WHAT IS ELEMENTARY GEOMETRY?

ALFRED TARSKI
Institute for Basic Research in Science,
University of California, Berkeley, California, U.S.A.

In colloquial language the term *elementary geometry* is used loosely to refer to the body of notions and theorems which, following the tradition of Euclid's *Elements*, form the subject matter of geometry courses in secondary schools. Thus the term has no well determined meaning and can be subjected to various interpretations. If we wish to make elementary geometry a topic of metamathematical investigation and to obtain exact results (not within, but) about this discipline, then a choice of a definite interpretation becomes necessary. In fact, we have then to describe precisely which sentences can be formulated in elementary geometry and which among them can be recognized as valid; in other words, we have to determine the means of expression and proof with which the discipline is provided.

In this paper we shall primarily concern ourselves with a conception of elementary geometry which can roughly be described as follows: *we regard as elementary that part of Euclidean geometry which can be formulated and established without the help of any set-theoretical devices.* [1]

More precisely, elementary geometry is conceived here as a theory with standard formalization in the sense of [9]. [2] It is formalized within ele-

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[1] The paper was prepared for publication while the author was working on a research project in the foundations of mathematics sponsored by the U.S. National Science Foundation.

[2] One of the main purposes of this paper is to exhibit the significance of notions and methods of modern logic and metamathematics for the study of the foundations of geometry. For logical and metamathematical notions involved in the discussion consult [8] and [9] (see the bibliography at the end of the paper). The main metamathematical result upon which the discussion is based was established in [7]. For algebraic notions and results consult [11].

Several articles in this volume are related to the present paper in methods and results. This applies in the first place to Scott [5] and Szmielew [6], and to some extent also to Robinson [3].

mentary logic, i.e., first-order predicate calculus. All the variables $x, y, z, \ldots$ occurring in this theory are assumed to range over elements of a fixed set; the elements are referred to as points, and the set as the space. The logical constants of the theory are (i) the sentential connectives — the negation symbol $\neg$, the implication symbol $\rightarrow$, the disjunction symbol $\vee$, and the conjunction symbol $\wedge$; (ii) the quantifiers — the universal quantifier $\bigwedge$ and the existential quantifier $\bigvee$; and (iii) two special binary predicates — the identity symbol $=$ and the diversity symbol $\neq$. As non-logical constants (primitive symbols of the theory) we could choose any predicates denoting certain relations among points in terms of which all geometrical notions are known to be definable. Actually we pick two predicates for this purpose: the ternary predicate β used to denote the betweenness relation and the quaternary predicate δ used to denote the equidistance relation; the formula $\beta(xyz)$ is read *y lies between x and z* (the case when y coincides with x or z not being excluded), while $\delta(xyzu)$ is read *x is as distant from y as z is from u.*

Thus, in our formalization of elementary geometry, only points are treated as individuals and are represented by (first-order) variables. Since elementary geometry has no set-theoretical basis, its formalization does not provide for variables of higher orders and no symbols are available to represent or denote geometrical figures (point sets), classes of geometrical figures, etc. It should be clear that, nevertheless, we are able to express in our symbolism all the results which can be found in textbooks of elementary geometry and which are formulated there in terms referring to various special classes of geometrical figures, such as the straight lines, the circles, the segments, the triangles, the quadrangles, and, more generally, the polygons with a fixed number of vertices, as well as to certain relations between geometrical figures in these classes, such as congruence and similarity. This is primarily a consequence of the fact that, in each of the classes just mentioned, every geometrical figure is determined by a fixed finite number of points. For instance, instead of saying that a point z lies on the straight line through the points x and y, we can state that either $\beta(xyz)$ or $\beta(yzx)$ or $\beta(zxy)$ holds; instead of saying that two segments with the end-points x, y and x', y' are congruent, we simply state that $\delta(xyx'y')$. [3]

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[3] In various formalizations of geometry (whether elementary or not) which are known from the literature, and in particular in all those which follow the lines of [1], not only points but also certain special geometrical figures are treated as

