results obtained by so generally influential a mathematician as Newton, and published with two of his most widely-read works, did not stimulate much further research along the same lines. Along different lines, however, his calculus did. The powerful new analytic methods ensured the continuance of the trend towards algebraic geometry.

Following la Hire's other work, even the theory of conic sections was recast in algebraic and analytic form, most notably by Euler. It might perhaps be supposed that when points at infinity arose they would direct attention back to synthetic methods. They did arise—in connection with Bezout's theorem, which asserts that two algebraic curves of degrees m and n, respectively, meet in $m.n$ points when counted with multiplicity—as Euler and Bezout himself were well aware. For example, in order to get the correct number of intersections of a parabola with a line parallel to its axis it is necessary to add a point at infinity to both the parabola and the line—but Euler and others satisfied themselves with *ad hoc* arguments, apparently feeling no need for a general theory. For them, the more important problems were to give the correct definition of the multiplicity of an intersection and to deal convincingly with imaginary points. These problems yielded to analytic techniques.

Again, equipped with hindsight, it may seem that some of Desargues' remarks might have spurred someone to advance a projective theory of quadric surfaces. However, it should be noted that when such a theory was put forward in the nineteenth century as a development of the projective geometry of Monge and his pupils, it was best developed in a way that relied upon a firm *algebraic* grasp of the idea of the plane at infinity. Since synthetic methods for quadrics require at least four dimensions for the ambient space, it is clear that Desargues' approach relies, perhaps too heavily, on a geometer's eye.

Other, less technical, factors may have played a greater role. The 50 copies circulated by Desargues probably represented, then as now, a reasonable distribution of a preprint. But the *Rough Draft on Conics* is notoriously difficult. Desargues' botanical vocabulary, which is botanical even in its profusion, his convoluted style, and his failure to separate statements of theorems from their proofs, certainly made the work difficult to read—even for René Descartes, if we read between the lines of his letter to Desargues of June 1639 (translated in Appendix 1). Desargues himself commented in the work on its lack of organization; one supposes he hoped thereby to elicit suggestions for improvements. For whatever reason, the printed copies do not seem to have got through to the generation beyond Desargues himself. Later mathematicians knew the work only through passing references to it in the *Nouvelle Méthode* of la Hire.

In the nineteenth century all these factors are changed. Monge's approach emphasized a systematic study of the projections of all curves, which for a time at least seemed to many people to be a fertile area. By now, moreover, geometry was a somewhat neglected subject, only recently revived by Legendre, and rising in importance. What the seventeenth century had had to escape as a deadening classical orthodoxy, the nineteenth century could re-embrace as a great source of genuine problems. Moreover, unlike Desargues, Monge was a great expositor, able, Chasles tells us, to make his students see the most complicated geometrical figures. Finally, whereas Desargues had addressed a mixed audience of mathematicians and practitioners, Monge could aim at engineers and Legendre at mathematicians, and both groups were by then much larger and more technically skilful than before. Complicated though even the nineteenth-century story is, it is certainly true that the Ecole Polytechnique and the central importance attached to mathematics in French education during and after the Revolution created an audience receptive to the new ideas. That this audience proved in the main to be German is only further testimony to the eloquence of the French example.

Chapter IV

The Mathematical Content of the *Rough Draft on Conics*

Desargues began with the remark that lines will be supposed to contain a point at infinity, which may be reached by travelling in either direction along the line. This enabled him to treat pencils of parallel and intersecting lines on a par; a pencil of parallel lines being regarded as a pencil of lines intersecting at a point at infinity. The use of points at infinity also simplified his treatment of points in involution, as we shall see. He then unleased a plethora of botanical names for simple configurations of points and lines which, taken together, only serve to obscure the text. Of those terms, we need only say that those which connote line segments, such as trunk, branch, shoot and limb, may be taken to mean 'line' or 'line segment', whereas those which suggest points, such as knot, butt, post and stump, mean 'point'. Then he reached his first useful term, the tree.

Three pairs of points B, H; C, G; D, F form a *tree* if there is a point A such that $AB.AH = AC.AG = AD.AF$. The point A was called the *stump* by Desargues. Two cases arise: either A separates the points of each pair, or it does not. It is does, then we shall assign different senses to the segments AB and AH, etc., and use signed magnitudes in our formulae. Desargues resorted to vague circumlocutions to suggest the need to treat each case separately. The difference shows up when, say, AB and AH have the same length. In the first case $BA = AH$, and in the second $AB = AH$ so B and H coincide. The first case is nowadays said to be *elliptic* since the map $P \to \mathrm{P}'$ where $AP.AP' = AB.AH$ has no fixed points, and the case when the map has two distinct fixed points is called hyperbolic. Desargues only introduced the parabolic involution, which has a repeated fixed point, much later, on p. 119. He found it much harder to understand than the other two.

Desargues does seem to suggest that given four points on a line, say B, H, G, C, one could define a pairing of points on the line, pairing F with D if F and D satisfy

$$\frac{GD.GF}{CD.CF}=\frac{GB.GH}{CB.CH},$$

but he does not emphasize this point of view.

Desargues then gave his first important result without warning (p. 76).

Lemma 1. *In a tree B, H, C, G, D, F with stump A*

$$\frac{GB.GH}{CB.CH}=\frac{GD.GF}{CD.CF}=\frac{AG}{AC},$$

with similar expressions for AB/AH and AD/AF. The lemma is easily proved, for, from the definition of a tree

$$\frac{AG}{AF}=\frac{AD}{AC}=\frac{GD}{CF}\ \textit{(by adding)},\qquad \textit{and}\qquad \frac{AF}{AC}=\frac{AG}{AD}=\frac{GF}{CD}\ \textit{(similarly)}$$

so a simple multiplication gives the result. Desargues then defined six points B, H, C, G, D, F to be in involution *if*

$$\frac{GD.GF}{CD.CF}=\frac{GB.GH}{CB.CH}.$$

The property of being six points in involution, or, rather of being three pairs of points in involution, is independent of the order in which the pairs are taken. Desargues deduced from Lemma 1 that six points in involution form a tree with respect to a point A which can be determined from the definition of the involution. Special sets of six points in involution occupied him for a while, when two points coincide, or one coincides with A and another is consequently at infinity. At length Desargues was free to bring forward the second crucial concept, that of four points in involution. He started from an equivalent formula for six points in involution:

$$\frac{BC.BG}{HC.HG}=\frac{BD.BF}{HD.HF},$$

and supposed that D and F coincide, and C and G coincide. In this case he said the four points B, H, G, F were four points *in involution*, and the following equality holds:

$$\frac{BG}{HG}=\frac{BF}{HF}\quad\text{or, equivalently}\quad\frac{BF.HG}{BG.HF}=1.$$

(The expression involving A becomes $AB/AH=(FB/FH)^2=(BG/GH)^2$.) This is sometimes described as B and H divide FG internally and externally in the same ratio; it is the case of four points separated harmonically discussed by Apollonius and Pappus. It is immediate that if B, H, G, F are four points in involution then so are G, F, B, H.

Desargues found the application of this definition when one of the points, say, H, is at infinity incomprehensible but valid (literally: 'Ce qui est incomprehensible...'. p. 82). It corresponds to the special case where B bisects FG, which he was to find useful in performing several geometric constructions (notably finding the fourth harmonic point—see p. 85 below).

Desargues derived a number of elementary equalities relating the four points in involution and A, the midpoint of FG, from the fact that the points B, H, F, G, A and the point at infinity on the line form a set of six points in involution. One simple fact which is worth recording for use later on relates the five points B, H, F, G and A (p. 88).

Lemma 2. *If BH and FG are four points in involution and A is the midpoint of FG, then $GA.HF = BF.HA$.*

Desargues' own argument to this end is vague, merely indicating that one calculates with ratios in the then standard (i.e. Euclidean) way, but he establishes *en route* a number of interesting results, such as B is the central point of an involution pairing A and H, and F and G; and similarly H is the centre of an involution. In turn the converses of such propositions supplied tests for determining when four points were in involution. For example, he remarked that it was also clear that given three points in involution the fourth was determined, but he did not give an explicit construction for it until later.

This discussion of involution concluded his analysis of configurations on the line, or, in modern terms, the geometry of the projective line. He began his discussion of the projective plane by proving Menelaus' theorem, which he regarded as a theorem about three transversals to a given line. (This is equivalent to modern presentations of the theorem in terms of a transversal to a triangle.) We state this important result in his unhelpful notation—see p. 91 below.

Theorem 1 (Menelaus) (See Fig. 4.1). *If hKH, $h4D$ and $K4G$ are transverse to HDG then*

$$\frac{Dh}{D4} = \frac{Hh.GK}{HK.G4}.$$

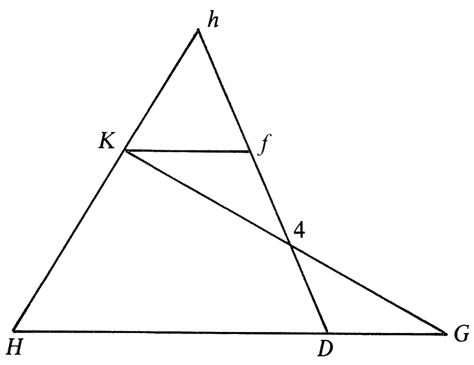

Fig. 4.1

Proof. Draw Kf parallel to HDG meeting hD at f, then

$$\frac{Dh}{D4} = \frac{Dh}{Df} \cdot \frac{Df}{D4},$$

but by the intercept theorem

$$\frac{Dh}{Df} = \frac{Hh}{HK} \quad \text{and}$$

$$\frac{Df}{D4} = \frac{GK}{G4},$$

so the theorem is true. This proof is essentially the same as the one in Ptolemy's *Almagest* (I,13, see Ptolemy, trans. Toomer, 1984, pp. 64–65).

The converse of Menelaus' theorem, which establishes that if H, D, G are so chosen on hK, $h4$ and $K4$, respectively, that

$$\frac{Dh}{D4} = \frac{Hh \,.\, GK}{HK \,.\, G4},$$

then H, D and G are collinear is also true. (The modern version would be: HGD transverse to $\triangle KH4$ implies $hH/HK \,.\, KG/G4 \,.\, 4D/Dh = 1$. In each case the three ratios are formed from three collinear points, the repeated letter being the one on the transversal.)

Menelaus' theorem is one of Desargues' chief means of finding equal ratios and equal products of ratios on different lines. He immediately used it to establish the important fact that if six points on a line are in involution then so are their images under projection from a point onto another line, or, more briefly, that being six points in involution is a projective property. We state this as a theorem.

Theorem 2 (See Fig. 4.2). *If BH, DF, CG are three pairs of points in involution on a line not passing through K, and b, h, d, f, c, g the points of intersection of BK, HK, etc., with another line, then bh, df, cg are also three pairs of points in involution.* (Note the rare use of a convenient notation by Desargues.)

Proof. He must show

$$\frac{dg \,.\, dc}{fg \,.\, fc} = \frac{db \,.\, dh}{fb \,.\, fh}.$$

To do this, he joined Df and supposed it met BK, CK, GK, HK in 2, 3, 4, 5, respectively.

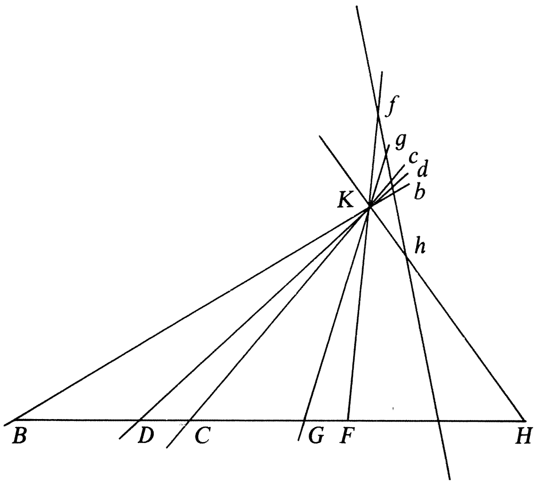

Fig. 4.2

Consider $\dfrac{dg\,.\,dc}{fg\,.\,fc}$.

$$\frac{dg}{gf}=\frac{Kd\,.\,4D}{KD\,.\,4f}\quad\text{(by Menelaus' theorem applied to } 4Kg \text{ and } \triangle Dfd).$$

and

$$\frac{dc}{fc}=\frac{Kd\,.\,3D}{KD\,.\,3f}\quad\text{(similarly } 3Kc \text{ and } \triangle Dfd),$$

so

$$\frac{dg\,.\,dc}{fg\,.\,fc}=\left(\frac{Kd}{KD}\right)^2\left(\frac{4D\,.\,3D}{4f\,.\,3f}\right).$$

But

$$\frac{4D}{4f}=\frac{GD\,.\,KF}{GF\,.\,Kf}\quad\text{(Menelaus on } \triangle\text{DFf and } K4G)$$

and

$$\frac{3D}{3f}=\frac{CD\,.\,KF}{CF\,.\,Kf}\quad\text{(Menelaus on } \triangle\text{DFf and } K3c),$$

so

$$\frac{dg\,.\,dc}{fg\,.\,fc}=\left(\frac{Kd}{KD}\right)^2\left(\frac{KF}{Kf}\right)^2\left(\frac{GD\,.\,CD}{GF\,.\,CF}\right).$$

But similarly $\dfrac{db.dh}{fb.fh}$ is equal to this last expression, so

$$\frac{dg.dc}{fg.fc} = \frac{db.dh}{fb.fh},$$

and the theorem is proved.

As Desargues observed, the theorem is trivial when K is at infinity, for then the lines BK, HK, etc., are all parallel (and the intercept theorem may be invoked).

He drew certain consequences of this important theorem at once, corresponding to special configurations. If one point, say d, is at infinity, the point f in involution with d is the stump of the tree bh, df, cg. So in the case of four points in involution, BH and DF, if DK is parallel to DF (sending d to infinity) then f bisects bh. The converse is also true and provided Desargues with a simple construction for the fourth harmonic point to three points F, B and G on the line FBG (see p. 95 below): take G, f, g on any other line through G, such that $Gf = fh$. Then G, f, h and the point at infinity on Gfh are in involution. Join Ff and Bh, let them meet at K. Draw the parallel to Gfh through K. Since it meets Gfh at infinity it meets FBG in the required point.

Desargues also considered the four lines KF, KB, KD and KG and showed that the pair KB, KH bisect the angles formed by the other pair KF, KD if and only if BKH is a right angle.

It follows from Theorem 2 that the property of being four points in involution is also preserved under projection, as Desargues observed. It also follows that, given any four points A, B, C, D, the quantity $\dfrac{AB.CD}{AD.CB}$ is an invariant, but Desargues did not point this out, nor did he ever work with the quantity $\dfrac{AB.CD}{AD.CB}$, except when A, B, C and D are four points in involution. Since this quantity is central in modern elementary projective geometry, where it is called the *cross-ratio* of A, B, C, D and denoted $(ABCD)$, it is worth commenting on this feature of Desargues' work. In fact, it closely resembles the Greek attitude to length which we discussed in Chapter I. Just as one often wants to know of two line segments that they do or do not have the same length, but only seldom if at all what that length is, so in projective geometry one often wants to know of two sets of four points that they do or do not have the same cross-ratio, but one seldom wants to evaluate it. Desargues expressed this point of view by working with six points in involution, and the special case of four points in involution. In this way he could investigate whether sets of collinear points were similarly disposed—projectively equivalent, as one might say—just as a classical geometer had techniques for deciding if pairs of points were similarly disposed in the sense of defining equal line segments. Like the classical geometer he avoided the paralogism of introducing a ruler to measure the length by not suggesting one could evaluate cross-ratios without first establishing a sense of projectively equivalent configurations.

However, it is easy to extract the invariant of cross-ratio from Theorem 2. Note first that if D, F; G, C; B, H are six points in involution, so

$$\frac{DG.DC}{FG.FC} = \frac{DB.DH}{FB.FH},$$

then

$$\frac{DG.FB}{FG.DB} = \frac{DH.FC}{FH.DC},$$

so in terms of cross-ratio

$$(DGFB) = (DHFC),$$

so a set of six points in involution has a unique cross-ratio. Now to prove invariance of cross-ratio it is enough to take say D, G, F and B and evaluate the four ratios, DG/dg, FB/fb, DB/db, FG/fg using Menelaus' theorem with the lines through K as transversals.

Desargues concluded the elementary half of the *Rough Draft on Conics* with his discussion of involution. He then turned to the projective study of conic sections, which is, as he was the first to see, unified by the fact that all non-degenerate conics are projectively equivalent to the circle. It was his profound idea that there are interesting properties of conic sections (he called them 'the most significant properties'), p. 110 which should be established once and for all by a single proof and the observation that the property is a projective invariant.

Desargues simultaneously considered the cone (as he properly insisted, following Apollonius, the double cone) and the circular cylinder. (The cylinder was not discussed by Apollonius, but it was by a fourth-century A.D. commentator, Serenus, extracts of whose work were included in Commandino's edition of the *Conics*. For Desargues, a cylinder is a species of cone.) Two plane sections of a cone give two conics which are related by a projective transformation (projection from the vertex), two plane sections of a cylinder give ellipses related by (we would say) an affine, and hence also projective, transformation (parallel projection from infinity). Of the conic sections, Desargues noted that some are closed and finite, i.e. the ellipses; that when the plane of section is parallel to a generator of the cone the curve meets itself at infinity, i.e. is a parabola; and that otherwise the curve has two parts and two separate points at infinity and is a hyperbola. The ellipses and hyperbolas come in various species according to their eccentricity, but there is only one kind of parabola (up to similarity, or uniform change of scale). So Desargues regarded all conics as equivalent to circles; those which do not meet the line at infinity in their plane being ellipses, those which touch it being parabolas, and those which cut it being hyperbolas. Desargues endeavoured throughout to treat finite and infinite points systematically on a par.

The sort of configuration Desargues considered which yields projective theorems is the pencil of lines through a point F. Let FDE meet a given conic in D and E, and R be the fourth harmonic point to F, D and E. Then as FDE runs through the pencil, R traces out a locus, which as Desargues said, is a line. (This recalls Apollonius, *Conics*, IV,9.) He called this locus the *transversal* to the pencil of *ordinates* (nowadays called the *polar* of F) but he did not immediately prove that the locus is a line. He did, however, consider the complete line, and not just the segment defined by the lines of the pencil which meet the conic. He then introduced one of the theorems which still bear his name:

Theorem 3 (Desargues' Involution Theorem). *Let B, C, D, E be the four vertices of a quadrilateral, whose pairs of opposite sides BC and ED meet at N, BE and CD at F, and BD and CE at R. Then the pairs of lines meet any other lines, l, in six points which are in involution. Furthermore, any conic through B, C, D and E, and any two pairs of sides of the quadrilateral also meet the line l in six points in involution.* (See Fig. 6.17, p. 107.)

Proof. Let NB, NE, FC, FB, RC, RB meet l at J, K, Q, P, H and G, respectively. We must show IK, PQ, HG are pairs in involution.

$$\frac{IQ}{IP}=\frac{CQ\,.\,BF}{CF\,.\,BP} \quad \text{(Menelaus on } \triangle QPF \text{ and } ICB\text{)},$$

$$\frac{KQ}{KP}=\frac{DQ\,.\,EF}{DF\,.\,EP} \quad \text{(Menelaus on } \triangle QPF \text{ and } KDE\text{)},$$

so

$$\frac{QI\,.\,QK}{PI\,.\,PK}=\frac{CQ\,.\,BF\,.\,DQ\,.\,EF}{CF\,.\,BP\,.\,DF\,.\,EP},$$

but this last expression is also equal to

$$\frac{QG\,.\,QH}{PG\,.\,PH}$$

(Menelaus on $\triangle PQF$ and BGD, and on $\triangle PQF$ and CHE) so

$$\frac{QI\,.\,QK}{PI\,.\,PK}=\frac{QG\,.\,QH}{PG\,.\,PH}$$

and the pairs IK, GH are in involution, with PQ. Suppose next that the conic through $BCDE$ is a circle, and l meets it at LM. Then

$$QL\,.\,QM = QC\,.\,QD \quad \text{by the secant theorem, and}$$

$$PL\,.\,PM = PB\,.\,PE \quad \text{similarly.}$$

So

$$\frac{QL.QM}{PL.PM}=\frac{QC.QD}{PB.PE}\cdot\left(\frac{FC.FD}{FC.FD}\right)$$

and

$$FC.FD = BF.EF \quad \text{by the secant theorem.}$$

So

$$\frac{QL.QM}{PL.PM} \quad \text{is also equal to} \quad \frac{QI.QK}{PI.PK}\left(=\frac{QC.QD}{PB.PE}\right)$$

which establishes that QP, LM and IK are six points in involution.

Finally, suppose the conic through $BCDE$ is arbitrary. It may be represented as a plane section of a cone which is mapped, by projection from the vertex, onto a second plane in which the conic is a circle. But, because involution is a projective invariant, the theorem is still true. As Desargues rightly insisted, 'This proof, naturally, is applicable on numerous occasions, . . .' (p. 110). Indeed, in his hands it at once unifies and simplifies the theory of conic sections.

As a corollary of his theorem, Desargues observed that if BC and DE are parallel, then

$$\frac{BI.CI}{KE.KD}=\frac{IQ.IP}{KQ.KP}.$$

This follows from the observation that $\triangle$'s PBI and PEK are similar, so $BI/KE = IP/KP$, and $\triangle$'s QIC and QKD are similar, so $CI/KD = IQ/KP$.

Desargues next established a series of results on line pairs and circles, which he used later to show how to find the foci of a given conic (see below), before turning to the proof that the transversal of the pencil of lines through a point with respect to a conic is indeed a straight line. His crucial result is:

Theorem 4. *If FCB and FDE are lines meeting a conic at B, C, and D, E respectively, BD meets CE at G, and BE meets CD at N, then GN is the transversal of the pencil of lines through F with respect to the conic.*

Proof. Suppose FG meets CD at X and BE at Y, then

$$\frac{GX}{GY}=\frac{DX.BN}{DN.BY} \quad \text{(Menelaus on } \triangle NXY\text{, with } DBG\text{),}$$

but

$$\frac{GX}{GY}=\frac{CX.EN}{CN.EY} \quad \text{(Menelaus on } \triangle NXY\text{, with } CGE\text{)}$$

and similarly

$$\frac{FX}{FY} = \frac{DX.EN}{DN.EY} = \frac{CX.BN}{CN.BY}$$

so

$$\frac{GX}{GY} = \frac{FX}{FY},$$

and so FG and XY are pairs in involution.

Similarly, if GN meets BC at O then FO and CB are pairs in involution. Again, if a general line of the pencil through F meets CD at I, BE at H, and GN at K then FK and IH are in involution. If this line meets the circle at L and M then by Theorem 3 (Desargues' involution theorem) FK and LM are pairs in involution, so QN is the transversal of F.

Desargues observed that this result makes it easy to construct the polar of F with respect to a conic. One merely forms a complete quadrangle inscribed in the conic with F as one diagonal point, then the line joining the other two diagonal points is the sought-for polar. Consequently it is easy to draw the tangents from F to the conic, they meet it where the polar of F meets the conic. It is also easy to construct the tangent to a conic at a given point P lying on it. For, draw an arbitrary tangent to the conic touching it at B, and let BC be a diameter of the conic. Let PQ be the line through P meeting BC at Q, which is parallel to tangent to the conic at B. Let R be the fourth harmonic point to B, C, and Q, then RP is tangent to the conic at P. (In general, it is enough to be able to find the point R which has a secant through P as its polar, for then RP must be tangent to the conic.)

These constructions are simpler than those in *Conics*, IV. Desargues then (p. 118) proved a theorem dual (as we would say today) to Theorem 4, and which, in the notation of that theorem, asserts:

Theorem 5. *The polars of points lying on NG all concur at F, and conversely the poles of lines through F lie on NG.*

Proof. The theorem is trivial for the circle when NG is a diameter, for the polars are all parallel, being perpendicular to NG. It remains true under projection.

Desargues defined conjugate in the context of projective geometry by considering a general projection of this configuration. Thus two lines are *conjugate* if each passes through the pole of the other. This agrees with Apollonius's definition of conjugate when the lines are diameters; conjugate diameters in the Apollonian sense are conjugate lines in Desargues'. The conjugate lines touching the conic at infinity are its asymptotes as Apollonius had shown; they are necessarily diameters since they meet at the centre of the conic.

A brief detour was then made to discuss coaxal circles before returning to the elaboration of more classical material on conics. The parameter was introduced as follows (p. 125). Let PH cut a conic at L and M, and draw BKC parallel to EDI, meeting PH at K and I, respectively. Also draw MT parallel to BIC, and let $LRST$ be any line through L such that $KS.KM = KD.KE$.

Then Desargues claimed that

$$\frac{KL.KM}{KE.KD} = \frac{ML}{MT},$$

and in particular, if LM is a diameter, MT is the parameter. To prove this, Desargues observed that

$$\frac{KL.KM}{KS.KM} = \frac{KL}{KS} = \frac{IL}{IR} = \frac{IL.IM}{IR.IM},$$

so

$$\frac{KS.KM}{IR.IM} = \frac{KL.KM}{IL.IM},$$

which, however, is equal to

$\dfrac{KD.KE}{IC.IB}$ (by the corollary to Desargues' involution theorem)

so

$$\frac{KL.KM}{KE.KD} = \frac{KL.KM}{KS.KM} = \frac{KL}{KS} = \frac{ML}{MT},$$

MT is the parameter corresponding to the diameter LM (cf. Apollonius' definition).

When furthermore LM is the principal diameter (major axis) the foci were obtained from the following result, which is also proved in *Conics* (III,42);

Lemma 3. *If a tangent to a conic at an arbitrary point L meets the two parallel tangents at the extremities E and C of a diameter at B and D, say, then $EB.CD$ is a constant, independent of L. In fact, if the bisector of $B\hat{E}C$ meets the conic at V and CV meets EB at F, then $EB.CD = \frac{1}{4}EC.EF$.*

Proof. Desargues first established

$$\frac{EB.CD}{7E.7C} = \frac{EF}{EC} \quad \text{where 7 is the centre of the conic,}$$

which means that $EB.CD$ is constant, as follows: draw LI parallel to BE, meeting EC at I. Let A be the fourth harmonic point of C, I and E, i.e. A is where the tangent of L meets EC. Then

$$\frac{A7}{AI} = \frac{7C.7E}{IC.IE},$$

since C, I, E and A are in a tree with stump 7.

Let $R7O$ be drawn parallel to B, E, then

$$\frac{A7}{AI} = \frac{7R.IL}{IM.IL} \quad \text{since} \quad \frac{A7}{AI} = \frac{7R}{IM}$$

and $IM = IL$. But $7R.IL = EB.CD$ since

$$\frac{EB}{IL} = \frac{AE}{AI}, \qquad \frac{7R}{CD} = \frac{A7}{AC}, \qquad \frac{AE}{AI} = \frac{A7}{AC}$$

since C, I, E, A are involution with stump 7, so

$$\frac{EB.CD}{7C.7E} = \frac{IM.IL}{IC.IE}.$$

But

$$\frac{IM.IL}{IC.IE} = \frac{EF}{EC} \quad \text{since } EF \text{ is the parameter,}$$

so

$$\frac{EB.CD}{7C.7E} = \frac{EF}{EC}$$

and $EB.CD$ is constant as claimed.

A simple calculation based on the fact that $7C = 7E = \frac{1}{2}CE$ then gives $EB.CD = \frac{1}{4}EC.EF$.

From this it follows that when EC is the major axis the circle on BD as diameter meets EC at P and Q which are the foci of the conic, that $PE.PC = QE.QC = \frac{1}{4}EC.EF$, that $PL + QL$ or $PL - QL$ is a constant, and that the tangent at L bisects the angles made by the focal radii at L. Desargues merely remarks that the proof of these assertions is 'an obvious converse to what we have proved', but Poudra and Coolidge professed themselves unable to see how. Taton (1951, p. 167) following Zacharias, makes the suggestion that they can be derived from the useful results on line pairs and circles referred to above. Even so, Desargues' argument is rather a consequence than a converse. As Apollonius had shown (*Conics*, III,45–52) they are all straightforward exercises in similar triangles.

The finale, pp. 138–141, is devoted to showing how, given a limited amount of information about a conic section, its foci can be found and indeed the cone constructed having the given section in a specifiable position. In this way Desargues showed how to return his description of conics to its Apollonian roots. The remainder of the *Rough Draft on Conics* consists of historically interesting remarks about the consequences of his work for the philosophically and the practically minded.

Chapter V

Translators' Preface

Few historians have undertaken a detailed study of Desargues' *Rough Draft on Conics*, although most histories of mathematics make at least a passing reference to the work. The reasons for this comparative neglect are not far to seek. The mathematical content of the *Rough Draft on Conics* is by no means elementary, and probably suffices to deter a majority of those whose primary training is not in science. On the other hand, mathematicians may well be discouraged by the language and style of Desargues' work. For example, statements and proofs of theorems are rarely separated from one another. Moreover, since the work is highly original—and Desargues himself was conscious of its originality—it required a new vocabulary. Desargues chose to construct one that was radically new, and even proposed new names for entities which already had accepted names—such as the parabola, ellipse and hyperbola.

The neologisms Desargues uses in renaming these three conics provided the translator with a rare opportunity for legitimately inventing words by way of translation. However, the most prominent feature of Desargues' new vocabulary presents no problem for the translator, but may present an obstacle to the reader. Desargues, writing in French instead of Latin, and thus, like Descartes, ostensibly addressing his work not only to professional mathematicians but also to educated laymen, apparently strove to give it a 'common touch' by employing a down-to-earth vocabulary drawn from botany—or, rather, mainly from arboriculture. Unfortunately, the effect of speaking of geometrical entities as, for example, trees, stumps, branches, boughs and shoots, seems to have been the reverse of what the author intended. When asked to express an opinion of the *Rough Draft on Conics*, Descartes advised Desargues to express himself in simpler terms and in a lighter style if he wished his work to appeal to laymen.[1] Indeed, it is not difficult to imagine that the very concise style and unusual vocabulary of

Desargues' work may have contributed to its being neglected even by mathematicians of the day.

There are, however, other notable features of Desargues' mathematical style which would have seemed quite natural to his contemporaries, but tend to make the text more difficult for a twentieth-century mathematician. For example, Desargues refers to the 'rectangle of the lines *AB*, *CD*' where a twentieth-century mathematician would refer to the 'product of *AB* and *CD*'. Moreover, Desargues prefers to deal in terms of ratios, which are 'compounded' (the equivalent of multiplying the corresponding fractions) or manipulated in various other more complicated ways. All these operations with ratios are to be found in Euclid's *Elements*—as too is the geometrical expression for the multiplication of line segments. In fact, these stylistic features are characteristic of the Greek tradition from which Desargues' work derives. We have accordingly translated Desargues' expressions as nearly as possible in the same words Thomas L. Heath used in his translation of Euclid. To have done otherwise would have involved a considerable modernization—and distortion—of Desargues' text. For similar reasons we have also followed Desargues in expressing equalities in terms of ordinary (if cumbersome) sentences, rather than re-writing his work in the neater form of equations. This 'modern' form was, in fact, used by some of Desargues' contemporaries, but he himself seems to have preferred the older, so-called 'rhetorical', mode. Modern notation is therefore confined to our notes.

Apart from the difficulties introduced by the stylistic features already noted, Desargues' text also lacks diagrams, so that, despite the fact that Desargues' results are now no longer unfamiliar, the translator often found it necessary to keep the place in the text with one hand while drawing diagrams and writing out the translation with the other. One reward of this plodding method, as applied to Desargues' perspective theorem, was the discovery that in Taton's edition (1951) the statement of the theorem was incorrect. Since the theorem concerns two triangles in perspective, its statement clearly requires *nine* original lines: the sides of the triangles and the joins of their corresponding vertices. Taton's text gave only eight lines. So, it turned out, did Poudra's text of 1864. So too did Abraham Bosse's text of 1648—the first surviving published account of the theorem (as far as is known). We feel it is an eloquent tribute to the obscurity of Desargues' style that such a simple omission could occur in so important a passage without attracting the attention of two highly competent mathematicians who edited the text. (Bosse probably would not have read his own proofs, so the omission may well have originated with his printer.) The fact that this error remained undetected for so long seems ample justification for the translation that follows.

Desargues' text is inherently difficult, and our translation cannot pretend to do more than merely make it more easily accessible to twentieth-century mathematicians and historians of science, many of whom will certainly wish to continue to refer to the original French text. In attempting to palliate the difficulties a modern mathematician may encounter with Desargues' text, we have not made any drastic alterations to his literary style. Those inversions of word order which are a feature of French style but not of English have been removed; but those which are a feature of the style of statements of geometrical theorems, as established by Euclid, have been retained. Occasional genuinely obscure passages in the original appear as obscure passages in the translation, the reader being referred to our notes. Desargues' botanical vocabulary has been translated consistently (since the words are in context technical terms) and the translations of the various terms are listed separately at the end of this chapter. Our list of Desargues' vocabulary tallies with that given by Poudra in 1864.

The Origin of Desargues' New Vocabulary

Desargues' new vocabulary has been much commented upon by historians. Some, indeed, have apparently regarded it as the element in his work most deserving of comment. However, only Ivins (1947), in a paper far too modestly titled 'A note on Desargues' theorem', has suggested a possible source of the botanical terminology which might also be a source of some of Desargues' ideas. Ivins suggested that Desargues' numerous arboricultural terms might be a reminiscence of Alberti's use of botanical imagery in his brief account of linear perspective in the first book of *Della Pittura* (1436).[2]

It is indeed true that Alberti uses botanical terms in connection with geometry, but he does so only in the course of one simile, whose length is only a few lines.[3] Moreover, Alberti's Italian text was not published until 1847 and if Desargues did read the work he must have read it in manuscript (not a very likely possibility) or in the Latin *editio princeps* of 1540[4] or in one of the two Italian translations apparently made from the Latin version and published in 1547 and 1568.[5] Unfortunately, the Latin text, as printed in 1540, appears to contain a solecism in the particular passage with which we are concerned—using the word *virgulata*, which we have not been able to locate (with a suitable sense) in any Latin dictionary. The sixteenth-century Italian translators give differing versions of the passage: one interpreting the word as feminine singular, the other taking it as neuter plural. Reference to Alberti's Italian text shows that the former was the happier choice. However, in neither case, as 'shoot' or 'shoots', does the word correspond to Desargues' usage of an equivalent French term in the *Rough Draft on Conics*. The same is true of the other botanical terms used by Alberti in this passage. As far as vocabulary is concerned, it would appear that Desargues' debt to Alberti is, at most, slight and ill-defined.

On the other hand, as will be clear from Chapter II above, we are in entire agreement with the further implication that is carried by Ivins' suggestion, namely that Desargues' work should be seen as indebted to the tradition of writings on linear perspective of which Alberti's work is the earliest surviving example. This is a matter of much more importance than tracing the origin of Desargues' botanical terms. We should, however, like to put forward a conjecture of our own as to their origin, namely that it seems possible that they developed from the basic term 'tree' (which is used to describe a fundamental concept) and that this term was derived from engineers' use of the word 'arbre' ('tree') to describe an axle or arbor (the second English term being, in fact, the Latin for 'tree'). This would link Desargues' unconventional vocabulary with his known activity as an engineer.

Texts Used

For the works of Desargues, and correspondence connected with them, we have worked primarily from the *Oeuvre mathématique de G. Desargues* edited by Taton (1951), except for the *Perspective* of 1636, which Taton did not reprint. For this work we used the reprint in Abraham Bosse's *Perspective* of 1648. The texts of Desargues' *Rough Draft on Conics* and *Perspective* have been checked against photocopies of the original editions supplied by the Bibliothèque nationale, Paris. In his edition of 1951, Taton adopted, without comment, all the modifications which Desargues made to his own text in an *Advertissement* attached to the first printed version of the *Rough Draft on Conics*. This *Advertissement* is much more than a huge list of errata, and since some of the changes are very considerable, we have noted all of them in our translation. It also appears from the 1639 edition that words printed in italics were intended as footnotes. They accordingly take this form in our translation.

The edition of Desargues' works by Poudra (1864) includes the *Perspective* of 1636, but was found to contain many editorial emendations, not usually indicated in the notes. Poudra's notes did, however, prove useful, since he was very interested in mathematical perspective (on which he wrote a textbook).

The first page and the important final page of Desargues' 1636 *Perspective* were reproduced in Ivins' paper 'A note on Girard Desargues' (Ivins, 1943). The complete 1636 text, which differs little from the reprint by Bosse of 1648, is reprinted in Appendix 5 below.

Desargues' *Universal Method of Setting up the Stylus and Drawing the Lines on a Sundial* (1640) was listed as lost by Taton (1951). A single copy of the work was found by A. J. Turner in 1983. Our translation was made from a photocopy of this, kindly supplied by Mr Turner, who has since published the French text (Turner, 1984).

For Pascal's *Essay on Conics* we have used the French text published by Taton in *Revue d'histoire des sciences*, Vol. VIII (Pascal, ed. Taton, 1955).

Vocabulary List for *Rough Draft on Conics* (1639)

Words are given in order of their appearance in the text.

	English	French
1.	ordinance of lines or planes	ordonnance de droictes ou plans
2.	butt of an ordinance	but d'une ordonnance

3. axle	essieu
4. trunk	tronc
5. knot(s)	nœud(s)
6. branch	rameau
7. branch springing from the trunk	rameau desployé au tronc
8. branch folded to the trunk	rameau plié au tronc
9. shoot of a branch	brin de rameau
10. crown	rameure
11. points engaged	poincts engagés
12. points disengaged	poincts desgagés
13. points mixed	poincts meslez
14. points unmixed	poincts démeslez
15. pair of points	couple de poincts
16. marker post	borne
17. marker line	bornale droicte
18. tree	arbre
19. stump	souche
20. limb	branche
21. mean limbs	branches moyennes
22. extreme limbs	branches extrémes
23. limbs paired with one another	branches couplées entre elles
24. pair of shoots related to another pair of shoots	couple de brins relative à une autre couple aussi de brins
25. twin pairs of shoots	couples de brins gemelles entre elles
26. simple mean knot	nœud moyen simple
27. double mean knot	nœud moyen double
28. inner extreme knot	nœud extréme interieur
29. outer extreme knot	nœud extréme exterieur
30. pair of knots	couple de nœuds
31. knots that correspond to one another	nœuds correspondans entre eux
32. stumps reciprocal to one another	souches reciproques entre elles
33. bough	ramée
34. branches that correspond to one another	rameaux correspondans entre eux
35. roll	rouleau
36. vertex of the roll	sommet du rouleau
37. flat basis or base of the roll	plate assiette ou base du rouleau
38. column or cylinder	colomne ou cylindre
39. cornet or cone	cornet ou cone
40. plane of section of the roll	plan de coupe du rouleau
41. section of a roll	coupe de rouleau
42. edge of the section of a roll	bord de la coupe de rouleau

43.	deficity [ellipse]	defaillement
44.	oval [ellipse]	ovale
45.	equalation [parabola]	égalation
46.	surpassing [hyperbola]	outrepassement
47.	excedency [hyperbola]	excedement
48.	transversal	traversale
49.	ordinates of a transversal	ordonnées d'une traversale
50.	transversal point of ordinate	[poinct] traversal d'une ordonnée
51.	ordinal	ordinale
52.	diametral	diametrale
53.	diametransversal	diametraversale
54.	conjugal	conjugale
55.	coadjutor	coadjuteur
56.	navel	nombril
57.	burning point	poinct bruslant

Vocabulary List for *Rough Draft on Conics* (1639)

Words in alphabetical order

(a) English

Word	*See number*	*Word*	*See number*
axle	3	flat basis	37
basis, base	37	folded	8
bough	33	inner	28
branch	6, 7, 8	knot(s)	5, 26–31
burning point	57	limb	20–23
butt	4	line, marker	17
coadjutor	55	marker line	17
column	38	marker post	16
conjugal	54	mean knot	26, 27
cornet	39	mean limbs	21
correspond	31	mixed, points	13
crown	10	navel	56
deficity	43	ordinal	51
diametral	52	ordinance	1
diametransversal	53	ordinates	49
disengaged	12	outer	29
double mean knot	27	oval	44
edge of section	42	pair of knots	30
engaged	11	pair of shoots	24
equalation	45	pairs of shoots, twin	25
excedency	47	plane of section	40
extreme	22, 28, 29	point, burning	57

Word	*See number*	*Word*	*See number*
point, transversal	50	stump	19
points	11–15	stumps, reciprocal	32
post, marker	16	surpassing	46
reciprocal stumps	32	transversal	48
related	24	transversal point	50
roll	35	tree	18
section of a roll	41	trunk	4
section, edge of	42	twin pairs	25
section, plane of	40	unmixed points	14
shoot	9	vertex of roll	36
simple mean knot	26		

(b) French

Word	*See number*	*Word*	*See number*
arbre	18	gemelles, brins	25
assiette	37	interieur	28
base du rouleau	37	meslez	13
bord de la coupe	42	moyen, nœud	26, 27
bornale	17	moyennes, branches	21
borne	16	nœuds	5, 26–31
branche(s)	20–23	nombril	56
brin	9	ordinale	51
but	2	ordonnance	1
coadjuteur	55	ordonnées	49
colomne	38	outrepassement	46
conjugale	54	ovale	44
cornet	39	plan de coupe	40
correspondans	31, 34	plate assiette	37
coupe, plan de	40	plié au tronc	8
coupe de rouleau	41	poincts	11–15
couple	15, 24, 25, 30	poinct bruslant	57
couplées, branches	23	poinct traversal	50
defaillement	43	rameau(x)	6–9, 34
démeslez	14	ramée	33
desgagés	12	rameure	10
desployé	7	reciproques, souches	32
double, nœud	27	relative	24
diametrale	52	rouleau	35
diametraversale	53	simple, nœud	26
égalation	45	sommet du rouleau	36
engagés	11	souche(s)	19, 32
essieu	3	traversal, poinct	50
excedement	47	traversale	48
exterieur	29	tronc	4
extrémes	22		

Vocabulary List for *Perspective* (1636)

Words are given in order of their appearance in the text

	English	French
1.	drawing in perspective	pratiquer la perspective
2.	distance point	point de distance
3.	perspective (noun)	perspective
4.	appearance	apparence
5.	representation	representation
6.	portrayal	pourtrait
7.	extremity	extremité
8.	edge	bord
9.	side	costé
10.	outline	contour
11.	represent	representer
12.	portray	pourtraire
13.	find the appearance	trouver l'apparence
14.	render or construct in perspective	faire ou mettre en perspective
15.	level	à niveau
16.	level parallel to the horizon	de niveau paralel à l'horizon
17.	vertically, upright	à plomb
18.	perpendicular to the horizon	perpendiculaire à l'horizon
19.	square to the horizon	quarrement à l'horizon
20.	on the square	à l'équière (esquerre)
21.	subject	sujet (suiet)
22.	geometric plan	plan geometral
23.	ground plan	plan de terre
24.	base of the subject	plante du sujet
25.	basis (of the subject)	assiete (du sujet)
26.	window [Leonardo window]	la transparence
27.	section	section
28.	picture	tableau
29.	front of the picture	devant du tableau
30.	back	derrière
31.	upper surface of the basis of the subject	dessus de l'assiete du sujet
32.	lower surface of the basis of the subject	dessous de l'assiete du sujet
33.	line of sight	ligne de l'œuil
34.	part (of a line)	piece (d'une ligne)
35.	the foot of the vertical through the eye	le pied de l'œuil
36.	area	espace
37.	distance scale	eschelle d'eloignement
38.	dimension scale	eschelle des mesures

39. side [vertical side]	élevation
40. axle	essieu

We have listed Desargues' perspective terms because he himself made them a matter for formal definition and there is no one standard vocabulary to be found in perspective treatises of the period. The vocabulary Desargues used seems, in fact, to have no significance beyond the work in which it occurs, except for the word 'axle' (item 40), used in the final section, which points forward to the *Rough Draft on Conics*.

Chapter VI

The *Rough Draft on Conics* (1639)

The original title of Desargues' work is *Brouillon proiect d'une atteinte aux evenemens des rencontres du Cone avec un Plan.*

Note: Numbers in the left-hand margin in [ ] are Desargues' page numbers (1639)

1639 With Privilege.[1]

ROUGH DRAFT FOR AN ESSAY ON the results of taking plane sections of a cone, By *L, S, G, D, L.*

In this work there will be no difficulty in making the necessary distinction between the giving of names (otherwise definitions), propositions, demonstrations, when they follow on, and passages of other kinds; neither will it be difficult to select from among the figures the one which refers to the sentence one is reading, or to construct these figures from what is said in the text.

Everyone will form his own opinion both of what we deduce and of the manner in which we deduce it, and will see that our reason is trying to grasp, on the one hand, infinite quantities, and together with these, quantities so small that their opposing extremities coincide;[2] and [will also see] that these quantities are beyond our understanding, not only because they are unimaginably large or small, but also because ordinary reasoning leads us to deduce from their sizes properties which it cannot comprehend.

In this work every straight line is, if necessary, taken to be produced to infinity in both directions.

We indicate this infinite extension in both directions by means of a row of dots, lined up with the line, lengthening it in both directions.

Ordinance of straight lines

To convey that several straight lines are either parallel to one another or are all directed towards the same point we say that these straight lines belong to the same ordinance, which will indicate that in the one case as well as in the other it is as if they all converged to the same place.[3]

Butt, *of an ordinance of straight lines*

The place to which several lines are thus taken to converge, in the one case as well as in the other, we call the *butt* of the ordinance of the lines.

To convey that we are considering the case in which several lines are parallel to one another we often[4] say that the lines belong to the same ordinance, whose butt is at an infinite distance along each of them in both directions.

To convey that we are considering the case in which all the lines are directed to the same point we say that all the lines belong to the same ordinance, whose butt is at a finite distance along each of them.

Thus any two lines in the same plane belong to the same ordinance, whose butt is at a finite or infinite distance.

In this work every Plane is similarly taken to extend to infinity in all directions.

We indicate that a Plane extends to infinity in all directions by means of a number of points scattered all round in this same Plane.[5]

Ordinance *of planes*

To convey that several planes are either parallel to one another, or are all directed to the same line, we say that all the planes belong to the same *ordinance*, which will indicate that in the one case as well as in the other it is as if they all converged to the same place.[6]

Axle *of an ordinance of Planes*

The place to which several Planes are thus imagined to converge, in the one case as well as in the other, we call the *Axle*[7] of the ordinance of the Planes.

To convey that we are considering the case in which several Planes are all parallel to one another we say that all the Planes belong to the same *ordinance* whose axle[8] is in each of them at an infinite distance in all directions.

To convey that we are considering the case in which several Planes are all directed to the same line we say that all the Planes belong to the same ordinance whose axle[9] is in each of them at a finite distance.

Thus any two planes belong to the same ordinance, whose axle[10] is in each of them at a finite or infinite distance.

If we imagine that an infinite straight line which has a fixed point moves the whole of its length, we see that in the various positions it occupies in the course of this motion it gives or represents, as it were, various lines belonging to the same ordinance, whose butt is the fixed point of the line.

When the fixed point of the straight line lies at a finite distance, and the line moves in a Plane, we see that in the various positions it occupies in the course of this motion it gives or represents, as it were, various lines belonging to the same ordinance, whose butt (the fixed point of the line) lies in each of them at a finite distance, and that every other point of the straight line apart from the fixed one traces out a simple uniform line, any two of whose parts have
[2] the same figure and are like one another; that is to say, it is curved completely round, that is, otherwise, it is circular.[11]

When the fixed point of the straight line lies at an infinite distance, and the line moves in a Plane, we see that in the various positions it occupies in the course of this motion it gives or represents, as it were, various lines belonging to the same ordinance, whose butt (the fixed point of the line) lies in each of them at an infinite distance in both directions, and that every other point of the straight line apart from the fixed one traces out a simple uniform line, any two of whose parts have the same figure[12] and are like one another, namely a straight line perpendicular to the moving one. If we follow the train of thought suggested by this idea, we come to see that there is, as it were, a kind of relation between the infinite straight line and the one with uniform curvature, that is, the relation of the infinite straight line and the circular one, so that they are like two species of the same genus, and the same words may be used in describing how to construct each of them.[13]

When through various points of a straight line there pass various other straight lines, in any manner, the line on which these points lie is called a *Trunk* [Fig. 6.1]. Trunk

The points on the trunk through which other straight lines pass in this manner are called *Knots*. Knots

Any other straight line which passes through one of the knots is called a *Branch* in relation to the trunk. Branch

When two Branches are parallel to one another they are called *parallel branches*, in relation to one another,[14] *right branches*. Right branches

When a branch cuts the trunk or diverges from the trunk it is called a *Branch springing from the trunk*. Branch springing from the trunk

Any part or segment of the trunk contained between any two knots of the trunk is called a *Branch folded to the trunk*. Branch folded to the trunk

Each part or segment of a branch contained between its knot Shoot of a Branch

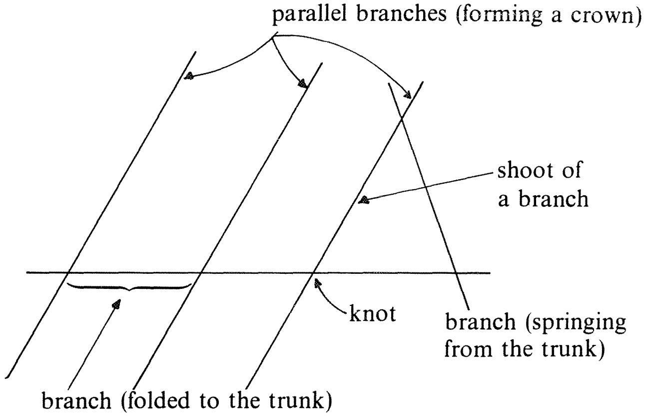

Fig. 6.1

and some other branch or its knot is called a *Shoot of the branch*.

