



Grothendieck: A Short Guide to His Mathematical and Philosophical Work (1949–1991)

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Abstract

We present a short survey of Grothendieck's "known" work (1949–1991), before his seclusion in the Pyrenees (1991–2014). Three main periods are

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distinguishable: (1) 1949–1957, with his work in functional analysis, homological algebra, categories, sheaves, and K -theory; (2) 1958–1970, with the foundations of algebraic geometry, schemes, toposes, and motives; and (3) 1981–1991, with topological algebra, anabelian geometry, stacks, derivators, on the one hand, and long philosophical reflections, on the other hand.

Keywords

Grothendieck · Functional analysis · Category theory · Algebraic geometry · Topological algebra

1 Introduction

The immensity and depth of Grothendieck’s work has prevented the community from obtaining a fair appraisal of his contributions. This is somewhat surprising, considering that Grothendieck himself often insisted in the “childish” flavor of his principal ideas. In this survey, organized in chronological order, we will emphasize those “simple” conceptual forces inscribed in his many mathematical creations, leaving some detailed technicalities for a wider monograph (see **Zal20**). Contrary to other approaches of Grothendieck’s work, this survey will emphasize (i) the coherence and *continuity* of the entire *opus*; (ii) the *multilayered* levels of Grothendieck’s thought, as it moves between abstraction and concretion, universality and particularity, archetypes and types, unity and multiplicity, structure and calculation, transits and obstructions; (iii) a permanent search for *smoothness* in Grothendieck’s mathematical practice; (iv) an *equilibrium* of inventive power in the three main periods, 1949–1957, 1958–1970, and 1981–1991, against the usual reception, where the IHES period (1958–1970) is unduly overemphasized.

2 First Period: 1949–1957

Grothendieck’s first period positions him as a mathematical *genius* (a term to be used sparingly in a century), with an incredible power of concentration, capable to navigate both the higher abstractions and the most concrete calculations, and to transit between widely different fields (functional analysis, homological algebra, categories, complex variables). After becoming the world’s greatest specialist in topological vector spaces, Grothendieck turns away to algebraic geometry and introduces sheaves at the very heart of his mathematical practice. In the space of just a few years, Grothendieck opens up entire new perspectives of research and begins to deploy a *grand vision* and a *free inventiveness*, unheard of in his generation.

2.1 Produits Tensoriels Topologiques et Espaces nucléaires (1949–1953)

The *Ph.D. Thesis* [g1949–1953] situates Grothendieck among the most brilliant mathematicians of his generation. (The references of the form [year(s), page] send to the bibliography of “Grothendieck Works,” at the end of this article. These years mark the *periods of creation* of the works, sometimes before publication.) The story according to which the rebellious young man who had spent the year 1948–1949 in the *Séminaire Cartan* is sent by an annoyed Cartan in 1949 to Nancy, where his mentors Schwartz and Dieudonné assign him 14 open problems to calm his frenzy, is well known. (Scharlau pp. 182–183; Bringer pp. 38–39.) Some months later, he has solved them all, and possesses, according to Schwartz’s testimony, many results which would suffice for multiple papers. The *Thesis*, a development of those researches, is defended in 1953 (February 28, doctoral committee consisting of Henri Cartan (*Président*), Laurent Schwartz (*Rapporteur*), and Gustave Choquet (*Examineur*)), and published in 1955 by the *American Mathematical Society*.

Tensor products and nuclear spaces (introduced by Grothendieck) are studied at the core of the work. In some subsequent accounts of the *Thesis*, Grothendieck emphasizes a characterization of *nuclearity* through a *smooth* coincidence of projectivity ($\hat{}$) and injectivity ($\check{}$) (later denoted $\check{}$), that is, E is nuclear if and only if $E \hat{\otimes} F \approx E \check{\otimes} F$ for all spaces F locally convex (or, alternatively, for all Banach [g1952a 100], [g1952b 196]). Nuclearity thus corresponds to a stability or *fixed-point* condition, but the *deviations* to the fixed-point situation provide, on the other hand, a very interesting expansion of intermediate variations (an illuminating case will be the 14 natural tensor norms in the case of Banach spaces, intermediate between the projective and the injective norms, see Subsection “Résumé de la théorie métrique des produits tensoriels topologiques (1953)” below).