A sentence formulated in our symbolism is regarded as valid if it follows (semantically) from sentences adopted as axioms, i.e., if it holds in every mathematical structure in which all the axioms hold. In the present case, by virtue of the completeness theorem for elementary logic, this amounts to saying that a sentence is valid if it is derivable from the axioms by means of some familiar rules of inference. To obtain an appropriate set of axioms, we start with an axiom system which is known to provide an adequate basis for the whole of Euclidean geometry and contains β and δ as the only non-logical constants. Usually the only non-elementary sentence in such a system is the continuity axiom, which contains second-order variables $X, Y, \ldots$ ranging over arbitrary point sets (in addition to first-order variables $x, y, \ldots$ ranging over points) and also an additional logical constant, the membership symbol $\in$ denoting the membership relation between points and point sets. The continuity axiom can be formulated, e.g., as follows:

$$\bigwedge XY\{\bigvee z \bigwedge xy[x \in X \wedge y \in Y \rightarrow \beta(zxy)] \rightarrow \bigvee u \bigwedge xy\ [x \in X \wedge y \in Y \rightarrow \beta(xuy)]\}.$$

We remove this axiom from the system and replace it by the infinite collection of all elementary continuity axioms, i.e., roughly, by all the sentences which are obtained from the non-elementary axiom if $x \in X$ is replaced by an arbitrary elementary formula in which x occurs free, and $y \in Y$ by an arbitrary elementary formula in which y occurs free. To fix the ideas, we restrict ourselves in what follows to the two-dimensional

individuals and are represented by first-order variables; usually the only figures treated this way are straight lines, planes, and, more generally, linear subspaces. The set-theoretical relations of membership and inclusion, between a point and a special geometrical figure or between two such figures, are replaced by the geometrical relation of incidence, and the symbol denoting this relation is included in the list of primitive symbols of geometry. All other geometrical figures are treated as point sets and can be represented by second-order variables (assuming that the system of geometry discussed is provided with a set-theoretical basis). This approach has some advantages for restricted purposes of projective geometry; in fact, it facilitates the development of projective geometry by yielding a convenient formulation of the duality principle, and leads to a subsumption of this geometry under the algebraic theory of lattices. In other branches of geometry an analogous procedure can hardly be justified; the non-uniform treatment of geometrical figures seems to be intrinsically unnatural, obscures the logical structure of the foundations of geometry, and leads to some complications in the development of this discipline (by necessitating, e.g., a distinction between a straight line and the set of all points on this line).

elementary geometry and quote explicitly a simple axiom system obtained in the way just described. The system consists of twelve individual axioms, A1–A2, and the infinite collection of all elementary continuity axioms, A13.

A1 [IDENTITY AXIOM FOR BETWEENNESS].

$$\bigwedge xy[\beta(xyx) \rightarrow (x = y)]$$

A2 [TRANSITIVITY AXIOM FOR BETWEENNESS].

$$\bigwedge xyzu[\beta(xyu) \wedge \beta(yzu) \rightarrow \beta(xyz)]$$

A3 [CONNECTIVITY AXIOM FOR BETWEENNESS].

$$\bigwedge xyzu[\beta(xyz) \wedge \beta(xyu) \wedge (x \neq y) \rightarrow \beta(xzu) \vee \beta(xuz)]$$

A4 [REFLEXIVITY AXIOM FOR EQUIDISTANCE].

$$\bigwedge xy[\delta(xyyx)]$$

A5 [IDENTITY AXIOM FOR EQUIDISTANCE].

$$\bigwedge xyz[\delta(xyzz) \rightarrow (x = y)]$$

A6 [TRANSITIVITY AXIOM FOR EQUIDISTANCE].

$$\bigwedge xyzuvw[\delta(xyzu) \wedge \delta(xyvw) \rightarrow \delta(zuvw)]$$

A7 [PASCH'S AXIOM].

$$\bigwedge txyzu \bigvee v[\beta(xtu) \wedge \beta(yuz) \rightarrow \beta(xvy) \wedge \beta(ztv)]$$

A8 [EUCLID'S AXIOM].

$$\bigwedge txyzu \bigvee vw[\beta(xut) \wedge \beta(yuz) \wedge (x \neq u) \rightarrow \beta(xzv) \wedge \beta(xyw) \wedge \beta(vtw)]$$

A9 (FIVE-SEGMENT AXIOM).

$$\bigwedge xx'yy'zz'uu'[\delta(xyx'y') \wedge \delta(yzy'z') \wedge \delta(xux'u') \wedge \delta(yuy'u') \wedge \beta(xyz) \wedge \beta(x'y'z') \wedge (x \neq y) \rightarrow \delta(zuz'u')]$$