Several right branches springing from the trunk in any manner are jointly called a *Crown*.*[15] Crown

When in a straight line AFD[16] a point A is common to each of the parts AF, AD, either these two parts are separate, one, AF, being on one side and the other, AD, on the other side of their common point A, which thus lies between them, or they both lie together on the same side of their common point A, which thus does not lie between them [Fig. 6.2].

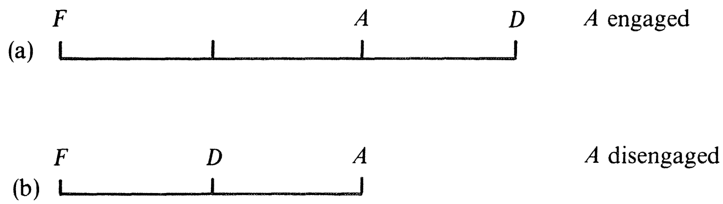

Fig. 6.2

To convey that we are considering the case in which the common point lies between the two parts we say that their common point A is *engaged* between the two parts. Engaged *common point*

To convey that we are considering the case in which the common point does not lie between the two parts we say their common point A is *disengaged* from the two parts. Disengaged *common point*

When in a straight line DF there are two pairs of points CG, DF, either one of the points, C, of one of the pairs, CG, lies

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* Note that up to page 10 everything lettered $A\ B\ C\ D$ refers to the simple lines in the plate.

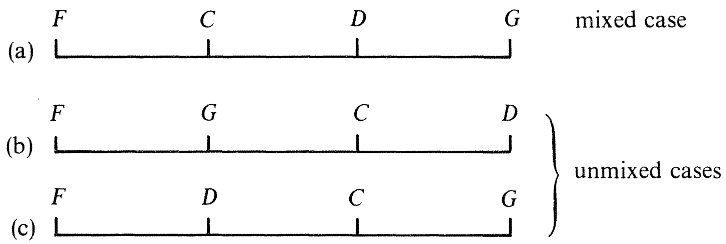

Fig. 6.3

between the two points of the other pair, DF, and the other point, G, of the same pair, CG, lies outside the same two points of the other pair, DF; or the two points of one pair, CG, either both lie between or both lie outside the two points of the other pair, DF [see Fig. 6.3(a), (b), (c), respectively].

Points of a pair mixed *with the points of another pair*

To convey that we are considering the case in which for two pairs of points one of the points of one pair, C, lies between the points of the other pair, and its partner, G, lies outside them, we say that the points of one of the pairs are *mixed* with the points of the other pair.

Points of a pair unmixed *with the points of another pair*

To convey that we are considering the case in which for two pairs of points the points of one pair either both lie between or both lie outside the points of the other pair, we say that the points of one pair are *unmixed* with the points of the other pair.

Marker post

When in a Plane four points do not all lie on the same straight line we say that each of these points is a *marker post* with respect to the others.

Marker line

Each straight line which passes through any two of the four marker posts is called a *marker line* in relation to these points.

Pair of marker lines

The two straight lines of which one passes through two of the marker posts and the other through the other two of the four, are taken as forming a pair and are called a pair of *marker lines*.

Each marker line can on occasion be a trunk.

Proposition comprising propositions 5 and 6 of the second book of Euclid's *Elements* and its converse.

Proposition comprising propositions 9 and 10 of the second book of Euclid's *Elements* and its converse.

Proposition comprising propositions 35 and 36 of the third book of Euclid's *Elements* and its converse.[17,1]

When[b] in a Plane we have three points, as knots, lying on a
[3] straight line, as trunk, and through these points there pass any three branches springing from the trunk, the two shoots of any of

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[a] The secant theorems.

[b] Menelaus' theorem.

these branches contained between their knot or trunk, and each of the other two branches bear to one another the same ratio as is formed by the product of the ratios of the two similar shoots of each of the other two branches taken in the appropriate order. Stated differently in Ptolemy.*

When in a straight line *AH* there is a point *A*, common to and similarly engaged between or disengaged from the two parts of each of the three pairs *AB*, *AH*; *AC*, *AG*; *AD*, *AF*, whose three rectangles[a] are equal to one another, this property in a straight line is called a *Tree*, and the straight line itself is a *Trunk* [Fig. 6.4].

Tree
Trunk

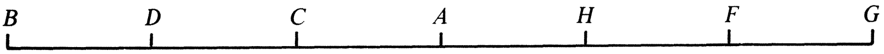

Fig. 6.4. A tree—engaged case.

The point such as *A* which is thus common to each of the six parts *AB*, *AH*, *AC*, *AG*, *AD*, *AF* is called a *Stump*.

Stump

Each of these same six parts *AB*, *AH*, *AC*, *AG*, *AD*, *AF* is called a *Limb*.

Limb

When the two limbs which contain one of the three rectangles are equal to one another we call them *Mean limbs*.

Mean Limbs

When the two limbs which contain one of the equal rectangles are not equal to one another we call them *Extreme limbs*.[19]

Extreme limbs

Two limbs such as *AG*, *AC*, or *AF*, *AD*, or *AH*, *AB*, whose rectangle is equal to each of the other two rectangles, are called *Limbs paired with one another*.

Paired limbs

Each of the separate ends *B C D F G H* of the limbs in each of the three pairs *AB*, *AH*; *AC*, *AG*; *AD*, *AF*, is called a *Knot*.

Knot

That is how the knots of the tree are spaced out along the trunk.

The knots of the mean limbs are called *Mean knots*.

Mean knots

The two knots of a pair of extreme limbs are called a pair of *Extreme knots*.[20]

Extreme knots

Knots such as *G* and *C*, which are formed on the trunk of the tree by the two limbs of any one pair *AG*, *AC*, are called *Knots paired with one another*.

Knots paired with one another

Two branches springing from the trunk which pass through the two knots of a pair are called a pair of *Branches springing from the trunk*.[21]

Pair of Branches springing from the trunk

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* Note that this proposition comes after the long one at the bottom of page 10.[18]

[a] Rectangle means product, so in a tree $AB.AH = AC.AG = AD.AF$

Each part of some one tree, such as the part GF, which is contained between one of the knots, G, belonging to any pair GC, and any other knot F, belonging to any one of the remaining pairs DF, is called a *Shoot of a branch folded to the trunk.*

Shoot of a Branch folded to its Trunk

That is how the same shoot of a branch folded to the trunk[22] is either the sum or the difference of two limbs of two distinct pairs.

Two shoots of branches folded to the trunk, such as GD, GF, which are both attached at one end to any limb AG at its knot G, and at the other end finish one at each of the two knots D, F of any other pair of limbs AD, AF, are called *Shoots of branches paired with one another.*

Shoots of branches paired with one another

The pair of shoots of branches such as CD, CF, which are both attached at one end to the knot C of the limb AC, which is partner to the limb AG, shoots which at the other end finish one at each of the same two knots D, F, at which the two shoots of the pair GD, GF, also finish, is called a *Pair of shoots related to the pair of shoots GD, GF.*

Pairs of shoots

Two pairs of shoots, such as the two pairs GD, GF, and CD, CF, one of which pairs is attached to the knots of each of the pair of limbs AG, AC, and which pairs of shoots moreover both[23] finish at each of the two knots of any of the other pairs DF, are called *Pairs of shoots related to one another.*

Pairs of shoots related to one another

The two rectangles of each of two pairs of related shoots, such as the rectangles of the shoots belonging to the pair GD, GF, and of the shoots belonging to the pair CD, CF, are called *Rectangles related to one another.*

Rectangles related to one another

Two distinct pairs of shoots, such as GD, GF; GB, GH,*[24] which are both attached to the knot G of the branch AG, and which moreover finish at another two distinct pairs of knots DF, BH, are called *Twin Pairs of shoots.*

Twin pairs of and shoots

The two rectangles of two twin pairs of shoots, such as those of the pair GD, GF and of the pair GB, GH are called *Twin rectangles.*

Twin rectangles

When in a tree, AH, the stump, A, is engaged between the two limbs of either of the pairs, AC, AG, it is clear that this stump, A, is also engaged between the two knots of each of the pairs CG, BH; and it is clear that CG, the two knots of each of the pairs, are mixed with the two knots of each of the other two pairs DF, BH.

* Note that the limbs and the knots are usually given in pairs, and that in the printed text we must separate the identifying letters of one pair from the letters of another when these letters follow immediately upon one another; and in the same way subordinate the order of the letters to that of the identifiers when such are present,[25] so that for convenience we might even in each of our figures always use the same letters for similar purposes and when similar properties are concerned.

And conversely, when in a tree AH the two knots of any of the pairs, CG [say], are mixed with the two knots of any of the other pairs, DF [say], then the stump A is engaged between the two limbs of each of the pairs AC, AG; AB, AH; AD, AF and between the two knots of each of the pairs CG, DF, BH.

When in a tree, AH, the stump, A, is disengaged from the two limbs of each of the pairs AC, AG; AF, AD; AB, AH, it is clear that this stump, A, is also disengaged from the two knots of each of the pairs CG, DF, BH, and the two knots of each of the pairs CG, DF, BH are also, clearly, disengaged from the two knots of each of the other pairs [Fig. 6.5].

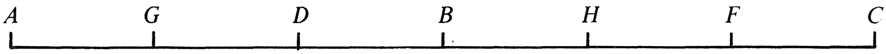


Fig. 6.5. A tree—disengaged case.

And, conversely, when in a tree the two knots of any one of the
[4] pairs, CG [say], are disengaged from the two knots of each of the other pairs [such as] DF, then the stump, A, is disengaged between the two knots and the two limbs of each of the pairs.

From the above it clearly follows that if in a tree HB we know the type of position of stump, limbs and knots in regard to one another [i.e. whether engaged or not] in one case [e.g. stump with limbs] then we also know the type of all the other positions among the remainder of the things.

And generally for either of these two forms of tree [Figs 6.4 and 6.5].

The ratio[a] of any limb, AG, to its partner, AC, is the same as the ratio of the rectangle of any pairs of shoots, GD, GF, attached to the limb AG, to its related rectangle CD, CF.

For, since the rectangles of each of the two limbs of each of the three pairs AB, AH; AC, AG; AD, AF are equal to one another, the four limbs AG, AF, AD, AC are proportional to one another, two by two, from which it follows that: as AG is to AF, or AD to AC, so is GD to CF; and that as AG is to AC, or AG to AD, so is GF to CD.

Consequently the ratio of the limb AG to its partner the limb AC is the product of the ratios of the shoot GD to the shoot CF and of the shoot GF to the shoot CD, which is the ratio of the rectangle of the shoots of the pair GD, GF to the rectangle of the shoots of its related pair CD, CF.

[a] Lemma 1, p. 48. $\dfrac{AG}{AC} = \dfrac{GD\,.\,GF}{CD\,.\,CF}$.

From which it follows that the ratio of the rectangle of the shoots GB, GH, twin to the rectangle GD, FG, to its related rectangle CB, CH, twin to the rectangle CD, CF, is the same as the ratio of the rectangle GD, GF, twin to the rectangle GB, GH, to its related rectangle CD, CF, twin to the rectangle CB, CH.

For, from what we have proved, the rectangle of the two shoots of the pair GB, GH is to its related rectangle CB, CH as the limb AG is to its partner AC.

Moreover, we have also proved that the rectangle of the shoots GD, GF is to its related rectangle CD, CF as the same limb AG is to its same partner AC.

Also, the rectangle of the shoots GB, GH, which is twin to the rectangle GD, GF, is to its related rectangle CB, CH as the rectangle GD, GF is to its related rectangle CD, CF.

From which it follows that the rectangle of the shoots FC, FG also is to its related rectangle DC, DG as the rectangle of the shoots FB, FH is to its related rectangle formed by the shoots DB, DH, that is, it is as the limb AF is to its partner AD.

From which it follows also that the rectangle of the shoots HC, HG is to its related rectangle BC, BG as the rectangle of the shoots HD, HF is to its related rectangle BD, BF, that is, as the limb AH is to its partner the limb AB.

And[a] when in a straight line AH there are three pairs of points, BH, CG, DF so disposed that the two points of each of these pairs are either both mixed with the two points of each of the other two pairs or both unmixed with them; and the related rectangles of the parts between these points are to one another as their twins, taken in the same order, are to one another: such an arrangement of these three pairs of points in a line is called here an *Involution*. Involution

That is to say that, as we say here, three lettered pairs of points on a line are arranged in involution with one another; that is to say that this arrangement of the three pairs of points satisfies all the conditions and shows all the properties which we have explained as appertaining to the knots of a tree of either of the two types, or that these three pairs of points are three pairs of knots of a tree of one of the two types described above.

From which it follows, in addition, that when in a straight line such as AH, four parts such as AG, AL, AD, AC, forming two pairs AG, AC; AL, AD, are not proportional to one another two by two, or their rectangles are not equal to one another, of the two parts of each of these two pairs AG, AC; AD, AL (whether their common end-point A is equally engaged between the two parts of each of the two pairs AG, AC; AD, AL or

[a] Six points in involution.

equally disengaged from them) any one of these parts, such as AG, is not to its partner AC as the rectangle GD, GL is to its related rectangle CD, CL, and the arrangement does not form a tree.

For, since the four parts AG, AL, AD, AC are not proportional to one another, so neither will the rectangles of each of the pairs of parts AL, AD; AG, AC be equal to one another.

Let us therefore take the line AF as partner to any one of these lines, AD [say], such that the pairs of parts AG, AC; AF, AD have
[5] equal rectangles.[a] This part AF, so defined, is not equal to the part AL, and the point F does not coincide with the point L, also, the ratio of the part FC to the part FG is not the same as the ratio of the part LC to the part LG.

So the ratio of the part AG to its partner AC, which is the ratio of the rectangle of the parts GD, GF to the rectangle of the parts CD, CF, is not the same as the ratio of the rectangle of the parts GD, GL to the rectangle of the parts CD, CL, so moreover unless the four parts AG, AL, AD, AC of the two pairs AG, AC and AL, AD are proportional to one another two by two, they do not form a tree of one of the types described above. And the ratio of any one of the parts AG to its partner AC is not equal to the ratio of the rectangle of the parts GD, GL to the rectangle of the parts CD, CL, to which it would necessarily be equal if the four parts were proportional to one another as are the parts AG, AF, AD, AC.[26]

From which it follows that given the positions of any two pairs of knots GC, DF in a tree AH, the position[b] of the stump A is also given, and this is equivalent to the fact that if the sum or difference of two quantities is given and their ratio is also given, then the magnitude of each of these two quantities is given.

For let us first place the stump A so that it is engaged between the knots of each of the two pairs GC, DF or disengaged from them, according as the knots of one of the pairs are mixed or unmixed with the knots of the other pair.

Then construct the limb AG to be to its associate the limb AF, or the limb AD to be to its associate the limb AC, as the shoot GD is to its corresponding shoot FC.

From this it follows that the limb AD is to the limb AC as the limb AG is to the limb AF, consequently the rectangles of the limbs of each of these pairs, the mean[27] limbs, AG, AC and the extreme ones AF, AD, are equal to one another.

Further, any one of these limbs, AG, is to its partner AC as the

[a] $AG.AC = AF.AD$.

[b] The centre of involution is determined by two pairs of points.

rectangle of the shoots GF, GD is to its related rectangle [formed by the shoots] CD, CF, therefore A is the stump of the tree of which GC and DF are two pairs of knots.

Or, again, construct the limb AF to be to its associate the limb AC, or the limb AG to be to its associate the limb AD, as the shoot GF is to its corresponding shoot CD. It follows that the limb AG is to the limb AF as the limb AD is to the limb AC. Consequently, the rectangles of the limbs of each of the two pairs AG, AC and AF, AD are equal to one another. Consequently, any one of these limbs, AG [say], is to its partner the limb AC as the rectangle of the shoots GD, GF is to its related rectangle CD, CF. Therefore A is the stump of the tree of which GC and DF are pairs of knots.

And we may note in passing that since similar bulks or solids contained by surfaces, sides or edges which are opposite, flat and parallel have to one another the same ratio as that compounded from the ratio of their bases and the ratio of their heights, it follows from what we have shown that the ratio of the bulk or solid composed of any one of these limbs, GA, times each of the paired shoots GD, GF [which are] attached to it,[28] to the similar bulk or solid composed of its partner the limb AC, times each of the paired shoots CD, CF attached to it,[29] and related to the pair GD, GF [this ratio] is the square of the ratio of the limb AG to its partner the limb AC, and what further can be deduced from this.[30]

It is again clear from the above that if we have a tree whose stump is engaged between the two limbs or knots of any one of the pairs then the two mean knots formed by a pair of mean limbs are separate from one another, and thus each of them is isolated, and for this reason we call it a *Simple mean knot*. Simple mean knot

And when in a tree of this kind there are two pairs of mean limbs which each form a pair of simple mean knots, each of the simple mean knots of either of these two pairs coincides with one of the knots of the other pair, this knot also being mean and simple; and for this reason there will be two different identifiers beside each of these simple mean knots.

But there are good reasons why this case of two pairs of mean limbs with a third pair of extreme limbs, all belonging to a tree of the kind whose stump is engaged between the limbs of a pair, should not also be included as one of the possibilities which constitute an involution of three pairs of knots with one another. All the same, there is much in this Rough draft to be revised, amended, explained, ordered, transposed, cut out, added to and smoothed out, and these two pairs of mean limbs together only give the same mean knots as are given by one of the pairs.[31]

Double mean knot

But if we have a tree whose stump is disengaged from the two limbs or two knots of any of the pairs, then the two mean knots formed by a pair of mean limbs both coincide at one point or knot, which for this reason we call a *Double mean knot*, and it can if necessary be designated by a single identifier to be understood as double or taken twice.

And in this kind of tree when there are two pairs of mean limbs
[6] one on one side of the stump and the other on the other side, each of the mean limbs forms one of these knots on the trunk of the tree.[32]

In the type of tree where the stump is disengaged from the limbs of a pair, the case of two pairs of mean limbs with a third pair of extreme limbs is included among the possibilities which constitute an involution of three pairs of knots with one another, where each of the two double mean knots is considered as a pair of knots which coincide at one point.

Extreme knots

Now, in either type of tree, the two knots formed by the two extreme limbs of the same pair are called *Extreme knots*.

Of the two extreme knots of a pair BH, one knot, B, is near the stump between two simple or double mean knots, and H, the other of the extreme knots of the same pair, lies outside the same simple or double mean knots [Figs 6.6(a), (b), (c)].

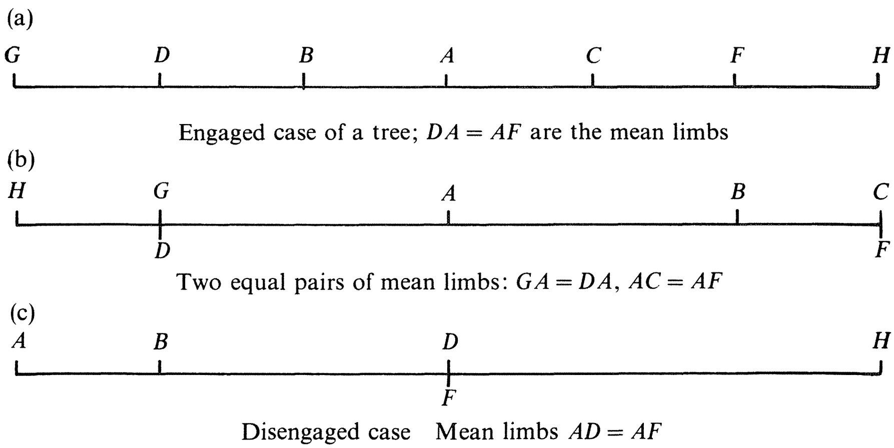
Inner extreme knot

In a pair of extreme knots BH, B, the knot which lies between the simple or double mean knots of the tree, is called an *inner extreme knot*.

Outer extreme knot

In a pair of extreme knots BH, H, the knot which lies outside the simple or double mean knots of the tree, is called an *outer extreme knot*.

In either of the two types of tree, in proportion as the smaller of a pair of extreme limbs is shorter than one of the mean limbs so



(a) Engaged case of a tree; $DA = AF$ are the mean limbs

(b) Two equal pairs of mean limbs: $GA = DA$, $AC = AF$

(c) Disengaged case Mean limbs $AD = AF$

Fig. 6.6

is the larger of this pair of extreme limbs proportionately larger than the same mean limb: And contrariwise.

Either, as B, the inner knot of a pair of extreme knots, BH, is nearer to the stump A, so the outer knot H of the same pair of extreme knots BH is further from the same stump A. And contrariwise.

So when B, the inner knot of a pair of knots, is distinct or separate from the stump of the tree, the outer knot of the same pair is at a finite distance on the trunk: And contrariwise.

And when the inner knot of a pair of extreme knots is coincident[a] or identified with the stump of the tree, the outer knot of the same pair is at an infinite distance on the trunk: And contrariwise.

Thus in a tree the stump and the trunk, from the stump to infinity, in both directions, form a pair of extreme limbs, the smaller limb being shrunk to the stump and the larger one extended to infinity.

Thus, also, the stump and the infinite distance are, in the tree, a pair of extreme knots, the stump being the inner knot and the infinite distance the outer one, and with any other pair of limbs they form an involution.

Now, trees of this form frequently occur in the figures which arise from taking particular [i.e. special] plane sections of a cone.[33]

And in a tree of the type in which the stump, A, is engaged between the two limbs of a pair, AC, AG, and there are two pairs of mean limbs AG, AC and AF, AD and any one of the simple mean knots, [say] G of the pair CG, is identified with one of the simple mean knots, [say] D, of the other pair DF, in that case there are a number of special properties.

Since the rectangles of each of the three pairs of limbs are equal (that is the rectangles of the two mean pairs AF, AD and AC, AG and the extreme pair AB, AH) that means that the three pairs of knots, two with simple mean knots, DF and CG, and one with extreme ones BH, are arranged so as to be in involution. First of all it is clear that each of the mean limbs is equal to each of the other three, and is a mean proportional[b] between the two limbs of any pair of extreme limbs AB, AH.

Moreover, the ratio between the rectangle of the shoots GD, GF and the related rectangle CD, CF is the same as the ratio of the rectangle of the shoots GB, GH (twin of the rectangle GD, GF) to the related rectangle CB, CH (twin of the rectangle CD, CF).

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[a] In an involution the infinite point is paired to the stump.

[b] $AG^2 = AB.AH$.

And, by interchanging,[34] the ratio between the rectangle GD, GF and its twin the rectangle GB, GH, is the same as the ratio between the rectangle CD, CF (related to the rectangle GD, GF) and its twin the rectangle CB, CH.

Now, it is clear that in this case the rectangle of the shoots GD, GF is equal to the rectangle of the shoots CD, CF, so we also have that the rectangle of the shoots GB, GH is equal to the rectangle of the shoots CB, CH.

This is again clear on other grounds, for by hypothesis and from what is proved here, it follows that the ratio between the shoot CH and its corresponding shoot BG is equal to the ratio between the shoot GH and its corresponding shoot BC, so that the rectangle of the two intermediate[35] shoots GB, GH is equal to the rectangle of the two extreme shoots CB, CH.

So again, the ratio[a] between the limb AG and its partner AC is equal to the ratio between the rectangle of the shoots GB, GH and its related rectangle CB, CH. So, since the limb AG is equal
[7] to its partner AC, the rectangle GB, GH is equal to its related rectangle CB, CH. This result is incomprehensible[36] in the case when B, the inner knot of the pair of extreme knots BH, is identified with the stump A, and H, the outer knot of the same pair, is at infinite distance.

So that what happens in this case is that three pairs of knots DF, CG, BH are reduced to giving only two pairs of points, on whose trunk there is a pair of extreme knots BH, and each of the points of the other pair represents two mean simple knots of two separate different pairs.

And these two pairs of points give us three consecutive parts of the trunk FC, B; B, DG [*sic*, should be DC]; DG, H whose sum FC [*sic*, should be FG], H is to the intermediate part GD [*sic*, should be CD], B as the end part of the side of the outer extreme knot H (that is, the part GD, H) is to the other end part on the side of the inner extreme knot B (that is, the part FC, B).

Thus in a tree of the form in which the stump is engaged, when there are two pairs of mean limbs, and the inner extreme knot is separate from the stump, or the outer extreme knot is at finite distance—that is, when three such pairs of knots thus only give two pairs of points on the trunk, which thus give three consecutive parts—in this case the intermediate part is not equal to either of the end parts, that on the side of the outer extreme knot or that on the side of the inner extreme knot.

And when the inner extreme knot is identified with the stump,

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[a] $\dfrac{AG}{AC} = \dfrac{GB.GH}{CB.CH}$ and $AG = AC \Rightarrow GB.GH = CB.CH$.

or the outer extreme knot is at infinite distance, in this case the intermediate part of these three consecutive parts is equal to the end part on the side of the inner extreme knot.

The case of a tree of this type has a number of other properties, with which our readers may divert themselves if they so wish, but it is not included among those which form an involution: And so, [*sic*]

Regarding the other type of tree, in which the stump *A* is disengaged from the limbs of a pair [Fig. 6.7].

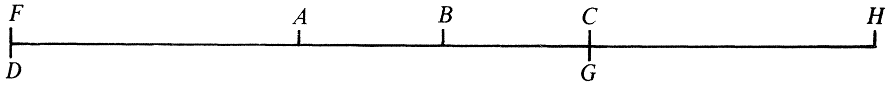

Fig. 6.7

When there are two pairs of mean limbs *AC*, *AG* and *AF*, *AD*, one on one side of the stump *A* and the other on the other,[37] and one pair of extreme limbs *AB*, *AH*, that is to say, the involution contains two double mean knots, *GC* and *DF*, and a pair of extreme knots *BH*, and these three pairs of knots give only two pairs of points on the trunk, then, apart from what this type of tree has in common with the other type in which the stump is engaged, which we need not repeat, there are other particular properties which are at once obvious, such as that

The ratio between the large extreme limb *AH* and any one of the mean ones, [say] *AG*, and the ratio between any of the mean limbs, [say] *AG*, to the small extreme limb *AB*, are the same as the ratio of the shoot *HG* to the shoot *BG*, that is to say, equal to half[a] the ratio of the rectangle of the shoots *HG*, *HC* to its related rectangle *BG*, *BC*.

And since we have shown that in a tree the ratio of the rectangle of the shoots *HF*, *HD* to its related rectangle *BF*, *BD* is equal to the ratio of the rectangle *HG*, *HC* (twin of the rectangle *HF*, *HD*) to its related rectangle *BG*, *BC* (twin of the rectangle *BF*, *BD*), and that the shoots *BF*, *BD* are equal to one another, and the shoots *HF*, *HD* are equal to one another, and that the shoots *HG*, *HC* are likewise equal to one another, and that the shoots *BG*, *BC* are equal to one another, it follows that the ratio of the shoot *HG* to the shoot *BG* is equal to the ratio of the shoot *HF* to the shoot *BF*.

It follows that the ratio of the large extreme limb *AH* to either of the mean limbs *AG* or *AF*, and the ratio of either of the mean limbs *AG* or *AF* to the small extreme limb *AB* are also equal to

[a] 'Half' means square root here.

half the ratio of the rectangle HF, HD to the rectangle BF, BD, that is [they are equal] to the ratio between the shoot FH and the shoot FB, and to the ratio between the shoot GH and the shoot GB, and inversely, by interchanging and alternation, dividing, compounding and the rest.[38]

That is to say that for this type of tree with a disengaged stump, and in the case of these two pairs of what are thus mean limbs,[39] with any pair of extreme limbs, these three pairs of limbs give four points on the trunk FD, B, CG, H, from which arise three consecutive parts FD, B; B, CG; CG, H such that the ratio of one at either of the ends [say] H, CG to the intermediate one CG, B is equal to the ratio of the sum of all three together H, FD [*sic*, should be FG] to the one at the other end, FD, B, for by alternation in the inverse we also have that the ratio of FD, B to B, CG is equal to the ratio of H, FD to H, CG, and so on interchanging, dividing and compounding and what follows from it.

And this same case clearly includes the situation in which we have these two pairs of what are thus mean limbs[40] with the stump and the trunk, from the stump out to infinity on one side, as a pair of extreme limbs, giving two mean knots, each of them double, each making a pair of mean knots. And the stump itself with the infinite distance on the trunk making another, third, pair of extreme knots, the whole making three pairs of knots in involution on the trunk of the tree; in which situation it is easy to distinguish the two double mean knots from the two knots of the
[8] extreme pair, because usually one of the extreme knots lies between the two double mean knots, or one of the double mean knots lies between the two extreme knots; and this case of an involution is usually described by first referring to the two knots of the extreme pair in this way: these two points are paired with one another in involution with two other points,[a] where the words *are paired with one another* imply that the two points so paired (which are separate and distinct from one another) are a pair of extreme knots, from which it follows that, since in this case the involution contains only four points, each of the other two is a double mean knot, and consequently one of the extreme knots lies between these two double mean knots, or one of the double mean knots lies between the two extreme knots.

Knots that correspond to one another *in the case of only four points in involution*

Furthermore, in this same case[41] we shall call the two double mean knots *Knots that correspond to one another*, and we shall also call the two extreme knots *Knots that correspond to one another*.

From this it is clear that if any three knots of such an

[a] Four points in involution.

involution are named and their positions given, then the position of the fourth is also given,[a] as will be seen still more clearly in what follows.

So, to describe this case of involution it will suffice to say that some four points are in involution with one another, or that some two points are paired in involution with some other two points, indicating which points correspond to which to form the pairs.

In this, the most important thing to note is that the case of four points in involution includes, as it were, two species of the same genus: the situation where four points on a straight line, each at finite distance, give three consecutive parts, such that the ratio of either of the end parts to the intermediate part is equal to the ratio of the sum of the three parts to the part at the other end. And the situation where three points on a straight line, each at finite distance, give two consecutive parts equal to one another, that is, when a point on a straight line bisects the interval between two other points,[42] in which circumstance or situation the point of either side of which the equal parts of the line lie is a stump; and, further, an extreme knot, paired with the infinite distance of the line, in involution with the two points which are the other ends of the two equal parts of the line, which in this case are each a double mean knot in the involution.

So by the words *four points in involution* we shall describe, as it were, two species of the same genus, one or other of these two situations; that is, the one in which four points on a straight line each at finite distance give three consecutive parts, such that the ratio of either of the end parts to the intermediate part is equal to the ratio of the sum of the three to the part at the other end: and the other [situation] in which three points at finite distance on a straight line with a fourth at infinite distance similarly give three parts [of the line], such that the ratio of either of the end parts to the intermediate part is equal to the ratio of the sum of the three to the part at the other end; which is incomprehensible and seems at first to imply (since in this case the three points at finite distance give two equal parts, between which the middle point is situated) the stump and the extreme knot [are] paired with the infinite distance. [Four points in involution][43]

So we shall take good note that a straight line divided into two equal parts by a point and understood to be extended to infinity is one of the forms of an involution of four points.

Now, in this same case of an involution of four points H, G, B, F, each at finite distance, since FG, the two knots that correspond to one another, are each a double mean knot, so BH, the other two that also correspond to one another, are a pair of

[a] Uniqueness of the fourth harmonic point.

extreme knots on the trunk of a tree, whose stump[a] A is the midpoint of the shoot GF.

Similarly, the two points HB are each a double mean knot, and the two points GF are a pair of extreme knots of a tree whose stump[b] L is the midpoint of the shoot BH [Fig. 6.8].

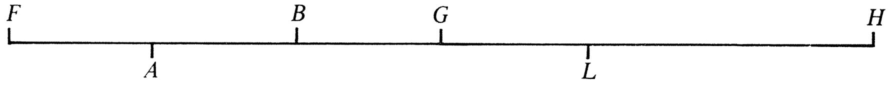

Fig. 6.8. Four points in involution $\frac{FB}{BG} = \frac{FH}{GH}$.

For since the ratio of BF to BG is equal to the ratio of HF to HG, that is to say the ratio of the rectangle FB, FB to the rectangle GB, GB is equal to the ratio of the rectangle FH, FH to the rectangle GH, GH.

Furthermore, if we consider the points GF as a pair of extreme knots, and each of the points H and B as a double mean knot;

Then the three pairs of knots FG, BB, HH, which are in involution with one another, are clearly unmixed with one another.

So, having disengaged L, the stump of the tree, from between the knots of each of these three pairs, it clearly lies between the points H and G.

In addition, since we have made the ratio of the rectangle FB, FB to the rectangle GB, GB, or the ratio of the rectangle FH, FH to the rectangle GH, GH, equal to the ratio of the limb LF to the limb LG, it will follow from the above that the rectangles of each of the three pairs of limbs LF, LG; LB, LB and LH, LH will be equal to one another, and so the stump L is the midpoint of the shoot GF, in a tree having LB, LB and LH, LH as pairs of mean limbs and LG, LF as a pair of extreme limbs, and all that follows from this.

So that when in a straight line four points, each at finite
[9] distance, form an involution, each of the points which bisects the shoot between any pair of the four points that correspond to one another is the stump of a tree, which has the four points as pairs of knots, L and A are two such stumps in such an involution of four points and we shall call them *Stumps reciprocal to one another* [Figs. 6.7 and 6.8].

Stumps reciprocal to one another *for four points in involution*

[a] $GA = AF$.
[b] $BL = LH$.

In future we shall leave out one of the identifiers when there are two of them for one of the four points.

From what we have said it also follows that the ratio[a] of BF to BA is equal to the ratio of BH to BG, and by inversion, alternation, interchanging, dividing, compounding and the rest.

So the rectangles of the extreme limbs BF, BG and of the mean limbs BA, BH are equal,[b] and the extreme knot B *makes a stump* for the two pairs of knots GF and AH, or for the limbs BF, BG and BH, BA, so the ratio of BF to BG is equal to the ratio[c] of the rectangle FA, FH to the rectangle GA, GH, that is, they are equal to the compound ratio of AF to AG and of HF to HG, that is, since the shoots FA and GA are equal, the ratio of HF to HG, that is half the ratio of the rectangle HF, HF to the rectangle HG, HG, and by inversion, alternation, interchanging, dividing, compounding and the rest. Makes a stump

From which it follows further that the ratio[d] of HG to HB is equal to the ratio of HA to HF, so the rectangles of the extreme limbs HG, HF and of the intermediate[44] ones HB, HA are equal, and the simple extreme knot H *makes a stump* for the two pairs of knots GF, BA or for the limbs HG, HF and HB, HA.

So the ratio of HF to HG is equal to the ratio of the rectangle FA, FB to the rectangle GA, GB, that is, it is the same as the compound ratio of FA to GA and FB to GB, that is, since the two limbs AF, AG are equal, it is equal to the ratio of BF to BG, which means that it is half the ratio of the rectangle BF, BF to the rectangle BG, BG, and by inversion, alternation, interchanging, compounding, dividing and the rest.

From which it follows that the ratio of BF to BG is equal to the ratio of the rectangle FA, FB to the rectangle GA, GB, and that the ratio of HF to HG is equal to the ratio of the rectangle FA, FH to the rectangle GA, GH, and what follows from this.

And that the ratio of the rectangle FA, FH to the rectangle GA, GH is equal to the ratio of the rectangle FA, FB to the rectangle GA, GB, and what follows from this.

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[a] $\dfrac{BF}{BA} = \dfrac{BH}{BG}$.

[b] $BF.BG = BA.BH$.

[c] $\dfrac{BF}{BG} = \dfrac{FA.FH}{GA.GH} = \left(\dfrac{FH^2}{GH^2}\right)^{1/2}$.

[d] $\dfrac{HG}{HB} = \dfrac{HA}{HF}$,

therefore $HG.HF = HB.HA$, making H a stump as defined above.

Further, the ratio[a] of BH to BA is equal to the ratio of the rectangle HF, HG to the rectangle of the limbs AF, AG, which are equal to one another, or to the equal rectangle AG, AG.

And the ratio[b] of HA to HB is equal to the ratio of the rectangle of the limbs AF, AG, which are equal to one another, or the equal rectangle AG, AG, to the rectangle BF, BG.[c]

From which it moreover follows that the ratio of FH to FA, or to AG (which is equal to FA) is equal to the ratio of BF to BA, and consequently equal to the ratio of BH to BG, and, by alternation, the ratio of FH to FB is equal to the ratio of AF, or AG (which is equal to AF), to AB, and what follows from this.[45]

So we have equal rectangles for the pair of extreme limbs FH, BA and the pair of intermediate[46] ones FA, FB; and for the extreme limbs HF, BG and the mean ones FA, FH.

Also, the ratio[d] of GA to BF is equal to the ratio of HA to HF, and therefore to the ratio of HG to HB, so we have equal rectangles for the pair of extreme limbs GA, HF and the pair of intermediate[47] ones BF, HA, and for the extreme limbs GA, HB and the mean ones BF, FG.

Further, the ratio[e] of BF to BH is equal to that of twice FA, which is FG, to twice GH, and by alternation in the inverse, interchanging, compounding and the rest.

Further, the ratio of FB to half of FG, which is FA,[48] is equal to the ratio of HB to HG. And therefore again to the ratio of HF to HA.

So we have equal rectangles for the pair of extreme limbs FB, HB and the pair of intermediate ones FA, HB, and for the extreme limbs FB, HA and the intermediate[49] ones, FA, HF,[50] and by inversion, alternation, interchanging, compounding and the rest.

Further, the ratio of the rectangle HB, HB to the rectangle HB, HA is equal to the ratio of the rectangle BA, BH (or the equal rectangle BG, BF) to the rectangle AB, AH (or the equal rectangle AG, AG or AF, AF), that is it is equal to the ratio of HB to HA, and what can be deduced from this.[51]

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[a] $\dfrac{BH}{BA} = \dfrac{HF.HG}{AF.AG}$, B stump.

[b] H stump.

[c] $\dfrac{HA}{HB} = \dfrac{AF.AG}{BF.BG}$, $AF = AG$, $\dfrac{HA}{HB} = \dfrac{AG.AG}{BF.BG}$.

[d] Lemma 2, p. 49.

[e] $\dfrac{BF}{BH} = \left(\dfrac{FA}{GH}\right)^2$.

From which it follows that the ratio compounded from the two ratios of BG to BA and BF to AH, which is the ratio of the rectangle BG, BF (or the equal rectangle BH, BA) to the rectangle AH, AB (or the equal rectangle AG, AG, or AF, AF), is the same ratio as that of HB to HA.

But the ratio of HB to HA is also the same as the ratio of the rectangle BG, BF to the rectangle AG, AF, that is, the ratio compounded from the ratios of BG to GA and of FB to FA.

So the ratio compounded from the ratios of BG to BA and of BF to AH is the same as that compounded from the ratios of GB to GA and of FB to FA, that is, the same as the ratio of HB to HA.

Anyone who wishes to pursue this discussion will find it productive of further interesting results.

[10] Furthermore, since the ratio of HB to HG is equal to the ratio of FB to FA, and the ratio is double, being compounded from the ratios of FG to FB and of FB to FA, that is the ratio[a] of FG *to* FA.

We also obtain a double ratio by compounding the ratios of FG to FB and of HB to HG (which is the same as that of FB to FA) or, which is the same thing, we obtain a double ratio by compounding the ratios of FG to HG and of HB to FB.

Similarly, since the ratio of BH to BG is equal to the ratio of FH to FA, and we obtain the double ratio by compounding the ratios of FG to FH and FG to FA, that is, the ratio of FH to FA.

We also obtain the double ratio by compounding the ratios of FG to FH and of BH to BG (the same as that of FH to FA). Or, which is the same thing, we obtain the double ratio by compounding the ratios of FG to BG and BH to FH. So the ratio compounded from the ratios of HB to HF and GF to GB is also double.[52]

And conversely from most of the properties we have stated here we conclude that four points are in involution.

For example, when in a straight line FH, three parts, say AB, AC, AH are in continued proportion, and a fourth part, say AF, is equal to the mean one AC, the four points H, C, B, F are clearly in involution.

When in a straight line FH, four parts, say BH, BG, BF, BA are proportional two by two, and one part, say AF, is equal to another part, say, AG, (that is to say that the point A bisects the part FG) then the four points H, G, B, F are clearly in involution.

When in a straight line FH, four parts, say HG, HB, HA, HF are proportional two by two, and one point, say A, bisects

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[a] $\dfrac{HB}{HG} = \dfrac{FB}{FA} = \dfrac{FG}{FB} \cdot \dfrac{FB}{FA} = \dfrac{FG}{FA}$.

another part, say FG, then the four points H, G, B, F are clearly in involution.

And it is clear that similar converses hold in the remaining cases, and could if necessary be deduced in full.

Similarly, it is clear from several passages in the above that, if the positions of any three of the four points of any involution are given, then the position of the fourth point of the involution,[a] corresponding to any one of the three given points, is also given.

When, in a straight trunk, three pairs of extreme knots, DF, CG and BH, are in involution with one another, and two other pairs of knots, mean, identified [with one another], double or simple, PQ, XY, as it were[53] make a four point involution with each of any two pairs of the three pairs of extreme knots, [say] CG and BH. These same two mean knots PQ, XY also as it were[54] make a four point involution with the third pair of extreme knots, DF*[55] [Fig. 6.9].

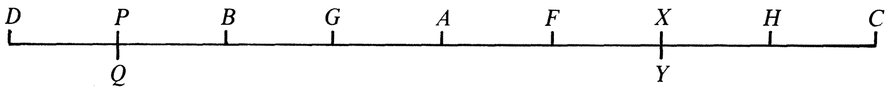

Fig. 6.9

For since the two mean knots PQ, XY as it were[56] make a four point involution with each of the pairs of extreme knots CG, BH, and the shoot PX has its mid-point at A, this point A is the stump for the pairs[57] of mean knots PQ, XY, and the extreme ones CG, BH. Therefore, the ratio of the limb AG to the limb AC is equal to the ratio of the rectangle GB, GH to the rectangle CB, CH, and, by hypothesis, the ratio of the rectangle GD, GF to the rectangle CD, CF is equal to the ratio of the rectangle GB, GH to the rectangle CB, CH; consequently, the ratio of the limb AG to the limb AC is equal to the ratio of the rectangle GD, GF to the rectangle CD, CF, and A is the stump for each of the pairs of mean knots PQ, XY and the extreme ones BH, CG and DF, which therefore are all in involution with one another, so the two pairs of mean knots PQ, XY are in involution with the third pair of extreme knots DF.

But for the purpose of this Draft it is enough to note some particular properties of this case, which is very rich in properties,[58] and if this proceeding seems unsatisfactory in

* And note, on this one occasion we do not mind including two pairs of simple mean knots in the term *involution*.

[a] Uniqueness of the fourth harmonic point.

Geometry, it is easier to leave the case out than to work through it in full and give it in its complete form.

The proposition which follows, set out at length, with its proof, is the same as that at the top of page 3, and which is mentioned as having been stated differently by Ptolemy.[59]

When*[60] in a straight line H, D, G which is a trunk through three points H, D, G, which are knots, there pass three straight lines which are branches springing from the trunk, the lines HKh, $D4h$, $G4K$, then the ratio of any shoot Dh of any of these branches**[61] $D4h$, contained between its knot, D, and either of the two other branches, [say] HKh, to its partner the shoot $D4$, contained between the same knot D and the third of the branches $G4K$, is the same as the ratio compounded of the ratios between the two shoots of each of the other two branches, in suitable order, that is, the ratio compounded of the ratio of, say, the shoot Hh, to the shoot HK and the ratio of the shoot GK to the shoot $G4$ [Fig. 6.10].

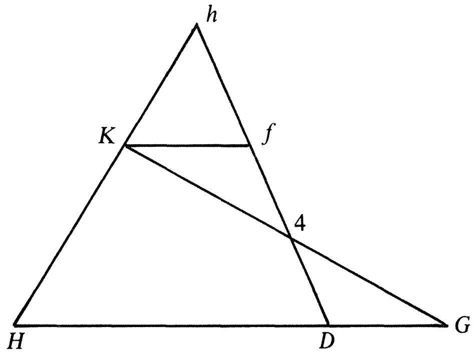

Fig. 6.10. Menelaus' Theorem $\dfrac{Dh}{D4} = \dfrac{Hh}{HK} \cdot \dfrac{GK}{G4}$.

For, if through the point K, the butt of the ordinance between the two other shoots Hh, $G4$, we draw a straight line Kf,
[11] parallel to the trunk HDG, to cut the branch $D4h$ in the point F, then take the shoot Df as intermediate[62] between the two shoots Dh and $D4$, and note that Kf and HG are parallel, then the ratio of the shoot Dh to the shoot $D4$ is the same as the

* Note that this proposition refers to fig. I [i.e. Fig. 6.10].

** In any one figure there are sometimes identifying letters which are the same letter but are of different kinds, and which must be referred to from the letterpress to the engraving; but in general all the letters K should be capitals.

ratio compounded of the ratios of the shoot Dh to the shoot Df (or of the shoot Hh to the shoot HK) and that of the shoot Df to the shoot $D4$ (or of the shoot GK to the shoot $G4$).

There are several things to note in this statement, when two of the three branches are parallel to one another, when two of the knots on the trunk coincide, and what follows from this, in which the understanding is at a loss.[63]

The converse of this proposition, which may be correctly stated as concluding that three points lie on a straight line, is also true.

When***[64] on a straight trunk GH there are three different pairs of knots BH, DF, CG, which are in involution, and through them there pass three pairs of branches springing from the trunk FK, DK; BK, HK; CK, GK, all belonging to the same ordinance, whose butt is K, these three pairs of branches all belonging to one ordinance are, taken together, called a *bough of a tree*, and on any other straight line cb, suitably[65] drawn in their plane, each of them gives one of the three pairs of knots of an involution gh, df, cg [Fig. 6.11]. Bough

When K, the butt of the ordinance of the three pairs of branches, or of the bough FK, DK, BK, HK, CK, GK, is at infinite distance, the proposition is obvious from the fact that the six branches are parallel to one another.

And when K, the butt of the ordinance of these three pairs of branches, that is, of the bough, is at finite distance. Firstly, the knots of each of the three pairs bh, df, cg, are clearly either mixed or unmixed with the knots of each of the other pairs according to whether on the trunk GH the knots of one pair are mixed or unmixed with the knots of the other pairs.

Further, the new, general, line cb, like the trunk GH, also belongs to an ordinance different from that of each of the branches FK, DK, forming any of the three pairs of branches of this bough.

Through[a] a point such as D, the butt of the ordinance containing the trunk GH and any one of the branches, [say] DK, belonging to any pair, FK, DK.

And through a point such as f, the butt of the ordinance containing the new general line cb and the other branch FK, belonging to the same general pair FK, DK, let there be drawn the straight line Df, which gives the points 2, 3, 4, 5 on BK, CK, GK, HK, the other four branches of the same bough.

Now on this new general line cb, the ratio of the rectangle of

*** Note that this proposition refers to figure II.
[a] See p. 50 for commentary.

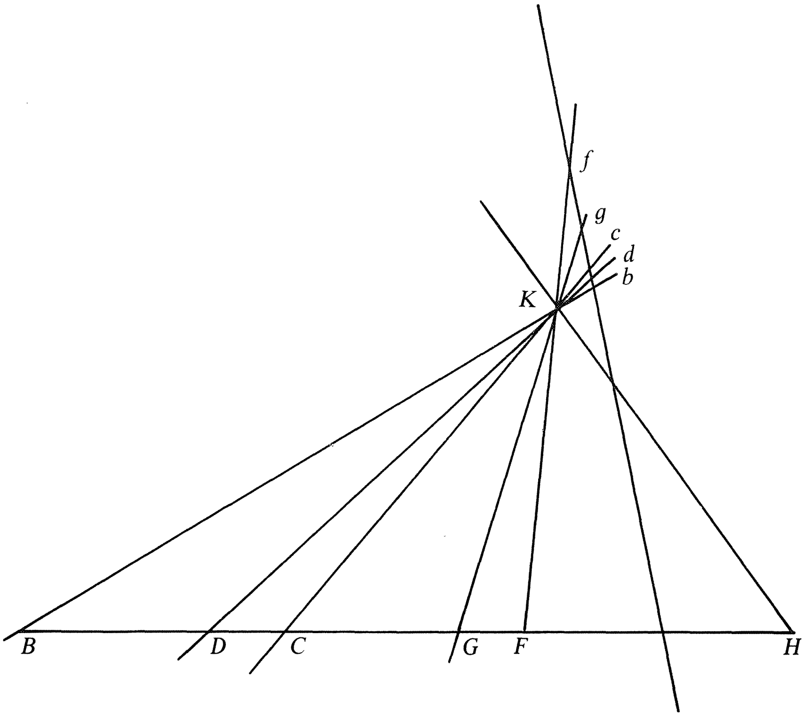

Fig. 6.11

any two shoots *dg*, *dc* to its related rectangle, that of the shoots *fg*, *fc*, is equal to the ratio compounded of the ratios of the shoot *gd* to the shoot *gf*, and of the shoot *cd* to the shoot *cf*.

And the ratio of the rectangle of the shoots *db*, *dh* (twin of the rectangle *dg*, *dc*) to its related rectangle *fb*, *fh* (twin of the rectangle *fg*, *fc*) is equal to the ratio compounded of the ratios of the shoot *bd* to the shoot *bf*, and of the shoot *hd* to the shoot *hf*.