The *combination* of diverse mathematical techniques is exemplary. Topologization, functionalization, algebrization, linearity, ordering, are within the *Thesis*. The technical exuberance is remarkable, if one further takes into account that, a mere couple of years before, Grothendieck did not know anything about modern mathematical structures (in *Récoltes et semailles*, he reckons “ignoring what was a topological space” in 1948, and “not having yet heard uttered, at least in a mathematical context, such strange or barbaric words as group, field, ring, module, complex, homology” [g1986 1.18–19]). The entirely novel practice of topological tensor products, the structural behavior of Fredholm’s theory, the use of topological projective and injective limits, the original connections with spaces of integrable functions, summable functions, or distributions, the emergence of incisive counterexamples, are all strong expressions of the fascinating *technical struggle* of the young *genius* with his more conventional mathematical surroundings.

The *Thesis* reveals a *multilayered strategy* which will become a trademark of Grothendieck’s *style*: a *back-and-forth* between (i) definitions, (ii) mathematical characterizations, (iii) permanence properties, (iv) derived theorems, (v) examples and counterexamples, (vi) specific applications, (vii) returns to definitions in

ascending or descending levels, etc. In particular, the *art of definition* (of which Grothendieck will be very proud, having introduced more than a thousand definitions!) stands as one of the central features of a very careful Grothendieckian practice. In a wider context, Grothendieck's *art of language*, an extremely chiseled French sometimes close to Proust's elaborate sentences, is something that will be stressed by Serre and Cartier. The *clearness* of the architectonics depends in fact on the definitional adjustments in the edifice. In that sense, the *Thesis* is paradigmatic, with its long chains of equivalences or with its progressive subdeterminations between concepts, which may wait hundreds of pages before being resolved.

2.2 Résumé de la théorie métrique Des Produits Tensoriels Topologiques (1953)

The *Résumé de la théorie métrique des produits tensoriels topologiques* [g1953c] can be seen as a concretion of the *Ph.D. Thesis*, where the general ideas for tensor products of locally convex spaces are specified to the case of Banach spaces. The restriction to the class of Banach spaces produces some outstanding *structural* results on the intermediate norms between the projective and injective norms, described systematically through topological, algebraic, and lattice-theoretic properties. Grothendieck achieves a *characterization* of (the category of) Hilbert spaces inside Banach spaces, through injective and projective properties, solving the Banach-Mazur problem. On the other hand, he inverts correlations between the projective norm and the other intermediate norms (Grothendieck's inequality), thanks to the emergence of a deep constant, still not completely known today. (For a complete presentation of the work, see **DiestelFS**.) The article's inventiveness, mathematical and symbolical, constitutes one of the most accomplished and harmonious texts of Grothendieck's output, and may be recommended as a first entry to his world.

After finishing his *Thesis*, the young apatrid has a difficult time finding a job in France, and **Schwartz and Dieudonné help him get a postdoc in São Paulo**. (Between the available biographies (Scharlau, Schneps, Bringuier, Douroux, Pradeau), only Schneps investigates with care Grothendieck's stay in Brazil. See **Scharlau1** pp. 185–186, **Schneps2** Chap. 4, pp. 1–7, **Bringuier** p. 41, **Douroux** p. 126, and **Pradeau** p. 74.) In the first year, between March and October 1953, he produces his monograph on topological vector spaces [g1953d] and the *Résumé* [g1953c], yielding another period of splendor. (The text *Espaces vectoriels topologiques* appears as a “Curso de extensão universitária da Faculdade de Filosofia, Ciências e Letras da Universidade de S. Paulo.” It would be fully published only in an English translation in 1973: see [g1953d].) The *Résumé* is submitted a year later to the *Boletim da Sociedade de Matemática de São Paulo* and published in that journal in 1956. (The reception acknowledgement reads, “Trabalho recebido pela Sociedade de Matemática de São Paulo em Junho de 1954.”) The second year in São Paulo will be, instead, a time of crisis (the only one reckoned by Grothendieck in his mathematical life), which will propel him to algebraic geometry (see *Subsection* “Sur

quelques points d'algèbre homologique (1955–1956)" below). ("At the moment in which I was still doing Functional Analysis, that is around 1954, it happened to me that I persisted without end on a question that I could not solve, I didn't have ideas and was content with turning around a circle of old ideas which, visibly, did not 'bite'. It was like this for a year, around the "approximation problem" for topological vector spaces, which was going to be solved only twenty years later by methods of a totally different order (...) It was a painful year - the only moment in my life that doing mathematics turned out painful for me! (...) From 1954 on, I took the habit in mathematics to have many irons on the fire at the same time" [g1986 1.163].)