A10 (AXIOM OF SEGMENT CONSTRUCTION).

$$\bigwedge xyuv \bigvee z[\beta(xyz) \wedge \delta(yzuv)]$$

A11 (LOWER DIMENSION AXIOM).

$$\bigvee xyz[\neg\beta(xyz) \wedge \neg\beta(yzx) \wedge \neg\beta(zxy)]$$

A12 (UPPER DIMENSION AXIOM).

$$\bigwedge xyzuv[\delta(xuxv) \wedge \delta(yuyv) \wedge \delta(zuzv) \wedge (u \neq v) \rightarrow \beta(xyz) \vee \beta(yzx) \vee \beta(zxy)]$$

A13 [ELEMENTARY CONTINUITY AXIOMS]. *All sentences of the form*

$$\bigwedge vw \ldots \{\bigvee z \bigwedge xy[\varphi \wedge \psi \rightarrow \beta(zxy)] \rightarrow \bigvee u \bigwedge xy[\varphi \wedge \psi \rightarrow \beta(xuy)]\}$$

where φ stands for any formula in which the variables $x, v, w, \ldots$, but neither y nor z nor u, occur free, and similarly for ψ, with x and y interchanged.

Elementary geometry based upon the axioms just listed will be denoted by $\mathscr{E}_2$. In Theorems 1–4 below we state fundamental metamathematical properties of this theory. [4]

First we deal with the *representation problem* for $\mathscr{E}_2$, i.e., with the problem of characterizing all models of this theory. By a model of $\mathscr{E}_2$ we understand a system $\mathfrak{M} = \langle A, B, D\rangle$ such that (i) A is an arbitrary non-empty set, and B and D are respectively a ternary and a quaternary relation among elements of A; (ii) all the axioms of $\mathscr{E}_2$ prove to hold in $\mathfrak{M}$ if all the variables are assumed to range over elements of A, and the constants β and δ are understood to denote the relations B and D, respectively.

The most familiar examples of models of $\mathscr{E}_2$ (and ones which can easily be handled by algorithmic methods) are certain Cartesian spaces over ordered fields. We assume known under what conditions a system $\mathfrak{F} = \langle F, +, \cdot, \leq\rangle$ (where F is a set, $+$ and $\cdot$ are binary operations under which F is closed, and $\leq$ is a binary relation between elements of F) is referred to as an ordered field and how the symbols 0, $x - y$, x^2 are defined for ordered fields. An ordered field $\mathfrak{F}$ will be called Euclidean if every non-negative element in F is a square; it is called real closed if it is Euclidean and if every polynomial of an odd degree with coefficients in F has a zero in F. Consider the set $A_{\mathfrak{F}} = F \times F$ of all ordered couples

[4] A brief discussion of the theory $\mathscr{E}_2$ and its metamathematical properties was given in [7], pp. 43 ff. A detailed development (based upon the results of [7]) can be found in [4] — where, however, the underlying system of elementary geometry differs from the one discussed in this paper in its logical structure, primitive symbols, and axioms.

The axiom system for $\mathscr{E}_2$ quoted in the text above is a simplified version of the system in [7], pp. 55 f. The simplification consists primarily in the omission of several superfluous axioms. The proof that those superfluous axioms are actually derivable from the remaining ones was obtained by Eva Kallin, Scott Taylor, and the author in connection with a course in the foundations of geometry given by the author at the University of California, Berkeley, during the academic year 1956–57.

$x = \langle x_1, x_2 \rangle$ with x_1 and x_2 in F. We define the relations $B_{\mathfrak{F}}$ and $D_{\mathfrak{F}}$ among such couples by means of the following stipulations:

$$B_{\mathfrak{F}}(xyz) \textit{ if and only if } (x_1 - y_1)\cdot(y_2 - z_2) = (x_2 - y_2)\cdot(y_1 - z_1),$$
$$0 \leq (x_1 - y_1)\cdot(y_1 - z_1), \textit{ and } 0 \leq (x_2 - y_2)\cdot(y_2 - z_2);$$
$$D_{\mathfrak{F}}(xyzu) \textit{ if and only if } (x_1 - y_1)^2 + (x_2 - y_2)^2 = (z_1 - u_1)^2 + (z_2 - u_2)^2.$$