Now the ratio of the shoot *gd* to the shoot *gf* is the same as the ratio compounded of the ratios of *Kd* to *KD*, and of 4*D* to 4*f*.

And the ratio of the shoot *cd* to the shoot *cf* is the same as the ratio compounded of the ratios of *Kd* to *KD*, and of 3*D* to 3*f*.

That is to say that the ratio of the rectangle *dg*, *dc* to the rectangle *fg*, *fc*, which is the ratio compounded of the ratios of *gd* to *gf* and of *cd* to *cf*, is the same as the ratio compounded of twice the ratio of *Kd* to *KD*, and of the two ratios of 4*D* to 4*f* and of 3*D* to 3*f*.

Now the ratio of 4*D* to 4*f* is the same as the ratio compounded of the ratios of *GD* to *GF* and of *KF* to *Kf*.

And the ratio of $3D$ to $3f$ is the same as the ratio compounded of the ratios of CD to CF and of KF to Kf.

That is to say, that the ratio compounded of the two ratios of 4D to $4f$, and of $3D$ to $3f$, is the same as the ratio compounded of twice the ratio of KF to Kf, and the two ratios of GD to GF and of CD to CF, that is to say, of the ratio of the rectangle DC, DG to the rectangle FC, FG.

That is to say that the ratio of the rectangle dg, dc to its related rectangle fg, fc, that is the ratio compounded of the ratios of gd to gf and of cd to cf, is the same as the ratio compounded of twice the ratio of Kd to KD, and of twice the ratio of KF to Kf, and of the two ratios of GD to GF and of CD to CF, that is to say, of the ratio of the rectangle DC, DG to its related rectangle FC, FG.

Similarly, the ratio of the shoot bd to the shoot bf is the same as the ratio compounded of the ratios of Kd to KD and of $2D$ to $2f$.

And the ratio of the shoot hd to the shoot hf is the same as the ratio compounded of the ratios of Kd to KD and of $5D$ to $5f$.

That is to say that the ratio of the rectangle db, dh to the
[12] rectangle fb, fh, that is the ratio compounded of the ratios of bd to bf and of hd to hf, is the same as the ratio compounded of twice the ratio of Kd to KD and of the two ratios of $2D$ to $2f$ and of $5D$ to $5f$.

Now the ratio of $2D$ to $2f$ is the same as the ratio compounded of the ratios of BD to BF and of KF to Kf.

And the ratio of $5D$ to $5f$ is the same as the ratio compounded of the ratios of HD to[66] HF and of KF to Kf.

That is to say that the ratio compounded of the two ratios of $2D$ of $2f$, and of $5D$ to $5f$, is the same as the ratio compounded of twice the ratio of KF to Kf, and the two ratios BD to BF and of HD to HF, that is, also of the ratio of the rectangle of the shoots DB, DH to its related rectangle FB, FH.

That is to say, the ratio of the rectangle db, dh to its related rectangle fb, fh, that is the ratio compounded of the ratios of bd to bf and of hd to hf, is the same as the ratio compounded of twice the ratio of Kd to KD and twice the ratio of KF to Kf, and the two ratios of BD to BF and of HD to HF, that is to say, and of the ratio of the rectangle DB, DH to its related rectangle FB, FH[a].

---

[a] $$\frac{db.dh}{fb.fh}=\left(\frac{Kd}{KD}\right)^2\cdot\left(\frac{KF}{Kf}\right)^2\cdot\frac{BD}{BF}\cdot\frac{HD}{HF}$$
$$=\frac{DB.DH}{FB.FH};$$ see p. 52.

Now, by hypothesis, the ratio of the rectangle *DB*, *DH* to its related rectangle *FB*, *FH* is equal to the ratio of the rectangle *DC*, *DG* to its related rectangle *FC*, *FG*, and by alternation and the rest.

That is to say, the ratio of the rectangle *dg*, *dc* to its related rectangle *fg*, *fc*, and also the ratio of the rectangle *db*, *dh* to its related rectangle *fb*, *fh*, are each equal to the ratio compounded of twice the ratio of *Kd* to *KD* and of twice the ratio of *KF* to *Kf*, and of the ratio of the rectangle *DB*, *DH* to its related rectangle *FB*, *FH*, or of the ratio (equal to the previous one by hypothesis) of the rectangle *DC*, *DG* to its related rectangle *FC*, *FG*.

Therefore, the ratio of the rectangle *dg*, *dc* to its related rectangle *fg*, *fc* is equal to the ratio of the rectangle *db*, *dh* (twin to the rectangle *dg*, *dc*) to its related rectangle *fb*, *fh* (twin to the rectangle *fg*, *fc*) and by alternation, interchanging, dividing, compounding and the rest.

So the three pairs of knots *df*, *cg*, *bh* are in involution.

And when the new general straight line *cb* is parallel to any one of the six branches of a bough, the partner of the parallel branch intersects the general line in the stump of the involution, [the stump] being the extreme knot paired with the infinite distance.

When we do not point out that there are various cases of a proposition, the proof applies to all the cases, otherwise we take note of the fact.

In this proposition,*[67] in the case of four points in involution, when the general line *cb* is parallel to one of the branches, [say] *DK*, the knot *f*, say, bisects the shoot *cg*, *bh* [Fig. 6.12].

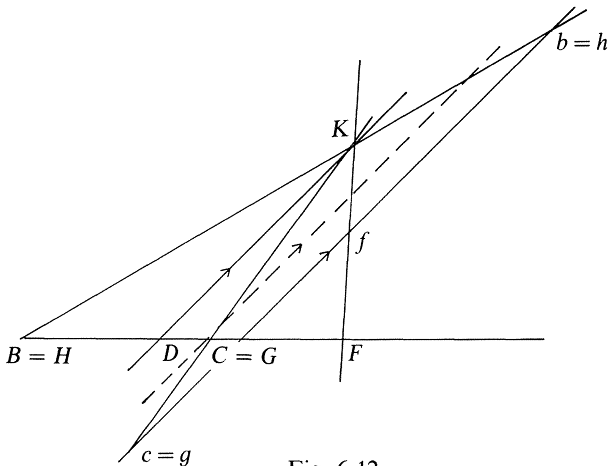

Fig. 6.12

* Note that this proposition refers to figure III.

For having taken this general line *cb*, or a line parallel to it, which comes to the same thing, to go through the point *CG*.

Since the lines *DK* and *cg*, *bh* are parallel to one another the ratio of *cg*, *bh* to *DK* is equal to the ratio of *BH*, *CG* to *BH*, *D*, and the ratio of *DK* to *cg*, *f* is equal to the ratio of *F*, *D* to *F*, *CG*. That is to say that the ratio of *cg*, *bh* to *cg*, *f* is equal to the ratio compounded of the ratios of *BH*, *CG* to *BH*, *D* and of *F*, *D* to *F*, *CG*, which has been proved to be the double ratio.

Therefore, *cg*, *bh* is double *cg*, *f*.

The converse is clearly also true, namely that when one of the branches, *FK*, bisects the shoot *cg*, *bh* say, the line *cb* is parallel to the branch *DK*, the partner of the branch *FK*; for since the four points in which the branches cut the line are in involution, and the point *f* bisects the shoot *cg*, *bh*, the fourth point, *d*, given by the branch *DK*, is at infinite distance.

There is another special proof of this converse theorem, namely, if there be drawn the straight line *C*, *N*, *L* parallel to *FK*, it is proved that the branch *KD* bisects this line *C*, *N*, *L* in *N*, and by hypothesis the branch *FK* bisects the line *cg*, *bh* in *f*, and because the lines *CL*, *FK* are parallel to one another the branch *FK* bisects *L*, *bh*, a side of the triangle *L*, *cg*, *bh*, at the point *K*; so the branch *DK* also bisects the same side of the same triangle at the point *K*, therefore the branch *DK* is parallel to *cg*, *bh*, the third side of the same triangle.

In the case where there are only four points *B*, *D*, *G*, *F* in involution on a straight line and through them there pass four springing branches *BK*, *DK*, *GK*, *FK*, which belong to an ordinance with butt *K*, the two branches such as *DK*, *FK*, or *BK*, *GK*, which pass through two points that correspond to one another *DF*, or *BG*, [these two branches] we shall call *Branches that correspond to one another*.

Branches that correspond to one another

And when in this case two corresponding branches *BK*, *GK* are perpendicular to one another, they each bisect one of the angles between the other two branches *DK*, *FK*, which also correspond to one another.

[13] For if we draw the line *Df* parallel to one of the branches *BK*, which is perpendicular to its corresponding branch *GK*, this straight line *Bf* [*sic*] is also perpendicular to this[68] branch *GK*.

In addition, since *BK* and *Df* are parallel the branch *GK* bisects *DF* [*sic*] at the point 3.

So the two triangles *K*3*D*, *K*3*f* each have a right angle at the point 3, and the sides 3*K*, 3*D* and 3*K*, 3*f*, which contain these equal angles *K*3*D* and *K*3*f*, are equal to one another.

Therefore the two triangles *K*3*D*, *K*3*f* are equal and similar.[a]

[a] Euclid's term for congruent.

So the branch GK bisects angle DKF, one of the angles between the corresponding branches DK, FK, and the branch BK clearly also bisects the other angle between the same corresponding branches DK, FK.

And when any one of these branches, GK [say], bisects the angle DKF between these two other branches that correspond to one another, DK, FK, this branch GK is perpendicular to its corresponding branch BK, which in turn bisects the other angle between the same corresponding branches DK, FK.

For if we draw a straight line such as Df perpendicular to any branch, GK say, the triangles $K3D$, $K3f$ each have a right angle at the point 3, and also an equal angle at the point K, and, in addition, a common side $K3$, therefore they are similar and equal, and a branch GK cuts Df at its midpoint, 3.

Consequently, the branch BK is parallel to the line Df, and therefore it is perpendicular to its corresponding branch GK.

When in a plane we have four straight lines BK, DK, GK, FK, belonging to an ordinance with butt K, and two of them, BK and GK say, are perpendicular to one another and each bisects one of the angles formed by the other two lines FK, DK. These four lines clearly intersect any line $BDGF$ drawn*[69] in their plane in four points B, D, G, F which are in involution.

When, in a plane, a straight line[a] FK bisects at f the side Gh of a triangle BGh, and through K, the point in which FK intersects another of the sides of the triangle, [say] Bh, there passes another straight line KD, parallel to the bisected side Gh, the four points such as B, D, G, F which are thus constructed on BG, the third side of the triangle, are in involution [Fig. 6.13].

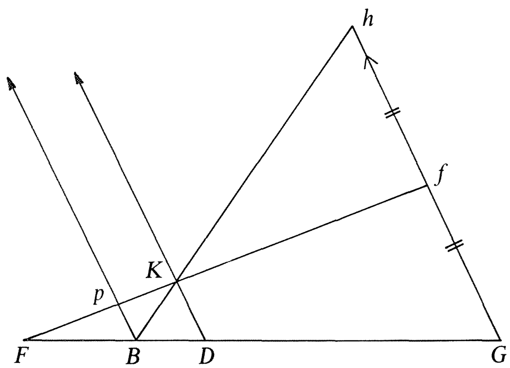

Fig. 6.13

* Sometimes one and the same point represents four points in involution.

[a] Construction of the fourth harmonic point.

And when through the angle B, subtended by the bisected side Gh, there passes another straight line Bp, parallel to the bisected side Gh, the four points F, f, K, p formed on the line FK by intersection with the three sides of the triangle BGh: BG, Gh, Bh, and the straight line Bp, are in involution.

For the first part,[70] the proposition is seen to be evident by drawing another straight line such as KG, since on the bisected line G, f, h the three points G, f, h and the infinite distance are four points in involution, through which there pass four branches of an ordinance with butt K, and therefore on the straight line BG they give four points in involution B, D, G, F.

And for the second part,[71] by drawing the line Bf, [so that] similarly, on the line Gh the three points at finite distance G, f, h and the infinite distance are in involution, and through them there pass four branches of an ordinance with butt B, which therefore on the straight line FK give four points in involution F, f, K, p.*[72]

When in a plane a line[a] FGB doubles hf one of the sides of a triangle hfK, and through the point B, in which it intersects either of the remaining sides of the triangle, hK, say, there passes a straight line Bp, parallel to the doubled side hf, this construction gives on Kf, the third side of the triangle, hfK, four points F, f, K, p [which are] in involution.

As is clear once we have drawn the line Bf.

And when through the angle K, subtended by the doubled side hf, there passes a straight line KD parallel to the doubled side hf, this construction gives on the doubling line F, B, four points F, G, D, B [which are] in involution, as is clear by drawing the straight line KG.

The material is rich in similar means of deducing that four points or three pairs of knots on a line are in involution, but the above is sufficient to start off an open-cast mine together with what follows.[73]

[Here begins Desargues' treatment of conics.]

When a straight line containing a fixed point moves round the edge, or circumference, of a circle:

The fixed point of the straight line is in the plane of the circle or it is not.

---

* When a straight line FGB cuts off a part Gf of the line hf such that it is equal to the part hf, say, which is a side of the triangle hfK, say, we shall say that the line FGH [*sic*: should be FGB] doubles the side hf of the triangle hfK.

[a] Reformulating the previous result.

When the fixed point of the line is in the plane of the circle it is either at finite or infinite distance.

And, whichever of the two kinds of position the fixed point is in, in the plane of the circle, the line as it moves remains in the plane of the circle, and the various places it occupies as it moves define an ordinance of lines which meet the circle, the butt of the ordinance being at finite or infinite distance.

[14] When the fixed point of the line lies outside the plane of the circle, it is at a finite or infinite distance, and whichever of the two kinds of position the fixed point is in, outside the plane of the circle, the line as it moves always remains outside the plane of the circle, and in its revolution it surrounds, encloses or describes a solid figure, which we shall call a *Roll*, the name being that of a genus which contains two subgenera. Roll

The fixed point of the line is called the *Vertex* of the roll. Vertex *of the roll*

The circle round whose edge the straight line moves is called the *Base* or *Flat basis* of the roll. Flat basis *or* base *of the roll*

The space through which the line passes as it moves is called the *Envelope*, or *Surface* of the roll. Envelope *or* Surface *of the roll*

When the fixed point of the line is at infinite distance, outside the plane of the circle round whose edge the line moves, the roll the line describes is of equal size everywhere along its length at any finite distance, and it is called a *Column*, or a *Cylinder*, which clearly may be of several species. Column *or* Cylinder

When the fixed point of the line is at finite distance, outside the plane of the circle round whose edge the line moves, the roll the line describes in its revolution narrows at its fixed point, where its size is only that of a single point, and on either side of this it widens out to infinity in the form of two cornets placed end to end at the fixed point, and on this account we shall call it a *Cornet*, or a *Cone*, which clearly may be of several species. Cornet or Cone

Thus the column or cylinder and the cornet or cone are two subgenera of a genus we have called the roll, which we shall mainly treat in general, and in our study the part of the cornet or cone which lies only on one side of its vertex, a part which elsewhere is taken to be a whole cone, here will be considered to pass for only half of a cornet or cone, and not a whole one.

And therefore the word *Cornet* or *Cone* will be used to signify the two parts taken together, at the same time, opposed at their vertex, otherwise the cone is not complete.

When a plane other than that of the circle, [which is the] basis or base of the roll, intersects the roll we shall call this plane a *Plane of section* of the roll. Plane of section of the roll

Such a plane of section cuts such a roll either in its vertex or not in its vertex, and, wherever it cuts it, it does so in one of two ways, either in such a way that the motion of the straight line which describes the roll never brings the line parallel to the plane of section, or [in such a way that it] sometimes does bring it parallel.

When such a plane of section cuts the roll in its vertex, in such a way that the motion of the straight line which describes the roll never brings it parallel to the plane of section.

If the vertex of the roll is at infinite distance, the outcome is impossible to imagine, and the understanding is incapable of comprehending how the results, which reasoning makes it deduce, are possible.

If[a] the vertex of the roll is at finite distance it is clear that the straight line gives only one point in the plane of section.

When a plane of section cuts a roll in its vertex, in such a way that the motion of the straight line which describes the roll sometimes brings it parallel to the plane of section.

If the vertex of the roll is at infinite distance, the motion of the straight line which describes the roll keeps it always parallel to the plane of section.

If the vertex of the roll is at finite distance, the motion of the straight line which describes the roll does not keep it always parallel to the plane of section.

And, in each of the two species of positions of the vertex of the roll, the motion of the line which describes it brings the line into the plane of section either once or two separate times.

When[b] the line is only in the plane of section once, it gives a straight line in the plane, which is thus joined to the roll along its length, or, to put it differently, [the plane] touches the roll in a straight line.

When[c] the line is in the plane of section two separate times, it gives two straight lines in the plane, which thus divides the roll along its length through the vertex.

When such a plane of section cuts a roll somewhere other than in its vertex, in such a way that the motion of the line which describes the roll never brings it parallel to the plane of section.

If they meet one another at infinite distance the outcome is impossible to imagine, and the understanding too weak to comprehend how the results, which reasoning makes it deduce, are possible.

If[d] they meet one another at finite distance, the motion of the straight line which describes the roll causes it to trace out in the

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[a] Point conic.
[b] Double line.
[c] Line pair.
[d] Ellipse.

plane of section a curved line which, at finite distance, doubles back and joins up with itself, the curved lines [so described] being of various species.

[15] When such a plane of section meets a roll somewhere other than in its vertex, in such a way that the line which describes the roll is sometimes parallel to the plane of section, the outcome in this case is quite impossible to imagine in regard to the kind of roll called a cylinder, and also for the kind called a cone when the plane meets it at infinite distance.

And when such a plane of section meets a cone somewhere other than in its vertex, in such a way that the line which describes the cone is sometimes parallel to the plane of section, the line is either parallel to the plane once or two separate times.

When[a] it is parallel only once it traces in the plane a curved line which, at infinite distance, doubles back and joins up with itself, the curve [so described] being of only one species.

When[b] the line is parallel to the plane two separate times it traces in the plane a curved line, which at infinite distance divides into two equal and similar halves, turned back to back to one another, and only one of these is taken here as being only half of the outcome of this relation between the plane of section and the roll it meets, and the curve [so described] is of various species.

Thus a plane of section and a roll, if we neglect its basis, meet either in a single point, or in a single straight line, or in two straight lines, in the same plane, or in a curved line.

If we except the types of meeting which result either in a single point or in a single straight line, and discuss only the other types, the part of the body of the roll occupied by the plane in these other types of meeting will be called a *Section of the roll*. Section *of the roll*

The straight or curved lines which the straight line that describes the roll traces out as it moves in the plane of section will be called the *Edge* of the section of the roll. Edge *of the section of the roll*

When the edge of a section of a roll is two straight lines, the butt of their ordinance is at finite or infinite distance [Fig. 6.14].

When the edge of a section of a roll is a curve which at finite distance doubles back and joins up with itself, the figure is called a *Circle*, or an *Oval*, otherwise an *Ellipse*, in the vernacular deficity.[74] Circle, Ellipse *or* Oval

When the edge of a section of a roll is a curve which at infinite distance doubles back and joins up with itself, the figure is called a *Parabola*, in the vernacular equalation.[75] Parabola

[a] Parabola.
[b] Hyperbola.

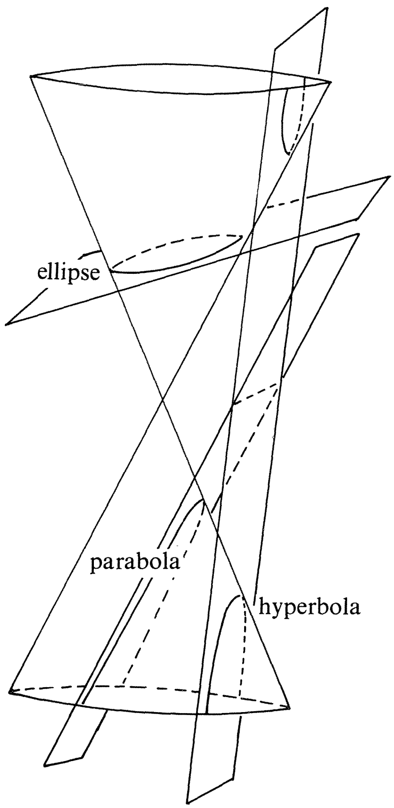

Fig. 6.14

When the edge of a section of a roll is a curve which at infinite distance divides into two halves, turned back to back, the figure is called a *Hyperbola*, in the vernacular surpassing or excedency.[76] Hyperbola

When in a plane a straight line meets any figure, the meeting is considered only in relation to the edge of the figure, and in a plane a straight line meets the edge of a figure in two points, which sometimes are identified with one another, in which case the straight line touches the figure.*[77]

When in a plane a figure NB, NC is met by several straight lines FCB, FIK, FXY belonging to the same ordinance, and if some one line $NGHO$ cuts the lines of the ordinance in points G, H, O, which with F, the butt of the ordinance, respectively form involutions with the [sets of] two points such as XY, IK, CB in

* The most remarkable properties of the sections of a roll are common to all types, and the names *Ellipse*, *Parabola* and *Hyperbola* have been given them only on account of matters extraneous to them and to their nature. Note that this definition refers to figure IIII.

which the line is intersected by the figure NB, NC, then we shall call such a straight line NGH a *Transversal*[a] of the lines of the ordinance with butt F, with respect to the figure NB, NC, and the lines of the ordinance with butt F are accordingly called *Ordinates* of the transversal N, G, H, with respect to the same figure NB, NC [Fig. 6.15].

Transversal to *the straight lines of an ordinance*

Ordinates *of a transversal*

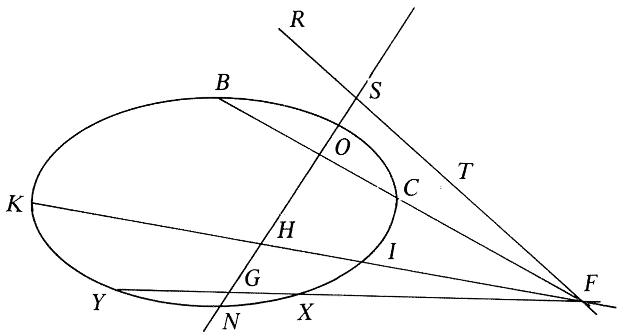

Fig. 6.15. The transversal $NGHO$ of F. Desargues has not yet shown that it is a straight line. Lines like $KHIF$ are called ordinates by Desargues, lines like $RSTF$ he called ordinals.

The point in which a transversal intersects its ordinate is called a *transversal point*.[78] [Transversal point]

Any line belonging to the set of ordinates of a transversal, and which does not meet, or only touches, the figure is called an *Ordinal* with respect to the transversal, as distinct from its ordinates, which cross the figure. [Ordinal]

Every straight line which bisects a figure is called a *Diametral* of the figure, and a *Diametransversal* with respect to its ordinates. [Diametral] [Diametransversal]

And for each of these ordinates we consider both the two parts or segments such as OC, OB contained between the transversal and each of the points where this straight line meets the edge of the figure. And the two parts or segments such as FC, FB contained between F, the butt of their ordinance, and each of the points where the line meets the edge of the figure.

So if in a plane the straight lines FB, FK, FY, belonging to an ordinance with butt F, meet a figure NB, NC, there are four kinds of greater and smaller [parts] to be considered for the straight lines belonging to the ordinance with butt F.

The greatest and the smallest of the parts which are contained between their common butt F and the points where they meet the

[a] Transversal = polar.

edge of the figure at a meeting point of the type [exemplified by] *B*.

The greatest and the smallest of the parts contained between their common butt *F* and the points where they meet the edge of the figure at a meeting point of the type [exemplified by] *C*.

And the one for which the part or segment such as *CB*, which is contained inside the figure, is the greatest or the least.

Or the one for which the sum or the difference of the two parts such as *FC*, *FB* and *OB*, *OC*, contained between their common butt *F* and their transversal *ON*, and each of the points where it meets the edge of the figure, is the greatest or the least.

[16] Consequently the words *Transversal* and *Ordinates* will be taken to indicate that the straight lines under consideration have these names in relation to a section of a roll which is in the same plane with the straight lines.*[79]

Such a result involving a transversal and ordinates often occurs when taking plane sections of a general roll.

And[a] the edge of the figure together with the butt of the ordinates and their transversal always give four points in involution on each of the ordinates, of these points, the two given by the edge of the figure correspond to one another, and the point given by the butt of the ordinance and the point given by the transversal also correspond to one another.

Either in each of the ordinates the butt of their ordinance is paired with the point given by their transversal in involution with the two points given by the edge of the figure, and conversely.

Or in each of the ordinates the two points given by the edge of the figure are paired in involution with these two other points, [namely] the butt of their ordinance and the point given by their transversal.

Now, as in an involution of four points the two which form the extreme pair are sometimes at a distance from one another in such a way that one is identified with the stump and the other is at infinite distance.

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* When, in a plane, none of the points of a straight line is at finite distance, then this straight line is at infinite distance.

Since in a plane the point called the centre of a section of a roll is only one case among the innumerable butts of ordinances of straight lines, we must never speak of the centre of a section of a roll.

Since every straight line which passes through the vertex of a roll and through any [point which is the] butt of an ordinance of straight lines in the plane of its base [i.e. the base of the roll] shares the property of the line which passes through the butt of the diametrals of the base of the roll, we must never speak of an axle of a roll.

Straight lines which are parallel to one another are identified by the same letter at each end, the letter representing the butt of their ordinance, at infinite distance.

[a] Harmonic properties of pole and polar.

On the other hand, the same two knots or points which form the extreme pair are sometimes close together to the extent of being identifed with one another and even with one of the other mean pair of knots which correspond to one another, in which case the four points of the involution are reduced to only two points, at one of which we imagine three points united in one.

There is much to be said on the subject of the four points in involution formed by an ordinance of straight lines with their transversal and the edge of the figure, but in this Draft it will suffice to say a little about the more general kinds of outcomes which make it clear what will happen in a particular case.

In the plane of any section of a roll, the butt of an ordinance of straight lines, otherwise [the butt] of a set of ordinates, is either on the edge of the figure or not on the edge of the figure, and in either case it is either at finite distance or at infinite distance.

In the plane of any section of a roll the transversal of an ordinance of straight lines, otherwise [that] of a set of ordinates, either meets or does not meet the edge of the figure, and in either case it is either at finite distance or at infinite distance.

When[a] the butt of a set of ordinates lies on the edge of the figure, at finite or infinite distance, the transversal of the ordinance [*sic*, should be 'ordinates'] itself belongs to the set of ordinates, and passes through the butt of the ordinance, at which point it touches the figure.

When the butt of the ordinates does not lie on the edge of the figure, at finite or infinite distance, and all the ordinates meet the edge of the figure, the transversal does not meet it, and if all the ordinates do not meet the edge of the figure, the transversal does meet it [Fig. 6.16(a), (b)].

Furthermore the two parts of each of the ordinates contained between their butt and each of the two points given on the ordinate by the edge of the figure, are either equal or unequal to one another.

When they are equal to one another, the two parts of each of these same ordinates contained between their transversal and each of the two points on the ordinate given by the edge of the figure, are equal to one another, and contrariwise.

And on the other hand, when the transversal of a set of ordinates, at finite or infinite distance, does not meet the edge of the figure, all the ordinates do meet it.

When the transversal of a set of ordinates, at finite or infinite distance, meets the edge of the figure, it meets it either in one or in two points.

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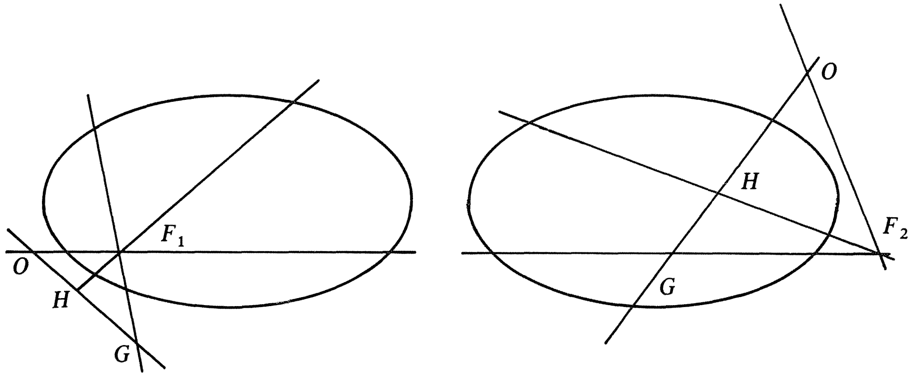
[a] The polar is the tangent in this case.



(a) Desargues' first case (b) Desargues' second case

Fig. 6.16. On the basis of this observation, one could say that any conic section divides the (projective) plane into an inside (points like F_1) and an outside (points like F_2). Desargues did not explicitly do so.

When it meets it in one point, this same point is the butt of its *Ordinates*.

When it meets in two points, all its ordinates do not meet it.

Furthermore, the two parts of each of the ordinates contained between their transversal and each of their [i.e. the ordinates'] two meets with the edge of the figure, are either equal or unequal to one another.

When they are equal to one another, the two parts of each of the same ordinates contained between the butt of their ordinance[80] and each of the two points on the ordinate given by the edge of the figure, are also equal to one another, and contrariwise.

When*[81,a] in a plane we have four points B, C, D, E, as marker posts paired three times among themselves, through which there pass three pairs of marker lines BCN, EDN, BEF, DCF, BDR, ECR, each of these three pairs of marker lines and the curved edge of any section of a roll which passes through the four points B, C, D, E gives on any other straight line in their plane, such as a trunk I, G, K, one of the pairs of knots of an involution IK, PQ, GH and LM, and if the two marker lines of one of the pairs BCN, EDN
[17] are parallel to one another, the ratios between the rectangles of their related pairs of shoots springing from the trunk are the same as those between their twin rectangles, [that is] the rectangles of the shoots folded to the trunk and in the same order [Fig. 6.17].

* This proposition concerns figure V.
[a] Theorem 3.

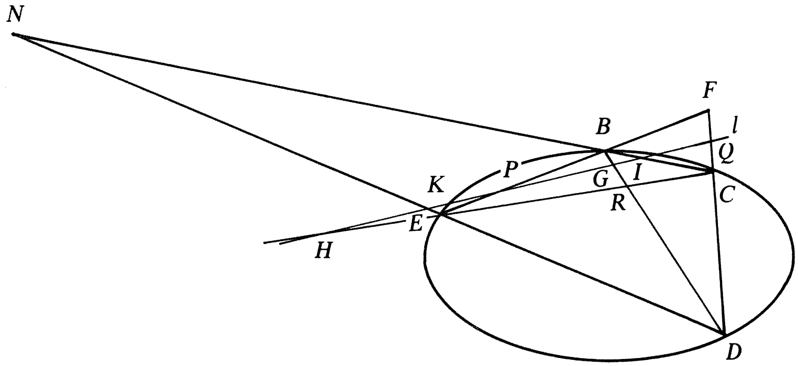


Fig 6.17. **Theorem 3.** *BCDEFRN is a complete quadrilateral and l is a transversal meeting the sides of the quadrilateral as shown.*

Desargues' Theorem 3 asserts that IK, PQ, HG are three pairs of points in involution. Moreover, if l meets the conic at L, M, he claims that L, M and any two of these pairs (say IK, HG) are also three pairs of points in involution. An outline of his proofs, which are actually quite elegant, is given on pp. 54–55.

For the ratio of the rectangle of any pair of shoots folded to the trunk QI, QK to its related rectangle PI, PK is the same as the ratio compounded of the ratios of IQ to IP and of KQ to KP.

Now the ratio of IQ to IP is the same as the ratio compounded of the ratios of CQ to CF and of BF to BP.

And the ratio of KQ to KP is the same as the ratio compounded of the ratios of DQ to DF and of EF to EP.

So the ratio of the rectangle QI, QK to its related rectangle PI, PK is the same as the ratio compounded of the four ratios of CQ to CF, of BF to BP, of DQ to DF and of EF to EP.

Similarly, the ratio of the rectangle QG, QH to the rectangle PG, PH is the same as the ratio compounded of the ratios of GQ to GP and of HQ to HP.

Now the ratio of GQ to GP is the same as the ratio compounded of the ratios of DQ to DF and of BF to BP.

And the ratio of HQ to HP is the same as the ratio compounded of the ratios of CQ to CF and of EF to EP.

So the ratio of the rectangle QG, QH to the rectangle PG, PH is the same as the ratio compounded of the four ratios of DQ to DF, of BF to BP, of CQ to CF and of EF to EP, which are the same four ratios which compound to form the ratio of the rectangle QI, QK to the rectangle PI, PK.

Consequently the ratio of the rectangle of the shoots QI, QK to its related rectangle PI, PK is the same as the ratio of the rectangle QG, QH (the twin of the rectangle QI, QK) to its related rectangle PI, PH (the twin of the rectangle PI, PK).

And consequently the three pairs of knots IK, PQ, GH are in involution with one another.

In this we see that the same property will hold for three pairs of branches springing from the trunk of a tree, both if they all belong to the same ordinance, and also when they are, as here, arranged with respect to the four points B, C, D, E in such a way that the butt of the ordinance of three pairs of branches is as it would be if these four points B, C, D, E were united in one point.[a]

And if the two marker lines of a pair BCN, EDN are parallel to one another, the ratio of the rectangle of the springing shoots IC, IB to its related rectangle KD, KE is equal to the ratio of the rectangle of any pair of shoots folded to the trunk, [say] IQ, IP (the twin of the rectangle IB, IC), to its related rectangle of the shoots folded to the trunk KQ, KP (the twin of the rectangle KD, KE), which is obvious from the fact that the branches or marker lines BC, DE are parallel to one another.

Which shows that when in a plane there are any five straight lines BE, DC, PK, BC and DE, any two of which, [say] BC, DE, are parallel to one another, and any one of the others, [say] KP, being considered as a trunk, and each of the others as branches springing from this trunk, letting the two parallel lines BC, DE be a pair, and the two others BE, DE another pair, the ratio of the rectangles of the pairs of shoots IC, IB and KD, KE which are the two branches of a pair, is clearly the same as the ratio of their twin rectangles taken in the same order, namely the rectangles of IQ, IP and of KQ, KP, formed by the shoots of the other pair of branches.

That is to say, also, that the ratio of the rectangle CI, CB to the rectangle DK, DE is clearly the same as the ratio of the rectangle CQ, CF to the rectangle DQ, DF.

And also that the ratio of the rectangle BI, BC to the rectangle EK, ED is equal to the ratio of the rectangle BF, BP to the rectangle EF, EP.

When the curved edge of any section of a roll passes through the four points B, C, D, E, anyone who would like to seek a proof in the same words for every kind of section can do so; however, here is the proof in two instances, firstly for when it is the edge of a circle which passes through the points, and next for any one of the other kinds of section of a roll.

[a] Six points in involution are also to be obtained on a transversal in the degenerate case when B, C, D and E coincide.

So, when the four marker posts B, C, D, E are on the edge of a circle which a seventh general line GPH meets in the points LM.

In this case taking the rectangle such as FC, FD as intermediate [i.e. a mean][82] between the rectangles QC, QD and PB, PE, that is to say between the rectangles equal to them, QL, QM and PL, PM, it is clear that the ratio of the rectangle QL, QM (or the equal rectangle QC, QD), to the rectangle PL, PM (or the equal rectangle PB, PE) is the same as the ratio compounded of the ratios of the rectangle QC, QD to the rectangle FC, FD (or the equal rectangle FB, FE), the ratio of the rectangle FC, FD (or the equal rectangle FB, FE), to the rectangle PB, PE.

[18] Now the ratio of the rectangle QC, QD (equal to the rectangle QL, QM) to the rectangle FC, FD (equal the rectangle FB, FE), is the same as[83] the ratio compounded of the ratios of CQ to CF and of DQ to DF.

And the ratio of the rectangle FB, FE (equal to the rectangle FC, FD) to the rectangle PB, PE (equal to the rectangle PL, PM) is the same as[84] the ratio compounded of the ratios of BF to BP and of EF to EP.

So the ratio of the rectangle QL, QM (equal to the rectangle QC, QD) to the rectangle PL, PM (equal to the rectangle PB, PE) is the same as the ratio compounded of the four ratios of CQ to CF, of DQ to DF, of BF to BP and of EF to EP, which are the four ratios which when compounded form both the ratio of the rectangle QI, QK to the rectangle PI, PK and the ratio of the rectangle QG, QH to the rectangle PG, PH.

And when the four marker posts B, C, D, E are on the curved edge of any other kind of section of a roll, we shall not draw so many figures for a mere Rough Draft, but if the reader cares for the amusement of drawing them for himself he will find that, if the roll of which the figure is a section is constructed from it [i.e. from the figure], and then on the basis or base of the roll we construct the general circle $BCDE$.

The four straight lines drawn through the vertex of the roll and the four marker posts on the edge of the general section, lie in the surface of the roll, and also give on the edge of the circle which is its base four marker posts B, C, D, E.

And the planes through the vertex of the roll and each of the marker lines of the three pairs drawn through the four marker posts of the general section, also give, in the plane of the circular base of the roll, three pairs of marker lines BC, ED, BE, CD, BD, CE, passing through the marker posts B, C, D, E.

And the plane through the vertex of the roll and the seventh general line drawn in the plane of the section, gives, in the plane of the circular base of the roll, a corresponding seventh general line K, G, H, which meets the edge of the circle B, C, D, E, which is the base or basis of the roll, in two points LM, and this same line, K, G, H also meets each of the marker lines of the three pairs in the plane of this circle, in points such as PQ, GH, IK.

And the straight lines drawn from the vertex of the roll through the points of LM, on its circular base, intersect the seventh, general, line in the plane of the section, in the same points as those given on this line by the edge of this general section.

And the straight lines drawn from the vertex of the roll through the points of each pair of knots QP, GH, IK on the seventh line GH, in the plane of the circular base of the roll, pass through the points given on the seventh line in the plane of the section by the three pairs of marker lines drawn in this plane.

Now it has been shown that the pairs of knots LM, QP, GH, IK in the plane of the circle are in involution with one another.

And the bough of the tree with three or four pairs of branches, all belonging to the same ordinance, whose butt is the vertex of the roll, gives on the seventh line in the plane of the section the same number of pairs of knots, also in involution. And consequently:

It will be understood that this proof is applicable on numerous occasions, and shows that each important straight line and point arises in the same way for every kind of section of a roll, and it is rare that a general line in the plane of a general section of a roll should have a significant property in relation to that section without a line corresponding to it in the plane of another section of the roll having its position and properties also given by a similar construction of the bough of an ordinance whose butt is at the vertex of the roll.*[85]

But, before going on to general propositions concerning general sections of a roll, it may not be amiss to give one more proposition peculiar to the plane of the circle [Fig. 6.18].

When**[86] in the diametral, $A7$, of a circle $LMEC$, any two points A, I are a pair in involution with the two points E, C, given on the diametral by the edge of the circle, and two straight lines LIS, LAM ordinates to a general point L on the edge of the circle, pass through the two points A and I.

And through each of the points E, C and the centre of the circle, 7, there passes a pair of straight lines $CP, CN, ER, EO, 7B, 7D$, conjugate[a] lines, and, because of the properties of the circle,

* Straight lines which are ordinates to the same butt, that is, which pass through or tend towards this same butt.

** This proposition refers to figure VI.

[a] Conjugate diameters of a circle are perpendicular.

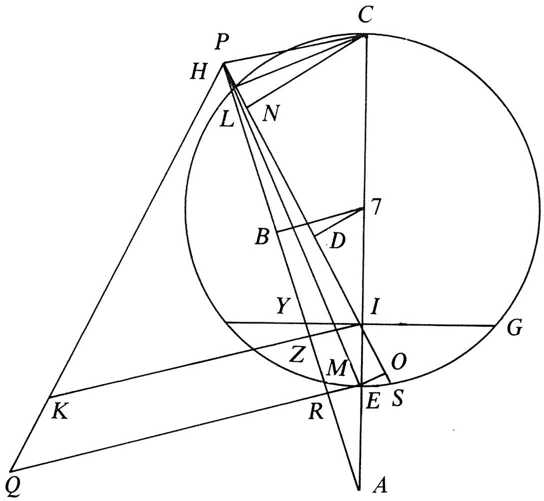


Fig. 6.18

in this case perpendicular to the two straight lines LA, LI, on which they give the points RO, PN and BD.[87]

The part. of either of the straight lines LA, LI contained between the two points given on it by its conjugal[88] lines drawn form the points E and C, is equal to the part of the other one of the lines LA, LI which is contained between the points given on the line by the edge of the circle.

Two lines each of which is parallel to one of two diameters of a section of a roll which are conjugate to one another are said to be *Conjugal* to one another.[89] [Conjugal lines]

That is to say[a] that the part NO of the straight line LI is equal to the part LM of the straight line LA, and the part PR of the straight line LA is equal to the part LS of the straight line LI.

And furthermore the rectangle of each of the pairs of
[19] conjugate lines drawn from the points E and C to each of the straight lines LA, LI is equal to the rectangle of the two parts of whichever of the lines LA, LI they [*sic*] are conjugate to, [the parts, that is,] contained between one of the points given on the line by the edge of the circle and each of the points given on the line by the two conjugate lines.

That is to say that the rectangle PC, RE is equal to the rectangle LP, LR, and the rectangle NC, OE is equal to the rectangle LN, LO.[b]

For[c] since $7B$ bisects EC in 7, in the same way, since CP, $7B$,

[a] Claims $NO = LM$, $PR = LS$, (1).
[b] $PC.RE = LP.LR$, and $NC.OE = LN.LO$, (2).
[c] Proof of (1) begins.

ER are parallel to one another, it bisects RP in B, but from the properties of the circle it also bisects LM in B, consequently the two parts LP, MR are equal to one another.

Similarly and by the same argument, since $7D$ bisects EC in 7, in the same way it bisects NO in D, but it also bisects LS in D, consequently the two parts NL, OS are equal to one another.

Furthermore having drawn the straight line LE, which gives the point H on PC, and the straight line LC, the two lines EL and CL are perpendicular to one another, since ECL is a semicircle.

And, since the four points C, I, E, A are in involution, the perpendicular lines CL, EL each bisect one of the angles which the straight lines LA, LI make with one another.

So the right-angled triangles CLP, CLN are similar, and since they have a common side, CL, they are equal to one another.

And similarly the right-angled triangles LER, LEO are similar, and since they have a common side, EL, they are equal to one another.

So the straight lines LR, LO are equal to one another, and the straight lines LP, LN are equal to one another.

But the straight lines LN, SO are equal to one another, therefore the straight lines LP, SO, MR are equal to one another, consequently the straight lines NO, LM are equal to one another, and the straight lines PR, LS are equal to one another.[a]

And,[b] producing the straight line MI to meet the edge of the circle at X, it is clear that the two parts IL, IX are equal to one another, and the two parts IM, IS are equal to one another; for having drawn the two straight lines ME, MC, since we have a semicircle they are perpendicular to one another, consequently[c] each of them bisects one of the angles which the two straight lines MAL, MIX make with one another.[90]

So[d] the two parts IL, IM are equal to the two parts IX, IS, or, alternatively, LS, the difference or the sum of the two parts IL, IM, is equal to RP, that is to say that RP is equal to the sum or to the difference of IL, IM.

Now,[e] because EL is perpendicular to LC, triangle CLH is right-angled, and through its right angle, at the point L, there passes the straight line AP, perpendicular to its side CH, the base of the right angle CLH, so the right-angled triangle LPH is

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[a] End of (1).
[b] Interlude: $IL = IX$, $IM = IS$.
[c] $L\hat{M}C = C\hat{M}X$.
[d] So $IL = IM$, implying $LI \pm IS = RP$.
[e] Proof of (2) begins.

similar to each of the two other right-angled triangles *CLH* and *CPL*.

And, since the straight lines *CPH* and *ERQ* are parallel to one another, the same triangle *LPH* is also similar to each of the two other right-angled triangles *LRE* and *LOE*.

Therefore the right-angled triangles *CPL*, *LRE*, *CLN* and *LOE* are similar to one another, consequently the ratio of *PC* or *RL* is equal to the ratio of *PL* to *RE*, and the rectangle of the two extreme lines *PC*, *RE*, which are the two perpendicular conjugate lines drawn from the points *C* and *E* to the straight line *LA*, is equal to the rectangle of the mean lines *LR*, *LP*, which are the two parts of the same straight line *LA* contained between *L*, one of the points given on the line by the edge of the circle, and each of the points, *R* and *P*, given by the two perpendicular conjugate lines[a] *ER*, *CP*; And by drawing the straight lines *MC*, *ME*, by the same argument we prove that the triangles *ERM*, *MPC* are similar, and producing the straight lines *CM*, *ER* we prove similarly that the rectangles *PC*, *RE* and *MP*, *MR* are equal to one another.[b]

Furthermore, since the triangles *CLN*, *LOE* are similar to one another, the ratio of *NC* to *LO* is equal to the ratio of *LN* to *OE*, and the rectangle of the extreme lines, the perpendicular conjugate lines *NC*, *OE*, is equal to the rectangle of the mean lines *LO*, *LN*, which are the two parts of *LI* contained between *L*, one of the points given on the line by the edge of the circle, and each of the points *N* and *O* given by its two perpendicular conjugate lines *CN*, *EO*.

And moreover using the same construction again, when through *I*, either one of the points *A*, *I*, there passes a straight line *IZ*, a perpendicular conjugate to the straight line *AL*, one of the two straight lines *LA*, *LI*, which passes through the other point *A*, of the two points *A* and *I*, in which straight line there is the general point *P*, either one of the points *P* and *R*, through which there passes the straight line *PQ*, which gives the point *K* on the line *IZ*, and the point *Q* on the perpendicular conjugate line *ER*, which passes through the other point, *R*, in such a way that the rectangle of the parts such as *ZK*, *ZR* is equal to the rectangle of twice the part such as *ZI*, that is to say, it is equal to the rectangle *ZI*, *ZI*.

Then the ratio of *RP* to a part of *ER*, such as *RQ*, is the same as the ratio of a rectangle, such as[91] *BR*, *BP*, to each of the equal rectangles *PC*, *RE* and *ZI*, *B*7 or *MP*, *MR*.

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[a] $PC.RE = PL.RL$ (2).

[b] $PC.RE = MP.MR$.

For,[a] taking ZR as the common height of each of the rectangles ZK, ZR and ZP, ZR, the ratio of the rectangle ZP, ZR to the rectangle ZK, ZR, that is to say to the rectangle
[20] equal to it ZI, ZI, is equal to the ratio of the base ZP to the base ZK, that is to say, since ZK and RQ are parallel, it is equal to the ratio of RP to RQ.

Furthermore, since the straight lines CP, $7B$, IZ and ER are parallel to one another, and the points A, E, I, C are in involution, and 7 is the mid point of EC, a shoot of the involution, it follows[b] that $7I$, $7E$, $7A$ are proportional to one another, and IA, IE, IC, $I7$ are proportional two by two, and AC, $A7$, AI, AE are proportional two by two, and CP, $7B$, IZ, ER are proportional two by two in the same ratio as that of the four lines AC, $A7$, AI, AE among themselves, and the four lines AP, AB, AZ, AR among themselves.[c]

From this it follows that the ratio of the rectangle AZ, AZ to the rectangle AR, AP, or the equal rectangle AZ, AB, that is to say the ratio of the limb AZ to its partner the limb AB, that is to say the ratio of the rectangle of the shoots ZR, AP to its related rectangle BR, BP, is[d] equal to the ratio of the rectangle ZI, ZI to each of the equal rectangles $ZI, B7$; RE, PC and LP, LR or MP, MR.*[92]

And by interchanging the ratio of the rectangle[e] ZR, ZP to the rectangle ZI, ZI, that is to say, the ratio of RP to RQ, is equal to the ratio of the rectangle BR, BP to each of the equal rectangles PC, RE; $ZI, B7$ and LP, LR or MP, MR.

But the ratio of RP to RQ is also equal to the ratio of the rectangle RP, RP to the rectangle RP, RQ, so the ratio of the rectangle BP, BR to each of the equal rectangles ER, CP; $ZI, B7$ and LP, LR; MP, MR is equal to the ratio of the rectangle RP, RP to the rectangle RP, RQ, and by interchanging the ratio of the rectangle BP, BR to the rectangle RP, RP is equal to the

* This proposition refers to figure IV.

[a] $\dfrac{RP}{RQ} = \dfrac{ZP.ZR}{(ZI)^2}$.

[b] $\dfrac{7I}{7E} = \dfrac{7E}{7A}$.

[c] $AC{:}AT{:}AI{:}AE = CP{:}7B{:}IZ{:}ER = AP{:}AB{:}AZ{:}AR$.

[d] $PZRA$ in involution with stump B,

$$\frac{AZ}{AB} = \frac{ZR.ZP}{BR.BP} = \frac{(ZI)^2}{B7.ZI}.$$

[e] $\dfrac{ZR.ZP}{ZI^2} = \dfrac{BR.BP}{ZI.B7} = \dfrac{RP}{RQ}$.

ratio of each of the equal rectangles[a] *ER*, *CP*; *ZI*, *B*7; *LP*, *LR*; *MP*, *MR* to the rectangle *RP*, *RQ*.

Now, since *PR* is bisected at *B*, we have that the rectangle[b] *BR*, *BP* is a quarter of the rectangle *RP*, *RP*, so, also, each of the equal rectangles *ER*, *CP*; *ZI*, *B*7; *LP*, *LR*; *MP*, *MR* is quarter of the rectangle *RP*, RQ.