A natural continuation of the *Thesis* [g1949–1953], the *Résumé* provides a new understanding of classes (not yet called "categories") of Banach and Hilbert spaces, from a triple perspective attentive to inequalities between tensor norms, analytic structures, and algebraic factorizations. Bourbaki's three central *structures-mère* enter simultaneously in play – order, topology, algebra – to solve difficult open problems (Banach-Mazur) and to open unexpected new roads (Grothendieck's inequality). (For an excellent introduction to Grothendieck's inequality and its unexpected uses in quantum mechanics, noncommutative geometry, graph theory, computation theory, etc., see **Pisier**.) The *back-and-forth* between the abstract – Banach and Hilbert spaces *at large* – and the concrete – spaces of type C (continuous functions, including c_0), L (integrable functions, including l^1), λ (L -subspaces), γ (C -quotients) – helps to anchor a general (archetypical) category around the particular spaces (types) which govern it (for precisions on "types" and "archetypes," see *Subsection* "Types and Archetypes" below). In these processes, a *harmonious smoothness* prevails, yielding naturally a solution of Banach-Mazur: every Hilbert space can be characterized as being *simultaneously* a subspace of type L and a quotient of type C , that is, simultaneously a λ and γ space.

The methodological strategy of the *Résumé* [g1953c] is fascinating: (i) describe comparable canonical norms, (ii) transform those norms, and (iii) cover the full spectrum thanks to a structural hierarchy. Those three general levels possess various sublevels of understanding: (A) following Bourbaki, we have (i) order, (ii) topology, and (iii) algebra; (B) semiotically, we have (i) syntax, (ii) semantics, and (iii) pragmatics; (C) phenomenologically, we have (i) separation, (ii) continuity, and (iii) *entrelacs*; and (D) musically, we have (i) theme, (ii) variations, and (iii) fugue. (The "entrelacs" in Merleau-Ponty summons the interweaving of multiple perception levels, something also reflected in the "chiasme," interconnection of vision perspectives in the mind. See **Merleau1**; **Merleau2**.) In particular, at level (i), we can recognize one of Grothendieck's major obsessions. In fact, from his *Thesis* until his generalized proof of Riemann-Roch, through the creation of abelian categories, Grothendieck detected ubiquitous appearances of *projectivity* and *injectivity* in many mathematical regions: projective and injective products (*Thesis* [g1949–1953]), projective and injective norms (*Résumé* [g1953c]), existence of enough injectives in abelian categories with a generator and infinitude axioms in order to reconstruct homology from derived functors (*Tôhoku* [g1955–1956]), inductive subspecializations (= "spatializations") of the generalized Riemann-Roch proof in the projective and injective cases (*Rapport Riemann-Roch* [g1957]).