The system $\mathfrak{C}_2(\mathfrak{F}) = \langle A_{\mathfrak{F}}, B_{\mathfrak{F}}, D_{\mathfrak{F}} \rangle$ is called the (two-dimensional) Cartesian space over $\mathfrak{F}$. If in particular we take for $\mathfrak{F}$ the ordered field $\mathfrak{R}$ of real numbers, we obtain the ordinary (two-dimensional) analytic space $\mathfrak{C}_2(\mathfrak{R})$. [5]

THEOREM 1 (REPRESENTATION THEOREM). *For $\mathfrak{M}$ to be a model of $\mathscr{E}_2$ it is necessary and sufficient that $\mathfrak{M}$ be isomorphic with the Cartesian space $\mathfrak{C}_2(\mathfrak{F})$ over some real closed field $\mathfrak{F}$.*

PROOF (in outline). It is well known that all the axioms of $\mathscr{E}_2$ hold in $\mathfrak{C}_2(\mathfrak{R})$ and that therefore $\mathfrak{C}_2(\mathfrak{R})$ is a model of $\mathscr{E}_2$. By a fundamental result in [7], every real closed field $\mathfrak{F}$ is elementarily equivalent with the field $\mathfrak{R}$, i.e., every elementary (first-order) sentence which holds in one of these two fields holds also in the other. Consequently every Cartesian space $\mathfrak{C}_2(\mathfrak{F})$ over a real closed field $\mathfrak{F}$ is elementarily equivalent with $\mathfrak{C}_2(\mathfrak{R})$ and hence is a model of $\mathscr{E}_2$; this clearly applies to all systems $\mathfrak{M}$ isomorphic with $\mathfrak{C}_2(\mathfrak{F})$ as well.

To prove the theorem in the opposite direction, we apply methods and results of the elementary geometrical theory of proportions, which has been developed in the literature on several occasions (see, e.g., [1], pp. 51 ff.). Consider a model $\mathfrak{M} = \langle A, B, D \rangle$ of $\mathscr{E}_2$; let z and u be any two distinct points of A, and F be the straight line through z and u, i.e., the set of all points x such that $B(zux)$ or $B(uxz)$ or $B(xzu)$. Applying some familiar geometrical constructions, we define the operations $+$ and $\cdot$ on, and the relation $\leq$ between, any two points x and y in F. Thus we say that $x \leq y$ if either $x = y$ or else $B(xzu)$ and not $B(yxu)$ or, finally,

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[5] All the results in this paper extend (with obvious changes) to the n-dimensional case for any positive integer n. To obtain an axiom system for $\mathscr{E}_n$ we have to modify the two dimension axioms, A11 and A12, leaving the remaining axioms unchanged; by a result in [5] ,A11 and A12 can be replaced by any sentence formulated in the symbolism of $\mathscr{E}_n$ which holds in the ordinary n-dimensional analytic space but not in any m-dimensional analytic space for $m \neq n$. In constructing algebraic models for one-dimensional geometries we use ordered abelian groups instead of ordered fields.

$B(zxy)$ and not $B(xzu)$; $x + y$ is defined as the unique point v in F such that $D(zxyv)$ and either $z \leq x$ and $y \leq v$ or else $x \leq z$ and $v \leq y$. The definition of $x \cdot y$ is more involved; it refers to some points outside of F and is essentially based upon the properties of parallel lines. Using exclusively axioms A1–A12 we show that $\mathfrak{F} = \langle F, +, \cdot, \leq \rangle$ is an ordered field; with the help of A13 we arrive at the conclusion that $\mathfrak{F}$ is actually a real closed field. By considering a straight line G perpendicular to F at the point z, we introduce a rectangular coordinate system in $\mathfrak{M}$ and we establish a one-to-one correspondence between points $x, y, \ldots$ in A and ordered couples of their coordinates $\bar{x} = \langle x_1, x_2 \rangle$, $\bar{y} = \langle y_1, y_2 \rangle$, $\ldots$ in $F \times F$. With the help of the Pythagorean theorem (which proves to be valid in $\mathscr{E}_2$) we show that the formula

$$D(xyst)$$

holds for any given points $x, y, \ldots$ in A if and only if the formula

$$D_{\mathfrak{F}}(\bar{x}\bar{y}\bar{s}\bar{t})$$

holds for the correlated couples of coordinates $\bar{x} = \langle x_1, x_2 \rangle$, $\bar{y} = \langle y_1, y_2 \rangle$, $\ldots$ in $F \times F$, i.e., if

$$(x_1 - y_1)^2 + (x_2 - y_2)^2 = (s_1 - t_1)^2 + (s_2 - t_2)^2;$$

an analogous conclusion is obtained for $B(xys)$. Consequently, the systems $\mathfrak{M}$ and $\mathfrak{C}_2(\mathfrak{F})$ are isomorphic, which completes the proof.