If we add to this the fact that since the straight line *EL* bisects the angle *MLS*, and the straight line *CM* bisects the angle *XML*, the parts of the edge of the circle *ES* and *EM* are equal to one another, and the parts *CX* and *CL* are equal to one another; from which it follows that the straight line *EIC* bisects one of the angles that the straight lines *IL*, *IM* make with one another, and that the straight line *IGY*, perpendicular to *EIC*, bisects[c] the other one of the angles the same straight lines *IL*, *IM* make with one another.

We shall soon see, as to their bulk, what kind of consequences and true converses follow from this in relation to the subject of this Draft, though it would expand it too much if we were to deduce them in detail.

When in a plane through the four points *B*, *C*, *D*, *E*, as marker posts in any plane section of a roll, there pass three pairs of marker lines *BCF*, *EDF*; *BEN*, *CDN*; *BDG*, *CEG*, and through *G* and *N*, the two butts of the two ordinances of any two of these[93] pairs of marker lines, there passes another straight line *GN*, in relation to the section of the roll whose edge passes through the four marker posts *B*, *C*, *D*, *E*, this straight line *GN* is a transversal of the straight lines of the ordinance of the third of these pairs of marker lines with butt *F*, that is to say that *F*, *X*, *G*, *Y* are in involution [Fig. 6.13].

For, as we have said, taking each of the two letters *X* and *Y* as representing double points:

The ratio of *GX* to *GY* is the same as the ratio compounded of the ratios of	*DX* to *DN* and of *BN* to *BY*.
And the ratio of *GX* to *GY* is the same as the ratio compounded of the ratios of	*CX* to *CN* and of *EN* to *EY*.
And the ratio of *FX* to *FY* is the same as the ratio compounded of the ratios of	*DX* to *DN* and of *EN* to *EY*.
And the ratio of *FX* to *FY* is the same as the ratio compounded of the ratios of	*CX* to *CN* and of *BN* to *BY*.

From which it follows that the ratio of *GX* to *GY* is equal to the

[a] $\dfrac{BR\,.\,BP}{RP^2} = \dfrac{ZI\,.\,B7}{RP\,.\,RQ}$.

[b] $BR\,.\,BP = \frac{1}{4}RP^2$.

[c] $L\hat{I}C = C\hat{I}X$.

ratio of FX to FY, that is to say the four points F, X, G, Y are in involution with one another.

And supposing we draw the straight line FN, the four straight lines NF, NX, NG, NY belong to the same ordinance, with butt N, and pass through the four points in involution $FXGY$, consequently on each of the general lines $FCOB, FIKH$ drawn in their plane they give four points in involution F, C, O, B; F, I, H, K.

And since the four straight lines GF, GC, GO, GB all belong to the same ordinance, with butt G, and they pass through the four points in involution F, C, O, B, which lie on the straight line FB, it follows that on a general straight line, FH, drawn in their plane they give four points in involution F, Q, H, P.

And when the four marker posts B, C, D, E lie on the curved edge of any section of a roll $CLDEMB$.

The same straight line GN is, with respect to this section of a roll, also a transversal of the straight lines ordinate to the butt F,[94] and the four points F, L, H, M given on any line of this ordinance by the butt of the ordinance, F, the transversal, GN, and the edge of the figure, LM, are in involution with one another: for by applying the same construction it has been proved that on the straight line FH the three pairs of points LM, IK, QP are three pairs of knots in involution.

[21] It has also been proved that the two points F and H are paired in involution both with the two points I, K and with two points Q, P.

Finally it has also been proved that in consequence the same two points F and H are also paired in involution with the two points L, M.

Or, if you will, since each of the two pairs of points IK and PQ is in involution with the two points F and H, it follows that each of them is a pair of extreme knots of a tree in which F and H are the two double mean knots.

And it is proved that the three pairs of knots IK, PQ and LM are in involution.

Consequently, the two points L, M are a pair of extreme[95] knots of the same tree in which F and H are the two double mean knots.

As a result, the same two points L, M are paired with one another in involution with the two points F and H.

The same thing can also be deduced and concluded in another way.

From which it follows that with respect to the section of a roll $CLDEM$, which contains the four marker posts B, C, D, E, the straight line GF is a transversal[a] of the straight lines ordinate to

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[a] The basis of the construction.

the butt N, and the straight line FN is a transversal of the straight lines ordinate to the butt G.

And when in the plane of a general[96] section of a roll, there is an ordinance of lines whose butt is at infinite distance, and the lines all cut the section, the parts of each of the ordinates contained between their transversal, and each of the points given on them by the edge of the figure, are equal to one another, and the same is true of the parts contained between the butt of the ordinance and each of the points given on them by the edge of the figure.

From which it follows that in the plane of a general section of a roll any straight line is, with respect to the section, a transversal of straight lines ordinate to some butt, a diametral,[a] otherwise a diametransversal, being only one case of this.[97]

And that any point is, with respect to this section, the butt of some straight lines, ordinates of a transversal, the butt of the diametrals being only one case of this.[98]

From which it also follows that if through a general point N, which lies on GN, the transversal of the straight lines belonging to an ordinance with butt F, there is drawn a general straight line NDC, which meets the edge of this section of a roll in points D and C, say, and then through F, the butt of the ordinance, and through D, one of these points, there is drawn another straight line, FDE, which gives the point E on the edge of this section of a roll.

Finally drawing the two straight lines such as NE, FC they are both ordinates to a butt B on the edge of the section of a roll.

For it has been proved that the points C and B, which the edge of the section of a roll gives on the straight line FCO, are paired in involution with the two points F and O, given on the same line by the two straight lines NG, NF.

It has also been proved that the points C and B, which the straight lines NYB, NXC give on the same straight line FO, are similarly paired in involution with the same two points F, O, given on the line by the two straight lines NG, NF, so the edge of the section of a roll and the straight line N, E [*sic*, should be NF] give the same point B on this straight line FO.

From which it follows,[b] in addition, that when in a plane two general straight lines FCB, FDE, meet the edge of a general section of a roll, in marker posts B, C, D, E, say, and through these points B, C, D, E, there pass two other pairs of marker lines BE, CD and BD, EC, [the points] N and G, the two butts of the two ordinances

[a] A diameter is a polar of a point at infinity.
[b] Construction of a polar.

of these two pairs of marker lines lie on *GN*, a transversal of the straight lines of the ordinance of these first two lines *FCB*, *FDE* whose butt is *F*.

From which it follows[a] that, in the plane of a general section of a roll *BCDE*, each of the straight lines *FO*, *FH*, *FG*, which belong to the same ordinance, is a transversal of the straight lines of an ordinance whose butt lies on their common transversal *GN*.

And, conversely, that *OF*, *HF*, *GF*, the transversals of the straight lines of the ordinances whose butt lies on a similar transversal or straight line *NG*, all belong to the same ordinance.

From which it follows that if in the plane of a section of a roll we are given the position of the butt, *F*, of a general ordinance of straight lines *FH*, *FG*, then the position of their common transversal *GN* is also given.

And if we are given the position of a general transversal or straight line *GN*, the position of the butt, *F*, of its ordinates is also given.

We see, moreover, that the straight lines such as *FS*, which we call tangents to a section of a roll, belong to the set of an ordinance of straight lines which do not all meet the figure, and each is only a [special] case of a [special] case.

From which it follows that the straight line of an ordinance
[22] drawn through the point that their transversal gives on the edge of a section of a roll touches that section.

And that if from [a point on] the edge of the section we draw an ordinate to any diametral of this section, and another straight line through the point on this diametral which is paired with the point given on it by the ordinate in involution with the two points given on it by the edge of the figure, then this last straight line touches the section; or, if through any point on the edge of a section of a roll we draw a general straight line across the figure as a transversal, and another straight line through the butt of the ordinates of this transversal, this second straight line touches the figure.[99]

Now on the straight line *NG*, a general transversal of a general ordinance of straight lines *FH*, *FO*, each of the pairs of points *NG*, *ZH*, *AR*, given on the line by the three butts of ordinates *A*, *Z*, *N* and their transversals *TGV*, *MHL* and *ERD*, is each one of three pairs of knots in involution in a tree whose stump is consequently given in position, that is to say the stump is at the one of its inner extreme knots which is paired with the infinite distance, or otherwise the point given on it by the transversal of

[a] Theorem 5.

the straight lines ordinate to infinite distance with the transversal NG.

If anyone wishes to follow Monsieur Pujoz's[100] example and amuse himself by giving one proof for all kinds of cases, in a general plane, he will anticipate the tidying-up of this Draft in which most things have been proved in the first instance by considering three dimensions.

However, in the present work we shall demonstrate this proposition in two instances, one in the plane and the other using three dimensions, that is in the plane of the circle, for which the proposition is obviously true, since the diametrals are perpendicular to their ordinates.

And for the other kinds of section, constructing the roll on this section, and then [considering] its circular basis, and next, using the bough of the tree ordinate to the vertex of the roll, from the property [just] proved, we see that the proposition is true.

Or, because the proposition is obviously true for any diametransversal, from which it can be deduced that it is true for any other line.[101]

From which it also follows that however many pairs of straight lines are ordinate to one of the points on the edge of the section of a roll, the lines passing through the two points on the same edge given by a general line of a general ordinance, these pairs of lines give on the transversal of that ordinance the same number of pairs of knots of an involution.

And conversely when a pair of branches springing from that trunk are ordinate to the same point on the edge of the figure, the two further points they give on the edge of the figure, and the butt of the ordinates of this transversal, lie on a straight line.[102]

It would be a lengthy matter to present here not all the properties, but even the properties that crowd in upon us, and which are properties common to all kinds of sections of a roll, and it will be sufficient to state only some of the most obvious ones, and ones which will provide the means of discovering less obvious properties.

However, we shall note that between the two kinds of structures for trees there is a third kind[a] in which one knot of each pair is always identified with the stump, a situation which like some others cannot be completely grasped by the understanding, and this kind of structure for the tree is intermediate between the other two, that in which the stump is engaged and that in which the stump is disengaged.

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[a] Parabolic involution.

When a transversal is at infinite distance, everything about it is unimaginable.

When it is at finite distance, either it meets the edge of the figure, or it does not meet it.

When it meets it, it does so either in two distinct points or in two points which coincide, at a single point at which the line touches the figure.

When it does not meet it [i.e. when the line does not meet the figure], the tree formed on it by the butts of the ordinates and their transversals is the kind with an engaged stump.

When it meets it in two distinct points, the tree is the kind with a disengaged stump.

When it meets it in two points which coincide, that is to say when it touches the figure, the tree is the intermediate kind, for which the understanding cannot grasp the properties which reasoning makes it deduce.

But here we have a proposition which as it were summarises all the preceding.

Given*[103,a] the magnitude and position of a general section of a roll with curved edge *E*, *D*, *C*, *B*, forming the base or basis of a general roll, whose vertex is also given in position, and let another plane be given, in a general position, to cut this roll, and let 4, 5, the axle of the ordinance of this cutting plane with the plane of the base or basis, also be given in position, then the figure resulting from this construction in the cutting plane is given in kind and in position, each of its diametrals with its distinguishing conjugates and axles, as also each kind of ordinates and tangents to the figure, and the nature of each, their ordinances, with the possible differences, all these are given as to type and in position.

For if through the vertex of the roll we draw a plane parallel to the plane of the section, this plane through the vertex gives in the plane of the base of the roll a straight line *NH*, parallel to the straight line 4, 5, which line *NH* is a transversal of an ordinance of straight lines, *ML*, *BC*, *TV*, whose butt, *F*, is given in position.

And the straight line drawn through the vertex of the roll and this butt *F* is the axle of the ordinance of planes which generate the diametrals of the figure which this construction produces in the plane of section, of which the straight line which passes through the vertex and the butt of the diametrals of the basis of the roll is only one case.[104]

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* This position refers to figure IV.

[a] It asserts that the properties of pole and polar are projective.

[23] Further,[a] if we draw two straight lines through each of the points of any pair HZ, given on the transversal NH[105] by Z, a general butt of an ordinance of straight lines, and their transversal HF, and through the vertex of the roll, the two planes which pass through the vertex of the roll and each of the straight lines FZ, FH, say, give in the figure which results from this construction in the plane of section one of the pairs of diametrals that we call conjugates, whose relation to one another is like the relation between the two straight lines through the vertex of the roll and each of the points Z, H, and the same straight lines through the vertex of the roll and the points Z, H are the axles of two ordinances of planes which each generate in the plane of section an ordinance of straight lines mutually ordinate or conjugate to one another which are the set of all the possible tangents to the figure, at finite or infinite distance, at the points which the edge of the figure gives on these conjugate diametrals, in which we see that the straight lines called *Asymptotes*, or lines which do not meet the edge of the figure at any finite distance, take their place as both diametrals of the figure and lines which touch its edge at infinite distance. All of these results are evident from the fact that the plane of section and the plane through the vertex are parallel to one another, and from the property of a bough of a tree ordinate to the vertex of the roll. [Asymptotes][106]

As an advantage of this procedure, its use can be extended to encompass and present as its initial results three other propositions.

One to include propositions 17 and 18 of Book 5 of Euclid's *Elements.*

Another to include proposition 19 and some others from the same book.

Another to include proposition 47 of the first book and propositions 12 and 13 of the second book of the *Elements.*

Nevertheless from what we have here we can already see quite clearly several properties which are common to all kinds of section of a roll.

As, among others, that on any one of these sections of a roll we may construct a roll which will have a section of any given kind.

And when, through the two points which the edge of a section of a roll gives any one of its diametrals, there pass two straight lines belonging to the set of ordinates, otherwise [called] *Ordinals*, of this diameter [*sic*], and another general straight line touches this section at some other point, the two parts of these two ordinates, otherwise [called] *Ordinals*, contained between Ordinals

[a] Asymptotes are tangents at infinity.

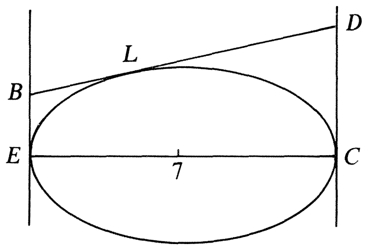

Fig. 6.19

the diametral and the other line, the tangent, contain a rectangle of constant magnitude, in that another constant magnitude always bears a constant ratio to each of them [Fig. 6.19].

And the rectangles of the two parts of any one of the ordinates to a diametral of a hyperbola contained between one of the points given on it by the edge of the figure and each of the points given on it by the asymptotes, or lines which do not touch the curve at any finite distance, are also all of the same magnitude, since another constant magnitude always bears a constant ratio to each of them, for which two results we shall later give an individual type of proof.

Stump *common to several trees*

When*[107] four points CG, BH are two pairs of knots of a tree HB, whose stump is A, and the part of the trunk contained between any two of these knots is the diameter of a circle, and the part contained between the other two remaining knots is the diameter of another circle, the edges of these two circles give on any other straight line, ordinate to the stump A, two similar pairs of knots, also belonging to a tree which has stump A, in common with the tree HB [Fig. 6.20].

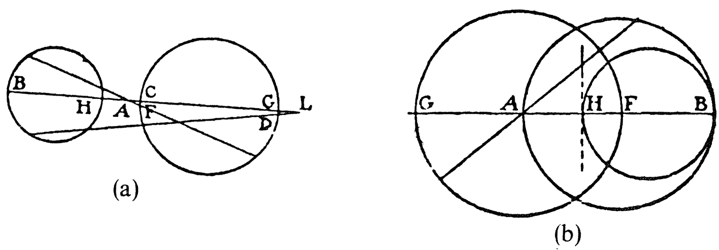

Fig. 6.20. From Taton (1951), p. 161.

The familiar proof of these results would add unnecessarily to the size of this Draft.

Moreover, if A does not remain the stump,[108] instead of circles we may have, on the same parts of lines between the same

* This section and those that follow refer to the simple lines in the plate.

twos of the four knots CG, BH, any two other sections of a roll arranged in such a way that their edges produce the same results as those that were clearly produced by the circles, by means of a bough of the tree HB.

And when in an involution of four points H, G, B, F, on a trunk BH, the two shoots such as GF and BH, contained between the two knots which correspond to one another, are each the diameter of a circle.

The edges of these two circles give on any other straight line they meet ordinate to either of the two reciprocal stumps, L and A, of this involution H, G, B, F, two points which also correspond to one another in a similar involution of four points, which has as stump whichever of the two reciprocal stumps to which, as butt, the straight line is ordinate, with the trunk BH.

And in some cases, when A does not remain the stump,[109] if on the shoots GF, BH, we construct not two circles but two general sections of a roll, in particular positions, their edges produce the same results as the edges of the circles.

And when on a trunk BH, four points H, G, B, F are in involution, and a shoot such as FG, the sum of two of the mean limbs of the tree AF, AG, is the diameter of a circle, and either
[24] one of AH or AB, the two extreme limbs of a pair in the same tree, is also the diameter of a circle, and through the extreme knot of the other remaining one of these two extreme limbs there passes a straight line belonging to the set of ordinates, otherwise an ordinal, to the common diametral of these two circles.

This ordinal gives on any other straight line which, with the diametral, is ordinate to the stump A, as butt, a point paired in involution with the point on it given by the edge of the circle on the extreme limb, and [in involution] with the two points on it given by the edge of the circle on the sum of the two mean limbs, for which the figure is easy to imagine for drawing, or the proof is obvious from what has been said.

And if, instead of two circles, there are two other sections of a roll, arranged in a certain way, the same result or a similar result[110] holds, as may be seen by [constructing] a bough.

There are several other similar properties common to all kinds of sections of a roll, which it would be tedious to include here.

The item which follows might have taken its place above, in the proposition relating to four marker posts on the edge of a general section of a roll, but there are reasons for treating it separately in this Draft.

When in a plane a straight line PH, as trunk, meets the edge of a general section of a roll B, C, D, E in the points L and M, and two other straight lines, parallel to one another, BC, DE, as

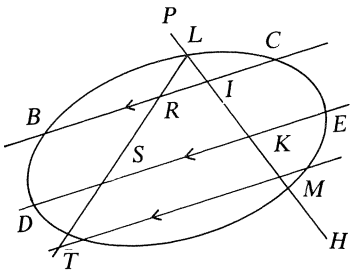

Fig. 6.21

branches, meet the edge of the same figure $BCDE$ in BC and DE, and also [meet] the trunk PH in K and I, and through L, one of the points the edge of the figure gives on the trunk, there passes another straight line L, R, S, which gives the points R and S on the two parallel springing branches BC, DE*[11] [Fig. 6.21]: [*sic*]

The ratio of the rectangle[a] of the two shoots, such as KS and KM, to the rectangle of the shoots such as KD, KE is equal to the ratio of the rectangle such as IR, IM (related to the rectangle KS, KM) to the rectangle such as IC, IB (related to the rectangle KE, KD).

In such a way that if the straight line LRS is so placed that the general rectangle of two shoots KS, KM, say, is equal to the rectangle KD, KE, say, then the other general rectangle IR, IM, say, is also equal to the rectangle IC, IB, say, with the result that if through M, the other point the edge of the figure gives on the trunk, we draw a straight line MT, belonging to the same ordinance as the two branches BC, DE, which are parallel to one another, to give the point T on the straight line such as L, R, S.

The ratio[b] of the rectangle of the general pair of shoots folded to the trunk KL, KM, contained between one of the knots K given on it by any one of the branches springing from the trunk, [say] ED, and each of the knots L, M given on it by the edge of the figure, [the ratio of this rectangle] to its twin the rectangle of the springing shoots KE, KD, say, contained between the same knot K and each of the points E, D given on it by the edge of the figure, is equal to the ratio of the shoot of the trunk ML, say, between the knots given on it by the edge of the figure, to the springing shoot such as MT, say, on the straight line MT.

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* This proposition refers to figure V.

[a] $\dfrac{KS.KM}{KD.KE} = \dfrac{IR.IM}{IC.IB}$.

[b] Claims $\dfrac{KL.KM}{KE.KD} = \dfrac{ML}{MT}$.

If the trunk *PH* is a diametral of the figure, and the branches *BC*, *DE*, its ordinates, the springing shoot, *MT*, say, is the line called elsewhere *Normal Side* and *parameter* but here called *coadjutor*. Normal side, parameter, coadjutor

For, taking *KM* as the common height of the rectangles *KL*, *KM* and *KS*, *KM*, and *IM* for the common height of the rectangles *IL*, *IM* and *IR*, *IM*.

The ratio of the rectangle *KL*, *KM* to the rectangle *KS*, *KM* is equal to the ratio of *KL* to *KS*, that is to say, since the branches *ED* and *BC* are parallel to one another, the ratio of *IL* to *IR*.

And the ratio of the rectangle *IL*, *IM* to the rectangle *IR*, *IM* is equal to the ratio of *IL* to *IR*, that is to say, since the branches *ED* and *BC* are parallel to one another, the ratio of *KL* to *KS*.

That is to say, the ratio of the rectangle *KL*, *KM* to the rectangle *KS*, *KM* is equal to the ratio of the rectangle *IL*, *IM* to the rectangle *IR*, *IM*.

And by alternation, the ratio of the rectangle *KL*, *KM* to the rectangle *IL*, *IM* is equal to the ratio of the rectangle *KS*, *KM* to the rectangle *IR*, *IM*.

And it is proved[a] that the ratio of the rectangle *KL*, *KM* to the rectangle *IL*, *IM* is also equal to the ratio of the rectangle *KD*, *KE* to the rectangle *IC*, *IB*.

Consequently, the ratio of the rectangle *KS*, *KM* to the rectangle *IR*, *IM* is equal to the ratio of the rectangle *KD*, *KE* to the rectangle *IC*, *IB*.

And, by alternation, the ratio of the rectangle *KS*, *KM* to the rectangle *KE*, *KD* is equal to the ratio of the rectangle *IR*, *IM* to the rectangle *IC*, *IB*.

In such a way that if the rectangle *KS*, *KM* is equal to the rectangle *KE*, *KD*, then the rectangle *IR*, *IM* is also equal to the rectangle *IC*, *IB*.

In consequence, the ratio of the rectangle *KL*, *KM*, say, to the rectangle *KS*, *KM*, say (or the rectangle which is equal to it, *KE*, *KD*), is equal to the ratio of *KL* to *KS*, that is to say, since the straight lines *KS* and *MT* are parallel to one another, the ratio of *ML* to *MT*.

Thus when the trunk *PH*, say, is a diametral of the figure, and
[25] the branches *ED*, *CB* are its ordinates, the shoot *MT*, say, is clearly the line called *Normal side*, *parameter* or *coadjutor*, and which is only a [special] case, of a [special] case, of a [special] case, and, moreover, its origin can be seen.*[112]

From what we have already said it will have been clear that to

* It would have been better if the following section had come before this one.

[a] By Desargues' involution theorem.

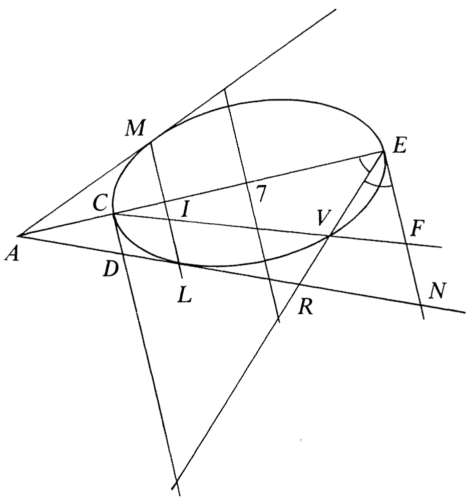

Fig. 6.22

draw through a general point a straight line which belongs to the same ordinance as two lines which are parallel to one another means that this straight line must be drawn so that it too is parallel to the two lines; and, in the same way, to draw through a general point a straight line through a point at infinite distance on another line means that this straight line must be drawn parallel to the one on which the assigned point is at infinite distance.

Although what follows appears obvious from what has already been proved, nevertheless:

When,*[113] in a plane, through the two points *E* and *C* which the curved edge of any section of a roll gives on any one of its diametrals *E7C*, there pass two straight lines *EB*,*CD*, each an ordinal of the diametral *E7C*, and any other line, *LR*, touches the section of a roll, in any other point *L* [Fig. 6.22].

We always have a rectangle of the same magnitude enclosed by the two parts of these ordinals *EB*, *CD* which are contained between their diametransversal *E7D* and the points *B* and *D* given on them by the other general line *LR*, tangent to the figure.

For since the two straight lines, the diametral *E7C* and the tangent[a] *LR*, are given in position in a plane, *A*, the butt of their ordinance, is also given in position.

And if through the point *L* we draw the line *LIM*, the

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* This proposition refers to figure VII.

[a] Tangent at *L*.

transversal of the ordinates of the point A, which gives the additional point M on the edge of the figure.

Since the straight line $E7C$ is a diametral of the figure, the transversal LIM is, together with the two [lines] EB, CD, ordinate to the diametral $E7C$.

Let us draw through the point 7, the midpoint of the part EC of the diametral $E7C$, yet another straight line, $7R$, which is also an ordinate or ordinal[114] of the diametransversal $E7C$, and which gives the point R on the tangent LR, then the butt of the four ordinates or ordinals[115] EB, CD, IL, $7R$ is at infinite distance, since $E7C$ is their diametransversal.

Let us also draw the straight line CGF, which gives on EB the coadjutor EF.

In order to construct on any tangent EB, including the general part EN, the coadjutor of the diametransversal $E7C$, one of the methods is to bisect the angle made by the tangent EN and the diametransversal EC on the side towards the figure, the bisector being a straight line EV which gives the additional point V on the edge of the figure, then to draw another straight line VC which gives the point F on EN, and EF is clearly the coadjutor.[116]

It has been proved that the four points $C\ I\ E\ A$ are in involution, and that 7 is the stump of a tree in which E, E; C, C and I, A are pairs of knots.

And that A is a stump common to three trees, in which EC, $7I$, BD, LR, HN and MO are pairs of knots.

And that the ratios of the four parts of lines[a] CD, $7R$, IL, EB and also those of the four parts $CH, 7O, IM$, and EN, two by two, are equal to the ratios formed among themselves by the four parts AC, $A7$, AI, AE and by the similar parts AH, AO, AM, AN or AD, AR, AL, AB.[117]

So the ratio of the rectangle of the two limbs[b] $A7, AI$ to the rectangle of its own height AI, AI, that is to say, the ratio of the base [*sic*] or limb $A7$ to its partner the base or limb AI, that is to say, the ratio of the rectangle of the pair of equal shoots folded to the trunk $7C, 7E$, to its related rectangle, that of the shoots IC, IE, is equal to the ratio of the rectangle $7R, IL$ to the rectangle of its own height IL, IL, or the equal rectangle of the pair of equal springing shoots IL, IM.

And, by interchanging, the ratio of the rectangle of the pair of equal shoots folded to the trunk[c] $7C, 7E$, to the rectangle $7R, IL$,

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[a] $CD{:}7R{:}IL.EB = CH{:}7O{:}IM{:}EN = AC{:}A7{:}AI{:}AE = AH{:}AO$, etc.

[b] $\dfrac{A7}{AI} = \dfrac{7C.7E}{IC.IE}$ (involution) $= \dfrac{7R}{IM}\cdot\dfrac{IL}{IL}$ $(IM = IL)$.

[c] $7R.IL = EB.CD, \quad \dfrac{IC.IE}{IL.IM} = \dfrac{EC}{EF}$.

or the equal rectangle *EB*, *CD*, is equal to the ratio of the folded shoots *IC*, *IE*, to its twin the rectangle of the springing shoots *IL*, *IM*, that is to say it is equal to the ratio of the part such as *EC*, of the diametral *E7C*, to the part such as *EF*, of its ordinal such as *EB*.

Consequently the rectangle of the same parts *7E*, *7C*, say, of the diametral *E7C*, bears the same ratio to each of the rectangles of the parts of its two ordinals *EB*, *CD*, contained between its two points such as *E* and *C* and the general straight line *LRBD* which touches the figure in the general point *L*[a].

And since 7 is the midpoint of the part *EC* of the diametral *E7C*, the rectangle of the equal parts[b] *7E*, *7C* is a quarter of the rectangle of *EC*, *EC*, or the square of *EC*.

And the ratio of *EC* to *EF* is equal[c] to the ratio of the rectangle *EC*, *EC*, or square of *EC*, to the rectangle *EC*, *EF*, with the same height *EC*, and is equal to the ratio of the rectangle *7E*, *7C*, a quarter of the rectangle *EC*, *EC*, to a quarter of the rectangle *EC*, *EF*.

So the rectangle *7E*, *7C* bears the same ratio both to a quarter of the rectangle of the parts such as *EC*, *EF*, and to the rectangle of the parts such as *EB*, *CD*, and in consequence the rectangle *EB*, *CD* is equal to a quarter of the rectangle[d] of the parts such as *EC*, *EF*.

And, by an obvious converse to what we have proved, when the diametral such as *E7C* is the large axle of the figure, [then] a shoot such as *BD*, belonging to the general tangent *LR*,[118] is the diameter of a circle whose circumference passes through two points such as *Q* and *P*, in the same diametral and axle *C7E*,[119] in such a way that the rectangle of the parts of the diametral *E7C* contained between either of these points *P*, [say,] and each of the points such as *E* and *C* given on the line by the edge of the figure,
[26] is again equal to a quarter of the rectangle *EC*, *EF*, and the part such as *EC* is equal either to the sum or to the difference of the two straight lines drawn from the point of contact such as *L* to each of the points such as *P* and *Q*, that is, it is equal to the sum or to the difference of the two straight lines drawn as are *LP* and *LQ*, and the tangent *LD* bisects one of the angles these two lines drawn like *QL*, *PL* make with one another.

That is to say, the two points such as *Q* and *P* are the points

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[a] So $\dfrac{7E.7C}{EB.CD} = \dfrac{EC}{EF}$ which is constant.

[b] $7E.7C = \frac{1}{4}EC^2$.

[c] $\dfrac{EC}{EF} = \dfrac{EC^2}{EC.EF} = \dfrac{7E.7C}{\frac{1}{4}EC.EF}$.

[d] $EB.CD = \frac{1}{4}EC.CF$.

called *Navels*, *burning points* or *foci* of the figure, about which there is much to be said.[121] [Navels, burning points, foci][120]

And particularly for the section of a roll called a hyperbola, in which the asymptotes $7X, 7Y$ are two tangents to the figure at infinite distance.

If we draw the two asymptotes $X7Z$ and $K7Y$, as tangents at infinite distance, we shall see, from the above, that the parts of the straight line IM contained mutually between the edge of the figure L and M and each of the two asymptotes are equal to one another, that is to say, IL and IM are equal to one another and IS, IT are equal to one another, and in consequence LS, MT are equal to one another, and MS, LT are equal to one another, as is obvious by [constructing] a bough.[122]

And then the ratio of the rectangle of the parts of a diametral $E7C$ contained between its ordinate through the points of contact with the figure of the asymptotes, at infinite distance, and each of the points such as E and C given on it by the edge of the figure, to the rectangle of the shoots, springing from this ordinate at infinite distance, contained between the diametral $E7C$ and the two points given on it by the edge of the figure, is equal to the ratio of the rectangle such as IE, IC to the rectangle such as IL, IM, that is to say, it is equal to the ratio of EC to EF, that is to say, it is equal to the ratio of the rectangle of the equal parts $7E, 7C$ to the rectangle of the equal parts EX, CZ of the straight lines EB, CD, ordinals of the diametral $E7C$, and contained between it [i.e. the diametral] and the general tangent at infinite distance $X7Z$.

That is to say that the ratio of $E7$ squared to EX squared, that is to say, the ratio of $I7$ squared to IS squared, is equal to the ratio of the rectangle of the folded shoots IE, IC to the rectangle of the equal springing shoots IL, IM.

Now, from the propositions which include propositions 5 and 6, and 9 and 10, of the second book of Euclid's *Elements*.

It is clear that the rectangle IE, IC plus $E7$ squared is equal to $I7$ squared.

And that the rectangle LS, ST plus IM squared is equal to IS squared.

Consequently, since the ratio of $I7$ squared to IS squared is equal to the ratio of the rectangle IE, IC to IM squared, it follows that the ratio of the remainder $7E$ squared to the remaining rectangle LS, LT is equal to the ratio of the rectangle IE, IC to the rectangle IL, IM, that is to say, it is equal to the ratio of EC to EF.

From which it follows that wherever we draw a straight line such as LIM, ordinate to a diametral such as $E7C$, the rectangle of the two parts of this ordinate contained between L, one of the points given on it by the edge of the figure, and each of the two asymptotes, is always of constant magnitude and is equal to a quarter of the rectangle of the two parts such as *EC* and *EF*, the coadjutor.

When two cones touch one another along a straight line, either the concave side of one touches the convex side of the other, or both touching sides are convex, and the straight line lies in a plane which joins or touches [along a line] in itself each of the two cones, which, in a cutting plane parallel to this plane that joins them, give two parabolas with a common axle, whose edges do not touch one another at any finite distance, but do touch one another at infinite distance,[123] and in every other cutting plane [the cones] give two figures whose edges touch one another at either finite or infinite distance.

When two cones touch one another along two separate and distinct lines, on either of these two lines it is either the convex side of one cone which touches the concave side of the other, or it is the convex side of each that touches, and each of the straight lines lies in a plane which touches each of the two cones, which, in a cutting plane parallel to either of the joining or touching planes, give two parabolas that touch one another at one point, and in every other plane they give two figures whose edges touch at two points at finite or infinite distance [Fig. 6.23].

When the two cones touch on the concave side of one of them, the two figures [in the cutting plane] touch on the concave side of one of them.

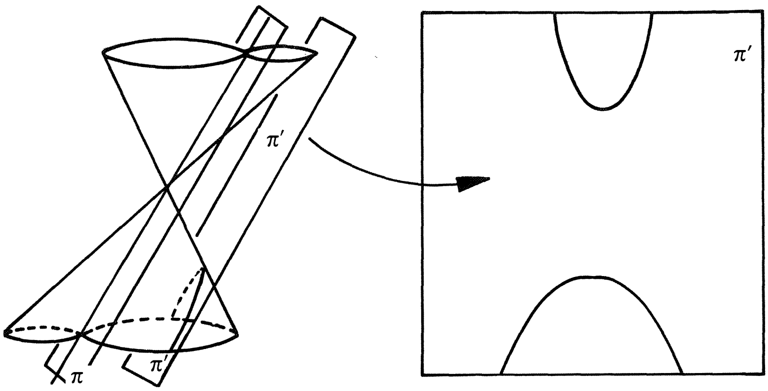
When the two cones touch on the convex side of each of them, the figures [in the cutting plane] touch on the convex side of each of them.

And when the cutting plane is parallel to the plane of the two straight lines in which the cones touch one another, we obtain two hyperbolas, either one inside the other or one outside the other, and they are said to be conjugate, each having the same asymptotes, and their edges do not meet at any finite distance or, alternatively, meet at infinite distance, and the properties of the resultant figures will become clear from our deductions as to how many ways and how the edges of the two general conic sections may meet one another.

Once we have grasped the concept of a straight line which is a transversal of the straight lines of an ordinance, we can easily grasp the concept of a plane also being a transversal of the straight lines of an ordinance as this applies to surface loci.[124]

When a ball [i.e. a sphere] and a plane are each fixed, the plane is a transversal, with respect to the ball, of an ordinance of
[27] straight lines whose butt is given in position, and if the butt is

Convex–convex case with one common line



common line

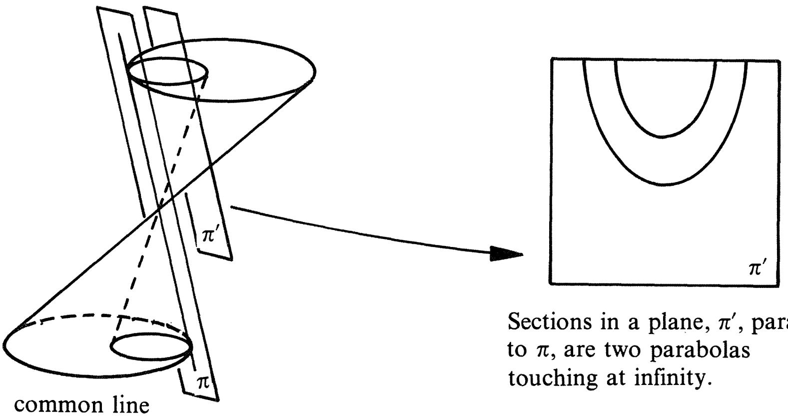
common tangent plane, π.
π' plane parallel to π.

Sections in a plane, π', parallel to π, are two parabolas touching at infinity.

The general section (not shown) would be two conics, one from each cone, touching where the plane of section cut the common line.

(a)

Convex–concave case with one common line



Sections in a plane, π', parallel to π, are two parabolas touching at infinity.

common line

common tangent plane
π' plane parallel to π

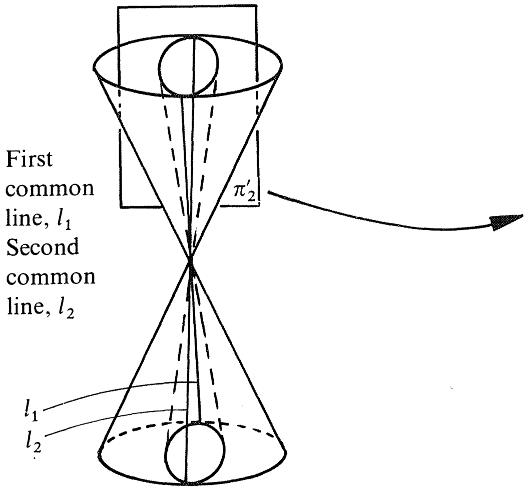
(b)

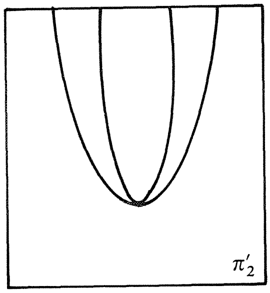
Fig. 6.23

given the position of the plane is given, all of which follows from the above.

And when several straight lines, each having a fixed point in this plane, move round the ball, the planes of the circles they

Convex–concave case with two common lines.





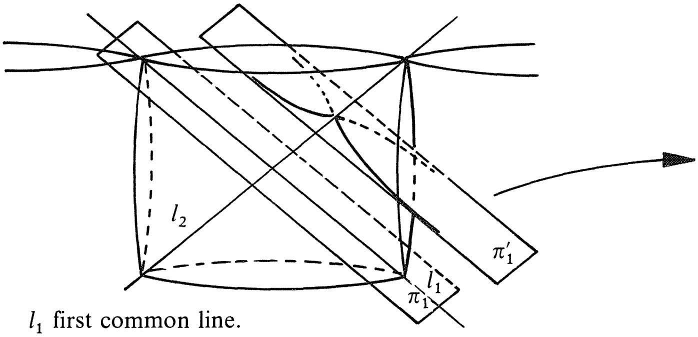
Sections in π'_2 are two parabolas touching where π'_2 meets l_1.

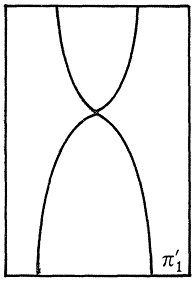
The common tangent planes, π_1 containing l_1, and π_2 containing l_2, are not shown, but π'_2 is a plane parallel to l_2 and π_2.

The general section, not shown, would be two conics one from each cone, touching where the plane of section cuts l_1 and l_2: it would look like the 'tops' of the cones in the figure.

(c)

Convex–convex case with two common lines.





l_1 first common line.
l_2 second common line.
π_1 first common tangent plane.
(π_2, the second common tangent plane, is not shown)
π'_1 a plane parallel to π_1.

Sections in π'_1 are two parabolas touching where π'_1 meets l_2.

The general section (such as the 'tops') is two conics touching at two points.

(d)

Fig. 6.23 (*contd*)

describe are each transversals of the straight lines ordinate to the fixed point of the line which describes them, and they all cut one another at the butt of the ordinates of this first plane.

A similar property exists for other solid figures which are related to the ball as the ovals, otherwise ellipses, are to the circle, but there is too much to say for it to be worth starting on the subject here.

When*[125] in the plane of a general section of a roll $5Y8GH$, on the general line AF, one of the ordinates of a transversal AV, the transversal point A is paired with F, the butt of the ordinates, in involution with any other two points X, Q, which[a] are taken as the two double mean knots of the involution, each of the pairs of branches springing from this trunk XQ which pass through one of these pairs of extreme knots, such as FH, AH, and RG, ZG, and are ordinate to the butts H and G, on the edge of the figure, in such a way that one of the two touches the figure, such as HA and ZGB, I say that each of such pairs of branches arranged in this way gives on the transversal VA[126] one of the pairs of knots DA, EB, belonging to a single tree whose stump, C, lies on the same straight line with 7 and P, the two stumps of the trunk 578 of the same ordinates to the butt F, which line [$7P$] is a diametral of the figure and of the other trunk AF [Fig. 6.24].

Now, in the first case of the hypothesis and of what we prove, in the straight line 7, F, T, a diametral of the figure and belonging to the set of lines ordinate to the butt F, the transversal point, T, is paired with the butt of the ordinance, F, in involution with 5 and 8, the two points given on the line by the edge of the figure, and if the shoot 58 is bisected in 7 then this point 7 is a stump in the involution[b] of the four points 5, F, 8, T.

Similarly, if P is the midpoint of the shoot X, Q, belonging to the involution of the points $XFQA$, this point P is the stump of the involution.[c]

It is furthermore clear from what has been proved above that the points X and Q are either both on the edge of the figure, or, as here, both on the same side, away from the edge of the figure, that is, both of them are on the concave side or both on the convex one.

And when[d] they are on the edge of the figure, the straight line PC drawn through the two stumps 7 and P is ordinate to a point

* This proposition refers to figure VIII.

[a] AV is polar of F; X, Q, F, A harmonic.

[b] F, T in involution, with 7 stump.

[c] $XPQA$ in involution with P stump.

[d] X, Q not on the conic implies $P7$ and HF meet AV in distinct points C, D.

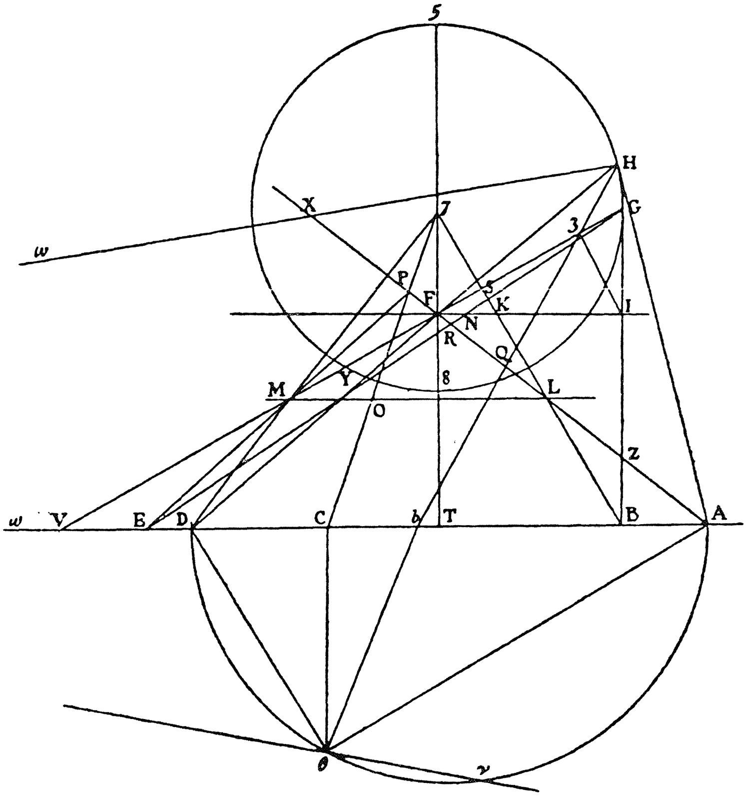


Fig. 6.24. From Taton (1951, p. 171).

of the transversal AV, with the straight line H, F, D which hence is a transversal of the lines ordinate to the butt A, which lies in the line AF and in the same transversal AT, and so in this way the two points C and D coalesce into a single point in the transversal AV, in consequence, apart from this case, the two points C and D lie in the same transversal AV as two distinct points.

Similarly,[a] and by the same reasoning, in the same case of two points X and Q on the edge of the figure, the same straight line 7, P, C is again ordinate to a point of the same transversal AV, with the straight line GRE, which thus is a transversal of the lines ordinate to the butt Z, which lies on the straight line AF, and so in this way the two points C and E coalesce into a single point in the transversal AV; in consequence, apart from this case, the two

[a] C and E are distinct.

points C and E lie in the same transversal A, V as two distinct points.

From which it is clear that in one of the two other cases the points such as D and E are always on the same side of the point such as C, that is to say, that the point such as C is similarly engaged or disengaged with respect to each of the two pairs of points DA and EB.

So, having drawn the straight line GF, which gives the point Y on the edge of the figure and the point V on the transversal AV.

The straight lines[a] $7D$ and $7B$ which the points M and L on the straight lines GF and AF, the straight line FN, parallel to the transversal AV, and which gives the points K, N, I on the straight lines $7B$, GRE and GB.

The straight line I, 3 [*sic*], parallel to the straight line $B7$, and which gives the point 3 on the straight line GF.

The straight line[b] L, M, which gives the point O on the straight line $7PC$.

And,[c] finally, the straight line EP, ordinate at some point with the straight line GF.

Now the method or order of this general Proof in the plane, in which M. Pujos[127] has a considerable share, is divided into two stages:

The first is to prove[d] that the straight line LM is parallel to the transversal AV,

And the second is to prove that the straight line EP is ordinate to the butt M, together with the three straight lines FV, $7D$, LM.

Having done this, we rapidly deduce what is stated in the proposition, namely that the rectangles contained by each of the pairs of limbs CD, CA and CE, CB are equal to one another.

[28] Concerning the first stage, proving that LM is parallel to the transversal AV.

We have shown above[e] that the ratio of the shoot $7T$ to the shoot $7F$ is equal to the ratio compounded of the ratio of the shoot DT to the shoot DV and the ratio of the shoot MV to the shoot MF.

And[f] by a similar argument the ratio of the same shoot $7T$ to

---

[a] Join $7D$, $7B$. Draw $FN \| AV$ meeting $7B$ in K and GB in I. $7B$ meets FA in L, $7D$ meets FV in M. Draw $I3 \| B7$, meeting GT in 3.

[b] LM meeting $7PC$ in O.

[c] Joint EP, meeting GF at M.

[d] Claims $LM \| AV$ and M lies on EP.

[e] $\dfrac{7T}{7F} = \dfrac{DT}{DV} \cdot \dfrac{MV}{MF}$ by Menelaus $\triangle VFT$ line $7MD$.

[f] $\dfrac{7T}{7F} = \dfrac{BT}{BA} \cdot \dfrac{LA}{LF}$ by Menelaus $\triangle TFA$, line $7LB$.

the same shoot $7F$ is equal to the ratio compounded of the ratio of the shoot BT to the shoot BA and the ratio of the shoot LA to the shoot LF.

So the ratio compounded of the ratios of DT to DV and of MV and MF

is the same as the ratio compounded of the ratios of BT to BA and of LA to LF.[128]

Now it has been proved[a] that the ratio of DT to DV is equal to the ratio of BT to BA.

So[b] the remaining ratio, that of MV to MF, is also the same as the ratio of LA to LF.

In consequence,[c] the two straight lines LM and AV are parallel to one another.

Concerning the second stage, proving that the straight line EP is ordinate to the butt M, together with the three straight lines LM, FV, $7D$.

We shall arrive at this by first showing[d] that the rectangle of the parts VE and FK is equal to the rectangle of the parts FI and FN, in this way.

From our hypothesis and our construction, it is clear that,[e] on the straight line GF, the four points G, F, Y, V are in involution, S being the stump of the involution, and SY, SY, SG, SG each being a pair of mean limbs, and SV, SF a pair of extreme limbs.

From which it follows[f] that the ratio of GV to GF is equal to the ratio of SG to SF.

And since FN and AV are parallel to one another, as are $B7$ and $I3$.

The ratio[g] of GV to GF is equal to the ratio of VE to FN, and to the ratio of GB to GI, and to the ratio of GS to $G3$.

From which it follows[h] that $G3$ is equal to SF, and $F3$ equal to GS, and thus $F3$ is also equal to FS.

---

a $\dfrac{DT}{DV} = \dfrac{BT}{BA}$.

b $\dfrac{MV}{MF} = \dfrac{LA}{LF}$.

c $LM \| AV$.

d Show $VE.EK = FI.FN$.

e $GFYV$ involution stump S.

f $\dfrac{GV}{GF} = \dfrac{SG}{SF}$.

g $\dfrac{GV}{GF} = \dfrac{VE}{FN} = \dfrac{GB}{GI}$.

h $\left.\begin{matrix} G3 = SF \\ F3 = GS \end{matrix}\right\} \Rightarrow F3 = FS$.

But the ratio of $F3$ to FS is thus also equal to the ratio of FI to FK.

In consequence the ratio of VE to FN is equal to the ratio of FI to FK.

Consequently, the rectangle of the two extreme [limbs] VE, FK is equal to the rectangle of the mean [limbs] FN, FI.

Furthermore, from the construction, the point P is a stump in the tree XQ, of which PA, PF, and PX, PR are two pairs of extreme limbs.