2.3 Sur Quelques Points d'algèbre Homologique (1955–1956)

Sur quelques points d'algèbre homologique (nicknamed *Tôhoku* for its place of publication) [g1955–1956] insures the *takeoff of category theory*, with a new arsenal of ideas, techniques, examples, and theorems, which go well beyond the initial emergence of the theory in Mac Lane and Eilenberg (1945–1948). Grothendieck introduces central categorical concepts (subobjects, representable functors, equivalences, adjunctions), defines additive and abelian categories, uses many examples of categories of sheaves, axiomatizes infinitary processes, elucidates a common frame for cohomology and derived functors, provides representation and exactness theorems, proves (using Cantor's transfinite recursion!) the existence of enough injectives in abelian categories with a generator and suitable infinitary axioms. (A (topological) sheaf is the simplest and deepest mathematical object to study the *transits* and *obstructions* between the *local* and the *global*. A *sheaf* is formed by two topological spaces E and S , and a projection $p : E \rightarrow S$ which is a local homeomorphism (i.e., for each point $e \in E$, there exists a neighborhood $V \ni e$ such that p is a homeomorphism from V into its image). In essence, the space E is then deployed as a series of “layers” locally similar to fragments of S : S is a sort of “basis” which *unfolds* along E , and E is a “multiple” space which is *folded* over a “unitary” space S . The unfolding of a point $s \in S (= p^{-1}(s))$ is by definition the *fiber* over s . We have then an image of a space E “*étalé*” (in the terminology of Cartan's Seminary in the 1950s), that is, a deployed space over its basis S . One should not confuse here *étalé* with *étale*, basically opposed concepts, distinguished by only a final accent (see Subsection “*Étalé and étale*” below). The local inverses of p (over neighborhoods in S) are the *local sections*; the global inverses of p (over all S) are the *global sections*. The fundamental objective of *sheaf theory* consists in finding gluing conditions (or proving their impossibility) for local sections, in order to produce global sections (or not). Sheaves emerge from Riemann's ideas on *analytical continuation* in complex variables, and the paradigm of a sheaf is given by the germs of holomorphic functions (for other crucial examples, see notes 18 and 22 below.) An *imaginative richness* and an extremely *precise technique* go hand in hand along a never-ending “article” (102 pp.), where the *invention* of a language (categorization, abelianization) and the *discovery* of deep mathematical structures (sheaves, monoid/group actions, injective objects, derived functors) reveal *natural* connections between algebraic geometry, topology, (co)homology, and complex variables. Still today, the *Tôhoku* article may be recommended as an enlightening (and mostly readable) introduction to the richness of Category Theory. Only *patience* is needed, as always with Grothendieck, while he explains *everything*, in painstakingly humble and accurate detail.

A postdoc in Kansas, in 1955, will be the occasion for Grothendieck's immersion in algebraic geometry. (“The year 1955 marks a crucial turn in my mathematical work: the shift from ‘analysis’ to ‘geometry’”. I still recall that strong impression, as if I were leaving arid and aloof steppes, to find suddenly myself in some sort of ‘promised Paradise’ with luxurious richnesses, multiplied to the infinite (...) There is something in mathematics that (forever without doubt) fascinates me beyond any

other thing, not ‘number’, nor ‘magnitude’, but **always form**. And between the thousand and one faces that form chooses to be revealed to us, the one that has fascinated me most, is the hidden structure of mathematical things” [g1986 P.26–27].) In that *crucial turn*, the role of Jean-Pierre Serre is fundamental. The correspondence with Serre [g1955–1987] presents the progressive emergence of the *Tôhoku* [g1955–1956]: (1) February 25, 1955, Grothendieck writes that he is exploring an analogy between the theory of derived functors and the cohomology with coefficients in a sheaf, without knowing yet “if it works well in the case of a non separated space (*Non-separation*, linked to general forms of *smoothness*, will become one of the pillars of Grothendieck’s work. In particular, the need to consider non-separated sheaves will be one of the roads that will lead naturally to the appearance of the *topos étale* of a *scheme*, key in the resolution of Weil’s conjectures. See *Subsections* “The Cohomology Theory of Abstract Algebraic Varieties (1958)”, “*Éléments de Géométrie Algébrique (EGA)* (1959–1964)”, “*Séminaire de Géométrie Algébrique (SGA)* (1960–69)” below.)” [g1955–1987 13–14]; (2) July 13, 1955, Serre informs that “your article on homological algebra was read with care [in Bourbaki’s congress] and converted everyone (even Dieudonné, who seems completely functorialized!)” [g1955–1987 17]; (3) a year later, September 191, 1956, Grothendieck indicates that he “has passed the best of last month redacting my *multiplodocus* of homological algebra (...) more than 100 pages (80 already drawn up) in big format” [g1955–1987 43]; (4) November 13, 1956, Grothendieck announces that he is going to propose the manuscript to Tannaka’s *Tôhoku*, where “it seems that the river-flowing articles do not discourage them” [g1955–1987 49]. (The “multiplodocus” (*multiplodoque*), a Bourbaki joke, evokes a gigantic treaty, worthy of the size of a diplodocus, long held to be the longest of the dinosaurs.) *Sur quelques points d’algèbre homologique* is finally received March 1, 1957, in the *Tôhoku Mathematical Journal*, and published in Japan in that same year. **We are here in presence of new global and extended mathematical practices, which overcome strong local obstructions and guarded traditions (in France and the USA).** (For a precise description of the intermediate events related to the publication, in particular some difficulties with Eilenberg and the American journals (*American Journal of Mathematics*, *Transactions of the AMS*), see **Kromer2** pp. 120–128.)