We turn to the *completeness problem* for $\mathscr{E}_2$. A theory is called complete if every sentence σ (formulated in the symbolism of the theory) holds either in every model of this theory or in no such model. For theories with standard formalization this definition can be put in several other equivalent forms; we can say, e.g., that a theory is complete if, for every sentence σ, either σ or $\neg\sigma$ is valid, or if any two models of the theory are elementarily equivalent. A theory is called consistent if it has at least one model; here, again, several equivalent formulations are known. If there is a model $\mathfrak{M}$ such that a sentence holds in $\mathfrak{M}$ if and only if it is valid in the given theory, then the theory is clearly both complete and consistent, and conversely. The solution of the completeness problem for $\mathscr{E}_2$ is given in the following

THEOREM 2 (COMPLETENESS THEOREM). (i) *A sentence formulated in* $\mathscr{E}_2$ *is valid if and only if it holds in* $\mathfrak{C}_2(\mathfrak{R})$;

(ii) *the theory* $\mathscr{E}_2$ *is complete* (*and consistent*).

Part (i) of this theorem follows from Theorem 1 and from a fundamental result in [7] which was applied in the proof of Theorem 1; (ii) is an immediate consequence of (i).

The next problem which will be discussed here is the *decision problem* for $\mathscr{E}_2$. It is the problem of the existence of a mechanical method which enables us in each particular case to decide whether or not a given sentence formulated in $\mathscr{E}_2$ is valid. The solution of this problem is again positive:

THEOREM 3 (DECISION THEOREM). *The theory $\mathscr{E}_2$ is decidable.*

In fact, $\mathscr{E}_2$ is complete by Theorem 2 and is axiomatizable by its very description (i.e., it has an axiom system such that we can always decide whether a given sentence is an axiom). It is known, however, that every complete and axiomatizable theory with standard formalization is decidable (cf., e.g., [9], p. 14), and therefore $\mathscr{E}_2$ is decidable. By analyzing the discussion in [7] we can actually obtain a decision method for $\mathscr{E}_2$.

The last metamathematical problem to be discussed for $\mathscr{E}_2$ is the *problem of finite axiomatizability*. From the description of $\mathscr{E}_2$ we see that this theory has an axiom system consisting of finitely many individual axioms and of an infinite collection of axioms falling under a single axiom schema. This axiom schema (which is the symbolic expression occurring in A13) can be slightly modified so as to form a single sentence in the system of predicate calculus with free variable first-order predicates, and all the particular axioms of the infinite collection can be obtained from this sentence by substitution. We briefly describe the whole situation by saying that the theory $\mathscr{E}_2$ is "almost finitely axiomatizable", and we now ask the question whether $\mathscr{E}_2$ is finitely axiomatizable in the strict sense, i.e., whether the original axiom system can be replaced by an equivalent finite system of sentences formulated in $\mathscr{E}_2$. The answer is negative:

THEOREM 4 (NON-FINITIZABILITY THEOREM). *The theory $\mathscr{E}_2$ is not finitely axiomatizable.*

PROOF (in outline). From the proof of Theorem 1 it is seen that the infinite collection of axioms A13 be can equivalently replaced by an infinite sequence of sentences $S_0, \ldots, S_n, \ldots$; S_0 states that the ordered field $\mathfrak{F}$ constructed in the proof of Theorem 1 is Euclidean, and S_n for $n > 0$ expresses the fact that in this field every polynomial of degree $2n + 1$ has a zero. For every prime number p we can easily construct an ordered field $\mathfrak{F}_p$ in which every polynomial of an odd degree $2n + 1 < p$ has a zero while some polynomial of degree p has no zero; consequently, if $2m + 1 = p$ is a prime, then all the axioms A1–A12 and S_n with $n < m$ hold in $\mathfrak{C}_2(\mathfrak{F}_p)$ while S_m does not hold. This implies immediately that the infinite axiom system A1, ..., A12, S_0, ..., S_n, ... has no finite subsystem from which all the axioms of the system follow. Hence by a simple argument we conclude that, more generally, there is no finite axiom system which is equivalent with the original axiom system for $\mathscr{E}_2$.