Thus the ratio[a] of the limb PA to its partner the limb PF is equal to the ratio of the rectangle of the shoots AR, AZ to its related rectangle, that of the shoots FR, FZ, that is to say, it is equal to the ratio compounded of the ratio of RA to RF (or the equal ratio of EA to FN) and the ratio of ZA to ZF (or the equal ratio of AB to FI) that is to say that the ratio of the limb PA to its partner the limb PF is equal to the ratio[b] of the rectangle of the parts AE, AB to the rectangle of the parts FN, FI, or the equal rectangle of the parts EV, FK, that is, it is equal to the ratio compounded of the ratio of EA to EV, and the ratio[c] of AB to FK, or the equal ratio of LA to LF, or the equal ratio of MV to MF, that is to say that, on the trunk EP,[129] the ratio of the shoot PA to the shoot PF is equal to the ratio compounded of the ratio of the shoot EA to the shoot EV, and the ratio of the shoot MV to the shoot MF.

And, by the converse of a result proved above, the three knots P, M, E lie on a single trunk PE, that is to say that the point M lies on the straight line EP, that is to say that the straight line EP is ordinate to the butt M, together with the three straight lines LM, $7D$ and SV.

So we have shown that the straight line LM is parallel to the transversal AV and that the three points $E\ M\ P$ lie in a straight line.

On account of this, finally, the ratio[d] of OL to OM is equal to the ratio of CA to CE.

And similarly, the ratio of OL to OM is equal to the ratio of CB to CD.[130]

In consequence, the ratio of CA to CE is equal to the ratio of CB to CD.

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a $\dfrac{PA}{PF} = \dfrac{AR.AZ}{FR.FZ} = \dfrac{EA.AB}{FN.FZ}$.

b $FN.FI = EV.FK$.

c $\dfrac{AB}{FK} = \dfrac{LA}{LF} = \dfrac{MV}{MF}$; $\dfrac{PA}{PF} = \dfrac{EA}{EV}.\dfrac{MV}{MF}$.

d $\dfrac{OL}{OM} = \dfrac{CA}{CE} = \dfrac{CB}{CD}$.

Consequently, the rectangle[a] of the pair of limbs CA, CD is equal to the rectangle of the pair of limbs CE, CB.

And thus, in the transversal AV, each of the pairs of points AD and EB is one of the pairs of knots of a tree in which the point C, given [on AV] by the straight line $7P$, is a stump.

From this it is clear that when the two points such as XQ lie away from the edge of the figure on its concave side, the tree this construction gives on the transversal, such as A, V, has an engaged stump.

When the points lie on the convex side the tree has a disengaged stump.

And when the points lie on the edge of the figure the tree is of the intermediate kind.

Now, from what has proved above, it follows that if the general section of a roll $5Y8$ is made the basis or base of a cone whose vertex is at a distance from the stump C, measured normal to the transversal AV, equal to one of the mean limbs of the tree on it [AV] resulting from the construction, and in a plane parallel to another plane which cuts this cone.

The two straight lines drawn through the vertex of this cone and each of the points X and Q, when they lie away from the edge of the figure on the concave side, give, in the figure which results on the cutting plane in this position, the two points called *navels*, *burning points*, otherwise *foci* of the figure.

[29] So that if for the basis of a cone we are given a general section of a roll with curved edge, the section being given in position, and in its plane a straight line such as AV as a transversal, and the angle the plane of the section makes with the plane defined by the vertex and this transversal, and in it [the transversal], either the stump of the tree resulting from the construction as, in our example, the point C, or alternatively two pairs of knots of the tree, or a point such as P not lying on the transversal, or alternatively one of the points such as X and Q, or alternatively two of the pairs of knots of the tree, such as XQ.

The vertex of this cone is given in position, and the cone is given in kind and in position, the figure of the section given by that position of the cutting plane is given in kind and in position, all the conjugate diameters of the figure of the section with their distinguishing properties, all the ordinates and tangents with their distinguishing properties, the coadjutant sides, the butt of the ordinance of its diametrals, and the focal points, each of these is given in its [type of] derivation, kind and position.

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[a] $CA \cdot CD = CE \cdot CB$.

Then if the vertex, the basis, the transversal, and the cutting plane are given in position, all the rest is similarly given in its [type of] derivation, kind and position.

And these circumstances disclose a special relation between the straight line and the circular line, and the points of each of them that are related to one another.

And for this purpose we must imagine that the trunk of a tree moves in a plane, in such a way that the point midway between the two paired knots is fixed, and we then consider what kind of line is described by each of the knots of this pair; and we shall find that when the midpoint is at finite distance, then each of the knots describes a line which is curved completely round, otherwise a circle, and that when the midpoint is at infinite distance, as for the pair of knots in which the extreme inner knot coincides with the stump and the extreme outer knot is at infinite distance, so the trunk is moving parallel to itself, the extreme inner knot, which, as we said, coincides with the stump, describes a straight line perpendicular to the trunk.

And the straight line described in this case has all the same properties as have the lines curved completely round [i.e. circles] or points which are described by the knots of each of the other pairs [i.e. in other cases].[131]

And this proposition alone would provide material for a whole book to anyone who wished to pick over all the obvious consequences of what we have proved in the above.

In which we see, further, various means of describing each of the kinds of section of a cone by points, and various designs of instruments for drawing them all, beginning with the point, following on with the straight line and eventually treating each of these curves, either [drawing them] by using the property of the coadjutor, or by using the properties of the foci, thus Monsieur Chauveau[132] has recently devised a method which is very simple and the more attractive on that account, but it would be a lengthy task to write down everything that follows on from what we have proved here.

To return to the propositions outlined at the bottom of the second page,[133] and which must come before all the rest, here is how they can be stated for the simple lines of the plate.

When a straight line AH is divided[a] at a general point B, the rectangle of the sum or total of the whole line AH with either one of its parts, AB say, that is to say the rectangle of HF into the other part HB, plus the square of the added part AB are together equal to the square of the whole of AH, [a result] which

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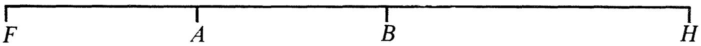
[a] From the *Elements* it is clear that F is taken so that A bisects FB, so $FA = AB$.

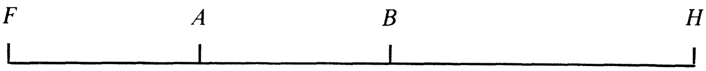


$FA = AB.$

Fig. 6.25. By the phrase 'the whole line AH' Desargues meant $AH + AB = FH$.

combines propositions 5 and 6 of Book 2 of Euclid's *Elements* [Fig. 6.25].

When a straight line AH is divided at a general point B, the square of the sum or total of the whole line AH with either one of its parts AB, that is to say, the square of HF, plus the square of the other part HB, are together equal to twice the squares of the whole line AH, and of the added part AB, [a result] which combines propositions 9 and 10 of Book 2 of Euclid's *Elements* [Fig. 6.26].



$FA = AB.$

Fig. 6.26. As above, Desargues meant $FH^2 + HB^2 = 2AH^2 + 2AB^2$.

The proofs of each of these propositions and their converses, derived from the fact that A is the midpoint of FH, are obvious.

When, in a plane, two straight lines of the same ordinance meet the same circle, the rectangles of the parts of each of the straight lines contained between the butt of their ordinance and each of the two points given on them by the edge of the circle are equal to one another, which combines propositions[a] 35 and 36 of Book 3 of Euclid's *Elements*.

And the proof of this is obvious from the preceding propositions, by drawing the straight line belonging to the same ordinance as the two given lines and which is a diametral of the circle, then the diametrals of the circle perpendicular to each of the two [given] lines, from this it will be seen that the tangent, if there is one, is included in the proof as one of the ordinates, and that when the butt of the ordinance of the [given] straight lines lies on the edge of the circle, the understanding cannot grasp the situation, and that one can construct a kind of converse to this [proposition].

There are propositions proved here, or consequences

[a] The secant theorems.

following from them, which combine together several of the propositions of the conics of Apolonius [*sic*], even [some] from the end of the third book. And after the lemmas or initial results, four of these propositions include a complete treatment of plane sections of the cone.

And since, when a straight line, which has one point absolutely fixed, moves in a plane, any other point on it, which
[30] moves simply with the line, describes a simple uniform line which is either straight or circular; we may imagine that this other point has also another motion, apart from the motion given it by the line, a motion backwards and forwards along the line, so that the point describes the edge of any other type of conic section.[134]

Definition.[135]

In this treatment of conics every plane section of a cone that has a curved edge is imagined equally as [possibly forming] the base of a cone.

Proposition.

If we are given the position, in any kind of plane base, of a cone cut by another plane, [and] the position and the axle of the ordinance between these two planes; In the plane of the base, in any line which touches it, we are given the part of that line that subtends the angle made at the vertex of the cone by two other straight lines, whose plane forms in the plane of section the coadjutor of the diameter of the figure given by this construction, the diameter formed by whichever suitable plane through the vertex of the cone passes through the point of contact [of the general tangent]. There are many more propositions of every kind to be constructed in connection with this subject, and as many entities to be named, for those who enjoy this entertainment.

Statement of belief.

In Geometry, when we reason about quantities, we do not distinguish them according to whether they exist either in actuality or only in potentiality, nor, when reasoning about natural things, do we decide that there are no things which the understanding cannot grasp.[136]

Concerning the infinite straight line.

The understanding feels itself wandering in space, not knowing whether that space continues for ever, or if at some place it ceases to continue. To decide the matter, the understanding may for example reason as follows; Either space continues for ever, or at some place it ceases to continue; if at some place it ceases to continue, wherever that place may be, the imagination can find it in time. Now imagination can never find any place in space at which this space ceases to continue; Therefore space and consequently the straight line continue for ever. The understanding again reasons that there must be quantities so small that their two opposite ends coincide with one another, and it feels itself incapable of grasping either of these kinds of quantity, without thereby giving itself grounds to conclude that one or the other does not exist in nature, nor is it able to come to that conclusion [i.e. non-existence] in respect of the properties which it has grounds to conclude that these things possess, or, again, that they seem to imply, merely on the grounds that it cannot grasp how they are such as, from its reasoning, it concludes them to be.[137]

From the contents of this Draft it follows that,

Concerning Perspective.[138]

For straight lines in the subject belonging to a general ordinance, the appearances [of the lines] in the flat picture are straight lines which all belong to the same ordinance, as does the straight line of the ordinance in the subject which passes through the eye, which is the axle of the ordinance between the planes through the eye and each of the straight lines in the subject.

In connection with Perspective.[139]

If we know the layout and position of any figure; with the sector,[140] and using the line of equal divisions, we draw this figure in plane perspective, of any size and in any position, and at any range, or distance from the eye; But few workmen know how to use a sector, and many know how to use a ruler and a pair of ordinary compasses,[141] to copy, reduce or draw the figure in proportion, or as they say, to scale, that is to say in the flat.[142] Now it is as easy to draw a figure in perspective with ruler and ordinary compasses as it is to draw it in the flat, because we draw it in perspective using scales of perspective measurements in the same way as we draw it in the flat using a scale of flat measurements, and in every case we use the ruler and ordinary compasses to make suitable scales of perspective measurements just as a suitable scale of flat measurements, so that we need only to refer to the scale of perspective measurements to draw the figure in perspective in the same way as we should refer to the scale of flat measurements to draw it in the flat, and there is an example of how this is done in practice which was printed in the year 1636.[143]

Concerning Sundials.

On any flat surface the straight hour-lines all belong to the same ordinance, as does the axle of the ordinance between the planes which give the division of the hours.[144]

Concerning the cutting of stone for facing.

On one wall surface the straight edges of the facing stones all usually belong to the same ordinance, as does the axle of the ordinance of the planes of the joints passing through these edges.

And the various ways of constructing each of these things are obvious.[145]

Those who do not find here all the propositions which they may have had communicated to them before will, I hope, allow that to have included them would have made the present work too voluminous.

Anyone who is interested in the matter of this Draft is invited to communicate what he thinks about it.

P.B.G.[146]

Chapter VII

The *Perspective* (1636)

The original title of Desargues' work is *Exemple de l'une des manieres universelles du S.G.D.L. touchant la pratique de la perspective sans emploier aucun tiers point, de distance ny d'autre nature, qui soit hors du champ de l'ouvrage.* Our translation follows the text of the first edition (1636, paginated from 1 to 12) but takes account of differences with the second edition, in Bosse's *La Perspective de Mr Desargues* (1648, pages 321 to 334), where it is followed almost immediately by the three geometrical propositions translated in Chapter VIII below. A list of Desargues' vocabulary is given in Chapter V above.

Note: Numbers at head of page, or in left-hand margin in [], are Desargues' page numbers (1636).

EXAMPLE OF ONE OF S.G.D.L.'S GENERAL METHODS concerning drawing in Perspective without using any third point, a distance point or any other kind, which lies outside the picture field.

Since this Example of a general method of drawing in perspective without using any third point, a distance point or any other kind, which lies outside the picture field, is written in French, the measures that are given are those current in France.

The words PERSPECTIVE, APPEARANCE, REPRESENTATION, & PORTRAYAL are all used to mean the same thing.

The words EXTREMITIES, EDGES, SIDES & OUTLINE of a figure are also all used to mean the same thing.

And the words REPRESENT, PORTRAY, FIND THE APPEARANCE, RENDER or CONSTRUCT IN PERSPECTIVE are all used with the same meaning as one another.

The words on the LEVEL, LEVEL, PARALLEL to the HORIZON are also all used to mean the same thing.

1 [1636]

The words VERTICALLY, PERPENDICULAR to the HORIZON, and SQUARE to the HORIZON are also all used to mean the same thing.

And the words SQUARELY, on the SQUARE, at RIGHT ANGLES and PERPENDICULARLY also in general are used to mean the same thing as one another.

That which one proposes to portray is called the SUBJECT.

What some people call *geometric plan*, others *ground plan*, and others the *base of the subject*, is called here the BASIS of the SUBJECT.

What some people call the *window*, and others the *section*, and others call by another name, that is the surface of the thing on which the perspective drawing is to be made, is here called the PICTURE, before the work is done as well as after.

The basis of the subject and the picture under discussion both lie in a flat surface, which is to say that we only discuss flat pictures and flat bases of subjects, which bases and pictures are considered as each having two faces.

The face of the picture which is visible to the eye is called the FRONT of the PICTURE while its other face, not visible to the eye, is called the BACK of the PICTURE.

When the basis of the subject is Level, the face which is turned upwards towards the Sky is called[1] the UPPER SURFACE of the BASIS of the SUBJECT, and the
[2] other face which is turned/ down towards the earth is called the LOWER SURFACE of the BASIS of the SUBJECT.

The expanse or flat and indeterminate surface in which the basis of the subject is imagined to lie is called the PLANE of the BASIS of the SUBJECT.[2]

The flat and also indeterminate expanse in which the picture lies is called the PLANE of the PICTURE.[3]

All lines are assumed straight.

In the single engraved plate [see Fig. 7.1], referring to this single example [*sc*. of perspective construction], there are three separate figures, marked with the same letters but in a different typeface for each figure.

The Letters used in the text when referring to these figures are of the same typeface as those in the figure referred to in each particular passage.

When the text contains references which employ the same Letters more than once, but in different typefaces, this mean that such passages refer equally to each of the figures in which the corresponding Letters occur.

When the two ends of a line in one of these figures are marked with the same Letters as those marking the two ends of a line in another of these figures, the two lines so marked correspond to one another, the one playing a part in one figure the same as that of the other in the other figure.

For the purpose of this Technique it is assumed that a single eye sees at a glance both the subject, with its basis, and the picture, disposed in line with one another, however this occurs: it does not matter whether it is by Emission of visual rays or by the receiving of species emanating from the subject, nor

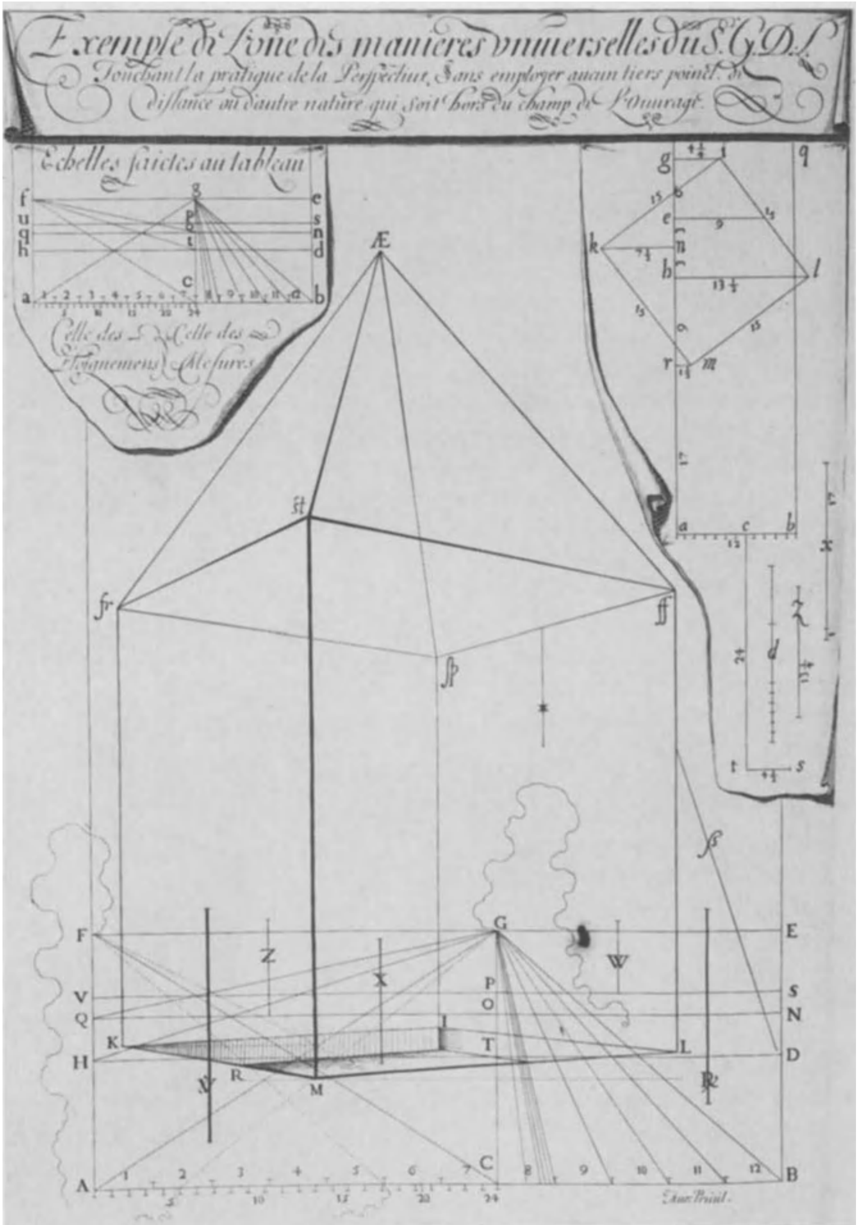


Fig. 7.1. The plate illustrating Desargues' *Perspective* (Paris 1636). Courtesy of the Metropolitan Museum, New York.

2 [1636]

from what position, or which of the two is seen in front of the other, provided both are easily seen by a single glance.

It is also assumed that a practitioner of this Technique understands the method and use of a scale rule in drawing up a basis [i.e. a ground plan] of the subject, together with its elevation; and in this example we shall assume that he understands what it is that is commonly called perspective.

And the manner of drawing in perspective that is described here, given the basis and the necessary elevations of a subject, with convenient intervals marked off in any size, or merely their alignment and their sizes recorded in a table, and knowing the disposition of the planes of the basis of the subject and the picture, will enable one, by means of a ruler and ordinary compasses,[4] to find and draw, immediately and easily, the perspective image of such a subject, in a picture of whatever size, without the aid of any point which lies outside the picture, at whatever distance and in whatever manner the subject, its basis and the picture are disposed among themselves and with respect to the eye.

The general rules of this procedure are expressed in a different style, they take in various general methods of procedure, they are applicable to a number of different cases and figures, and only two obvious and familiar propositions
[3] are required to demonstrate them/ to those who wish to understand them.

But for the present, and for those who only know how to follow the old rules for practising this technique, this example, which is expressed in simple terms and involves a subject which is treated by the old rules,[5] is a purely practical one.

In this, in view of circumstances which will be mentioned, we begin with three kinds of preparatory procedure.

One relates to the subject and is carried out in the plane of its basis, or somewhere else.

The other two concern the appearance of the subject and are commonly carried out on the picture itself.

The subject of this example is a cage simply built up of lines, square in plan and of even thickness up to a certain level, after which it terminates in a solid point, in the manner of a construction in the form of a roofed pavilion, put down in level country, with all its sides rising vertically from the ground up to roof level, set into the earth on a level lower than the surrounding terrain, with the measures of several vertical and sloping lines at various places outside and inside the cage, in the earth, on the earth and suspended over the earth, each line parallel to the picture, which hangs vertically.

At the top of the engraved plate on the right

The square figure m, l, i, k, whatever size it may happen to be, is the basis of the cage, the basis here being placed on the level.

The line x is the height of the vertical sides, straight legs or rising members

3 [1636]

of the cage, which are understood to be placed vertically on its basis one at each of the four corners of the square *m*, *l*, *i*, *k*.

The line *d* is the length of three units[6] of the scale with which the edges of the basis of the cage and its sides were measured, we shall call it the SCALE of the SUBJECT.

The line *ts* is the measure of the perpendicular height of the eye above the plane of the basis of the subject, which eye height [i.e. the line *ts*] meets the plane at the point *t*.

In the same plane of the basis of the subject, at the place where the plane of the picture meets it, there is drawn a line *ab*, called the LINE of the PLANE of the PICTURE, so that here the eye sees the picture in front of the subject, or the eye sees the subject behind the picture.

The line *tc* is the perpendicular distance from the foot of the vertical line through the eye to the picture, that is, the perpendicular distance of the eye from the picture.

Through one of the points, *a* or *b*, of the line *ab*, as here through the point *a*, we draw an intermediate line *ag*, in the same plane and on the same side of *ab* nearer the basis of the subject, making it parallel to the line *tc*.

Then from each of the points marked on the basis of the subject—from the four corners, from the midpoint of one of the sides of the square *m*, *l*, *i*, *k*—we draw to the line *ag* lines parallel to the line *ab*, such as the lines *mr*, *lh*, *kn*, *e*15 [*sic*] and *ig*.

[4] Through *b*, the other point of the line *ab*, is drawn the line *bq*, whose length is as yet indeterminate, parallel to the lines *ag* and *tc*.

The length of each of these lines or [of a] marked segment of them is measured with the scale of the subject, *d*, and their measurement is committed to memory or is written down on the line as a reminder or is written in a table.

So the numbers 15 written beside the edges of the square *m*, *l*, *i*, *k* indicate that each of the sides of that figure is fifteen feet long.

And the numbers 1, 17 written beside the line of the vertical sides, *x*, indicate that each of the vertical sides of the subject is eighteen feet long, that is seventeen feet above ground and one foot below ground.

So the number 12 written beside the line *ab* indicates that in this example that line is twelve feet long.

So the number 17 indicates that the part of the line *ag* contained between the lines *rm* and *ab* happens to be seventeen feet long, and by this means, or this method of measuring, we know that in this example it chances that the subject lies behind the picture, seventeen feet away from it, which means also that it chances that the picture is situated in front of the subject, seventeen feet away from it.

Similarly, the number $4\frac{1}{2}$ on the line *st* shows that in this example the height of the eye is four and a half feet perpendicularly above the plane of the basis of the subject [the base plane of the subject].

4 [1636]

In the same way, the number 24 signifies that in this example the foot of the vertical line through the eye, or the eye itself, is at a perpendicular distance of twenty-four feet from the picture in front of it.

In the same way, the number $13\frac{1}{2}$ indicates that the line *lh* is thirteen and a half feet in length.

In the same way, one of the numbers 9 indicates that the part of the line *ag* contained between the lines *rm*, *lh* is nine feet long.

The same applies to the numbers 3 and also to all the other similar numbers.[7]

And that completes the one of the three preparatory procedures which is concerned with the subject.

Now the whole engraved plate is taken to be a wooden panel, a wall or something similar, suitable and prepared to make a picture of whatever size it may be, assumed to be hanging vertically over the plane of the basis of the subject, which plane it as it were touches in the line *ab*. It is in this imagined picture that we propose to represent the cage by means of a perspective figure, whose size suits with that of the picture, without employing for our work any point which lies outside the picture, nor first making another perspective construction elsewhere the same width as the [length of the] line *ab* and then transferring it to the picture in scaled form, using a grid[8] or a scale rule.

At the bottom of the engraved plate

For this purpose we draw the line *AB*, level and as long as can be drawn at the bottom of the picture, corresponding to the line *ab*.

Then at the end points of the line, *A* and *B*, we draw, on the same side of the
[5] line *AB*/ two more lines, *AF* and *BE*, parallel to one another and, as shown here, perpendicular to the line *AB*.

Then we divide the line *AB* into as many equal parts as there are feet in the length of *ab*.

In this example, the line *ab* is twelve feet long, so the line *AB* is divided into twelve equal parts, marked above it, which make a scale of that number of feet, one of which, in this example the seventh[9] or half or a quarter of it is subdivided into inches, and into lines if they are needed.[10]

Furthermore, we take account of the height of the eye above the plane of the basis of the subject, which height is, in our example, four and a half feet; and this measure of four and a half feet is taken in the feet marked on the scale drawn as we described on the line *AB*, and it is applied to each of the lines *AF* and *BE*, that is from *A* to *F*, and from *B* to *E*;[11] then we draw the line *FE*, which by this means is parallel to the line *AB*.

In addition, on this line *FE* we mark the point such that the eye is perpendicularly opposite it, at its known distance in front of the picture [i.e. the point where the line from the eye perpendicular to the picture plane

5 [1636]

intersects the picture], in our example this is the point G, opposite which the eye is meant to be placed, in front of the picture at a distance of twenty-four feet measured perpendicularly.

Through this point G we then draw the line GC, parallel to each of the lines AF and BE, that is, in this example, at right angles to the line AB, so that the area $AFEB$ happens to be divided into two areas whose opposite sides are in each case parallel lines, that is, in this example, the areas $GCAF$ and $GCBE$.

Next, either in the whole area, $ABEF$, or in one or other of the two smaller areas, $GCAF$ and $GCBE$, as, in this example, in the area $GCAF$, we draw the two lines AG and CF [i.e. diagonals of the parallelogram].

Through the point of intersection of the lines AG and CF we draw the line HD, parallel to the line AB, which line HD meets the line BE in the point D, the line GC in the point T, and the line AF in the point H.

Then, from one or other of the points H and T, we draw a line, in the same area $GCAF$, to whichever of the points G or F lies diagonally opposite it.

If, as in the lower part of the engraved plate, this line is drawn from the point G to the point H, it is the line GH.

And if, as in the upper part of the engraved plate on the left, the line is drawn from the point **f** to the point **t**, it is the line **ft**.

And let us imagine that through the points **f** and **t** we have drawn the line **ft**, and next through the point of intersection of this line **ft** with the line **ag** we have drawn the line **nq**, parallel to the line **ab**.

Then through the point of intersection of this line **nq** with the line **cg**, in our example, the point **o**, and through the point **f**, we draw the line **fo**/.

[6] Then through the point of intersection of this line **fo** with the line *ag* we draw the line **su**, parallel to the line **ab**.

And the same process is repeated as many times as we need.

Now, supposing that we have carried out this process using the lines CF and AF, the lines NQ and SU will always be in the same position on the picture as they would have been if they had been constructed by using the lines AG and CG.

Finally, the part of the line **ab**, AB which happen to lie along the area in which this procedure has been carried out, as in this example the part **ac**, AC, is divided into as many equal parts as there are [feet] in the distance from the eye to the picture.

In our example, the distance from the eye to the picture is twenty-four feet, so the part **ac**, AC of the line **ab**, AB is divided into twenty-four equal parts, marked below it, which are like so many feet, one of which, or half or a quarter of it, can if necessary, again be subdivided into inches and lines.

This completes one of the two preparatory procedures which relate to the perspective construction; this procedure forms a figure we shall call a DISTANCE SCALE, though others may call it an optical scale or some other name.

Further, from any point of the line FGE, **fge**[12] that is convenient for the

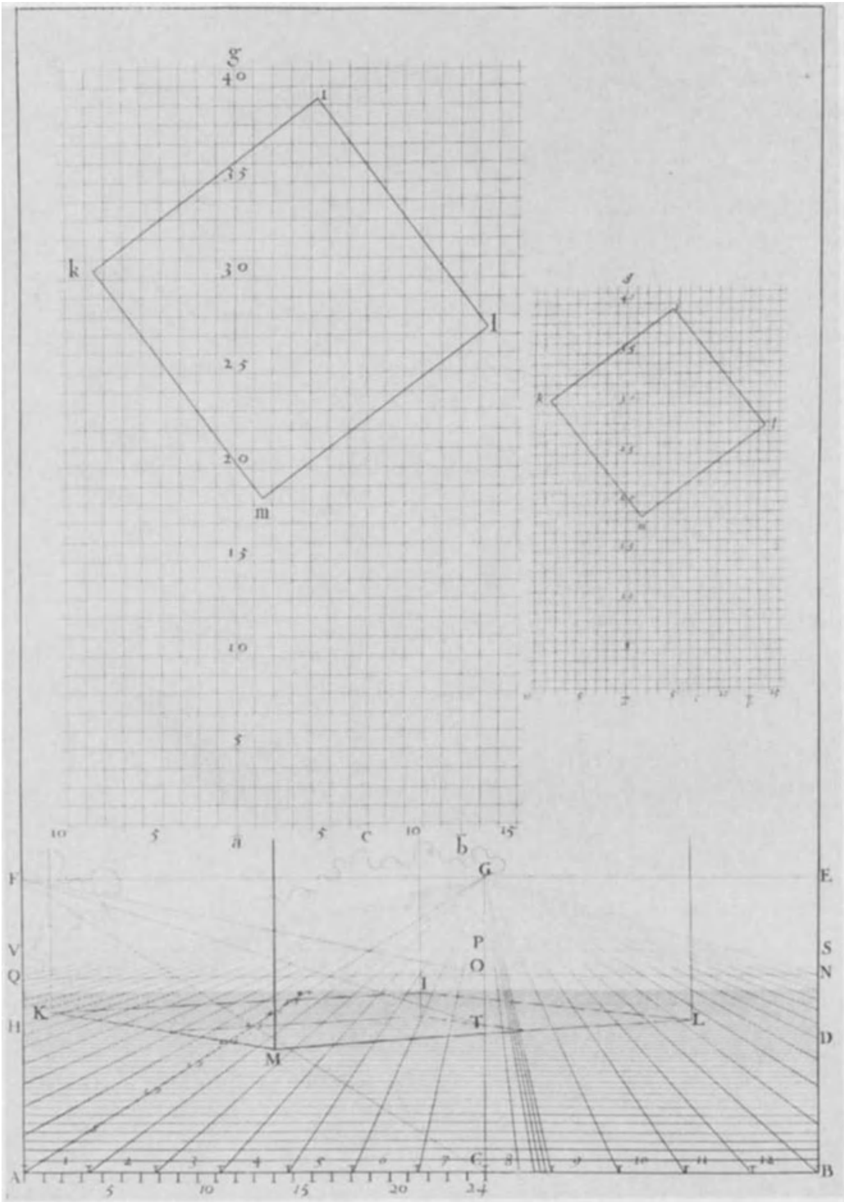


Fig. 7.2. Plate from Desargues' treatise on the cutting of stones (Paris 1640), showing ground plans and grid for the example treated in his *Perspective* (Paris 1636). Courtesy of the Metropolitan Museum, New York.

6 [1636]

purpose, as in our example the point *G*, **g** we draw lines to the points which mark the original division of the complete line *AB*, **ab** into twelve equal feet.

In this example, these lines are drawn from the point *G*, **g** only to the points marking the division in the part of the line *AB*, **ab** which lies along the area *GCBE*, **gcbe**[13], which in our example is the part *BC*, **bc**, since it is sufficient to use this smaller number of points. And in the same way we draw from the point *G*, **g** lines to the points marking the subdivision of one of these twelve feet into inches, in this example the seventh foot,[14] or half or quarter of it.

This completes the other of the two preparatory procedures which relate to the perspective construction; this procedure forms a triangular figure *GCB*, **gcb** which we shall call a DIMENSION SCALE, which may be called Geometrical or something else, and which in this method of drawing in perspective serves the draughtsman in the same way as proportional compasses.[15] [See Figs 7.3, 7.4.]

If necessary, these two scales, of distance and dimension, for the perspective drawing, can be constructed elsewhere, and arranged differently on the picture, even in any number, in different ways which all come to the same thing.

And by means of the ratio or correspondence between either one of these scales and the other, we can construct whatever we please in perspective.

For with the distance scale we find the positions on the picture of the appearances of each significant point of the plane of the basis of the subject and of the subject itself.

And with the dimension scale we find the various dimensions of each of the/
[7] lines of the subject parallel to the picture, according to the various distances of these lines in relation to the picture itself, and the angle which they subtend at the eye.

Now, taking the lines *AB*, **ab** and *ab* as being the same line, as a result of these preparatory procedures, the appearance of the line *ag* is the line *AG*, **ag** and the appearance of the line *bg*[16] is the line *BG*, **bg**.

Further, it happens that the line *AG*, **ag** is shortened at the end *G*, **g** firstly by cutting off a half, then by cutting off a third part, then by cutting off a quarter and so on by cutting off an aliquot part corresponding to the number of times we carry out the procedure which gives the distance scale.

In addition, it happens that the point marking the end of the first of these segments that are cut off from the line *AG*, **ag**, which is the point in which it meets the line *HD*, **hd**, is the appearance of a point on the line *ag*, at a distance of 24 feet behind the picture, that is as far behind the picture as the distance of the eye in front of the same picture.

And the point marking the end of the second of these segments that are cut off from the line *AG*, **ag**, which is the point in which the line *NQ*, **nq** meets it, is the appearance of another point of the line *ag*, at a distance of 48 feet behind the picture, that is twice as far behind the picture as the distance of the eye in front of the same picture.

7 [1636]

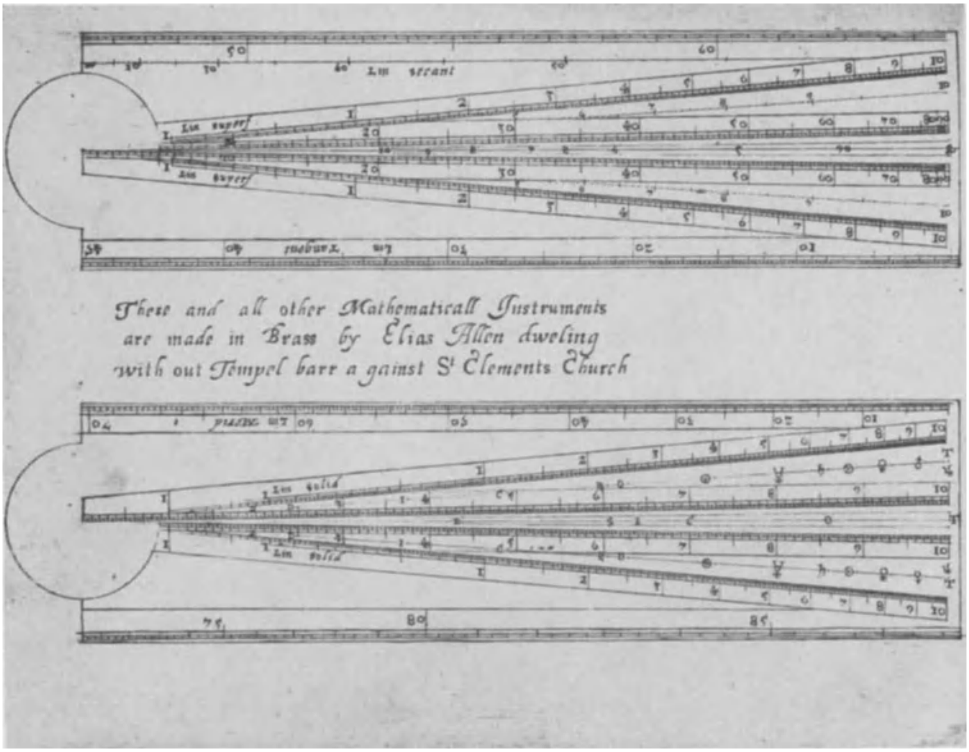


Fig. 7.3. A sector (called in French *compas de proportion*, see notes 4 and 15). The instrument consists of two flat pieces, hinged together at one end. From E. Gunter, *The description and use of the sector . . .*, second edition, London 1636, plate opposite page 1. Courtesy of the Trustees of the Science Museum, London.

And the point marking the end of the third of these segments that are cut off from the line *AG*, **ag**, which is the point in which the line *SV*, **su** meets it, is the appearance of another point of the line *ag*, at a distance of 72 feet behind the picture, that is three times as far behind the picture as the distance of the eye in front of the same picture.

And similarly for the other similar lines when we repeat the process which gives us the distance scale.

Furthermore, it happens that the same lines of the dimension scale radiating from the point *G*, **g** to the points which define the original division of the line *AB*, **ab** into 12 feet, mark out and divide five of the 12 feet on the part *BC*, **bc** of the line *AB*, **ab**; and the same lines mark out and divide each of the parts of the lines *HD*, **hd**, *NQ*, **nq**, *SV*, **su** which they[17] meet, and lines parallel to them, into five equal feet, making the same number of different scales for the

7 [1636]

various dimensions of the appearances of lines of the subject parallel to the picture and situated at various distances from it.[18]

It happens, finally, as a result of these preparatory procedures, that since the line AB, **ab** is 12 feet long, the line HD, **hd** is 24, the line NQ, **nq** 36 and the line SV, **su** 48, that is, when the length of each is measured in the feet which the dimension scale defines on the part it meets.

From these things it is clear that the line HD is the appearance of a line in the plane of the basis of the subject [the base plane of the subject] parallel to
[8] the line *ab* and lying 24 feet/ behind the picture. But the point *m* only lies 17 feet behind the picture, so this point *m* lies on a line such as *rm*, parallel to the line *ab* and lying 7 feet less far behind the picture than the line represented by the line HD.

The appearance of the point *m* is thus found in this way.

Firstly, using the distance scale we find a point on the line AG which is the appearance of a point on the line *ag* lying 17 feet away from the picture, that is, we first find the appearance of the point *r*, and to do so we draw from the point F a line to the point which marks the 17th division and separates it from the 18th of the 24 equal parts into which the line AC is divided, and the point in

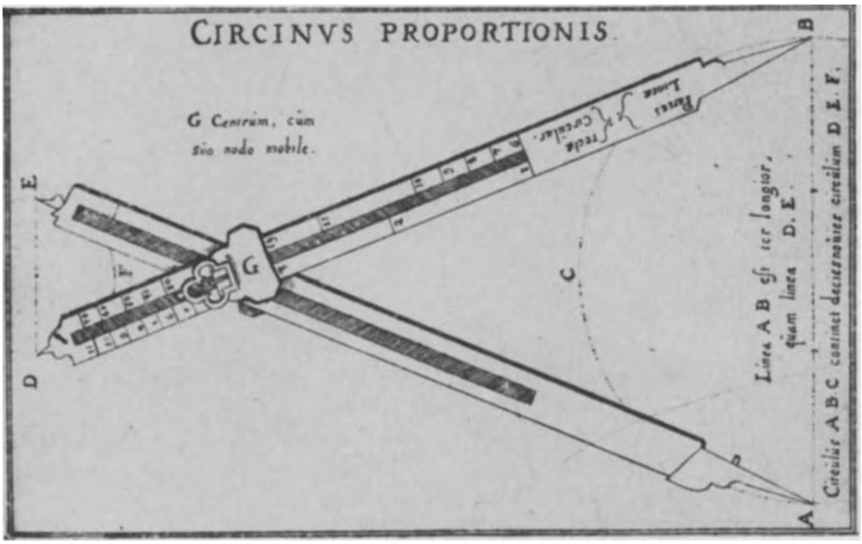


Fig. 7.4. A pair of proportional compasses (called in French *compas de proportion*, see notes 4 and 15). The instrument consists of two flat pieces, pointed at either end, hinged in a cross shape. The illustration shows 'adjustable' proportional compasses, in which the hinge can take a range of positions along the arms. From L. Hulsius, *Tractatus . . . instrumentorum mechanicorum*, Frankfurt 1605, titlepage of third treatise. Courtesy of the Trustees of the Science Museum, London.

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which the line drawn in this way meets the line *AG*, in this example the point *R*, is the appearance of a point on the line *ag* lying 17 feet away from the picture, that is, the point *R* is the appearance of the point *r*; then through the point *R* we draw the line *RM*, parallel to the line *AB*, which line *RM* is the appearance of the line *rm*, on which the point *m* lies, so the appearance of the point *m* lies on the line *RM*.

And since the point *m* lies on the line *rm*, to the right of the line *ag*, a foot and a half away from the point *r*, if the line *RM* is produced to cut the dimension scale, and with ordinary compasses we take the length one and a half feet, as measured on the line *RM* by the dimension scale, and, with the compasses opened by this amount, we place one foot of the compasses in the point *R* and direct the other foot to the right of the line *AG*, turning it to meet the line *RM*, it will meet it in the point *M*, which is the appearance of the point *m*.

The appearance of the point *k* is found as follows.

Since the line *ar* is 17 feet long, the line *rh* is 9 and the line *hn* 3; adding up these numbers, 17, 9 and 3 gives 29, so the point *k* lies on a line *nk*, parallel to the line *ab* and 29 feet behind the picture, that is five feet further away than the line represented by the line *HD*.

In this case, Firstly, using the distance scale we find on the line *AG* the appearance of a point *n* of the line *ag*, 29 feet away from the picture, that is, five feet further away than the line represented by the line *HD*; and in order to do this, we draw through the point *G* a line to the point which marks the fifth division of the line *AC* and divides it from the sixth of the 24 equal parts into which the line *AC* has been divided. Through the point in which this newly constructed line meets the line *HD* we draw another line, to the point *F*, and the point in which this last line meets the line *AG* is the appearance of the point *n*; then through this appearance of the point *n* we draw a line parallel to the line *AB*, which is the appearance of the line *nk*, on which the point *k* lies, so the appearance of the point *k* lies on this last line.

And since the point *k* lies on the line *nk*, to the left of the line *ag*, seven and a
[9] half feet away from the point *n*, if we produce the last line drawn in the/ picture parallel to the line *AB*, that is, the line which is the appearance of the line *kn*, so that it cuts the dimension scale; then if with ordinary compasses we take the length 7 and a half feet, as measured on this line by the dimension scale, and, with the compasses opened by this amount, we place one foot of the compasses in the appearance of the point *n* and direct the other foot to the left of the line *AG*, turning it to meet the line just drawn [i.e. the appearance of *nk*], it will meet it in the point *K*, which [being constructed] by this means is the appearance of the point *k*.

If one wished to find, on the line *AG*, the appearance of a point of the line *ag*, at a distance of 53 feet behind the picture, that is, 5 feet further away than the line represented by the line *NQ*. In that case, we should draw through the

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point *G* a line to the point which marks the fifth division of the line *AC* and divides it from the sixth of the 24 equal parts into which *AC* has been divided, then from the point in which this newly constructed line meets the line *NQ* we should draw a line to the point *F*, which line would meet the line *AG* in a point which is the appearance of a point on the line *ag* 5 feet further from the picture than the line represented by the line *NQ*, and similarly for other such points.

The points *L* and *I*,[19] the appearances of the points *l* and *i*, are found in the same manner.

Then we may draw through these points the lines *ML*, *MK*, *KI* and *LI*, each of which is the appearance of the corresponding side of the square *mlik*, namely *ml*, *mk*, *ki* and *li*.

Now, to find the appearance of a point 17 feet vertically above the point *m*. Through the point *M* we draw, on the same side as the line *FE*, a line *Mst*, perpendicular to the line *AB*, and this line *Mst* is made of length equal to 17 feet as measured by the dimension scale on the line *MR*, so the line *Mst* is the appearance of the vertical side of the subject, rising to a height of 17 feet vertically above the point *m*.

The lines *Lff*, *Kfr* and *Isp*, the vertical sides of the subject rising from *l*, *k*, *i* the remaining points of its square basis *mlik*, each of these sides also having a length of 17 feet, they are found in the same way as the appearance *Mst*, it being understood that the lengths of 17 feet for each of these appearances are as measured by the dimension scale on the line drawn through the lower end of the side parallel to the line *AB*.

To find the appearances of the parts of the subject that lie one foot below the points *m*, *l*, *i*, *k*, using these same lines of the vertical sides, we extend each of these appearances of the sides downwards by a foot, measured each according to the appropriate scale; and through the lower points of the feet whose appearances have been extended we draw lines which show the outside edge in the basis of the subject, although it is seen as in the figure at the bottom of the engraved plate.

Further, the line *Z* [in the figure at the upper right in the plate], 13 and a quarter feet long, being the vertical measurement of this [roof],[20] whose
[10] vertex, where the ribs of the cover meet, lies above/ the central point of the basis of the subject, at a greater height than each of its corner-pieces, the appearances of these ribs are found in the same way.

For having, by the method described above, found the point *Æ*, the appearance of the point where the ribs join at the apex of the subject, then when we draw through each of the upper points of the appearances of the vertical sides of the corner-pieces (in this example the points *st*, *ff*, *fr* and *sp*) lines to the point *Æ*, these lines *stÆ*, *ffÆ*, *frÆ* and *spÆ*, are each the appearance of the corresponding rib.

The lines *Ꝟ*, *Z*,[21] *W* and *R* are the measures of the heights of some human figures standing at various points in the plane of the basis of the subject.[22]

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The line X[23] is the measure of the height of a person standing on the bottom of the hollowed out base of the cage, which base is assumed to be level, like that of a fountain basin.

The line β is the appearance of a line 12 feet long one of whose ends rests on the plane of the basis of the subject, on the line *hl* produced, 4 feet 9 inches from the point *l*, and whose other end is leaning on the vertical edge represented by the line *Lff*.

The line ★ is the appearance of a line 5 feet long, suspended or hanging vertically from the middle of the upper edge of one of the vertical faces of the subject.

These appearances—those of each of the ornamental members of the architecture, those of the fall of shadows, and, generally, the appearances of everything of a kind that may be represented in art—are, by means of the knowledge of the appropriate intervals [i.e. distances from the picture, from the base plane, etc.], in this way found for any flat picture, in whatever manner or at whatever angle it be disposed, hanging vertically as a backdrop or at an angle to the line of sight to one side or another,[24] whether the point usually known as the vanishing point lies in the picture or lies outside it;[25] but in each of these different arrangements, there is room for a number of different examples and several figures [i.e. several forms of the figure]: further, knowing this method of making flat pictures will make it easy to see how to make pictures on any other kind[26] of surface, and threads attached to the points *F* and *G* will help to avoid drawing incorrect lines.[27]

There are rules, also, for where one should use strong and where delicate colours. The proof of these rules depends partly on Geometry and partly on Physics, and is not yet explained in any book published in France.[28]

For the various circumstances encountered in perspective drawing, there are individual methods of achieving success easily in each case, in the manner of this example and otherwise, or by means of instruments whose design derives from Geometrical theorems, of which instruments there are various kinds.

Some instruments, for making a careful copy of any flat subject in a smaller, equal or larger size, and also rendering it in perspective, together with its elevations, in whatever manner, at whatever angle and at whatever distance one may choose, all this as quickly as one might have made a [simple] copy.

Others for drawing the subject exactly, while looking at it, through a figure that is smaller, of equal size or larger, and placed in the same way as the figure
[11] that would appear/ on the plane to which the instrument is applied. Such instruments, or one of them, formed the subject of a treatise published in Rome about two years after [*sic*] the publisher's privilege for the present work was granted in France. This Roman treatise does not include the method of obtaining the figure in perspective in equal size and disposed in the same way as that on the plane to which the instrument is applied.[29] [See Figs 7.5, 7.6.]

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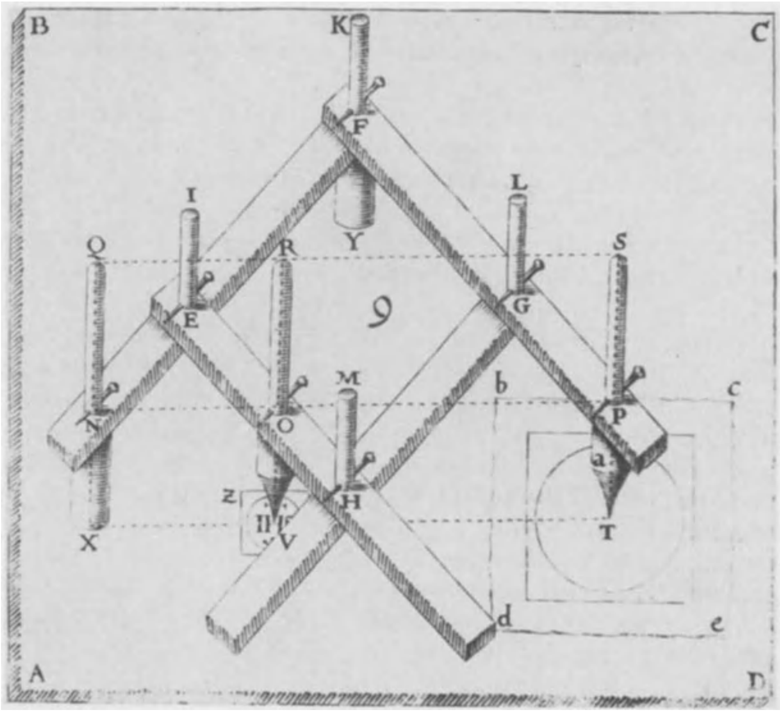

Fig. 7.5. A pantograph being used to make a scale copy of a drawing. From C. Scheiner, S. J., *Pantographice*, Rome 1631, Part I, page 29. Courtesy of the Trustees of the Science Museum, London.