From the proof just outlined we see that $\mathscr{E}_2$ can be based upon an axiom system A1, ..., A12, S_0, ..., S_n, ... in which (as opposed to the original axiom system) each axiom can be put in the form of either a universal sentence or an existential sentence or a universal-existential sentence; i.e., each axiom is either of the form

$$\bigwedge xy \ldots (\varphi)$$

or else of the form

$$\bigvee uv \ldots (\varphi)$$

or, finally, of the form

$$\bigwedge xy \ldots \bigvee uv \ldots (\varphi)$$

where φ is a formula without quantifiers. A rather obvious consequence of this structural property of the axioms is the fact that the union of a chain (or of a directed family) of models of $\mathscr{E}_2$ is again a model of $\mathscr{E}_2$. This consequence can also be derived directly from the proof of Theorem 1.

The conception of elementary geometry with which we have been concerned so far is certainly not the only feasible one. In what follows we shall discuss briefly two other possible interpretations of the term "elementary geometry"; they will be embodied in two different formalized theories, $\mathscr{E}_2'$ and $\mathscr{E}_2''$.

The theory $\mathscr{E}_2'$ is obtained by supplementing the logical base of $\mathscr{E}_2$ with a small fragment of set theory. Specifically, we include in the symbolism of $\mathscr{E}_2'$ new variables X, Y, ... assumed to range over arbitrary finite sets of points (or, what in this case amounts essentially to the same, over arbitrary finite sequences of points); we also include a new logical constant, the membership symbol $\in$, to denote the membership relation between points and finite point sets. As axioms for $\mathscr{E}_2'$ we again choose A1–A13; it should be noticed, however, that the collection of axiom A13 is now more comprehensive than in the case of $\mathscr{E}_2$ since φ and ψ stand for arbitrary formulas constructed in the symbolism of $\mathscr{E}_2'$. In consequence the theory $\mathscr{E}_2'$ considerably exceeds $\mathscr{E}_2$ in means of expression and power. In $\mathscr{E}_2'$ we can formulate and study various notions which are traditionally discussed in textbooks of elementary geometry but which cannot be expressed in $\mathscr{E}_2$; e.g., the notions of a polygon with arbitrarily many vertices, and of the circumference and the area of a circle.

As regards metamathematical problems which have been discussed and solved for $\mathscr{E}_2$ in Theorems 1-4, three of them — the problems of representation, completeness, and finite axiomatizability — are still open when referred to $\mathscr{E}_2'$. In particular, we do not know any simple characterization of all models of $\mathscr{E}_2'$, nor, do we know whether any two such models are equivalent with respect to all sentences formulated in $\mathscr{E}_2'$. (When speaking of models of $\mathscr{E}_2'$ we mean exclusively the so-called standard models; i.e., when deciding whether a sentence σ formulated in $\mathscr{E}_2'$ holds in a given model, we assume that the variables $x, y, \ldots$ occurring in σ range over all elements of a set, the variables $X, Y, \ldots$ range over all finite subsets of this set, and $\in$ is always understood to denote the membership relation). The Archimedean postulate can be formulated and proves to be valid in $\mathscr{E}_2'$. Hence, by Theorem 1, every model of $\mathscr{E}_2'$ is isomorphic with a Cartesian space $\mathfrak{C}_2(\mathfrak{F})$ over some Archimedean real closed field $\mathfrak{F}$. There are, however, Archimedean real closed fields $\mathfrak{F}$ such that $\mathfrak{C}_2(\mathfrak{F})$ is not a model of $\mathscr{E}_2'$; e.g., the field of real algebraic numbers is of this kind. A consequence of the Archimedean postulate is that every model of $\mathscr{E}_2'$ has at most the power of the continuum (while, if only by virtue of Theorem 1, $\mathscr{E}_2$ has models with arbitrary infinite powers). In fact, $\mathscr{E}_2'$ has models which have exactly the power of the continuum, e.g., $\mathfrak{C}_2(\mathfrak{R})$, but it can also be shown to have denumerable models. Thus, although the theory $\mathscr{E}_2'$ may prove to be complete, it certainly has non-isomorphic models and therefore is not categorical. [6]

Only the decision problem for $\mathscr{E}_2'$ has found so far a definite solution:

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[6] These last remarks result from a general metamathematical theorem (an extension of the Skolem-Löwenheim theorem) which applies to all theories with the same logical structure as $\mathscr{E}_2'$, i.e., to all theories obtained from theories with standard formalization by including new variables ranging over arbitrary finite sets and a new logical constant, the membership symbol $\in$, and possibly by extending original axiom systems. By this general theorem, if $\mathscr{T}$ is a theory of the class just described with at most β different symbols, and if a mathematical system $\mathfrak{M}$ is a

THEOREM 5. *The theory $\mathscr{E}_2'$ is undecidable, and so are all its consistent extensions.*

This follows from the fact that Peano's arithmetic is (relatively) interpretable in $\mathscr{E}_2'$; cf. [9], pp. 31ff.

To obtain the theory $\mathscr{E}_2''$ we leave the symbolism of $\mathscr{E}_2$ unchanged but we weaken the axiom system of $\mathscr{E}_2$. In fact, we replace the infinite collection of elementary continuity axioms, A13, by a single sentence, A13$'$, which is a consequence of one of these axioms. The sentence expresses the fact that a segment which joins two points, one inside and one outside a given circle, always intersects the circle; symbolically:

A13$'$. $\bigwedge xyzx'z'u \bigvee y'[\delta(uxux') \wedge \delta(uzuz') \wedge \beta(uxz) \wedge \beta(xyz) \rightarrow \delta(uyuy') \wedge \beta(x'y'z')]$

As a consequence of the weakening of the axiom system, various sentences which are formulated and valid in $\mathscr{E}_2$ are no longer valid in $\mathscr{E}_2''$. This applies in particular to existential theorems which cannot be established by means of so-called elementary geometrical constructions (using exclusively ruler and compass), e.g., to the theorem on the trisection of an arbitrary angle.

With regard to metamathematical problems discussed in this paper the situation in the case of $\mathscr{E}_2''$ is just opposite to that encountered in the case of $\mathscr{E}_2'$. The three problems which are open for $\mathscr{E}_2'$ admit of simple solutions when referred to $\mathscr{E}_2''$. In particular, the solution of the representation problem is given in the following

standard model of $\mathscr{T}$ with an infinite power α, then $\mathfrak{M}$ has subsystems with any infinite power γ, $\beta \leq \gamma \leq \alpha$, which are also standard models of $\mathscr{T}$. The proof of this theorem (recently found by the author) has not yet been published; it differs but slightly from the proof of the analogous theorem for the theories with standard formalization outlined in [10], pp. 92 f. In opposition to theories with standard formalization, some of the theories $\mathscr{T}$ discussed in this footnote have models with an infinite power α and with any smaller, but with no larger, infinite power; an example is provided by the theory $\mathscr{E}_2'$ for which α is the power of the continuum. In particular, some of the theories $\mathscr{T}$ have exclusively denumerable models and in fact are categorical; this applies, e.g., to the theory obtained from Peano's arithmetic in exactly the same way in which $\mathscr{E}_2'$ has been obtained from $\mathscr{E}_2$. There are also theories $\mathscr{T}$ which have models with arbitrary infinite powers; such is, e.g., the theory $\mathscr{E}_2'''$ mentioned at the end of this paper.

THEOREM 6. *For $\mathfrak{M}$ to be a model of $\mathscr{E}_2''$ it is necessary and sufficient that $\mathfrak{M}$ be isomorphic with the Cartesian space $\mathfrak{C}_2(\mathfrak{F})$ over some Euclidean field $\mathfrak{F}$.*

This theorem is essentially known from the literature. The sufficiency of the condition can be checked directly; the necessity can be established with the help of the elementary geometrical theory of proportions (cf. the proof of Theorem 1).

Using Theorem 6 we easily show that the theory $\mathscr{E}_2''$ is incomplete, and from the description of $\mathscr{E}_2''$ we see at once that this theory is finitely axiomatizable.