There are, similarly, general mathematically demonstrated methods for drawing the lines necessary for cutting stones for architectural purposes, with checks to enable one to tell if one has carried out the procedure accurately.[30]

There are, further, general and also mathematically demonstrated methods for drawing sundials with an ordinary ruler, ordinary compasses, an ordinary plumb line and an ordinary set square, for all flat surfaces in general, in which the axis (*essieu*) of the world is set up appropriately[31] in whatever direction and at whatever angle the surface is set up.[32]

In the remainder of this work scholars will find some propositions which might be enunciated differently in other connections but are stated here in a manner relating to perspective; and the proofs of these propositions can be understood well enough without a figure because all the lines are again

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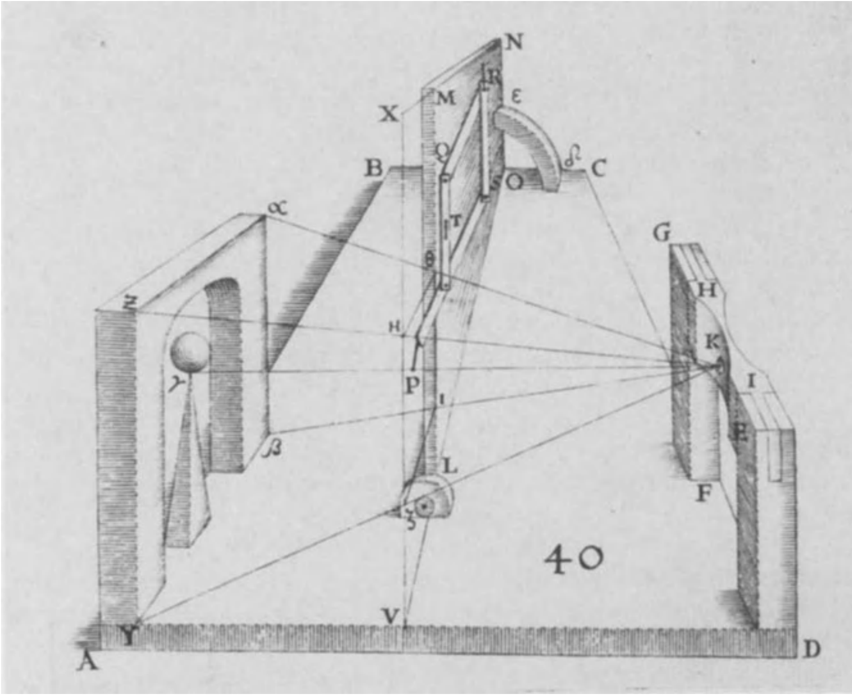

Fig. 7.6. A pantograph being used as an aid to drawing in perspective. The eye is at K. The point P of the pantograph is taken round the perceived outline of the object, causing the point T, which carries a pen, to trace a similar outline on the drawing surface $LMNO$. From C. Scheiner, *Pantographice*, Rome 1631, Part II, page 99. Courtesy of the Trustees of the Science Museum, London.

understood to be straight, and all the pictures flat. In fact we have a host of important propositions occurring in this area.

Let us imagine that through the fixed centre of the eye there passes a line indeterminate in position and movable along its whole length in every direction; we shall call this line the LINE OF SIGHT, which can, as necessary, be drawn parallel to any other line.

When the subject is a point, and when from the points of the subject and of the eye we draw lines parallel with one another, producing them to meet the picture, the appearance of the subject lies on the line through the points in which these parallel lines meet the picture, since these parallel lines and the line drawn in this way on the picture all lie in the same plane.

When the subject consists of lines, they are either parallel or inclined at an angle to one another.

When lines of the subject are parallel to one another, the line of sight drawn

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parallel to them[33] is either parallel or not parallel to the picture, but each of the lines of the subject always lies in a plane with this line of sight, in which all the planes intersect one another, as in their common axle.

When lines of the subject are parallel to one another, and the line of sight drawn parallel to them is parallel to the picture, the appearances of these lines of the subject are lines parallel to one another, to the lines of the subject and to the line of sight, because each of these lines of the subject lies in the same plane with the line of sight, in which all the planes intersect one another, as in their common axle, and all these planes are cut by one other plane, that of the picture./

[12] When the lines of the subject are parallel to one another, and the line of sight drawn parallel to them is not parallel to the picture, the appearances of these lines of the subject are lines which all converge to the point where the line of sight meets the picture, since each of these lines of the subject lies in a plane with the line of sight, in which all the planes intersect one another, as in their common axle, and all these planes are cut by one other plane, that of the picture.

When lines of the subject are inclined to one another and all converge at a point such that the line of sight drawn through this point is parallel to the picture, the appearances of these lines of the subject are lines parallel to one another and to the line of sight, because each of these lines of the subject lies in a plane with the line of sight, in which all these planes intersect one another, as in their common axle, and all these planes are cut by one other plane, that of the picture.

When lines of the subject are inclined to one another and all converge at a point such that the line of sight drawn through this point is not parallel to the picture, the appearances of these lines of the subject are lines which all converge at the point where this line of sight meets the picture, since each of these lines of the subject lies in a plane with this line of sight, in which all these planes intersect one another, as in their common axle, and all these planes are cut by one other plane, that of the picture.

The proposition which follows cannot be explained so briefly as the preceding ones.

Given to portray a flat section of a cone, draw two lines [in it] whose appearances will become the axles of the figure which will represent it.

Paris, May 1636. With Privilege[34]

Chapter VIII

The Three Geometrical Propositions of 1648

These three propositions come from Abraham Bosse, *La Perspective de Mr Desargues* (1648, pp. 340–343). Bosse's book is essentially a detailed reworking of Desargues' brief *Perspective* of 1636 and its style throughout is discursive and diffuse. These three propositions, which occur at the very end of the book, form a contrast with the rest both in their content and in their style. There is general agreement that they are due to Desargues rather than to Bosse. They are not known to have been printed before 1648, but it is conceivable that they may have formed part of the lost *Leçons de ténèbres* (apparently written some time after the *Rough Draft on Conics*).

The first of the three propositions is Desargues' famous theorem on two triangles in perspective. Desargues gave two proofs of this theorem and its converse: the elegant one for the case in which the figure lies in three dimensions, and a proof based on Menelaus' theorem for the deeper two-dimensional case, which requires a more elaborate proof. When some lines in the figure are parallel, Desargues gives an interesting argument to show how the two- and three-dimensional figures are related.

Desargues wrote $AB - AC$ for what would today be written as AB/AC, and

$$AB - AC\begin{cases} DE - DF \\ \overline{bH - bK} \end{cases}$$

for what would today be written as

$$\frac{AB}{AC} = \frac{DE}{DF}\frac{bH}{bK}\,.$$

[First] Geometrical Proposition

When straight lines *HDa*, *HEb*, *cED*, *lga*, *lfb*, *Hlk*, *DgK*, *EfK*, [*cab*,][1] which either lie in different planes or in the same one, cut one another in any

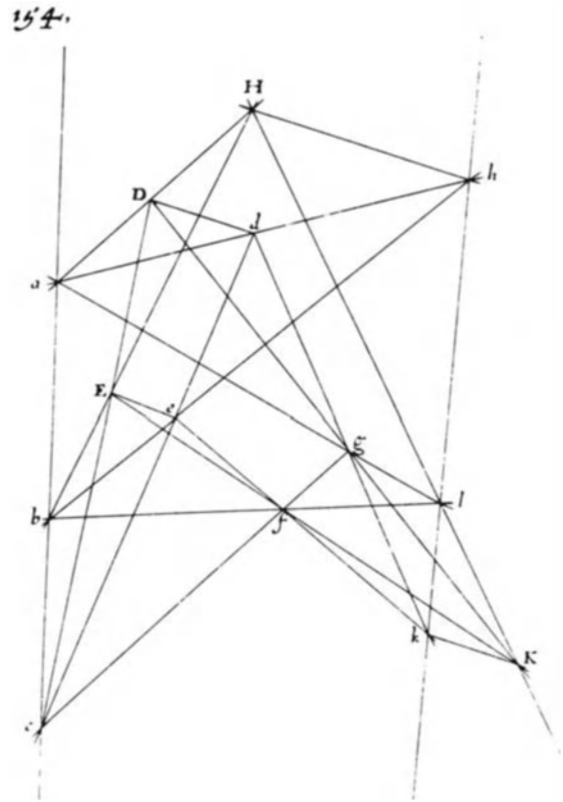


Fig. 8.1(a). Original figure for the first geometrical proposition, from Bosse, *La Perspective de Mr Desargues*, Paris 1648, plate 154 (opposite page 340). Courtesy of the British Library Board.

order and at any angle in such points [as those implied in the lettering]; the points c, f, g lie on a straight line cfg. For, whatever form the figure takes, in every case; if the straight lines lie in different planes, the lines abc, lga, lfb lie in a plane; the lines DEc, DgK, KfE lie in another; and the points c, f, g lie in each of these two planes; consequently they lie on a straight line cfg. And if the same straight lines all lie in the same plane,

$$\frac{gD}{gK} = \frac{aD}{aH}\frac{lH}{lK},$$

and

$$\frac{fK}{fE} = \frac{lK}{lH}\frac{bH}{bE},$$

and

$$\frac{aD}{aH} = \frac{cD}{cE}\frac{bE}{bH},^2$$

therefore

$$\frac{cD}{cE} = \frac{gD}{gK}\frac{fK}{fE}.$$

Consequently c, g, f lie on a straight line.

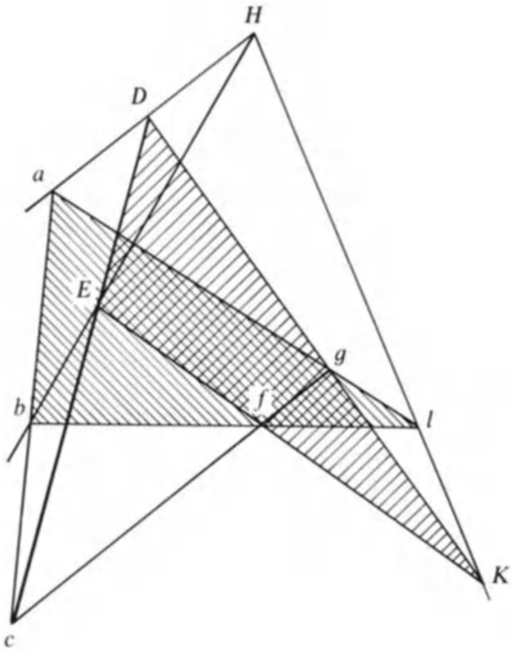
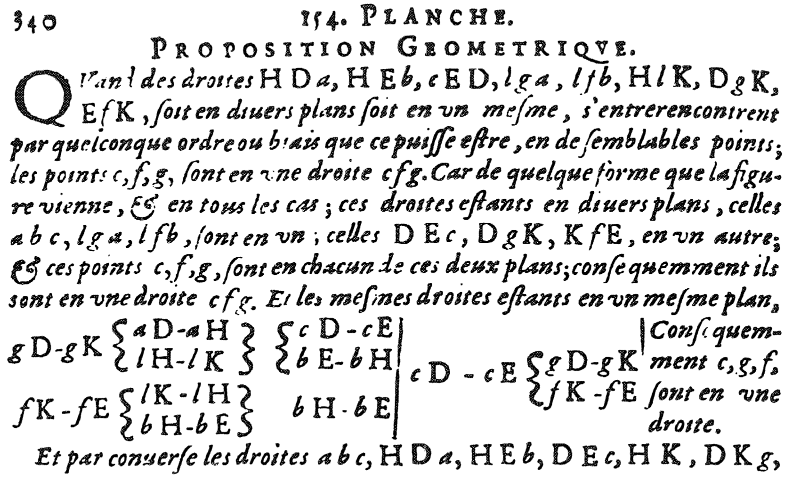


Fig. 8.1(b). Figure 8.1(a) redrawn to eliminate superfluous lines and accentuate the two triangles, *DEK* and *abl*, in perspective from *H*. The picture works best if you can see the line *HEb* coming out of the plane *HaK* and towards you.

And, conversely, if the straight lines *abc*, *HDa*, *HEb*, *DEc*, *HK*, *DKg*, *KEf* meet one another in any manner and at any angles, in points such as those [that are given], the lines lying either in different planes or in the same one; the lines *agl*, *bfl* will always meet at a butt *l* which lies on the line *HK*.[3] For if the straight lines lie in different planes, one of these planes is *HKgDag*; another is *HKfEbf*; and another *cbagf*: and the straight lines *HlK*, *bfl*, *agl* are the lines of intersection of these three planes; therefore they all meet at the butt *l*. And if the same straight lines all lie in one plane; if we draw through the point *a* the line *agl* to meet the line *HK*, and then draw the line *lb*, it has just been proved that this line meets the line *EK* in a point such as *f* which is collinear with the points *c* and *g*, which is to say that the line [*lb*] passes through *f*, and consequently that the two lines *ag*, *bf* meet at a butt *l*, on the line *HK*. And if, again, the same lines lie in different planes, if through points on them, *H*, *D*, *E*, *K* there pass other straight lines *Hh*, *Dd*, *Ee*, *Kk* which all meet at some butt at an indeterminate distance, or, to put it another way, are parallel to one another; and these lines meet one of the planes, *cbagfl*, in points such as *h*, *d*, *e*, *k*; the points *h*, *l*, *k* lie on a straight line; the points *h*, *d*, *a* lie on one; the points *h*, *e*, *b* lie on one; the points *k*, *g*, *d* lie on one; the points *k*, *f*, *e* lie on one; and the points *c*, *e*, *d* lie on one. For by this construction the straight lines *Hh*, *Kk*, *HlK* all lie in a plane; the lines *abc*, *bfl*, *klh* lie in another; and the points *h*, *l*, *k* lie in


340 154. PLANCHE.

PROPOSITION GEOMETRIQVE.

QVand des droites HD*a*, HE*b*, *c*ED, *lga*, *lfb*, H*l*K, D*g*K, E*f*K, soit en diuers plans soit en vn mesme, s'entrerencontrent par quelconque ordre ou biais que ce puisse estre, en desemblables points; les points *c*, *f*, *g*, sont en vne droite *cfg*. Car de quelque forme que la figure vienne, & en tous les cas; ces droites estants en diuers plans, celles *abc*, *lga*, *lfb*, sont en vn; celles DE*c*, D*g*K, K*f*E, en vn autre; & ces points *c*, *f*, *g*, sont en chacun de ces deux plans; consequemment ils sont en vne droite *cfg*. Et les mesmes droites estants en vn mesme plan,

*g*D-*g*K { *a*D-*a*H / *l*H-*l*K } { *c*D-*c*E / *b*E-*b*H | *c*D - *c*E { *g*D-*g*K / *f*K-*f*E | Consequemment *c*, *g*, *f*, sont en vne droite.

*f*K-*f*E { *l*K-*l*H / *b*H-*b*E } *b*H-*b*E

Et par conuerse les droites *abc*, HD*a*, HE*b*, DE*c*, HK, DK*g*,

Fig. 8.2. The first few lines of the first geometrical proposition, showing the incorrect statement of the theorem (see note 1), from Bosse, *La Perspective de Mr Desargues*, Paris 1648, p. 340. Courtesy of the British Library Board.

each of the two planes. Consequently they lie on a straight line; and similarly for every other set of three points [in the proposition]. And all these straight lines lie in a plane, *cgabfl*, and each of them is divided by the parallel lines through the points H, D, E, K in the same way as the corresponding line in the three-dimensional figure.[4] So the figure which these parallel lines have defined in the plane *hdabcedgfkl* corresponds straight line for straight line; point for point; and ratio for ratio; to the three-dimensional figure *abcEHlkgf*. And one can discuss their properties in the same way in the one figure as in the other, and so do without the solid figure,[5] by using instead the figure in the plane.

[Second] Geometrical Proposition

When through each of the four coplanar points O, A, D, B, as marker posts or links,[6] there pass three of the six straight lines $DO2, DA4, BO3, BA7, OA59, BD89$; and one of the four lines OoK, AaK, DdK, BbK which all meet at any other butt K; the six lines define six points 2, 3, 4, 7, 5, 8 on any other line 857432 which they meet. So,[7] if through three of these points, such as 2, 3, 5 there pass three straight lines such as $2o$, $3o$, $5o$, which meet at a butt o on the straight line OoK, which belongs to the set of four lines to the butt K which these are related to; these lines [$2o, 3o, 5o$] intersect the other three lines through the butt K in three points a, b, d, each being a point of the line the particular one of the three is related to; that is, the line $2o$ passes through d on DK; $3o$ passes through b on BK; $5o$ passes through a on AK; which points a, b, d, together with the three remaining points 4, 7, 8, on the line 234758, lie by three on three lines: that is, the points 7, a, b lie on one line; 4, a, d on another; and 8, d, b on another. For when the butt K does not lie in the plane of the

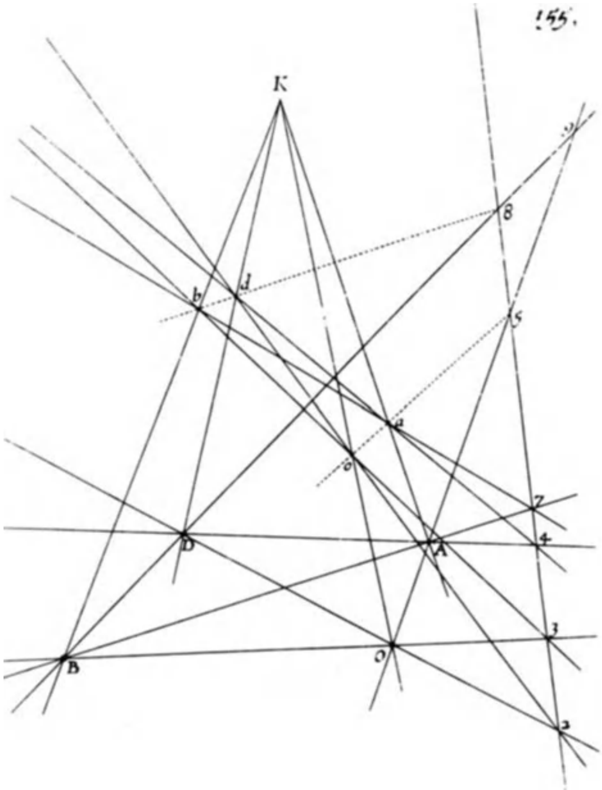


Fig. 8.3(a). Original figure for the second geometrical proposition, from Bosse, *La Perspective de Mr Desargues*, Paris 1648, plate 155 (opposite page 341). Courtesy of the British Library Board.

links O, A, D, B, the figure involves several planes, one of which is $BD872OA$; $bd872oa$ is another; $KoOA5a$ another; $KbBD8d$ another; considering these planes, the thing is obvious, since the intersections of any three planes all meet at a single butt, at a determinate or indeterminate distance. And when the butt K lies in the plane of the links O, A, D, B, it has just been shown, from the position of the lines given by the construction, and considering the figure as being made up of several versions of the previous one, [all] on the straight line 857432, and each separately in relation to each of the four links O, A, D, B; that if from a point d on the line DK we draw lines $do2$, $da4$, the points 5, a, o, lie on a straight line: And when through a we draw the line $7ab$ to meet the straight line BbK, while the line $b3$ passes through the point o: And having drawn through a, a point on the straight line AK, the straight lines $a4d$, $a7b$, the points 8, d, b lie on a straight line: And by this reasoning we see all this complete figure lies in a single plane, and it corresponds in the manner we have described with the three-dimensional figure; And by this means one can, in such a case, again manage without a representation of the solid figure for the

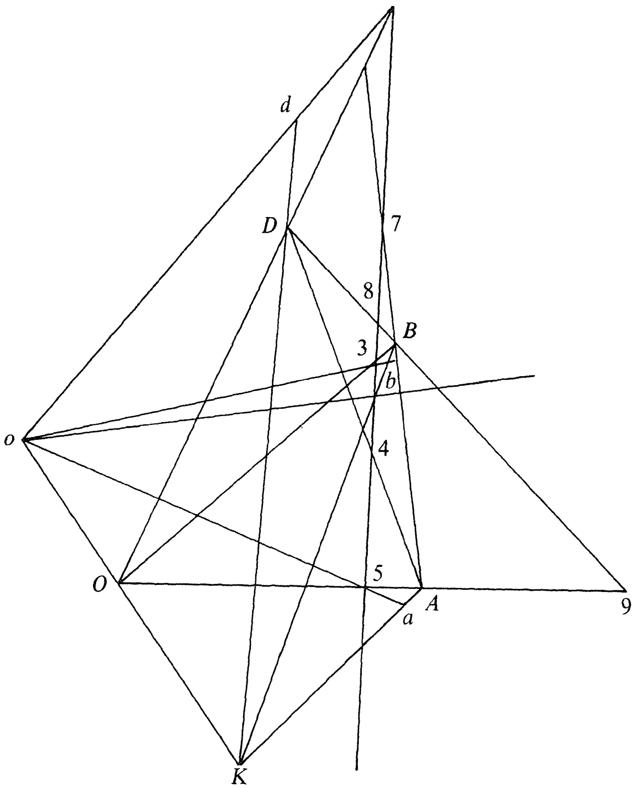


Fig. 8.3(b). Figure 8.3(a) redrawn. The points 7*ab* on a line because triangles *BAK*, 35o are in perspective from *O*. The points 4*ad* lie on a line because triangles *ADK*, 250*o* are in perspective from *O*. The points 8*db* lie on a line because triangles *BDK*, 32*o* are in perspective from *O*.

purpose of investigating its properties, and substitute the corresponding plane figure.

The figure for the last but one proposition clearly changes its appearance according to the different positions of the points *a*, *b*, *c*; and *H*, *l*, *K*: And the figure for the last proposition changes in the same way, depending on the type of relation of the straight line 234758 to the marker posts or links *A*, *O*, *D*, *B*; But in order to save space, I have not been able to show the various forms of the figures, nor to discuss them at length, any more than I shall do for the one that follows.

[Third] Geometrical Proposition

Let us take the straight lines *OD*2, *OB*3, *AD*4, *AB*7, 2347, *AaK*, *OoK*, *BbK*, *DdK*, 7*ab*, 4*ad*, 3*ob*, 2*od* to be arranged as in the previous proposition; if on each of the four lines $AB7$, $AD4$, $OB3$, $OD2$ there are, in addition to the two links and the point on the straight line 2347 that the line passes through, the same number of general pairs of other points, such as, on the line $OB3$, the two pairs HP and FE; on the line $OD2$, the two pairs GR, LY; on the line $AB7$, the two pairs NM, IQ; on the line $AD4$, the two pairs ST, VX; And through these points there pass straight lines which again, like the four previous lines, meet at the butt K; these last lines make on each of the lines $2od$, $3ob$, $4ad$, $7ab$ just the same number of pairs of further points, in addition to the points o, a, d, b, 2, 3, 4, 7; the points of the pairs on one line having letters corresponding to those of pairs on the other line to which it is related, such as those points on the straight line $2od$, corresponding to the points on the line $2OD$ to which the

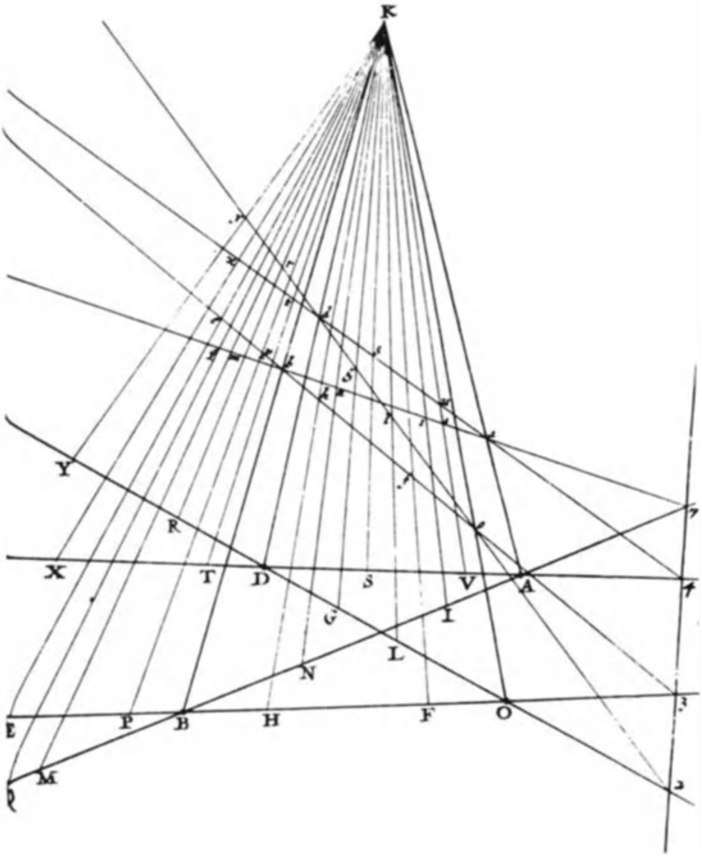


Fig. 8.4. Original figure for the third geometrical proposition, from Bosse, *La Perspective de Mr Desargues*, Paris 1648, plate 156 (opposite page 342). Courtesy of the British Library Board.

first line is related:[8] And similarly for the others: And the sum of the ratios between parts of the two straight lines such as *OB*3, *OD*2, and such as *HO* to *HB*, and *PO* to *PB*, and *FO* to *FB*, and *EO* to *EB*, and *GD* to *GO*, and *RD* to *RO*, and *LD* to *LO*, and *YD* to *YO*; differs from the sum of the ratios of two other straight lines such as *AB*7, *AD*4 and such as *NA* to *NB*, and *MA* to *MB*, and *IA* to *IB*, and *QA* to *QB*, and *SD* to *SA*, and *TD* to *TA*, and *VD* to *VA*, and *XD* to *XA*; by the same ratio by which the sum of the ratios between parts of straight lines such as *ob*3, *od*2 and such as *ho* to *hb*, and *po* to *pb*, and *fo* to *fb*, and *eo* to *eb*, and *gd* to *go*, and *rd* to *ro*, and *ld* to *lo*, and *yd* to *yo*; differs from the sum of the ratios between parts of two other straight lines such as *ab*7, *ad*4; and such as *na* to *nb*, and *ma* to *mb*, and *ia* to *ib*, and *qa* to *qb*, and *sd* to *sa*, and *td* to *ta*, and *ud* to *ua*, and *xd* to *xa*.

This is proved on the following page, in symbols, without using words, for greater ease both in seeing and grasping the argument; however, to make it even more clear what I mean I shall explain what the symbols used there signify. Thus

$$HO\text{-}HB \begin{cases} HO\text{-}H3 \begin{cases} KO\text{-}Ko \\ ho\text{-}h3 \end{cases} \\ H3\text{-}HB \begin{cases} h3\text{-}hb \\ Kb\text{-}KB \end{cases} \end{cases}$$

means that the ratio of *HO* to *HB* is equal to the ratio compounded from the ratio of *HO* to *H*3 and the ratio of *H*3 to *HB*; and that the ratio of *HO* to *H*3 is equal to the ratio compounded from the ratio of *KO* to *Ko* and the ratio of *ho* to *h*3; and that the ratio of *H*3 to *HB* is equal to the ratio compounded from the ratio of *h*3 to *hb* and the ratio of *Kb* to *KB*; and so on similarly: and symbols like *ho-hb* mean that having taken out the whole column of 32 previous ratios the ones which come to the same thing because they are equal and those which contain terms equal to others in the same column;[9] there remain of the four ratios from there to the right only that of *ho* to *hb*.

$$\begin{array}{l|l}
HO\text{-}HB \left\{ \begin{array}{l} HO\text{-}H3 \begin{cases} KO\text{-}Ko \\ ho\text{-}h3 \end{cases} \\ H3\text{-}HB \begin{cases} h3\text{-}hb \\ Kb\text{-}KB \end{cases} \end{array} \right\} ho\text{-}hb & NA\text{-}NB \text{ -}\left\{ \begin{array}{l} NA\text{-}N7 \begin{cases} KA\text{-}Ka \\ na\text{-}n7 \end{cases} \\ N7\text{-}NB \begin{cases} n7\text{-}nb \\ Kb\text{-}KB \end{cases} \end{array} \right\} na\text{-}nb \\
PO\text{-}PB \left\{ \begin{array}{l} PO\text{-}P3 \begin{cases} KO\text{-}Ko \\ po\text{-}p3 \end{cases} \\ P3\text{-}PB \begin{cases} p3\text{-}pb \\ Kb\text{-}KB \end{cases} \end{array} \right\} po\text{-}pb & MA\text{-}MB \left\{ \begin{array}{l} MA\text{-}M7 \begin{cases} KA\text{-}Ka \\ ma\text{-}m7 \end{cases} \\ M7\text{-}MB \begin{cases} m7\text{-}mb \\ Kb\text{-}KB \end{cases} \end{array} \right\} ma\text{-}mb
\end{array}$$

$$FO\text{-}FB\begin{cases}FO\text{-}F3\begin{cases}KO\text{-}Ko\\ fo\text{-}f3\end{cases}\\ F3\text{-}FB\begin{cases}f3\text{-}fb\\ Kb\text{-}KB\end{cases}\end{cases} fo\text{-}fb \qquad IA\text{-}IB\begin{cases}IA\text{-}I7\begin{cases}KA\text{-}Ka\\ ia\text{-}i7\end{cases}\\ I7\text{-}IB\begin{cases}i7\text{-}ib\\ Kb\text{-}KB\end{cases}\end{cases} ia\text{-}ib$$

$$EO\text{-}EB\begin{cases}EO\text{-}E3\begin{cases}KO\text{-}Ko\\ eo\text{-}e3\end{cases}\\ E3\text{-}EB\begin{cases}e3\text{-}eb\\ Kb\text{-}KB\end{cases}\end{cases} eo\text{-}eb \qquad QA\text{-}QB\begin{cases}QA\text{-}Q7\begin{cases}KA\text{-}Ka\\ qa\text{-}q7\end{cases}\\ Q7\text{-}QB\begin{cases}q7\text{-}qb\\ Kb\text{-}KB\end{cases}\end{cases} qa\text{-}qb$$

$$GD\text{-}GO\begin{cases}GD\text{-}G2\begin{cases}KD\text{-}Kd\\ gd\text{-}g2\end{cases}\\ G2\text{-}GO\begin{cases}g2\text{-}go\\ Ko\text{-}KO\end{cases}\end{cases} gd\text{-}go \qquad SD\text{-}SA\begin{cases}SD\text{-}S4\begin{cases}KD\text{-}Kd\\ sd\text{-}s4\end{cases}\\ S4\text{-}SA\begin{cases}s4\text{-}sa\\ Ka\text{-}KA\end{cases}\end{cases} sd\text{-}sa$$

$$RD\text{-}RO\begin{cases}RD\text{-}R2\begin{cases}KD\text{-}Kd\\ rd\text{-}r2\end{cases}\\ R2\text{-}RO\begin{cases}r2\text{-}ro\\ Ko\text{-}KO\end{cases}\end{cases} rd\text{-}ro \qquad TD\text{-}TA\begin{cases}TD\text{-}T4\begin{cases}KD\text{-}Kd\\ td\text{-}t4\end{cases}\\ T4\text{-}TA\begin{cases}t4\text{-}ta\\ Ka\text{-}KA\end{cases}\end{cases} td\text{-}ta$$

$$LD\text{-}LO\begin{cases}LD\text{-}L2\begin{cases}KD\text{-}Kd\\ ld\text{-}l2\end{cases}\\ L2\text{-}LO\begin{cases}l2\text{-}lo\\ Ko\text{-}KO\end{cases}\end{cases} ld\text{-}lo \qquad VD\text{-}VA\begin{cases}VD\text{-}V4\begin{cases}KD\text{-}Kd\\ ud\text{-}u4\end{cases}\\ V4\text{-}VA\begin{cases}u4\text{-}ua\\ Ka\text{-}KA\end{cases}\end{cases} ud\text{-}ua$$

$$YD\text{-}YO\begin{cases}YD\text{-}Y2\begin{cases}KD\text{-}Kd\\ yd\text{-}y2\end{cases}\\ Y2\text{-}YO\begin{cases}y2\text{-}yo\\ Ko\text{-}KO\end{cases}\end{cases} yd\text{-}yo \qquad XD\text{-}XA\begin{cases}XD\text{-}X4\begin{cases}KD\text{-}Kd\\ xd\text{-}x4\end{cases}\\ X4\text{-}XA\begin{cases}x4\text{-}xa\\ Ka\text{-}KA\end{cases}\end{cases} xd\text{-}xa$$

So the 8 ratios in the first column are equal to the 32 in the third; and the 8 in the fifth are equal to the 32 in the seventh; and cancelling from the 32, on one side and the other, two times four identical ratios; and 16 others 16 times making the ratio unity; there remain the 8 ratios of the fourth column, and 8 in the eighth column. Consequently, the difference between the sums of the ratios in the first column and those in the fifth column is the same as that between the sums of those in the fourth column and those in the eighth, as the proposition requires. And since the first [sums] are equal to one another; the others are also: as a whole and in every numerical respect. From which, if the said pairs of points on one side lie in a plane on a general line; and those on the other side lie in another plane and on other lines; these lines correspond to one another; and their circumstances make their relationship clear.

PRAISE BE TO GOD

Chapter IX

The Sundial Treatise (1640)

In the early seventeenth century, the correct setting out of shadows was considered an important element in perspective drawing, as may be seen, for example, in Stevin's treatment of the matter in his work on perspective (1605), (see also Sinisgalli, 1978). Indeed, shadows seem to have been recognized as presenting some kind of analogy to perspective representation in general, for Willebrord Snel's Latin translation of Stevin's work (1605) has the title *De Sciagraphia* (literally: On shadow-drawing). Since sundials are designed to take account of the way the shadow of the gnomon moves in the course of the day, it is not surprising to find that a similar title, *Sciographia or the Art of Shadowes*, was also used by John Wells (1606–1635) for his treatise on the making of sundials (1635). In fact, laying out sundials seems to have been generally accepted as a part of the study of perspective. Jonas Moore (1617–1679) makes the connection between the two in explicit terms in his 'Epistle to the reader' at the beginning of *Mr De Sargues' Universal Way of Dyalling* ... (1659) (which is a translation by Daniel King of Bosse's expanded version of Desargues' work, published in Paris in 1643):

> Dyalling I accompt one kind of Perspective, for that glorious Body the *Sun*, the Eye of the world, traceth out the lines and hour-points by his Diurnal Course, and upon the resubjected *Plane* by the laws of Picture, *Scenographically* delineates the Dyal.

A little later in his Epistle, Moore gives a summary of Desargues' procedure for constructing dials, indicating its connection with conic sections:

> ... this point *B* [the tip of the gnomon, whose shadow marks the time], you must imagine to be the Center of the Earth (for the vast distance of the Sun, maketh the space betwixt the Center and superficies of the

Earth to be insensible) and from it at all times of the year (excepting the Æquinoctial day) the Sun in its course forms two Cones, whose apex is the point *B*, that next the Sun termed *Conus luminosus* or the light Cone, the other whereof our Author makes use, termed *Conus umbrosus* the dark Cone, now this dark Cone, if by any three points equally distant from the Apex *B*, the Cone be cut, the Section will be a Circle parallel to the Equinoctial [i.e. the celestial equator]: And thereby, as the Author shews many wayes [in the Bosse version], the position of the Axis or Gnomon may be found out, and the Dyal easily made.

Given the context, it is not surprising that Moore should remark upon the originality and practical utility of Desargues' work, but it is perhaps of interest to note that he regards the French as the leading students of perspective, and concludes his Epistle with the hope that King, 'who out of his desire to serve his Country, hath caused this piece to speak English', may go on to translate more French works on perspective:

Then prosper King, *untill thy worthy hand,*
The Gallick *learning make us understand.*

Although Desargues' work seems severely practical, it will be noted that he is concerned only with finding the positions of gnomon and lines, not with the actual business of setting up a dial. The dial he describes is actually drawn on the ground.

Desargues does not explain any of his geometrical procedures, but it is clear that the work is connected with his study of conics. Moreover, in its first line, Desargues tells us that the work on sundials was originally intended to form part of something larger, and it is very tempting to suppose that this larger work may have been the lost *Leçons de ténèbres*. The *Leçons de ténèbres* (literally: Lessons on shadows), probably published in 1640 (see above and Taton, 1951), also included a revised version of the work we find in the *Rough Draft on Conics*.

The text we have used for our translation of Desargues' work on sundials is that of 1640, recently rediscovered by A. J. Turner, who kindly supplied us with a copy of it. This French text has been reprinted in *Archives internationales d'histoire des sciences*, vol. 34 (Turner, 1984).

The original title of Desargues' sundial treatise is *Brouillon proiet du S.G.D.L. touchant une maniere universelle de poser le Style & tracer les lignes d'un Quadran aux rayons du Soleil, en quelquonque endret possible, avec la Reigle, le Compas, l'equiere & le plomb.*

In 1640. With Privilege [i.e. limited copyright]

SKETCH PLAN OF S.G.D.L. CONCERNING A UNIVERSAL METHOD OF setting up the Stylus[1] *& drawing the lines of a Sundial, in whatever situation possible, using Ruler, Compasses, square and plumb-line.*

It is a common error to suppose that people are not mistaken in thinking that in the Crafts practical work or mechanical execution verify the proofs given by their theory, for on the contrary it is the proofs given by their theory that are the mainstay of what may be accomplished by the most excellent practical work or mechanical execution, which is always so crude that it rarely achieves the precision of the precepts and proofs of theory, which is why it [the practical work] *is usually accompanied by proofs in each case.*

This method of setting up the stylus of a Sundial was intended to form part of a larger proposition;[2] but it is here detached from it for a very good reason: And despite its here being given in a form appropriate for the use of mechanical techniques. [*sic*] Competent scholars[3] will take good note that we assume all that is usually assumed in this matter, both how it can be stated purely theoretically and that it is so manifestly obvious from a plain account, without any kind of figure, that the mere thought of attaching a figure to this text would cast doubt on their competence [*sc.* that of the 'competent scholars' from the beginning of this sentence], whereas they are all entreated to direct their attention to the geometry of this sketch. And, by way of answer to those who raise petty quibbles about nomenclature, we shall here use the word Stylus to indicate only the rod, or part of the rod, whose entire shadow continually indicates the time of day, any other rod or part of a rod which is joined to it is, here, only an auxiliary or prop for the Stylus.[4]

There have already been published several methods of setting up the stylus and drawing the lines of a Sundial, by taking account of the altitude of the pole,[5] and using the magnetic compass, a horizontal dial, or some other special instrument: But apart from the fact that not everyone can easily learn to use these instruments, those who have to draw Sundials do not always have the means to come by them, as they do by compasses, square and plumb-line, which anyone may learn to use.

Further, one requires several rigid rods, of suitable length, and each having a sharp ridge[6] running in a straight line along its length, and, in a place suitable in relation to the [surface which will become the] dial, on its Western side, there must be dug out a sufficiently deep hole, facing to the East.[7]

Then on a bright sunny day, some time after dawn when the light has become clear and sharp, one end of one of the rods must be inserted into the hole, turning the sharp ridge of the rod to the Sun and pointing the other end of the rod at the Sun, so that the shadow of its ridge does not fall outside the rod anywhere along its length; and then hold the rod firm or fix it, in that position, to the body of the [surface that will become the] Dial.[8]

Then, in the course of the same day, a considerable time later, set the centre of a star of three adequately long lines on the surface of the Dial exactly at the point where you see the tip of the shadow cast by the ridge of the rod that has been fixed. And, still on the same day, again a considerable time later, again set the centre of another similar star on the surface of the Dial, again exactly at the point where you see the tip of the shadow cast by the ridge of the rod that has been fixed. And the further apart are the centres of these stars and the fixed end of the rod, the more accurate the [ensuing] procedure will be.[9]

Then close to whichever of the stars is further from the fixed end of the first rod, and in the half of the star which is further from the fixed end of the first rod, you must make in the Dial another sufficiently deep hole, one side of whose opening lies on one of the lines of the star that has been divided in half, and in this second hole you must insert one of the ends of another rod, turning its ridge towards the ridge of the first rod, and fitting this ridge on the one hand to the centre of the star that has been divided in half, and on the other hand to the protruding end of the ridge of the first fixed rod, and then hold the second rod firm or fix it, in that position, to the body of the Dial. Then you must fit the ridge of one more rod or ruler at one end to the centre of the star that remains complete on the surface of the Dial, and at the other to the combined ends of the two ridges of the fixed rods; and when the three ridges of the rods are in this position, starting from the point where they all meet, you must measure out three parts equal to one another, one on each of the three ridges. Then you must take the three straight-line distances between the three separate ends of these three equal parts, and somewhere draw a triangle whose sides are these three lengths, then bisect two sides of this triangle and through the midpoints of the sides draw straight lines, each one perpendicular to the side that is bisected.[10]

Then, from the point of intersection of these perpendiculars to the bisected sides of the triangle, you must draw a straight line to one of the vertices of the triangle, and through the same point of intersection of the perpendiculars you must draw another straight line perpendicular to the one drawn from the point of intersection to the vertex of the triangle.[11]

Then taking this vertex as centre, and with radius equal to one of the equal parts cut off on the ridges of the fixed rods, you must draw a circular arc to cut the line drawn perpendicular to the line through the vertex.[12]

Then, returning to the two fixed rods, on the side away from that where the third ridge was placed, you must twist one end of a good piece of wire round each of the fixed rods, so that on each rod the free end [of the wire] comes away from the twisted part just at the point at the end of the equal part marked off on the ridge of the rod, then measuring from the ridges of the fixed rods, you must mark off on each of these wires a part equal to the straight-line distance to the corner of the triangle and afterwards, when joining the wires together at these marks, you must twist them together by their free ends.[13]

Then, laying along their length (*adjoignant en leur mesme sens*) another

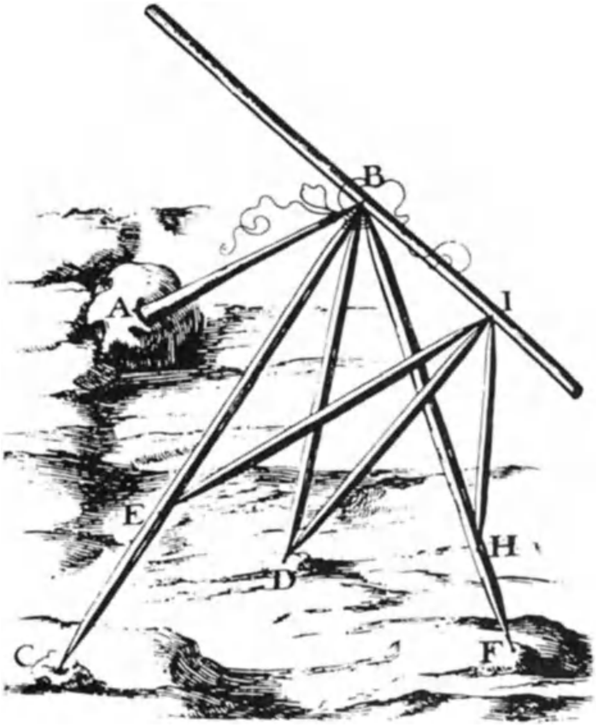


Fig. 9.1. Positioning the stylus of a sun-dial, from Bosse's work of 1643.

straight, smooth and rigid rod, you must attach these two wires firmly to this last rod exactly at the point where they join one another, and starting from this meeting point you must mark off on this last rod, on the side towards the two fixed rods, a part equal in length to the part of the straight line perpendicular to the line drawn to a corner of the triangle which is contained between the perpendiculars to the bisected sides of the triangle and the above-mentioned arc.[14]

Then, at this last marked point, fit this last rod to the point at which the two ridges of the fixed rods meet, and rotate it about this point so that the two wires are each stretched out into a straight line, and then this last rod, once fixed in that position, is suitably positioned as the axle[15] [i.e. stylus] of a Dial in this place, and if the rods are correctly cut and have holes made in them, one can use wire to make them fast and hold them properly together in that position, after which one can remove the wires which were earlier twisted round the two fixed rods.[16]

After the Stylus (*Style*) is in position, the method of drawing the lines for flat Dials is to be found at the end of the sketch of a way of drawing lines for cutting stones,[17] and if the surface of the Dial is not flat, all you need to do is to place a straight ruler across the surface of the Dial, and cause the six o'clock line to fall on one of the ridges of the ruler,[18] and further, hanging a plumb-line beside the Stylus, mark the twelve o'clock point on the same ridge, then take the points and lines which give the six o'clock line and its twelve o'clock point over onto another straight line in a plane surface, and next draw the triangles on it, and the circle of the equator, and mark the hour points, then by some convenient method take over the hour points of the line onto the ridge of the ruler, then, with a thread running from the Stylus in the plane of the equator, take the hour points from the ridge of the ruler over onto the surface of the Dial. And finally, having removed the ruler, running alongside the Stylus several times with each of the rays to the hour points on the equator, mark the hour lines on the surface of the Dial, and in the same way the other types of line.

P.B.G.[19]

Appendix 1

Letter from Descartes to Desargues[1] (19 June 1639)

Sir,

The openness I have observed in your temperament, and my obligations to you, invite me to write to you freely what I can conjecture of the *Treatise on Conic Sections*, of which the R[everend] F[ather] M[ersenne] sent me the *Draft*.[2] You may have two designs, which are very good and very praiseworthy, but which do not both require the same course of action. One is to write for the learned, and to instruct them about some new properties of conics with which they are not yet familiar; the other is to write for people who are interested but not learned, and make this subject, which until now has been understood by very few people, but which is nevertheless very useful for Perspective, Architecture etc., accessible to the common people and easily understood by anyone who studies it from your book. If you have the first of these designs, it does not seem to me that you have any need to use new terms: for the learned, being already accustomed to the terms used by Apollonius, will not easily exchange them for others, even better ones, and thus your terms will only have the effect of making your proofs more difficult for them and discourage them from reading them. If you have the second design, your terms, being French, and showing wit and elegance in their invention, will certainly be better received than those of the Ancients by people who have no preconceived ideas; and they might even serve to attract some people to read your work, as they read works on Heraldry, Hunting, Architecture etc., without any wish to become hunters or architects but only to learn to talk about them correctly. But, if this is your intention, you must steel yourself to writing a thick book, and in it explain everything so fully, so clearly and so distinctly that these gentlemen, who cannot study a book without yawning and cannot exert their imagination to understand a proposition of Geometry, nor turn the page to look at the letters on a figure, will not find anything in your discourse which seems to them to be less easy of understanding than the description of an enchanted palace in a novel. And, to this end, it seems to me that, to make your proofs less heavy, it would not be out of the question to employ the terminology and style of calculation and of Arithmetic, as I did in my *Geometry*;[3] for there are many more people who know what multiplication is than there are who know about compounding ratios, etc.

Concerning your treatment of parallel lines as meeting at a butt[4] at infinite distance, so as to include them in the same category as lines which meet at a point, this is very good, provided you use it, as I am sure you do, rather as an aid to understanding what is difficult to see in one of the types, by comparing it with the other, where it is clear, and not conversely.

I have nothing to add on what you have written about the centre of gravity of a sphere: for I have already expressed my opinion sufficiently fully to the R[everend] F[ather] M[ersenne] and your comment at the end of your corrections shows that you have understood what I said. But I ask your pardon if I have allowed myself to be carried away by my enthusiasm in recounting my thoughts so freely, and I beg you to believe me[5], . . .

Appendix 2

Letter from Beaugrand to Desargues[1] (25 July 1639)

Sir,

I am astonished that you have not received solutions to your problems from those to whom you addressed yourself before, since I find nothing in them that cannot easily be constructed by the propositions of Apollonius. It must be that they are not familiar with the work of this author since they have assured you that they did not see how they could use it; I want to show you how to do so by using the figures of [proposition] 54 [of book] 3, so that you may be left in no doubt.[2] Let there be three points A, H, C, two of which, that is A, C, you suppose to lie on the straight lines AD, CD, whose position is given; you wish to construct a conic section which passes through the point H and touches the straight lines AD, DC at the points A, C. Divide AC into two equal parts at the point E, and draw the line DE, which will be the diameter of the conic that is to be constructed. Draw the straight lines AF, CG, the first parallel to DC, the other to AD, and also the lines AH, HC which meet these lines at the points F, G. Then find on the infinite straight line DE a point or two such as B, such that the ratio of the square of BF to the square of DB is equal to the ratio of the rectangle AF, CG to the rectangle $4AD, DC$. Now, if these four rectangles are equal you will only find one point B between D and E which defines, in relation to these points, two lines in the proportion I have prescribed; in this case the [conic] section you require will be a parabola, which you can construct by proposition 20 [of book] 1.[3] But you will find two such points B, K if the rectangles are unequal, and if rectangle AF, CG is greater than rectangle $4AD, CD$ the conic section will be a hyperbola, and an ellipse if it [the first rectangle] is smaller. And since you know the size and position of the chord[5] BK and one of the ordinates, AE, you can describe the ellipse, by proposition 21 [of book] 1.

The other problem was to describe a conic passing through four points so arranged that the infinite straight line drawn through two of these points passes through the point of intersection of the straight lines which touch the conic at the other two points. Let the four given points be Q, A, H, C and let the line QH pass through the point of intersection of the lines which will touch the required conic at the points A, C. Construct the points R and D on the line QH such that the ratio of QR to RH is equal to the ratio of QD to DH; the point D is the point of intersection of the tangents mentioned above.[4] Having drawn the tangents, it only remains to describe a conic section which passes through the point H and touches the straight lines AD, DC at the points A, C, [which can be done] in the manner I have explained above.

However, since, as you see, the second problem reduces to the first, I shall add a second solution to the first problem, a solution which is short and not very difficult. Having drawn the straight lines AF, CG, CH, AH as above, take on the line AF any point Z, and on the line CG cut off CY, a length such that its ratio to CG is the same as that of FA to AZ; joining the lines AY, CZ [i.e. producing them until they intersect] their point of intersection, P, will lie on the required conic section, which you can describe by finding as many more such points as you consider appropriate. I could easily give you other solutions to the same problems, either taken from Apollonius or from my own writings on conics, and I could equally render them more general by making certain changes, but I have solved them in the form in which you proposed them, in order to give you greater satisfaction. I am Sir. [*sic*]

Proof of the second construction of the first problem:

$$af,\ cg \quad \begin{matrix} ac'' \\ \end{matrix} \begin{cases} cb'' \\ 4a''d, \end{cases} \begin{matrix} & bd'' \\ dc & ac'' \end{matrix} \begin{cases} af, cg \\ az, cy \end{cases} \begin{matrix} 4ad, & dc \\ 4ad, & dc \end{matrix} \begin{cases} cb'' & bd'' \\ cb'' & bd'' \end{cases}$$

$$af, cg = az, cy$$

$$\begin{matrix} af & cg \\ af & cg \end{matrix} \begin{cases} 4ad, dc & ac'' \\ 4ad\ dc & \end{cases} \quad cb'' \quad db''$$

Your very humble servant

De Beaugrand

Paris 25 July 1639

Appendix 3

Pascal's *Essay on Conics* (1640)

The original title of Pascal's work is *Essay pour les Coniques*.

ESSAY ON CONICS By B.P.

First Definition

When several straight lines meet at one point, or are all parallel to one another, all these lines are said to belong to the same order or ordinance [*ordonnance*, the word used by Desargues], and the set of lines is called an order of lines or an ordinance of lines.

Definition II

By the term conic section we mean the circumference of the Circle, the Ellipse, the Hyperbola, the Parabola and the Pair of Straight lines, since a Cone, cut parallel to its base, or through its vertex [*sommet*] or in the three other ways which give the Ellipse, the Hyperbola and the parabola, produces in the Conical surface either the circumference of a circle, or a pair of straight lines, or the Ellipse, or the Hyperbola, or the parabola.

Definition III

By the word straight [*droite* = straight], used alone, we mean straight line [*ligne droite*].

Lemma I

[Figure A3.1]—If in the plane M, S, Q we have two lines MK, MV through the point M and two lines SK, SV through the point S, and K is the point of intersection of the straight lines MK, SK and A the point of intersection of the lines MK, SV[1] and μ the point of intersection of the straight lines MV, SK and through any two of the four points A, K, μ, V which do not lie on the line MS, such as the points K, V, there passes the circumference of a circle which

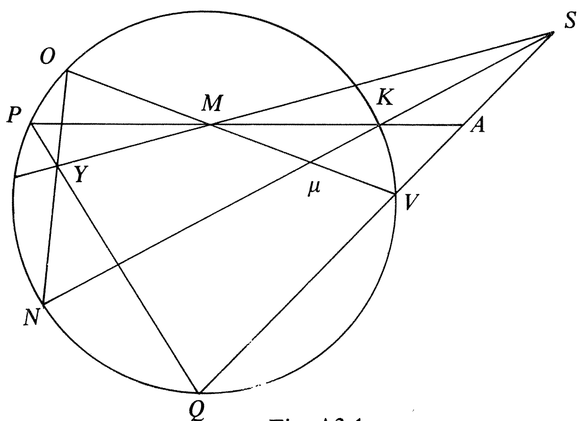


Fig. A3.1

cuts the straight lines MV, MK^2, SV, SK, in the points O, P, Q, N, I say that the straight lines MS, NO, PQ belong to the same order.[3]

Lemma II

If through a straight line there pass several planes, cut by another plane, all the lines of intersection of these planes belong to the same order as the straight line through which the said planes pass.

Assuming these two lemmas, and some simple results that follow from them, we shall show that, given the same points and lines as in the first Lemma, if through the points K, V there passes any conic section which cuts the straight lines MK, MV, SK, SV in the points P, O, N, Q, the straight lines MS, NO, PQ belong to the same order.[4]

Following these three lemmas, and some results that follow from them, we shall give a complete elements of Conics, that is, all the properties of the diameters and latera recta [*costez droits*—see Desargues, pp. 103, 125], of tangents, etc., the almost complete reconstruction of the Cone from all the data, the description of conic sections by constructing points [which lie on them], etc.

[Figure A3.2]—In doing this, we state the properties we mention in a more general manner than is usual. For example, the property that if in the plane MSQ, given the conic section PKV, we draw the straight lines AK, AV which cut the conic at the points PK, QV, and through two points which are not collinear with A, such as K, V, and through two points N, O on the conic, we draw the four straight lines KN, KO, VN, VO which cut the straight lines AV, AP at the points S, T, L, M,[5] I say that the ratio compounded from the ratio of the straight line PM to the straight line MA, and the ratio of the straight line AS to the straight line SQ, is the same as the ratio compounded from the ratio of the straight line AT to the straight line TQ.[6]

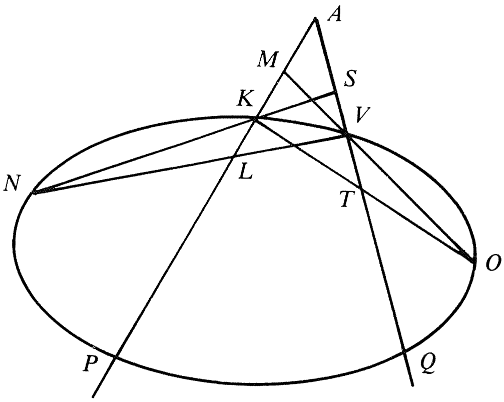


Fig. A3.2

We shall also show that if there are three straight lines DE, DG, DH, which the straight lines AP, AR cut in the points F, G, H, C, γ, B, and that on the straight line DC there lies a given point E, the ratio compounded from the ratio of the rectangle EF into FG to the rectangle EC into $C\gamma$, is the ratio of the straight line $A\gamma$ to the straight line AG, is the same as the ratio compounded from the ratio of the rectangle EF into FH to the rectangle EC into CB, and the ratio of the straight line AB to the straight line AH. And is also equal to the ratio of the rectangle of the straight lines FE, FD to the rectangle of the straight lines CE, CD;[7] moreover, if through the points E, D there passes a conic section which cuts the straight lines AH, AB in the points P, K, R, ψ, the ratio compounded from the ratio of the rectangle of the straight lines EF, FG[8] to the rectangle of the straight lines EC, $C\gamma$, and the ratio of the straight line γA to the straight line AG, will be the same as the ratio compounded from the ratio of the rectangle of the straight lines FK, FP to the rectangle of the straight lines $CR, C\psi$, and the ratio of the rectangle of the straight lines $AR, A\psi$ to the rectangle of the straight lines AK, AP.[9]

We shall also show [Fig. A3.3] that if four straight lines AC, AF, EH, EL cut one another in the points N, P, M, O and a conic section cuts the said straight lines in the points C, B, F,[10] D, H, G, L, K, the ratio compounded from the ratio of the rectangle of MC into MB to the rectangle of the straight lines PF, PD and the ratio of the rectangle of the straight lines AB, AC, is the same as the ratio compounded from the ratio of the rectangle of the straight lines ML, MK to the rectangle of the straight lines PH, PG and the ratio of the rectangle of the straight lines EH, EG to the rectangle of the straight lines EK, EL.[11]

We shall also prove this proposition, which was first discovered by M. Desargues of Lyon, one of the best minds of our day and extremely well-versed in Mathematics, and, among other things, in the study of Conics. His writings on the subject, though few in number, have given ample evidence of the fact to those who cared to look into the matter; and I am happy to

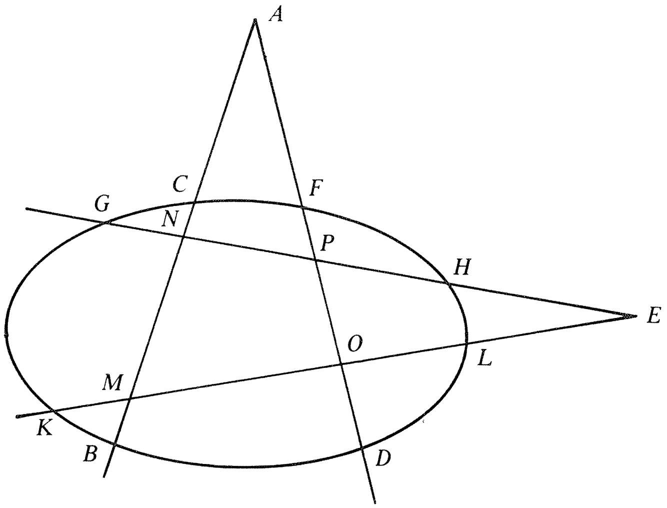


Fig. A3.3

acknowledge that I owe the little I have discovered on the subject to my study of his writings and that I have tried as far as I could to imitate his method of approaching this material, which he has treated without using the triangle through the axis of the cone. Now, considering the general case of any section of the cone, the marvellous proposition with which we are concerned is that: if in the plane MSQ there is a conic section PQV, and through four points K, N, O, V of this conic section we draw straight lines KN, KO, VN, VO such that no more than two straight lines pass through any one of these four points, and another straight line cuts the circumference of the section in points $xyZ\delta$, then the ratio of the rectangle of the lines ZR,[12] $Z\psi$ to the rectangle of the lines yR, $y\psi$[13] is equal to the ratio of the rectangle of the lines δR, $\delta\psi$ to the rectangle of the lines XR, $x\psi$.[14]

We shall also show that if in the plane of the hyperbola, or the ellipse, or the circle AGE, whose centre is C, we draw the straight line AB, tangent to the section at A, and draw the diameter CA, and then take the straight line AB whose square is to be equal to a quarter of the rectangle of the figure [i.e. the product of the parameter and the latus rectum—see Chapter I for calling this 'rectangle of the figure'] and draw CB, then if we draw any line, such as DE, parallel to the line AB, to cut the section in E and the lines AC, CB in the points D, F, if the section AGE is an ellipse or a circle the sum of the squares of the lines DE, DF will be equal to the square of the line AB, and in the hyperbola, the difference of the squares of the same lines DE, DF will be equal to the square of the line AB.

We shall also deduce the solutions to several problems, for example, from a given point to draw a tangent to a given conic section.

To find two conjugate diameters which cut one another at a given angle.

To find two diameters which meet one another at a given angle and whose lengths are in a given ratio. We have several more Problems and Theorems and some consequences of the preceding ones, but the lack of confidence I feel on account of my small experience and talent does not permit me to proceed further with this work until it has been scrutinised by the competent, who will oblige us by taking that trouble; after which, if the thing is judged to be worth continuing, we shall try to proceed with it as far as God will give us strength to go.

Paris, 1640

Appendix 4

Kepler's Invention of Points at Infinity

As a thoroughgoing Platonist and a devout Christian, Kepler believed that all mathematical entities existed eternally in the mind of God. Hence, he would presumably have considered that they could not be invented, but merely awaited discovery by the mathematician. However, he introduced points at infinity in an entirely *ad hoc* way in 1604, in a context not of pure mathematics but of geometrical optics, to provide the parabola with a second focus. As a result, the reader may well take them to be an ingenious invention rather than a profound mathematical discovery. That Kepler himself made little fuss about the matter is probably not significant—he made equally little fuss about his other mathematical discoveries—but his unemphatic treatment may perhaps have encouraged casual readers to suppose that they were merely concerned with another example of the imprecise use of the phrase 'at an infinite distance'. It seems unlikely that this is actually the case. Kepler's famous arithmetical lapses (often detected later by Kepler himself) are spectacular at least partly by their unexpectedness. He is in fact a highly competent mathematician.

Curiously enough, Kepler's concern with unifying the theory of conics, which led him to invent points at infinity, seems to have had nothing to do with the discovery for which he is best remembered, namely that the orbits of the planets are elliptical. It was early in 1605 that he convinced himself that the orbit of Mars was an ellipse. His work on conics—which was to remain his only work on conics in general—had been published in the previous year. It occurs in a work with the unwieldy title *Ad Vitellionem paralipomena quibus astronomiae pars optica traditur* (1604),[1] a work which apparently had its origin in an investigation of the images formed by the *camera obscura*. However, Kepler's characteristic thoroughness soon widened its scope to include physiological optics (the eye being the astronomer's observing instrument) and a general investigation of the nature and properties of light.

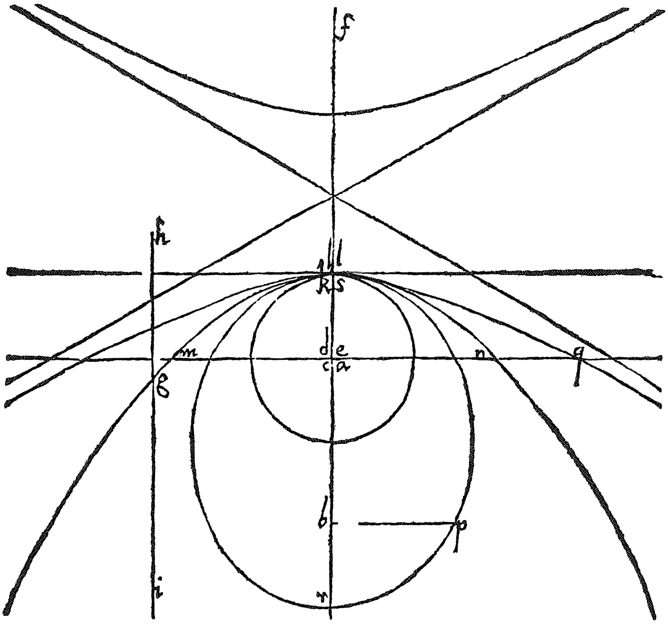


Fig. A4.1. Kepler's system of conics. *Ad Vitellionem paralipomena* (Frankfurt 1604), Chapter IV, sect. 4, p. 94. Courtesy of the Trustees of the Science Museum, London.

The geometry of conic sections is introduced to elucidate the properties of burning mirrors.

As Davis (1975) has shown, Kepler's treatment of the geometry of conics is designed to relate the curves one to another, all the curves being in the same plane. The resultant system of conics is shown in Fig. A4.1. Kepler begins with a brief account of the shapes of the curves, and then turns to the foci:

> Moreover, for these curves there are some points of special importance, which have a precise definition, but no name, unless you take their definition or property for a name. For straight lines drawn from these points to meet tangents to the curve at their point of contact [with the curve], make with the tangents angles equal to those formed if the opposite points are joined up with these points of contact [see Fig. A4.2]. On account of [the fact that we are concerned with] light, and with our sights set on Mechanics, we shall call these points Foci.[2] We should have called them 'centres', because they lie on the axes of the conic sections, if it were not that [previous] authors have used the name centre for another point of the Hyperbola and the Ellipse. So, in the circle there is one focus, A [see Fig. A4.1],[3] which is the same point as the centre: in the Ellipse there are two foci B, C equidistant from the centre and further from it as the figure is more elongated. In the Parabola one focus, D, is inside the conic section, the other is to be imagined (*fingendus*) either inside or outside, lying on the axis [of the

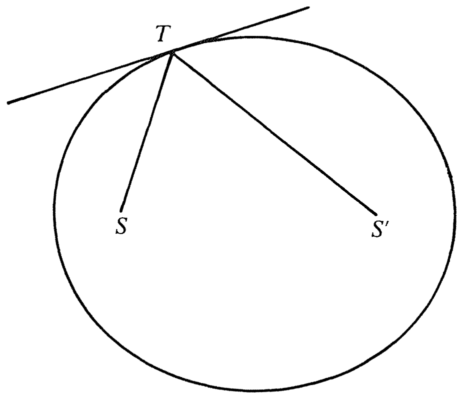


Fig. A4.2. Ellipse with foci S and S'. The lines $ST, S'T$ make equal angles with the tangent to the curve at T.

> curve] at an infinite distance from the first (*infinito intervallo a priore remotus*), so that if we draw the straight line HG or IG from this blind focus (*ex illo caeco foco*) to any point G on the conic section, the line will be parallel to the axis DK. In the Hyperbola, the outer focus, F, will be nearer the inner one, E, as the Hyperbola is less sharply curved. And the focus which lies outside one of the pair of sections [i.e. outside one branch] lies inside the other, and conversely.[4]

Kepler next considers the foci of the straight line:

> It follows therefore by analogy that in the straight line (we speak thus of the straight line, without the authority of usage (*sine usu*), merely to complete the analogy) the two foci coincide and lie on the straight line; so there is one focus, as for the circle.[5]

There follows a short summary of the relations between foci and curve in each type of section. Its purpose seems to be to emphasize both the systematic nature of Kepler's arrangement of the curves and the self-consistency of his use of analogy. After a brief account of the chord and the sagitta, Kepler again returns to the subject of analogical reasoning, commending its usefulness, particularly in Geometry, and illustrating his point by giving methods of drawing the hyperbola and the parabola which are analogous to the well-known method of drawing an ellipse by means of a loop of thread running round two pins. (This method is now usually, somewhat pejoratively, known as 'the gardeners' method', but Kepler correctly points out that it is in fact based upon Apollonius *Conics* III,51.)

As Davis (1975) has noted, Kepler's use of analogy in constructing a system of conics bears a considerable resemblance to the employment of a 'Principle of Continuity' in nineteenth-century studies of these curves. The same is true of his use of analogy in his treatment of the foci. In that respect, it may be seen as forward-looking. However, analogical reasoning is highly characteristic of much of the natural philosophy of Kepler's day, and of Kepler's own natural philosophy. In fact, Kepler's use of analogy in constructing a theory of the refraction of light, especially in the particular work with which we are concerned, namely *Ad Vitellionem paralipomena*, has been analysed in considerable detail by Buchdahl (1972). Mathematical analogy of the type we have been discussing here is, of course, different in many ways from the physical analogies found in Kepler's treatment of refraction (for which see Buchdahl, 1972). However, Kepler seems always to have seen himself not as what we should now call a pure mathematician but as a natural philosopher who concerned himself with mathematics.[6] That is, his work was conceived as applicable mathematics. So it comes as no surprise to find that the analogical methods used elsewhere in his work on topics should also be used in its mathematical part. It is of some interest, however, that in contrast to his use of physical analogies, Kepler's use of mathematical analogy suggests that he considered it much more reliable, in fact sufficiently reliable to justify his introducing a notion so seemingly fanciful (in physical terms) as a point 'at an infinite distance'.[7] Nevertheless, it is clear that the origin of the analogical treatment of the foci lies in a consideration of the reflection of light from one focus of a burning mirror to another—the parabola being the commonest 'practical' example of such a mirror in optical treatises.[8] Kepler's pure mathematical discovery seems to have its roots very firmly fixed in Optics.

We have found no reason to suppose that Desargues was familiar with Kepler's work in general, or with *Ad Vitellionem paralipomena* in particular. The repeated connection between points at infinity and conic sections seems to be a rational consequence of the mathematics rather than a historical consequence of one mathematician knowing the work of another.

Appendix 5

The French Text of Desargues' *Perspective* (1636)

I

EXEMPLE DE L'VNE DES MANIERES VNIVERSELLES DV S. G. D. L. TOVCHANT LA PRATIQVE DE LA PERSPECTIVE SANS EMPLOIER AVCVN TIERS POINT, DE DISTANCE NY D'AVTRE nature, qui ſoit hors du champ de l'ouurage.

COMME cét Exemple d'vne maniere vniuerſelle de pratiquer la perſpectiue ſans emploier aucun tiers point, de diſtance ou d'autre nature, qui ſoit hors du champ de l'ouurage, ſe manifeſte en langue Françoiſe, auſſi les meſures y ſont de l'vſage de la France.

Les Mots PERSPECTIVE, APARENCE, REPRESENTATION, & POVRTRAIT, *y ſont chacun le nom d'vne méme choſe.*

Les Mots, EXTREMITEZ, BORDS, COSTEZ & CONTOVR, *d'vne figure y ſont auſſi chacun le nom d'vne méme choſe.*

Et les Mots, REPRESENTER, POVRTRAIRE, TROVVER L'APARENCE, FAIRE *ou* METTRE *en* PERSPECTIVE *y ſont emploiez en méme ſignification l'vn que l'autre.*

Les Mots à NIVEAV, *de* NIVEAV, PARALEL *à* L'HORISON, *y ſignifient auſſi chacun vne méme choſe.*

Les Mots à PLOMB, PERPENDICVLAIRE *à* L'HORISON, & QVARREMENT *à* L'HORISON *y ſignifient auſſi chacun vne méme choſe.*

Et les Mots QVARREMENT, *à* L'EQVIERE, *à* DROITS ANGLES, & PERPENDICVLAIREMENT *y ſignifient encor en general vne méme choſe l'vn que l'autre.*

Ce qu'on ſe propoſe à pourtraire y à nom SVIET.

Ce qu'aucuns nomment plan geometral, *autres* plan de terre, *autres* la plante du ſujet, *y à nom* ASSIETE *du* SVIET.

Ce qu'aucuns nomment la tranſparence, *autres* la ſection, *autres d'vn autre nom, à ſçauoir la ſurface de la choſe en laquelle on fait vne perſpectiue s'y nomme* TABLEAV, *deuant comme apres l'ouurage acheué.*

L'aſiete du ſujet, & le tableau dont il eſt icy parlé ſont en des ſurfaces plates, c'eſt à dire qu'il n'eſt icy parlé que des tableaux plats, & des aſietes de ſujet plates, leſquelles aſietes & tableaux ſont conſiderez comme aians deux faces chacun.

La face du tableau qui ſe trouue expoſée à l'œil s'y nomme le DEVANT *du* TABLEAV *comme ſon autre face laquelle n'eſt pas expoſée à l'œil, s'y nomme le* DERRIERE *du* TABLEAV.

Quand l'aſiete du ſujet eſt eſtenduë à Niueau, celle de ſes faces qui ſe trouue tournée du coſté du Ciel, y à non le DESSVS *de* L'ASSIETE *du* SVIET, *comme l'autre face de la meſme aſiete qui ſe trouue tournée*

A

du côté de la terre y à nom le DESSOVS *de* L'ASSIETE *du* SVIET.

L'étendue ou la surface plate & indeterminée, en laquelle est figurée l'assiete du sujet s'y nomme PLAN *de* L'ASSIETE *du* SVIET.

L'étendue plate & indeterminée aussi, dans laquelle est le tableau s'y nomme le PLAN *du* TABLEAV.

Toutes les lignes y sont entendues drétes.

En vne seule & méme stampe, & pour ce méme & seul exemple il y a trois figures separées & cotées de Caracteres d'vn méme nom, mais de forme diferente en chacune de ces figures.

Les Caracteres de renuoi sont de la méme forme en l'impression, qu'en celle de ces trois figures à laquelle se raporte le discours en châque endrét.

Quand en l'mpression il y à pour renuoi plus d'vne fois en suite des Caracteres de méme nom; mais de forme diferente entr'eux, cela signifie que le discours en cét endrét là, s'adresse également à chacune des figures ou les semblables Caracteres sont estampez.

Quand les deux bouts d'vne ligne en l'vne de ces figures sont cotés de Caracteres de méme nom que les deux bouts aussi d'vne ligne en vne autre de ces figures, ces deux lignes ainsi cotées ont de la corespondance entre elles, & sont l'vne en sa figure & en son espece, la méme chose que l'autre en sa figure & en son espece.

En cét Art il est supose qu'vn seul œil voit d'vne méme œillade le sujet auec son assiete & le tableau, disposez l'vn au drét de l'autre, comme que ce soit: il n'importe si c'est par Emission de raions visüels, ou par la reception des especes émanées du sujet, ny de quel endrét, ou lequel des deux il voit deuant ou derriere l'autre, moïenant qu'il les voïe tous deux facilement d'vne méme œillade.

Il est encore supose que celuy qui pratique cét Art, entend la façon & l'vsage de l'échelle à faire vne assiete du sujet auec son éleuation; & dans cét exemple il est supose qu'il entend qu'elle chose c'est qu'on nomme communement la perspectiue.

Et par cette maniere icy de la pratiquer aiant l'assiete & les éleuations necessaires d'vn sujet auec les interuales conuenables tracés en telle grandeur que ce soit, ou seulement leur route & leurs mesures écrites en vn deuis, & la disposition des plans de l'assiete du sujet & du tableau cogneuë; auec la régle & le compas communs on trouue & fait au premier coup facilement le trait de la perspectiue d'vn tel sujet, en ce tableau de telle grandeur qu'il puisse étre, sans ayde aucune de point qui soit hors de son étendue en telle distance & de telle façon, que le sujet son assiete & le tableau soient disposez entre eux & deuant l'œil.

Dont les régles generales s'expriment en autre langage, enuelopent diuerses manieres vniuerselles de pratique, s'apliquent à nombre de cas & de figures dissemblables, & se demonstrent auec deux seules propositions manifestes &

3

familieres à ceux qui ſont diſpoſez à les conceuoir.

Mais quand à preſent, & pour ceux qui ſçauent ſeulement executer les anciennes régles de la pratique de l'art, cét exemple ſimple en langage, & de ſujet commun à ces régles anciennes, eſt de pure pratique.

Où pour circonſtances de remarque on commence par trois eſpeces de preparations.

L'vne qui regarde le ſujet & ſe fait au plan de ſon aſsiete, ou bien autre part.

Les deux autres concernent l'aparence du ſujet, & ſont faites communement au tableau méme.

Le ſujet en cét exemple eſt vne cage baſtie ſimplement de lignes, quarrée & d'égale groſſeur iuſqu'à certain endrét depuis lequel elle aboutit en pointe maſsiue, à la maniere d'vn baſtiment couuert en pauillon, aſsiz en raze campagne, éleué ſur terre à plomb iuſqu'au toit, creuzé dans œuure plus bas que le niueau du terrain d'alentour, auec les meſures de quelques lignes debout & penchantes en diuers endréts hors & dans cette cage dans terre, ſus terre, & ſuſpenduës hors terre, chacune paralelle au tableau qui pend à plomb.

Au haut de la Stampe à main droite.

La figure quarrée, m, l, i, k, de telle étenduë qu'elle ſe rencontre, eſt l'aſſiete de cette cage, laquelle aſsiete eſt icy poſée de niueau.

La ligne, x, eſt la hauteur des éleuations, pieds dréts, ou montans de la méme cage, entendus poſez à plomb à ſon aſsiete vn à chacun des quatre coins du quarré, m, l, i, k.

La ligne, d, eſt la longueur de trois thoiſes de l'échelle, à laquelle ont eſté meſurez les bords de l'aſsiete de cette cage, & ſes éleuations, ici nommée ESCHELLE *du* SVIET.

La ligne, t s, eſt la meſure de la hauteur perpendiculaire de l'œil au deſſus du plan de l'aſsiete du ſujet, laquelle hauteur d'œil rencontre ce plan au poinct, t..

Par le méme plan de cette aſsiete du ſujet, à ſçauoir à l'endrét auquel eſt entendu que le plan du tableau le rencontre eſt menée vne ligne, a b, nommée LIGNE *du* PLAN *du* TABLEAV, *de façon qu'ici l'œil voit le tableau deuant le ſujet, ou bien l'œil voit le ſujet derriere le tableau.*

La ligne, t c, eſt la diſtance perpendiculaire du pied de l'œil au tableau, c'eſt à dire, la diſtance perpendiculaire de l'œil au méme tableau.

Par vn des poincts, a, ou, b, de cette ligne, a b, comme ici par le poinct, a, dans le méme plan, & de la part de l'aſsiete du ſujet eſt menée vne ligne indeterminée, a g, paralelle à la ligne, t c.

Puis de chacun des points remarquables en l'aſsiete du ſujet ici des quatre coins, & du milieu de l'vn des cotez du quarré, m, l, i, k, ſont menées iuſqu'à cette ligne, a g, des lignes paralelles à la ligne, a b, comme les lignes, m r, l h, k n, e 15, & i g.

4

Par l'autre poinct, b, de la méme ligne, a b, est menée la ligne encore indeterminée, b q, paralelle aux lignes, a g, t c.

La longueur de chacune de ces lignes ou piece remarquable d'icelles, est mesurée auec l'échelle du sujet, d, & leur mesure est retenuë en memoire, ou pour memorial est écrite sur elle, ou en vn deuis.

Ainsi les nombres 15. écrits aupres des bords du quarré, m, l, i, k, denotent que chacun des cotez de cette figure a quinze pieds de long.

Et les nombres 1, 17, écrits aupres de la ligne des éleuations, x, denotent que chacune des éleuations du sujet a dix-huict pieds de long, à sçauoir dix-sèpt pieds hors terre, & vn pied dans terre.

Ainsi le nombre 12. écrit aupres de la ligne, a b, denote qu'en cét exemple, cette ligne a douze pied de long.

Ainsi le nombre 17. denote que la piece de la ligne, a g, contenuë entre les lignes, r m, & a b, se rencontre auoir dix-sept pieds de longueur, & par ce moien, ou selon cette façon de mesurer, ici dauanture le sujet est derriere le tableau a dix-sept pieds loin de lui, ce qui veut dire encore qu'ici dauanture le tableau se rencontre deuant le sujet a dix-sept pieds loin de lui.

Semblablement le nombre 4½ de la ligne, s t, monstre qu'ici l'œil est éleué quatre pieds & demi de hauteur perpendiculaire au dessus du plan de l'assiete du sujet.

De méme le nombre 24. signifie qu'ici le pied de l'œil, ou l'œil méme, est éloigné quarrement à vingt-quatre pieds loin du tableau deuant lui.

De méme le nombre 13½ denote que la ligne, l h, a treize pieds & demi de long.

De méme l'vn des nombres 9. denote que la piece de la ligne, a g, contenuë entre les lignes, r m, l h, a neuf pieds de long.

Tout de méme des nombres 3. comme encore de chacun des autres semblables.

Et voila celle des trois preparations qui regarde le sujet, acheuée.

Maintenant, la Stampe entiere est comme vne planche de bois, vne muraille, ou semblable chose accommodée & preparée à faire vn tableau de telle étenduë qu'il puisse estre, entendu pendant à plomb sur le plan de l'asiete du sujet, auquel plan il touche comme en la ligne, a b, dans lequel tableau supposé que l'on se propose à representer cette cage par vne figure en perspectiue, de grandeur proportionnée à celle du tableau, sans aide pour cela d'aucun poinct qui soit hors de lui, ny faire premierement ailleurs vne autre perspectiue de largeur égale à la ligne, a b, pour apres la contretirer dans ce tableau proportionnellement, au moien du treillis ou du petit pied.

Au bas de la Stampe.

A cette fin est menée la ligne, A B, de niueau si longue, qu'il est posible au bas du tableau corespondante à la ligne, a b.

De suite aux bouts, A, & B, d'vne méme part de cette ligne, A B, sont menées

menées deux autres lignes, A F, & B E, paralelles entr'elles, & communément comme ici perpendiculaires à cette ligne, A B.

Puis cette ligne, A B, eſt diuiſée en autant de parties égales, que la ligne, a b, contient de pieds.

Ici la ligne, a b, contient douze pieds de long, partant la ligne, A B, eſt diuiſée en douze parties égales marquées au deſſus d'elle, qui ſont vne échelle dautant de pieds, l'vn deſquels ici le ſeptiéme, ſa moitié, ou ſon quart eſt ſouſdiuiſé en ſes pouces, & lignes s'il en eſt beſoin.

D'abondant eſt conſiderée la hauteur de l'œil au deſſus du plan de l'aſſiete du ſujet, laquelle hauteur d'œil eſt ici de quatre pieds & demi, & cette meſure de quatre pieds & demi, eſt lors priſe des pieds de l'échelle ainſi faite en la ligne, A B, & portée ſur chacune des deux lignes, A F, & B E, ſçauoir d'A en F, & de B, en E, puis eſt menée la ligne, F E, paralelle par ce moïen à la ligne, A B.

Dauantage en cette ligne, F E, eſt marqué le poinct au drét duquel on entend que l'œil eſt au bout de ſa diſtance, pointé deuant le tableau, comme ici le poinct, G, au drét duquel on entend que l'œil eſt vingt-quatre pieds loin à l'équiere deuant le tableau.

Par ce poinct, G, d'vne ſuite eſt menée la ligne, G C, paralelle à chacune des lignes, A F, & B E, ſçauoir ici quarrement à la ligne, A B, de façon que l'eſpace, A F E B, ſe trouue diuiſé d'auenture en deux autres eſpaces, dont les bords opoſez ſont en chacun, des lignes paralelles entr'elles, ſçauoir ici les eſpaces, G C A F, & G C B E.

Lors, ou dans tout l'eſpace, A B E F, ou bien dans l'vn ou dans l'autre des deux moindres eſpaces, G C A F, & G C B E, comme ici dans l'eſpace, G C A F, ſont menées les deux lignes, A G, & C F.

Par le poinct auquel ces deux lignes, A G, & C F, ſe rencontrent, eſt menée la ligne, H D, paralelle à la ligne, A B, laquelle ligne, H D, rencontre la ligne, B E, au poinct, D, la ligne, G C, au poinct, T, & la ligne, A F, au poinct, H.

Puis de l'vn ou de l'autre des poincts, H, ou, T, eſt menée vne ligne dans le méme eſpace, G C A F, à celui des poincts, G, ou, F, qui lui eſt opoſé diagonalement.

Si cette ligne eſt menée comme au bas de la Stampe du poinct, G, tendant au poinct, H, c'eſt la ligne, G H.

Que ſi cette ligne eſt menée comme au haut de la Stampe à main gauche, du poinct, f, tendant au poinct, t, c'eſt la ligne, f t.

Et ſupoſé que par les poincts, f, & t, l'on ait mené la ligne, f t, lors par le poinct auquel cette ligne, f t, rencontre la ligne, a g, eſt menée la ligne, n q, paralelle à ligne, a b.

Puis par le poinct auquel cette ligne, n q, rencontre la ligne, c g, icy le poinct, o, & par le poinct, f, eſt menée la ligne, f o.

B

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Puis par le poinct auquel cette ligne, f o, *rencontre la ligne*, a g, *est menée la ligne*, s u, *paralelle a la ligne*, a b.

Et semblable operation est continuée autant de fois qu'il en est besoin.

Supposé maintenant qu'on ait pratiqué cette operation au moïen des lignes, C I *&* A F, *les lignes*, N Q *&* S V, *sont toujours au méme endrét du tableau qu'elles seroient aiant esté menées au moïen des lignes*, A G, *&* C G.

Finalement la piece de la ligne, a b, A B, *laquelle se rencontre du côté de l'espace auquel on a fait une semblable operation, comme ici la piece*, a c, A C, *est diuisée en autant de parties égales qu'en contient la distance de l'œil au tableau.*

Ici la distance de l'œil au tableau contient vingt-quatre pieds de longueur, partant cette piece, a c, A C, *de la ligne*, a b, A B, *est diuisée en vingt-quatre parties égales marquées sous elle, qui sont comme autant de pieds, l'vn desquels sa moitié ou son quart peut au besoin étre encore sous-diuisé en ses pouces & lignes.*

Lors est acheuée l'vne des deux preparations qui concernent la perspectiue entreprise, laquelle preparation forme vne figure ici nommée ECHELLE *des* ELOIGNEMENS, *dira qui voudra d'optique ou autrement.*

Dauantage, de tel poinct que ce soit commode pour l'ouurage, en la ligne, A B, a b, *comme ici du poinct*, G, g, *sont menées des lignes aux poincts de la premiere diuision en douze pieds égaux de la ligne entiere*, A B, a b.

Dans cét exemple ces lignes sont menées du poinct, G, g, *seulement aux poincts de cette diuision, qui sont en la piece de cette ligne*, A B, a b, *qui se rencontre du côté de l'espace*, G C B F, g c b f, *laquelle est ici la piece*, B C, b c, *dautant qu'il sufit de cela, voire de moindre nombre: Et de méme du poinct*, G, g, *sont menées des lignes aux poincts de la sous-diuision de l'vn de ces douze pieds, ici le septiéme, sa moitié ou son quart en ses pouces.*

Lors est acheuée l'autre des deux preparations qui concernent la perspectiue entreprise, laquelle preparation forme vne figure en triangle, G C B, g c b, *ici nommée* ECHELLE *des* MESVRES, *dira qui voudra Geometrique ou autrement, & qui dans cette maniere de pratiquer la perspectiue, est à l'ouurier vn outil de méme vsage que le compas de proportion.*

Ces deux échelles des éloignemens & des mesures pour la perspectiue, peuuent au besoin étre faites ailleurs, & disposées autrement au tableau méme en nombre comme innombrable, de manieres diferentes qui reuiennent toutes à méme chose.

Et au moien du raport ou de la corespondance qu'il y a de l'vne de ces deux échelles à l'autre, on fait ce que l'on desire en perspectiue.

Car auec l'échelle des éloignemens on trouue les places au tableau des aparences de chaque poinct remarquable du plan de l'asiete du sujet, & du sujet méme.

Et auec l'échelle des mesures on trouue les diuerses mesures de chacune des *lignes du sujet qui sont paralelles au tableau, suiuant leurs diuers éloignemens au regard du tableau méme, & l'angle sous lequel elles sont veuës.*

Maintenant, les lignes, A B, a b, *& a b, considerées comme vne seule & méme ligne, il auient de ces preparations que l'aparence de la ligne, a g, est en la ligne,* A G, a g, *& que l'aparence de la ligne, b q, est en la ligne,* B G, b g.

Dauantage, il auient que la ligne, A G, a g, *se trouue retranchée du côté du bout,* G, g, *premierement en sa moitié, puis en sa troisiéme, puis en sa quatriéme partie, & ainsi de suite en autant de parties que l'on continuë de fois l'operation qui fait l'échelle des éloignemens.*

De plus, il auient que le poinct du premier de ces retranchemens de la ligne, A G, a g, *qui est le poinct auquel la ligne,* H D, h d, *la rencontre, est l'aparence d'vn poinct en la ligne, a g, reculé 24. pieds derriere le tableau, sçauoir aussi loin du tableau derriere lui, que l'œil est éloigné du méme tableau deuant lui.*

Et que le poinct du deuxiéme de ces retranchemens de la ligne, A G, a g, *qui est celuy auquel la ligne,* N Q, n q, *la rencontre, est l'aparence d'vn autre poinct en la ligne, a g, reculé 48. pieds derriere le tableau, sçauoir deux fois aussi loin du tableau derriere lui, que l'œil est éloigné du méme tableau deuant lui.*

Et que le poinct du troisiéme de ces retranchemens de la ligne, A G, a g, *qui est celui auquel la ligne,* S V, ſ u, *la rencontre est l'aparence d'vn autre poinct de la ligne, a g, reculé 72. pieds derriere le tableau, sçauoir trois fois aussi loin du tableau derriere lui, que l'œil est éloigné du méme tableau deuant lui.*

Et semblablement des autres semblables lignes quand on continuë plus de fois l'operation qui fait l'échelle des éloignemens.

D'abondant, il auient que les mémes lignes de l'échelle des mesures qui venans du poinct, G, g, *aux poincts de la premiere diuision en 12. pieds de la ligne,* A B, a b, *marquent & diuisent cinq de ces 12. pieds en la piece,* B C, b c, *de cette ligne,* A B, a b, *les mémes lignes marquent & diuisent les pieces qu'elles rencontrent des lignes,* H D, h d, N Q, n q, S V, ſ u, *& de leurs paralelles chacune de méme en cinq pieds égaux entr'eux, qui sont autant d'échelles diferentes pour les diuerses mesures des aparences des lignes du sujet, paralelles au tableau, & situées à diuers éloignemens au regard du tableau méme.*

Il auient finalement de ces preparations, que la ligne, A B, a b, *contenant 12. pieds de long, la ligne,* H D, h d, *en contient 24. la ligne,* N Q, n q, *36. & la ligne,* S V, ſ u, *48. cét à sçauoir chacune de ceux que l'échelle des mesures marque en la piece qu'elle en rencontre.*

Desquelles choses il est euident que la ligne, H D, *est l'aparence d'vne ligne du plan de l'asiete du sujet, paralelle à la ligne, a b, & reculée 24. pieds*

B ij

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derriere le tableau. Mais le poinct, m, n'est reculé que 17. pieds derriere le tableau méme, donc ce poinct, m, est en vne ligne, comme, r m, paralelle à la ligne, a b, & reculée 7. pieds moins du tableau derriere lui, que n'en est reculée celle que la ligne, H D, *represente.*

L'aparence de ce poinct, m, est donc trouuée en cette façon.

Premierement, auec l'échelle des éloignemens est trouué vn poinct en la ligne, A G, *qui soit l'aparence d'vn poinct en la ligne, a g, reculé 17. pieds loin du tableau, c'est à dire, est premierement trouuée l'aparence du poinct, r, & pour ce faire, du poinct,* F, *est menée vne ligne au poinct qui marque la 17ᵉ, & la separe d'auec la 18ᵉ des 24. parties égales de la ligne,* A C, *& le poinct auquel cette ligne ainsi menée rencontre la ligne,* A G, *ici le poinct,* R, *est l'aparence d'vn poinct en la ligne, a g, reculé 17. pieds loin du tableau, c'est à dire, que le poinct,* R, *est l'aparence du poinct, r, puis par le poinct,* R, *est menée la ligne,* R M, *paralelle à la ligne,* A B, *laquelle ligne,* R M, *est l'aparence de la ligne, r m, en laquelle est le poinct, m, partant l'aparence du poinct, m, est en cette ligne,* R M.

Et dautant que le poinct, m, est en la ligne, r m, à dréte de la ligne, a g, vn pied & demi loin du poinct, r, la ligne, R M, *alongée qu'elle trauerse l'échelle des mesures, lors auec vn compas commun est prise la longueur d'vn pied & demi, de ceux que l'échelle des mesures marque en cette ligne,* R M, *& le compas ouuert de cette mesure, vne de ses iambes est aiustée au poinct,* R, *& son autre iambe est tournée à dréte de la ligne,* A G, *& arrestée sur la méme ligne,* R M, *& comme au poinct,* M, *lequel est l'aparence du poinct, m.*

L'aparence du poinct, k, est trouuée en la façon qui suit.

Consideré que la ligne, a r, à 17. pieds de long, la ligne, r h, en à 9. & la ligne, h n, en à 3. aiant aiousté ces trois nombres 17, 9, & 3, leur somme est 29. de façon que ce poinct, k, se rencontre en vne ligne paralelle à la ligne, a b, & reculée 29. pieds loin du tableau derriere lui, sçauoir est cinq pieds dauantage loin que n'en est reculée celle que la ligne, H D, *represente.*

En ce cas, Premierement auec l'échelle des éloignemens est trouuée en la ligne, A G, *l'aparence d'vn poinct en la ligne, a g, reculé 29. pieds loin du tableau, c'est à dire, cinq pieds dauantage loin que n'en est reculée la ligne que la ligne,* H D, *represente; & pour ce faire, du poinct,* G, *est menée vne ligne au poinct qui marque la 5ᵉ & la separe d'auec la 6ᵉ des 24. parties égales de la ligne,* A C. *Par le poinct auquel la ligne ainsi menée rencontre la ligne,* H D, *est menée vne autre ligne au poinct,* F, *& le poinct auquel cette derniere ligne rencontre la ligne,* A G, *est l'aparence du poinct, n, puis par cette aparence du poinct, n, est menée vne ligne paralelle à la ligne,* A B, *laquelle est l'aparence de la ligne, n k, en laquelle est le poinct, k, partant l'aparence du poinct, k, est en cette derniere ligne.*

Et dautant que le poinct, k, est en la ligne, n k, à gauche de la ligne, a g, sept pieds & demi loin du poinct, n, aiant alongé la ligne derniere menée au tableau

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tableau paralelle à la ligne, A B, c'est à dire celle qui est l'aparence de la ligne k n; afin qu'elle trauerse l'échelle des mesures; lors auec vn compas commun sont pris 7. pieds & demi de ceux que l'échelle des mesures y marque, & le compas ouuert de cette mesure, vne de ses iambes est aiustée à l'aparence du poinct, n, & son autre iambe tournée à gauche de la ligne, A G, & arrestée sur la méme ligne ainsi derniere menée, & comme au poinct, K, lequel par ce moïen est l'aparence du poinct, k.

Si l'on vouloit auoir en la ligne, A G, l'aparence d'vn poinct en la ligne, a g, reculé 53. pieds loin derriere le tableau, sçauoir 5. pieds dauantage loin que n'en est reculée la ligne que represente la ligne, N Q. En ce cas aiant mené la ligne du poinct, G, au poinct qui marque la 5ᵉ, & la separe d'auec la 6ᵉ des 24. parties égales de la ligne, A C, lors du poinct auquel cette ligne ainsi menée rencontre la ligne, N Q, l'on meneroit vne ligne au poinct, F, laquelle rencontreroit la ligne, A G, en vn poinct lequel est l'aparence d'vn poinct en la ligne, a g, reculé 5. pieds dauantage loin du tableau que n'en est reculée la ligne, que la ligne, N Q, represente, & ainsi des semblables.

Les poincts, L, & I, aparences des poincts, l, & i, sont trouués en la méme façon.

Apres sont menées conuenablement de poinct en poinct les lignes, M L, M K, K I, & L I, qui sont les aparences chacune de sa corespondante des côtez, m l, m k, k i, & l i, du quarré, m, l, i, k.

Maintenant pour trouuer l'aparence d'vn poinct éleué 17. pieds à plomb au dessus du poinct, m. Par le poinct, M, est menee de la part de la ligne, F E, vne ligne, M ST, perpendiculaire à la ligne, A B, & cette ligne, M ST, est faite égale à 17. des pieds que l'échelle des mesures marque en la ligne, M R, ainsi la ligne, M ST, est l'aparence de l'éleuation du sujet, haute de 17. pieds à plomb sur le poinct, m.

Les lignes, L ff, K fr, & I sp, aparences des éleuations du sujet sur les autres poincts, l, k, i, de son assiete quarrée, m, l, i, k, & longues aussi chacune de 17. pieds, sont trouuées de méme façon que l'aparence, M ST, bien entendu que les 17. pieds dont chacune de ces aparences est longue, sont de ceux que l'échelle des mesures marque en la ligne menée par son bout d'embas paralelle à la ligne, A B.

Pour auoir les aparences des abaissemens du sujet vn pied sous les mémes poincts, m, l, i, k, & par les mémes lignes des éleuations, on alonge par embas les aparences de ces éleuations chacune vn pied de long de sa mesure propre & particuliere; & par les poincts bas du pied dont ces aparences là sont alongées, on méne des lignes conuenables desquelles on marque ce que le dehors œuure en l'assiete du sujet, n'empesche pas d'étre veu comme le montre la figure du bas de la Stampe.

Dabondant la ligne, Z, longue de 13. pieds vn quart, étant la mesure à plomb de ce, dont le poinct auquel aboutissent les aréstiers du couuert, est éleué

C

dessus le poinct milieu de l'aßiete du sujet plus haut que chacune de ses encoigneures, les aparences de ces arétiers sont trouuées en la méme façon.

Car aiant au moien ci-dessus trouué le poinct, æ, aparence du poinct auquel aboutissent les aretiers au feste du sujet, lors de chacun des poincts hauts des aparences des éleuations des encoigneures ici des poincts, st, ff, fr, & sp, sont menées a ce poinct, æ, les lignes, st æ, ff æ, fr æ, & sp æ, lesquelles sont les aparences chacune de sa corespondante des lignes de ces arétiers.

Les lignes, Ꝟ, z, W, & Ꝁ. sont les mesures des hauteurs de quelques personnes debout en diuers endréts du plan de l'aßiete du sujet.

La ligne, x, est la mesure de la hauteur d'vne personne debout sur le fonds du creux de la cage, lequel fonds est supposé de niueau comme celui d'vn bassin de fontaine.

La ligne, ß, est l'aparence d'vne ligne de 12. pieds de long, qui pose d'vn bout sur le plan de l'aßiete du sujet en la ligne alongée, h l, 4. pieds 9, pouces loin du poinct, l, & apüie de l'autre bout au montant que la ligne, L ff, represente.

La ligne, ⁎, est l'aparence d'vne ligne de 5. pieds de long, suspenduë ou pendante à plomb du milieu de la cime de l'vn des flancs du sujet.

Ces aparences là, celles de chacun des membres des ornemens de l'architecture, celles de la cheute des ombres, & generalement les aparences de toute chose telle qu'elle puisse estre de nature à representer en portraiture, moiennant les interuales conuenables coneüs sont ainsi trouuez en vn tableau plat de quelque façon & biais qu'il soit disposé, pendant à plomb en plat fonds, ou penchant d'vn ou d'autre côté deuant l'œil, soit que le poinct qu'on nomme à l'ordinaire poinct de veuë, se rencontre dans ce tableau, soit qu'il en soit hors; mais en chacune de ces diferentes circonstances, il y a matiere de nombre d'exemples diferens comme de plusieurs figures: outre que l'intelligence de cette maniere de faire les tableaux plats, conduit aisément au moien de faire les tableaux en toute autre espece de surface, & des filets atachez aux poincts F & G, releuent l'ouurier de beaucoup de lignes fausses.