On the other hand, the decision problem for $\mathscr{E}_2''$ remains open and presumably is difficult. In the light of the results in [2] it seems likely that the solution of this problem is negative; the author would risk the (much stronger) conjecture that no finitely axiomatizable subtheory of $\mathscr{E}_2$ is decidable. If we agree to refer to an elementary geometrical sentence (i.e., a sentence formulated in $\mathscr{E}_2$) as valid if it is valid in $\mathscr{E}_2$, and as elementarily provable if it is valid in $\mathscr{E}_2''$, then the situation can be described as follows: *we know a general mechanical method for deciding whether a given elementary geometrical sentence is valid, but we do not, and probably shall never know, any such method for deciding whether a sentence of this sort is elementarily provable.*

The differences between $\mathscr{E}_2$ and $\mathscr{E}_2''$ vanish when we restrict ourselves to universal sentences. In fact, we have

THEOREM 7. *A universal sentence formulated in $\mathscr{E}_2$ is valid in $\mathscr{E}_2$ if and only if it is valid in $\mathscr{E}_2''$.*

To prove this we recall that every ordered field can be extended to a real closed field. Hence, by Theorems 1 and 6, every model of $\mathscr{E}_2''$ can be extended to a model of $\mathscr{E}_2$. Consequently, every universal sentence which is valid in $\mathscr{E}_2$ is also valid in $\mathscr{E}_2''$; the converse is obvious. (An even simpler proof of Theorem 7, and in fact a proof independent of Theorem 1, can be based upon the lemma by which every finite subsystem of an ordered field can be isomorphically embedded in the ordered field of real numbers.)

Theorem 7 remains valid if we remove A13′ from the axiom system of $\mathscr{E}_2''$ (and it applies even to some still weaker axiom systems). Thus we see that every elementary universal sentence which is valid in $\mathscr{E}_2$ can be proved without any help of the continuity axioms. The result extends to all the sentences which may not be universal when formulated in $\mathscr{E}_2$ but which, roughly speaking, become universal when expressed in the notation of Cartesian spaces $\mathfrak{C}_2(\mathfrak{F})$.

As an immediate consequence of Theorems 3 and 7 we obtain:

THEOREM 8. *The theory $\mathscr{E}_2''$ is decidable with respect to the set of its universal sentences.*

This means that there is a mechanical method for deciding in each particular case whether or not a given universal sentence formulated in the theory $\mathscr{E}_2''$ holds in every model of this theory.

We could discuss some further theories related to $\mathscr{E}_2$, $\mathscr{E}_2'$, and $\mathscr{E}_2''$; e.g., the theory $\mathscr{E}_2'''$ which has the same symbolism as $\mathscr{E}_2'$ and the same axiom system as $\mathscr{E}_2''$. The problem of deciding which of the various formal conceptions of elementary geometry is closer to the historical tradition and the colloquial usage of this notion seems to be rather hopeless and deprived of broader interest. The author feels that, among these various conceptions, the one embodied in $\mathscr{E}_2$ distinguishes itself by the simplicity and clarity of underlying intuitions and by the harmony and power of its metamathematical implications.

Bibliography

[1] HILBERT, D., *Grundlagen der Geometrie*. Eighth edition, with revisions and supplements by P. BERNAYS, Stuttgart 1956, III+251 pp.

[2] ROBINSON, J., *Definability and decision problems in arithmetic*. Journal of Symbolic Logic, vol. 14 (1949), pp. 98–114.

[3] ROBINSON, R. M., *Binary relations as primitive notions in elementary geometry* This volume, pp. 68–85.

[4] SCHWABHÄUSER, W., *Über die Vollständigkeit der elementaren euklidischen Geometrie*. Zeitschrift für mathematische Logik und Grundlagen der Mathematik, vol. 2 (1956), pp. 137–165.

[5] SCOTT, D., *Dimension in elementary Euclidean geometry*. This volume, pp. 53–67.

[6] SZMIELEW, W., *Some metamathematical problems concerning elementary hyperbolic geometry*. This volume, pp. 30–52.

[7] TARSKI, A., *A decision method for elementary algebra and geometry*. Second edition, Berkeley and Los Angeles 1951, VI+63 pp.

[8] —— *Contributions to the theory of models*. Indagationes Mathematicae, vol. 16 (1954), pp. 572–588, and vol. 17 (1955), pp. 56–64.

[9] —— MOSTOWSKI, A., and ROBINSON, R. M., *Undecidable theories*. Amsterdam 1953, XI+98 pp.
[10] —— and VAUGHT, R. L., *Arithmetical extensions of relational systems*. Compositio Mathematica, vol. 13 (1957), pp. 81–102.
[11] VAN DER WAERDEN, B. L., *Modern Algebra*. Revised English edition, New York, 1953, vol. 1, XII+264 pp.