Il y a régle außi de la place du fort & du féble coulory, dont la demonstration est mélée en partie de Geometrie, en partie de Phisique, & ne se troue en France encore expliquée en aucun liure public.

Pour les diuers rencontres en cét art, il y a des moïens particuliers de les expedier chacun aisément à la façon de cét exemple & autrement, ou bien auec des instrumens fondés en demonstration Geometrique, desquels il y a diuerses façons.

Les vns pour copier diligemment tout sujet plat en plus petit, égal, ou plus grand, & le métre de méme en perspectiue auec ses éleuations, de quelque façon, biais & distance que ce soit, aussi promptement qu'on l'auroit copié.

Les autres pour desiner exactement le sujet en le voïant par vne figure plus petite, égale, ou plus grande, & semblablement posée que celle qui vien-

droit au plan méme auquel l'instrument est apliqué, desquels instrumens, ou de l'vn d'eux, a été fait à Rome vn traité deux ans enuiron apres le priuilege des presentes seelé en France, lequel traité de Rome ne contient pas le moïen d'auoir la figure d'aparence, égale & disposée comme celle qui se fait au méme plan auquel l'instrument est apliqué.

Il y a de méme des manieres vniuerselles & demonstrées, touchant la pratique du trait pour la coupe des pierres en l'Architecture, auec les preuues pour conétre si l'on a procedé bien exactement à l'execution.

Il y a de suite des manieres vniuerselles aussi demontrées pour tracer les quadrans solaires auec la régle, le compas, le plomb & l'equiere communs, en toutes les surfaces plates generalement, ou l'essieu du monde est conuenablement apliqué, de quelque sens ou biais qu'elles soient étenduës.

En ce reste de place les contemplatifs auront quelques propositions lesquelles peuuent étre enoncées autrement pour diuerses matieres, mais elles sont accommodées ici pour la perspectiue, & la demonstration en est assez intelligible sans figure, puis que toutes les lignes y sont encore entenduës droites, & les tableaux toujours plats. Il est vrai qu'en fin c'est vne fourmiliere de grandes propositions, abondante en lieux.

Aiant imaginé qu'au centre immobile de l'œil passe vne ligne indeterminée & mobile ailleurs de son long en tous sens, vne telle ligne est ici nommée LIGNE DE L'OEIL, *laquelle au besoin est menée paralelle à telle autre ligne que ce soit.*

Quand le sujet est vn poinct, & que des poincts de sujet & de l'œil, sont menées iusqu'au tableau des lignes paralelles entre elles, l'aparence du sujet est en la ligne menée par les poincts ausquels ces paralelles rencontrent le tableau, dautant que ces paralelles, & cette ligne ainsi menée au tableau, sont en vn méme plan entr'elles.

Quand le sujet est des lignes, elles sont, ou bien paralelles, ou bien inclinées entr'elles.

Quand des lignes sujet sont paralelles entr'elles, la ligne de l'œil menée paralelle à icelles, est ou bien paralelle, ou bien non paralelle au tableau, mais toujours chacune de ces lignes sujet, est en vn méme plan auec cette ligne de l'œil, en laquelle tous ces plans s'entre-coupent ainsi qu'en leur commun essieu.

Quand des lignes sujet sont paralelles entr'elles, & que la ligne de l'œil menée paralelle à icelles est paralelle au tableau, les aparences de ces lignes sujet sont des lignes paralelles entr'elles, aux lignes sujet, & à la ligne de l'œil, à cause que chacune de ces lignes sujet est en vn méme plan auec cette ligne de l'œil, en laquelle tous ces plans s'entre-coupent ainsi qu'en leur commun essieu, & que tous ces plans sont coupez d'vn autre méme plan le tableau.

C ij

Quand des lignes ſujet ſont paralelles entr'elles, & que la ligne de l'œil menée paralelle à icelles n'eſt pas paralelle au tableau; les apparences de ces lignes ſujet, ſont des lignes qui tendent toutes au poinct auquel cette ligne de l'œil rencontre le tableau, dautant que chacune de ces lignes ſujet eſt en vn méme plan auec cette ligne de l'œil, en laquelle tous ces plans s'entre-coupent ainſi qu'en leur commun eſsieu, & que tous ces plans ſont coupez d'vn autre méme plan le tableau.

Quand des lignes ſujet inclinées entr'elles tendent toutes à vn poinct, la ligne de l'œil menée à ce poinct eſt, ou bien paralelle, ou bien non paralelle au tableau, mais toujours chacune de ces lignes ſujet eſt en vn méme plan auec cette ligne de l'œil, en laquelle tous ces plans s'entre-coupent ainſi qu'en leur commun eſsieu.

Quand des lignes ſujet inclinées entr'elles tendent toutes à vn poinct, auquel aiant mené la ligne de l'œil elle eſt paralelle au tableau, les aparences de ces lignes ſujet ſont des lignes paralelles entr'elles, & à la ligne de l'œil à cauſe que chacune de ces lignes ſujet eſt en vn méme plan auec cette ligne de l'œil, en laquelle tous ces plans s'entre-coupent ainſi qu'en leur commun eſsieu, & que tous ces plans ſont coupez d'vn autre méme plan le tableau.

Quand des lignes ſujet inclinées entr'elles tendent toutes à vn poinct, auquel aiant mené la ligne de l'œil elle n'eſt pas paralelle au tableau, les aparences de ces lignes ſujet ſont des lignes qui tendent toutes au poinct auquel cette ligne de l'œil rencontre le tableau, dautant que chacune de ces lignes ſujet eſt en vn méme plan auec cette ligne de l'œil, en laquelle tous ces plans s'entre-coupent ainſi qu'en leur commun eſsieu, & que tous ces plans ſont coupez d'vn autre méme plan le tableau.

La propoſition qui ſuit ne ſe deuide pas ſi briéuement que celles qui precedent.

Aiant à pourtraire vne coupe de cóne plate, y mener deux lignes, dont les aparences ſoient les eſsieux de la figure qui la repreſentera.

A Paris, en May 1636. Auec Priuilege.

Ces Exemplaires ſont és mains de Monſieur Bidault H. du Roy, demeurant au gros Pauillon des Tuylleries, au bout de la grande Galerie du Louure.

Notes

Chapter II

1. Marin Mersenne (1588–1648), *Harmonie Universelle* (1636), vol. II, Book VI, Proposition I, pp. 332–342 [333].

2. This work is translated in Chapter VII. The French text, which has not been reprinted since 1648, is given in Appendix 5.

3. See Lindberg (1976).

4. The treatment of geometrical optics in terms of beams sent out by the eye (extramission of rays) goes back to Euclid. However, in the Renaissance it was commonplace to point out that the mathematics would be the same if one adopted the alternative theory that sight took place by means of intromitted rays, that is by light being received in the eye, as had been suggested by other authors, most notably Alhazen (Ibn al-Haitham, A.D. 965–1039). General acceptance of intromission seems to have followed quite quickly after Johannes Kepler's publication of his version of the theory in 1604 in his *Ad Vitellionem paralipomena quibus astronomiae pars optica traditur*. On this theory of vision, see Lindberg (1976) and Field (1987). *Ad Vitellionem paralipomena* also contains a discussion of points at infinity—see Appendix 4.

5. On Alberti's description, see Gadol (1969). The evidence for artists' use of various techniques of perspective construction is discussed in Kemp (1986b).

6. For recent accounts, see White (1967), Kemp (1985) and Kemp (1986b).

7. See Tormey and Tormey (1982).

8. The ground plan of the scene shown in the picture has been reconstructed by Wittkower and Carter (1953). See also Pirenne (1970) and Kemp (1986b).

9. Giacomo Barozzi da Vignola, *Le Due regole della prospettiva prattica* . . . (1583).

10. Danti's history of perspective treatises is discussed in Field (1985).

11. See Field (1985). Viator's mathematics has been analysed in some detail by Ivins (1938).

12. See Deforge (1981).

13. The eye's tolerance is discussed in Pirenne (1970). See also Kemp (1986b). Piero della Francesca discussed the minimum viewing distance in his *De prospectiva pingendi*, see Field (1986b).

14. See Thomson (1978). Since this projection was in use throughout the Middle Ages it is difficult to take very seriously the suggestion by Edgerton (1975) that conical projection was rediscovered when Ptolemy's *Geographia* became available in the fifteenth century.

15. Benedetti's mathematically rigorous and highly original treatise has been unaccountably neglected by historians. It is discussed in some detail in Field (1985).

16. Giovanni Battista Benedetti, *De gnomonum umbrarumque solarium usu liber* (1574), Appendix 'Novi instrumenti conoidalis . . . descriptio et usus', f 118v, Theorem 5. The total length of the appendix is nine folios, much more than half of the space being taken up with large diagrams. The work is discussed in more detail in Field (1986a).

17. The relationship of the treatise on sundials to Benedetti's other mathematical concerns is discussed in Field (1986a).

18. Daniele Barbaro came from a patrician family and was well known as a humanist scholar. He was Venetian ambassador to the court of King Edward VI of England and later became Patriarch Elect of Aquileia, a position of some political importance. He was a founder-member of the Accademia Olimpica of Vicenza, in 1551, and a personal friend and patron of Andrea Palladio (1508–1580), who designed the Barbaro family villa, and Paolo Veronese (1528–1588), who painted Daniele's portrait. Barbaro's interest in mathematics is thus an interesting indication of the intellectual climate of his times.

19. See Sinisgalli (1978).

20. See Carter (1970) and Andersen (1984).

21. *Harmonie Universelle* (1636), vol. II, Book VI, Proposition I, p. 332.

22. See Blum (1924) for further details.

23. See Blum (1924).

24. It should be pointed out, however, that in the fifteenth and sixteenth centuries the observer's two eyes were generally believed to act exactly as one. The theoretical limitations imposed by the usual assumptions of perspective

construction seem to have worried Leonardo da Vinci (see Kemp, 1977) and many later writers on perspective for artists (see Frangenberg, 1986).

25. See Pirenne (1970).

26. Bosse, *Traité des practiques geometrales*, p. 124, 11.2 f.

Chapter V

1. Descartes to Desargues, 19 June 1639 (Taton 1951, p. 185), translated in Appendix 1.

2. Alberti's Italian version is now believed to be later than his Latin text *De pictura* (1435). See Alberti, ed. Grayson (1972).

3. Alberti ed. Janitschek 1877, p. 57, ll.–11 ff.

4. *De pictura* (Basel 1540).

5. Alberti *Della pittura*, trans. Domenichini (Florence 1547); and Alberti *Opusculi morali*, trans. and ed. Bartoli (Venice 1568), in which the translation of *De pictura* is dedicated to Giorgio Vasari.

Chapter VI

The text of the Rough Draft is $29\frac{1}{2}$ pages long, each page having 61 lines. These pages are numbered 1 to 30. They are followed by three unnumbered pages of similar format, headed Notice (*Advertissement*) and giving corrections to the main text. Some of these corrections deal with mere typographical errors, but many make significant alterations, by changes of wording or the addition of several extra lines. It is clear that these changes must be the author's own so we have followed Taton (1951) in adopting them. Since the 1639 text survives only in a single copy and has not been reprinted as it stands, we have indicated all these changes (apart from the typographical errors) in our text.

1. Privilege was a limited form of copyright, usually lasting for fifteen years.

2. Desargues originally wrote '. . . [quantities] which decrease so as to reduce their two opposing extremities to one' ('. . . qui s'apetissent jusques à reduire leurs deux extremitez opposées en une seule . . .'). We have adopted his revised version, given in the Notice appended to the work, which reads '. . . si petites que leurs deux extremitez opposées sont unies entre elles'. It will be noted that this second version removes the reference to change.

3. The earlier version of this sentence ended '. . . in the one case as in the other they all converge to the same point' ('. . . elles tendent toutes à un mesme endroict'). In his Notice Desargues adds 'comme' before 'toutes' giving the version we have adopted in our translation.

4. The word 'often' ('souvent') is added in the Notice.

5. The earlier version had 'points scattered all round the edges of the Plane'.

6. In his Notice Desargues adds 'comme', as in the passage discussed in note 3 above, giving the version we have adopted.

7. 'Axle' ('Essieu') is given in the Notice, to replace the earlier 'butt' ('but'). We have made this change in the marginal note also. The use of the same term for lines and planes was presumably intended to emphasize the analogy between the two different forms of ordinance (i.e. pencil).

8. Again 'axle' (essieu') is given in the Notice to replace the earlier 'butt' ('but').

9. See previous note.

10. See note 8 above.

11. The earlier version adds at the end of this sentence 'always at the same distance from the fixed point' ('tousiours égallement éloignée du poinct immobile'). The Notice calls for these words to be deleted.

12. The earlier version includes at this point the same phrase as in the passage referred to in note 11 above. The Notice again calls for it to be deleted.

13. The Notice simplifies the first part of this sentence, which originally began 'If we follow the train of thought suggested by this idea, we come to see that there is, as it were, a kind of relation between the infinite straight line, which is perpendicular to several other different straight lines, and the line with uniform curvature, which is always at the same distance from the butt of several lines belonging to a single ordinance [the butt being] at a finite distance; that is, the relation of the infinite line and the circular one curved completely round, so that they are like . . .' ('Et suivant la poincte de cette conception, finalement on y void comme une espece de raport entre la ligne droicte infinie, qui est perpendiculaire à plusieurs autres diverses droictes, & la ligne courbée d'une courbure uniforme & qui est tousiours également éloignée du but de plusieurs droictes d'une mesme ordonnance à distance finie; c'est à dire, le rapport de la ligne droicte infinie avec la circulaire en sa pleine rondeur, en façon qu'elles parroissent . . .').

14. The words '*parallel branches*, in relation to one another' ('*Rameaux paralels* entre eux') are added in the Notice.

15. The preceding sentence and its note are added in the Notice. No copies of Desargues' original plate seem to have survived.

16. *AFH* in Desargues' text.

17. The references to the converses of these three propositions are all added in the Notice.

18. The note is added in the Notice. The proposition to which Desargues refers is on his page 10, line 49 ff (p. 91 in our translation). Compare Ptolemy, *Almagest* I, 13 (Ptolemy, trans. Toomer, 1984, pp. 64–65).

19. This sentence is added in the Notice, with the introductory formula 'By definition' ('Par definition') which we have omitted, substituting a marginal note in the style of those accompanying other definitions. In so doing we are following Taton (1951).

20. Sentence added in the Notice. We have proceeded as described in note 19 above.

21. Sentence added in the Notice. We have proceeded as described in note 19 above.

22. The phrase translated 'folded to the trunk' ('plié au tronc') is added in the Notice.

23. The words translated 'both finish' are 'aboutissent ensemble'. The word 'ensemble' is added in the Notice.

24. The note is added in the Notice, introduced by 'Et par avis' which we have omitted.

25. This seems to mean that we may depart from the usual convention of employing letters for new points in alphabetical sequence in order that we may instead always use particular letters to designate points of a particular kind. An example of this latter practice would be the use of *S* to denote a focus of a conic, a convention which seems to have become securely established only after the publication of Newton's *Principia* (1687) (where the second focus is usually called *H*). However, the choice of the letter *S* clearly derives from the position of the Sun (*Sol*) in one focus of Kepler's elliptical orbit for Mars (*Astronomia Nova*, 1609).

26. The phrase translated 'as are the parts *AG*, *AF*, *AD*, *AC*' ('comme *AG*, *AF*, *AD*, *AC*') is added in the Notice.

27. The word translated 'mean' is originally 'mitoyennes' but is corrected to 'moyennes' in the Notice, presumably for the sake of consistency of vocabulary, since both words have the same significance in this context. (See also note 35 below.)

28. The words translated 'attached to it' ('qu'elle porte') are added in the Notice.

29. See previous note.

30. The final phrase ('& ce qui s'en peut davantage déduire') is not grammatically consistent with what has preceded it. Desargues seems to intend to convey that the example he has given will yield further similar results.

31. The final clause ('& ces deux couples de branches moyennes ensemble ne donnent que les mesmes nœuds moyens d'une seule d'elles') is added in the Notice.

32. We have adopted the version of this paragraph given in the Notice. The earlier version was 'And in this kind of tree when there are two pairs of mean limbs each of them forms on the tree one of these double mean knots, one [knot] on one side of the stump the other on the other' ('Et quand en cette espece d'arbre il y a deux couples de ces branches moyennes chacune d'elles y donne un de ces nœuds moyens doubles, l'un d'une part, l'autre de l'autre part de la souche').

33. Desargues' text reads 'Or l'évenement de semblables especes de conformation d'arbre est frequent aux figures qui viennent de la rencontre d'un Cone avec des Plans en certaine disposition entre eux'. The phrasing of the second part of the sentence echoes that of the title of the work.

34. The French phrase is 'en changeant'. This refers to a standard procedure in handling ratios. In modern notation the result of the previous paragraph may be written

$$\frac{GD.GF}{CD.CF}=\frac{GB.GH}{CB.CH}.$$

The left-hand side denominator and the right-hand side numerator may be 'interchanged' giving Desargues' next result

$$\frac{GD.GF}{GB.GH}=\frac{CD.CF}{CB.CH}.$$

35. Desargues uses the word 'mitoyen'. In an earlier passage his Notice replaces the original 'mitoyen' with 'moyen', presumably for consistency (see note 27 above).

36. Desargues' sentence begins 'Ce qui est incomprehensible...'. Presumably he means that no commonsense interpretation is possible. He goes on to show that no such interpretation is necessary, thus justifying his avoidance of philosophical discussion of the notion of infinity at this juncture. Both the infinitely small and the infinitely large presented intractable problems to philosophers at this time.

37. The phrase which starts after '*AD*', and ends here is added in the Notice.

38. These are all standard operations with ratios. For interchanging see note 34 note.

39. We have adopted the version of this phrase given in the Notice ('& au cas de ces deux couples de branches ainsi moyennes'), which is rather clearer than the original version 'and in the case of two pairs of mean limbs' ('& au cas de deux couples de branches moyennes').

40. In the Notice Desargues adds 'ces' and 'ainsi' in the same way as in the passage at the beginning of the last paragraph (referred to in note 39 above). We have adopted his revised version.

41. The phrase 'in this same case' ('en ce mesme cas') is added in the Notice.

42. We have adopted the version of this phrase given in the Notice. The earlier version was 'when a point bisects the straight-line interval between two other points' ('qu'un poinct mypartit l'intervalle droicte d'entre deux autres poincts').

43. No marginal note appears at this point in the 1639 edition of the Rough Draft. We have followed Taton (1951) in supplying a note like those that accompany Desargues' other definitions.

44. On the use of 'intermediate' for 'mean' see note 35 above.

45. The section from 'and, by alternation . . .' to the end of this sentence is added in the Notice.

46. On the use of 'intermediate' for 'mean' see note 35 above. Later in this sentence the original 'intermediate' is replaced by 'mean' in the Notice.

47. For the use of 'intermediate' for 'mean' see note 35 above.

48. We have adopted the version given in the Notice. The earlier version of this sentence began 'Further, the ratio of FB to FA is equal to . . .' ('Davantage FB, est à FA, comme . . .').

49. On the use of 'intermediate' for 'mean' in two places in this paragraph see note 35 above.

50. At this point Desargues' earlier version included 'and the ratio of HB to HG is equal to the ratio of FB to FA, which is half of FG' ('& HB, est à HG, comme FB, est à FA, moitié de FG'). The Notice calls for these words to be deleted, as being 'superfluous'.

51. At this point Desargues' earlier version included the paragraph: 'That is, the ratio of the rectangles BH, BA (or the equal rectangle BG, BF) to the rectangle AH, AB (or the equal rectangle, AG, AG or AF, AF) is equal to the ratio of HB to HA'. ('C'est à dire, que le rectangle BH, BA, ou son égal le rectangle BG, BF, est au rectangle AH, AB, ou à son egal le rectangle AG, AG, ou AF, AF, comme HB, est à HA.') The Notice calls for the whole paragraph to be deleted.

52. The earlier version had the last two sentences in reverse order. We have adopted the order called for in the Notice.

53. 'As it were' ('comme') is added in the Notice.

54. 'As it were' ('comme') is added in the Notice.

55. The note is added in the Notice, introduced by 'Et par avis', which we have omitted.

56. 'As it were' ('comme') is added in the Notice.

57. The earlier version had 'for the four mean knots' ('aux quatre nœuds moyens'). We have adopted the wording given in the Notice ('aux couples de nœuds moyens').

58. 'Qui en fourmille', which might be translated more literally as 'which is crawling with them'.

59. See passage referred to in note 18 above, p. 74.

60. The note is added in the Notice, introduced by 'Et par avis', which we have omitted.

61. The note is added in the Notice, introduced by 'Et par avis', which we have omitted.

62. On the use of 'intermediate' for 'mean' see note 35 above.

63. The last clause of this sentence is added in the Notice.

64. The note is added in the Notice, introduced by 'Et par avis', which we have omitted.

65. The word 'suitably' ('convenablement') is added in the Notice.

66. We have followed Taton (1951) in emending Desargues' 'de' to the 'à' which is required by the sense.

67. The note is added in the Notice, introduce by 'Par avis', which we have omitted.

68. 'This' is added in the Notice (by substituting 'à ce' for 'au').

69. The note is added in the Notice, introduced by 'Et par avis', which we have omitted.

70. This phrase is added in the Notice.

71. This phrase is added in the Notice.

72. The note is added in the Notice, introduced by 'Et par avis', which we have omitted.

73. We have translated Desargues' metaphor literally. Although Desargues does not mark any break in his text at this point, the next section begins his discussion of the cone and conic sections.

74. Desargues wrote 'en francez, defaillement'. Since his neologism seems to alter the ending of 'défaillance', ours does the same with 'deficiency'.

75. Desargues wrote 'en francez, égalation'. Since his neologism seems to be based on 'égal', ours has been based on 'equal'.

76. Desargues wrote 'en francez, outrepassement ou excedement'. Since his neologism apparently makes a noun from the verb 'excéder', ours does the same for 'exceed'.

77. The note is added in the Notice, introduced by 'Par avis', which we have omitted.

78. This definition and those of the ordinal, diametral and diametransversal, are added in the Notice, introduced by 'Par definition', which we have omitted. We have followed Taton (1951) in supplying marginal notes like those that accompany Desargues' other definitions.

79. The note is added in the Notice, in italic like most of Desargues' other notes, introduced by 'By way of explanation' ('Par forme d'éclaircissemens') which we have omitted.

80. We have adopted the version given in the Notice, which substitutes 'the butt of their ordinance' for the earlier 'their transversal' ('leur traversale').

81. The note is added in the Notice, introduced by 'Par avis', which we have omitted.

82. On the use of 'intermediate' for 'mean' see note 35 above.

83. The words 'the same as' ('mesme que la') are added in the Notice.

84. See note 83 above.

85. The note is added in the Notice, introduced by 'Et par avis', which we have omitted.

86. The note is added in the Notice, introduced by 'Par avis', which we have omitted.

87. 'And *BD*' is added in the Notice.

88. The Notice substitutes 'conjugal' ('coniugales') for the 'conjugate' ('coniuguées') of the earlier version. The Notice had already given the definition of 'conjugal' referred to in note 89 below.

89. This definition is added in the Notice, in connection with the correction of a typographical error in the first line of the paragraph which precedes it in our translation. We have added a marginal note like those that accompany Desargues' other definitions.

90. The part of the sentence following the semicolon is added in the Notice.

91. The words 'such as' ('tel que') are added in the Notice. They are presumably intended to indicate that this is a typical example.

92. The note is added in the Notice, in a reference to an earlier line of this paragraph in which a typographical error is to be corrected. The note is introduced by 'Par avis', which we have omitted.

93. 'Of these' is added in the Notice.

94. We have translated the last phrase literally: Desargues refers to 'droictes ordonnées au but *F*'. The context shows that he means 'straight lines

which belong to an ordinance with butt F'. The turn of phrase recurs repeatedly later in the text, and we have continued to translate it as here.

95. 'Extreme' is added in the Notice.

96. The Notice substitutes 'a general section' ('une quelconque coupe') for 'such a section' ('une semblable coupe').

97. The phrase following 'butt' is added in the Notice.

98. The phrase following 'transversal' is added in the Notice.

99. The part of the sentence following the semicolon is added in the Notice.

100. On Pujoz see Chapter III above, and the passage referred to in note 127 below.

101. This sentence is added in the Notice.

102. This sentence is added in the Notice.

103. The note is added in the Notice, introduced by 'Par avis', which we have omitted.

104. The part of the sentence following the comma is added in the Notice.

105. The specific reference to NH is added in the Notice. The earlier version was '. . . any pair H, Z given on it by . . .' ('. . . la quelconque couple H, Z, qu'y donnent . . .').

106. We have followed Taton (1951) in adding a marginal note like those which accompany other definitions.

107. The note is added in the Notice, introduced by 'Par avis', which we have omitted.

108. The words 'if A does not remain the stump' ('A ne demeurant pas souche') are added in the Notice.

109. The words 'in some cases, when A does not remain the stump' ('quelquefois A ne demeurant pas souche') are added in the Notice.

110. The reference to 'a similar result' ('ou semblable [chose]') is added in the Notice.

111. The note is added in the Notice, introduced by the words 'Par avis', which we have omitted. The adjectives 'parallel springing' ('déployez et paralels') are added in the Notice.

112. The words following 'and, moreover . . .' ('& visible d'ailleurs en sa generation') are added in the Notice, as also is the note, the latter being introduced by 'Par avis', which we have omitted.

113. The note is added in the Notice, introduced by the words 'Par avis' which we have omitted.

114. The words ‘or ordinal’ (‘ou ordinal’) are added in the Notice.

115. The words ‘or ordinals’ (‘ou ordinales’) are added in the Notice.

116. This paragraph is added in the Notice, introduced by ‘Par occasion’ which we have omitted.

117. The phrase ‘or AD, AR, AL, AB’ is added in the Notice.

118. This phrase (‘de cette touchante LR’) is added in the Notice.

119. This phrase (‘en la mesme diametrale & essieu $C7E$’) is added in the Notice.

120. We have followed Taton (1951) in adding a marginal note like those which accompany other definitions.

121. The final phrase (‘au sujet desquels il y a beaucoup à dire’) is added in the Notice.

122. The final phrase (‘évidemment au moyen d’une ramée’) is added in the Notice.

123. This clause (‘mais se touchent à distance infinie’) is added in the Notice.

124. This paragraph is added in the Notice, introduced by ‘For clarification’ (‘Par éclaircissement’) which we have omitted.

125. The note is added in the Notice, introduced by ‘Par avis’ which we have omitted.

126. Our version follows that given in the Notice. In the earlier version the passage ‘. . . each of the pairs of branches springing from this trunk XQ . . . on the transversal VA’ read ‘. . . each of the pairs of branches of this tree which pass through the pairs of extreme knots of this tree, such as FH, AH and RH, ZG, which spring from the trunk XQ and are ordinate to the butts H and G, on the edge of the figure, and of which one of each pair, such as HA and GZB (*sic*), touches the figure, I say that each of such pairs of branches gives on the transversal VA . . .’ (‘. . . chacune des couples de rameaux de cét arbre qui passent aux couples de nœuds extrémes de cét arbre, comme FH, AH, & RG, ZG, déployez à ce tronc XQ, & ordonnez à des buts H, & G, au bord de la figure, & desquels un en chaque couple comme HA, & GZB, touche la figure, chacune dis-ie de semblables couples de rameaux donne en cette traversale VA . . .’).

127. For Pujoz, whose name is here spelled Pujos, see Chapter III above, and the passage referred to in note 100 above.

128. We have followed Taton (1951) in emending the layout of this sentence. In Desargues’ text (p. 28) it is

So the ratio compounded of the ratios of	DT to DV, and of MV to MF.
Is the same as that compounded of the ratios of	BT to BA, and of LA to LF.

129. 'On the trunk *EP*' is added in the Notice.

130. We have followed Taton (1951) in emending the layout of the last two paragraphs. In Desargues' text (p. 28) it is

On account of this, finally, the ratio of	OL to OM is equal to the ratio of CA to CE.
And similarly the ratio of	OL to OM is equal to the ratio of CB to CD.

131. We have translated literally. Presumably the points are described by points on the trunk which coincide with the fixed point.

132. For Chauveau see Chapter III.

133. These are the three propositions stated in the passage referred to by note 17 above (see p. 73).

134. This combination of circular motion with radial seems to be unconnected with the rest of Desargues' treatise. It is reminiscent of Kepler's account of elliptical planetary orbits in *Epitome Astronomiae Copernicanae*, Book IV (1621), a work which was a popular textbook at the time (only its last few books being specifically concerned with heliocentric astronomy). See Aiton (1972).

135. The following four paragraphs are added in the Notice, apparently as a single paragraph. The italicized titles may have been intended as marginal notes, though the outer margin on each of Desargues' pages carries the heading 'Names given' ('Noms imposez') which would seem inappropriate in these instances. The titles, not all reprinted in Taton (1951), are 'Par definition', 'Par proposition', 'Par declaration de sentiment' and 'A propos de la droicte infinie'.

136. Note that Desargues connects mathematicians' reasoning about geometrical entities with natural philosophers' reasoning about concrete things. See also note 36 above and note 137 below.

137. It is clear that Desargues is concerned with what we may call natural space and natural infinitesimals rather than merely with mathematical notions. He shows no inclination to take refuge in mathematics and give a purely axiomatic treatment of infinity and infinitesimals. Such a treatment had, for instance, been given for the roots of negative numbers by Bombelli in his *Algebra* (1572).

138. For the vocabulary in this section, see our vocabulary for Desargues' *Perspective* (1636) in Chapter V.

139. The following section is added in the Notice, where it is marked as being an addition to page 32. There is no page 32 so Desargues' intentions are unclear.

140. The French for the instrument known in English as the sector is 'compas de proportion'. This French name is also used of another, different,

instrument called in English 'proportional compasses'. The sector is illustrated in Fig. 7.3 in Chapter VII.

141. Desargues distinguishes ordinary compasses ('le compas commun') from the sector ('compas de proportion') mentioned earlier (see note 140 above).

142. The words we have translated as 'in the flat' are 'en geometral'. There is no English equivalent for this expression. It refers to drawing plans and elevations not perspective views.

143. Desargues refers to his *Perspective* (1636), which is translated in Chapter VII.

144. Desargues was to publish a *Rough Draft* on sundials in 1640. This work was believed to have been lost but a copy of it came to light in 1983 (Turner, 1984). See Chapter IX.

145. Desargues was to publish a *Rough Draft* on the cutting of stones in 1640 (see Bibliography and Ivins (1942)). It is concerned with three-dimensional geometry, but is practical rather than theoretical and makes no use of the ideas in the *Rough Draft on Conics*.

146. The initials P.B.G. stand for 'Praise Be to God'. Desargues had 'L.S.D.' for 'Loué Soit Dieu'.

Chapter VII

1. Reading *y a nom* for *y a non*.

2. The more usual name is 'base plane of the subject'.

3. More usually 'picture plane'.

4. That is, not proportional compasses or a sector, both of which are called *compas de proportion* in French. See Figs 7.3 and 7.4.

5. That is, it appears in the old treatises, see Chapter II.

6. The word translated 'units' is *toises*. This is cognate with a verb that means 'to measure' (toiser), but usually signifies a particular length, about equivalent to the English fathom (a little under two metres). This sense seems to be impossible in the present context.

7. That is, all the numbers denote lengths in feet.

8. *Treillis*. See note 29 below.

9. 'Eighth' in the 1648 edition.

10. There were twelve lines to the inch in France, and ten in Britain. The plate does not show divisions into lines. The plate in the 1648 edition does not even show division into inches.

11. That is, $AF = 4\frac{1}{2}$ ft in the scale shown on AB. It all seems much clearer when one is accustomed to graphs. Compare Fig. 7.2.

12. Given as AB, **ab** in the 1636 edition, corrected to FGE, **fge** in 1648.

13. Given as $GCBF$, **gcbf** in 1636, corrected to $GCBE$, **gcbe** in 1648.

14. Corrected to 'eighth' in 1648, but no subdivisions are shown in the accompanying diagram. See note 10 above.

15. The French 'compas de proportion' denotes two different instruments, known in English as 'proportional compasses' and 'sector' (see Figs 7.3, 7.4). The context does not make it possible to decide which is intended in this passage.

16. Given as *bq* both in the 1636 edition and in the 1648 edition.

17. *Elles* in 1636, misprinted as *elle* in 1648.

18. That is, presumably, the same number of scales as the number of lines they meet. The French is ambiguous.

19. '*I* and *L*' in the 1648 edition.

20. A word, of masculine gender, is clearly missing in both the texts of 1636 and 1648. I have supplied the word *couvert*, which is found elsewhere and seems to make sense of this passage.

21. Given lower case in 1636, corrected to upper case in 1648.

22. If the figures were all imagined as the same height as the person viewing the picture, and standing at ground level, their heads, or rather their eyes, should all be at the level of the horizon. It was usual to deal with figures of this type in treatises on perspective. Desargues' descriptions of figures are too vague to be useful to prospective draughtsmen.

23. Given as lower case in 1636, corrected to upper case in 1648.

24. For example, like the side pieces in stage scenery. Desargues is usually credited with being the first writer to consider the case where the picture plane is not perpendicular to the line of sight. It was, however, treated by Benedetti in his 'De rationibus operationum perspectivae' in *Diversarum speculationum . . . liber* (1585). See Field (1985).

25. *Point de veuë*, that is the foot of the perpendicular from the eye to the picture. It can lie outside the picture if the picture is designed to be viewed at a steep angle, as is the case for side pieces in stage scenery, to which we have already referred. Drawing stage scenery seems to have been a standard problem in treatises on perspective. The mechanical method used in its solution is shown in several such works, for example the treatise by Vignola of 1583. The illustration of the method is reproduced in Field (1985).

26. *Espece* (1636) misprinted as *espace* in 1648.

27. Desargues shows such threads as curly lines in the figure. In the 1648 figure, presumably drawn by Bosse, the only *F* thread that is shown seems to be that used to draw the dashed line from *F* to the division between the fifth and sixth feet on the line *AB*. Desargues' practical point comes as a reminder that mechanical aids were usual in drawing such pictures. See Baltrušaitis (1976) and Field (1985).

28. This lacuna was filled by the publication of an edition of Leonardo da Vinci's treatise on painting in 1651, see Chapter II above.

29. 'Privilege' was a limited form of copyright, granted at publication.
Although the word *apres* appears in both the editions of 1636 and 1648, it seems that Desargues must mean that the Roman treatise appeared *before* his own, and it seems likely that the work to which he refers is the *Pantographice* of Christoph Scheiner (1573–1640), published in Rome in 1631. Scheiner describes the use of his 'pantograph' to make copies of drawings—larger, smaller or the same size as the original—and, in the second part of the work, as an aid to drawing in perspective (see Figs 7.5, 7.6). Earlier in the *Perspective* (p. 7) Desargues has referred to a copying instrument called a *treillis* (see p. 149, referring to note 8 above) a name which might perhaps have been applied to the pantograph, but which, in view of the reference in the present passage, we have preferred to translate as 'grid'.
The second type of instrument which Desargues describes (and which is not mentioned by Scheiner) is probably the type which sets up measuring devices in the picture field. Examples of this type of machine are shown in many treatises addressed to painters, e.g. that by Vignola (see Chapter II above, and Field (1985) for an illustration).

30. This was published in Desargues' *Brouillon project d'exemple d'une maniere universelle du S.G.D.L. touchant la practique du trait a preuves pour la coupe des pierres en l'Architecture*... (1640). See Ivins (1943), Taton (1951, p. 68), and Chapter II.

31. That is, the gnomon is set parallel to the axis of rotation of the Universe or of the Earth—depending on whether one accepts Tychonic or Copernican astronomy (Tychonic is more likely at this date).

32. This is the subject of Desargues' *Brouillon project du S.G.D.L. touchant une maniere universelle de poser le style & tracer les lignes d'un Quadran aux rayons du Soleil, en quelconque endret possible, avec la Reigle, le Compas l'equiere et le plomb* (1640). One copy of the original edition is now known (see Turner (1984) and Chapter IX). A reconstruction of the work was printed by Poudra in 1864 (see Taton (1951, p. 68)). Desargues' single-page work was rewritten in expanded form by Abraham Bosse, whose edition was published in English translation (by Daniel King) as *Mr De SARGUES Universal Way of Dialling*... (1659).

33. *Icelles* (1636), misprinted *icelle* in 1648.

34. See note 28 above.

Chapter VIII

1. The last line is omitted in the edition of 1648 (see Fig. 8.2), and in Poudra's edition of 1864, and in Taton's of 1951.

2. The original text has $bH - bE$ (equivalent to bH/bE) on the left-hand side of this equation.

3. The butt of any number of concurrent lines is, in modern terms, the vertex of the pencil (see our notes on the *Rough Draft on Conics* and item 2 in the vocabulary list in Chapter V).

4. *la figure de divers plans*, literally 'the figure in several planes'.

5. *celle de relief.*

6. *Bornes ou liens*. For the former term, see our notes on the *Rough Draft on Conics* above and item 16 in the vocabulary list in Chapter V.

7. Reading *Donc* for *Dont*.

8. That is, the point O corresponds to the point o, D to d, and so on.

9. That is, terms which will cancel.

Chapter IX

1. Desargues uses the word 'style' in this passage, though he later also refers to the gnomon by the term 'essieu'. In English, the term 'stylus' for the gnomon seems to have become current only in the eighteenth century. In King's translation of Bosse's revised version of Desargues' work (Bosse: 1643b; King: 1659) we find the words 'axis' or 'axletree', which clearly correspond to Desargues' 'essieu' (literally: axle—the word Desargues also uses in referring to the axes of conics, see Chapter V).

2. It seems possible that the larger proposition (*plus large proposition*) is the elusive *Leçons de ténèbres*. See introductory remarks at the beginning of this chapter.

3. *Les habiles Contemplatifs*. The final paragraphs of Desargues' *Perspective* (1636) were also addressed to 'contemplatifs'. We have adopted the translation 'scholars' in both passages.

4. For our use of the word 'stylus' see note 1 above.

5. Desargues refers to 'l'eslevation du lieu'. This must presumably be a contraction, referring not to the elevation of the place above sea-level but to the height of the pole, i.e. the geographical latitude, which affects the correct position for the gnomon.

6. The word we have translated 'ridge' is 'arreste' (*sic*). Later in the text it appears with the more usual spelling 'areste'.

7. That is, the earth must be cut away to form a hole which will accommodate the first of the system of rods which Desargues proposes to set up. The arrangement is shown in the plates supplied by Bosse for his expanded version of Desargues' work (1643). These same plates seem to have been used in King's translation (1659).

8. The rod, call it $a = AG$, is stuck in the ground at A and oriented so that it casts its shadow, at that moment, at the point A.

9. Mark the shadow of G at B and later at C.

10. Stick rods b along BG, c along CG, by means of the construction with auxiliary rods. Mark equal lengths, r, along GA, GB, GC at A', B', C', respectively.

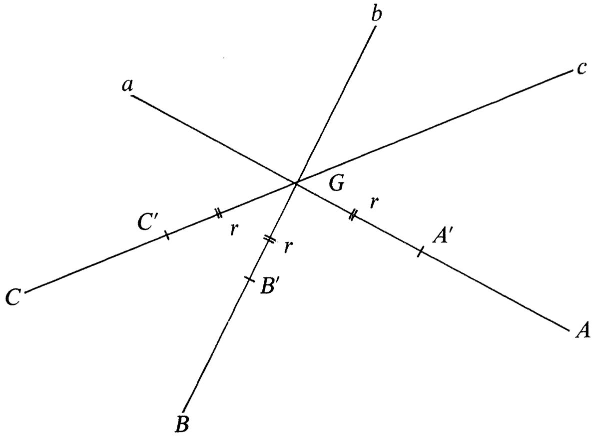

Fig. 9.1. $GA' = GB' = GC' = r$.

Draw $A'B'C'$ on a plane and locate the circumcentre of this triangle by drawing the perpendicular bisectors of its sides. They will meet at the circumcentre.

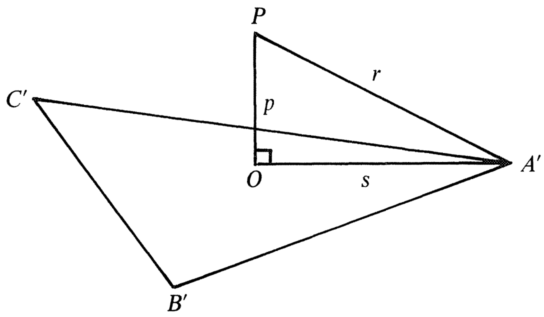

Fig. 9.2. $OA' = s$.

OG is the axis of the cone, let $OA' = s$. Desargues now gives a physical construction of the axis.

11. Join OA', draw OP perpendicular to OA'.

12. Locating P such that $A'P = r$. Let $OP = p$.

13. Fasten wires at A', B' of length $OA' = s$, join them at X.

14. Fasten a rod of length $OP = p$ at X.

15. The word we have translated 'axle' is 'essieu'. See note 1 above.

16. With its other end at G, rotate until the wires AX, $B'X$ are taut. GX is the gnomon of the sundial, since X is now at O (not in the only other possible position, which is obviously wrong).

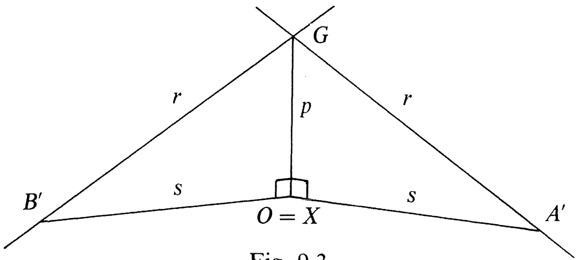

Fig. 9.3

Proof.
p, r, s are sides of a right-angled triangle—by the construction. So $A'\hat{G}X = B'\hat{G}X = C'\hat{G}X$.

17. The work to which Desargues refers is presumably his *Brouillon project d'exemple d'une maniere universelle du S. G. D. L. touchant la practique du trait à preuves, pour la coupe des pierres en l'Architecture* ... (August 1640). The French text of this work is reprinted in Desargues ed. Poudra (1864) vol. I, pp. 305–358. See also Taton (1951, p. 68) and Ivins (1942).

18. This suggests that for Desargues a normal ruler resembled the modern 'triangular' type.

19. P.B.G. stands for 'Praise Be to God'. The French had 'L.S.D.', for 'Loué Soit Dieu'. This phrase also occurs at the end of Desargues' *Rough Draft on Conics* (see Chapter VI).

Appendix 1

1. We have used the French text published in Descartes' *Correspondance*, eds. C. Adam and G. Milhaud, **3** (1940, pp. 228–229). This is reprinted in Taton (1951, pp. 185–186).

2. Descartes seems to have been mistaken in supposing that Desargues had already written a fuller version of the work described in the *Draft*.

3. Descartes' *Geometrie* had been published in 1637.

4. Descartes uses Desargues' term *but*, which we have translated 'butt' as in Desargues' work (see Chapter VI).

5. The French text ends in mid-sentence: 'je vous prie de me croire', which is clearly the beginning of one of the elaborate formulae used to finish letters.

Appendix 2

1. We have used the French text published in Taton (1951, pp. 187–190).

2. Apollonius' *Conics* III,54 asserts

$$\frac{AF\,.\,AG}{AC^2} = \frac{BE^2}{DB^2}\cdot\frac{AE\,.\,DC}{AE^2} \quad \text{or} \quad \frac{AF\,.\,AG}{4AD\,.\,AC} = \frac{BE^2}{DB^2}\,.$$

3. Apollonius' *Conics* I,20 is his use of the method of application of areas to construct the equation of a parabola. That theorem, continued in I,21, provided him with the tripartite division of the conics as well as the choice of names for them according as one areas equalled, exceeded, or fell short of another.

4. Beaugrand's construction is imperfectly described. He must first make R the point of intersection of AC and QH, then define D as the fourth harmonic point of Q, R and H, so that he can appeal to Apollonius' *Conics* II,37.

Appendix 3

1. Reading MK, SV for MA, SA in the original, as did Taton.

2. Reading MK for MP in the original, as did Taton.

3. I.e. MS, NO and PQ are concurrent. This is Pascal's famous theorem. The usual formulation today is obtained as follows. Let the sides NO and PQ meet in Y, then the points S, M and Y are collinear. No trace remains of Pascal's original proof, unless it be the redundant letters A and μ, and the choice of a circle as conic through the six points O, P, N, Q, V, K. The choice is immediately made redundant by Lemma II, which explains how the result is invariant under any projection.

4. I.e. are concurrent.

5. Reading S, T, L, M for L, M, T, S in the original, as did Taton, who pointed out that the original statement was mathematically incorrect but Pascal's figure was correct.

6. This statement asserts

$$\frac{PM}{MA}\cdot\frac{AS}{SQ}=\frac{PL}{LA}\cdot\frac{AT}{TQ}, \quad \text{i.e.} \quad \frac{PM.LA}{MA.PL}=\frac{QS.TA}{SA.QT},$$

which can be restated as the equality of the cross-ratios ($PMLA$) and ($QSTA$). Since the cross-ratios of the lines VP, VO, VN, VQ and KP, KO, KN, KQ are equal—Steiner's criterion, which it is easy to prove when the conic is a circle—the result is immediate: $(PMLA) = (VP, VO, VN, VQ) = (KP, KO, KN, KQ) = (ATSQ) = (TAQS)$.

7. I.e.

$$\frac{EF.FG}{EC.C\delta}\cdot\frac{A\delta}{AG}=\frac{EF.FH}{EC.CB}\cdot\frac{AB}{AH}$$
$$=\frac{FE.FD}{CE.CD}.$$

As Taton noted, the first and third terms are equal in virtue of Menelaus' theorem applied to triangle AFC with $G\delta D$ as transversal, and the second and third terms are likewise equal using triangle AFC and HBD as transversal.

8. Reading FG for FC, as did Taton.

9. I.e.
$$\frac{EF.FG}{EC.C\delta}\cdot\frac{\delta A}{AG}=\frac{FK.FP}{CR.C\psi}\cdot\frac{AR.A\psi}{AK.AP}.$$

10. Reading F for E as did Taton, who also observed that Pascal's figure was, however, correctly labelled.

11. I.e.

$$\frac{MC.MB}{Pf.PD}\cdot\frac{AD.AF}{AB.AC}=\frac{ML.MK}{PH.PG}\cdot\frac{EH.EG}{EK.EL}, \quad \text{a theorem of Menelaus.}$$

12. Reading R for r throughout to accord with Pascal's figure, as did Taton.

13. Reading y for γ as did Taton.

14. I.e.

$$\frac{ZR.Z\psi}{yR.y\psi}=\frac{Sr.S\psi}{xR.x\psi}.$$

Appendix 4

1. Literally 'Things omitted by' (or 'Supplements to') 'Witelo with which the optical part of astronomy is concerned'. The work is reprinted, with introduction and notes, in *Werke*, II. Witelo's *Perspectiva*, probably written in

the 1270s, appeared in several new editions in the sixteenth century, and seems to have been the standard textbook on Optics.

2. Literally 'hearths'. Since light was reflected to the focus, as shown in Fig. A4.2, the focus of the mirror was the position in which one would place the material one wished to burn. In the present context, 'Mechanics' probably refers to the making or use of actual mirrors. (It was only many years after Kepler's day that the word 'mechanics' could be applied to an account of planetary motion.)

3. Although points are indicated by lower case letters in Kepler's figure they are referred to by upper case letters in his text.

4. *Ad Vitellionem paralipomena quibus astronomiae pars optica traditur*, Chapter IV, pp. 93–94, *Werke*, II, p. 91, l. 11—p. 92, l. 7. It will be noted that Kepler regards the hyperbola not as a single section with two branches, but as a pair of separate sections. Moreover, the limiting case of the hyperbola is taken to be a single straight line, since Kepler makes the plane of section move in such a way that when it passes through the vertex of the cone it contains only one generator (see Davis 1975)—see the following quotation below and line *mq* in Fig. A4.1.

5. *Ad Vitellionem paralipomena* . . ., Chapter IV, p. 94, *Werke*, II, p. 92, ll. 8–10.

6. The *locus classicus* for his opinion on the matter is in a work on cosmology: *Harmonices mundi libri V* (1619), Book I, Introduction, p. 6, *Werke*, VI, p. 20, ll. 1–2.

7. A fuller discussion of Kepler's heuristic use of mathematical analogy is given in Field (1987).

8. One cannot help suspecting that, given the problems encountered in figuring a mirror, actual burning mirrors may well have been more nearly spherical than parabolic.

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1648 The Three Geometrical Propositions published in A. Bosse, *Maniere universelle de Mr. Desargues, etc*, full citation below.

The printing histories of these works are as follows:

The *Perspective* (1636) was reprinted, almost unaltered, by Bosse (1648) and in an emended form by Poudra (1864). It appears here in its original form (Appendix 5) and for the first time in English translation (Chapter VII).

The *Rough Draft on Conics* (1639) was reprinted, from a manuscript copy by Philippe de la Hire, by Poudra (1864) and, from the then newly-rediscovered printed text, by Taton (1951). It appears here in its entirety in English translation for the first time (Chapter VI).

The treatise on stone cutting (1640) was reprinted by Poudra (1864).

The treatise on sundials (1640) was reconstructed by Poudra (1864), from the expanded version of Bosse (1643b), and reprinted, from the then newly-rediscovered printed text, by Turner (1984). It appears here in English translation for the first time (Chapter IX).

The three geometrical propositions (1648) were reprinted by Poudra (1864) and Taton (1951). They are translated in Chapter VIII.

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