Tôhoku’s technical inventiveness pushes out into many levels of mathematical complexity: (i) **the first construction, *ex nihilo*, of a general axiomatic categorical environment (abelian categories with additional axioms) where sophisticated mathematics could be achieved** (See the following judgments (taken from Krömer, **Kromer2** p. 158): Mac Lane (1981): “The first papers on categories had no immediate sequels, because for this period they provided just a language. The notion of category theory as a subject of study in its own right appears only in the third phase of the abstract algebra movement (...) [*i.e.* the period 1957–1974 under Grothendieck’s influence]”; Barr & Wells (1985): “[The *Tôhoku*] demonstrated that categories could be a tool for actually *doing* mathematics and from then on the development was rapid.”); (ii) the elevation and axiomatic proof (via AB5 + generator) of the existence of enough injectives in a universal frame, a result which would allow a reconstruction of cohomology as a particular case of an even more

plastic, smooth, flexible technique (derived functors); (iii) the use of a *multiplicity of functors* (representable, exact, sheaf-like, derived, resolvent) to establish a precise architectonics of *transits and obstructions* between mathematical regions (topologies, algebras, differential varieties, arithmetical fragments); and (iv) the beginning of a *new vision* of algebraic geometry (connections with Serre’s work [g1955–1956 120, 137, 171], intuition of new perspectives on Riemann and “arithmetical varieties” [g1955–1956 165–166, 199]).

The *definitions* are clean and simple, the *examples* are new and multifaceted, the *theorems and corollaries* are finely hierarchized and phased, and the *proofs* are economical and perfectly fitted in. The *genius is at work*, and a (non-exhaustive) list of the definitions introduced by Grothendieck is really bewildering: subobjects [123], (Numbers between brackets refer here to pages in [g1955–1956].) equivalence (and implicitly adjunction) [125], kernels and cokernels [126], additive and abelian categories [126–127], infinitary axioms (AB3-AB6) [128–129], categories of diagrams [130], categories of actions [132], (co)generators [134], injective and projective objects [135], quotient categories [137], (co)bording functors [139], derived functors [143], filtrations [145], presheaves and sheaves (functorial characterizations) [153], spectral sequences (functorially) [148, 171], representable functors [185], *G*-sheaves [195]... As mentioned by Barr and Wells, an *actual practice of mathematics inside Category Theory* became a reality with the *Tôhoku*.

2.4 Classes de Faisceaux et théorème de Riemann Roch (1955–1957)

Continuing with Grothendieck’s amazing creativity in his first decade, his generalization of the Riemann-Roch theorem introduces another *deep projection from the archetypical to the spectra of types*. (Riemann-Roch (1857) unifies two very different approaches to obtain a natural invariant for a complex surface. On one hand, the genus g of the surface is obtained by counting the number of holes of the surface, or, equivalently, the number of cuts (minus one) with which the surface becomes disconnected (e.g., the genus of the sphere is 0, the genus of the torus is 1, etc.). On the other hand, one can think about the “good” functions (holomorphic) and the “bad” functions (meromorphic) definable on the surface. If we fix n points on the surface, consider the (vector) space Hol of holomorphic functions with zeroes on those points, and the (vector) space Mer of holomorphic functions with poles on those points. The Riemann-Roch theorem says that $n - g + 1 = \dim(Mer) - \dim(Hol)$. In this way, an intrinsic geometric invariant (the genus) is related to an extrinsic complex variable invariant (harmonic difference between Mer and Hol). At the very heart of profound connections between geometry, topology, complex variables, differential geometry, and algebra, Riemann-Roch marks the beginning of modern algebraic geometry.) In this case, the K -theory group invented by Grothendieck ($K(X) =$ free group of coherent sheaves over a variety X) [g1957] is projected onto the cohomology groups, providing a new, higher understanding of *forms*. (Given a sheaf of rings over a topological space X (that is, where the fibers are

rings, and where we denote the total space of those fibers by O_X), one can then take O_X as a basis for a new sheaf, whose fibers are modules over each $R \in O_X$. The resulting object (with due initial topologies) is a *sheaf of modules over a sheaf of rings*. If good properties of finite type moreover hold in the vertical transitions (modules of finite type) and in the horizontal transitions (kernels of finite type), we have the notion of *coherent sheaf* (Serre, FAC 1955). The article **Serre2** had a strong influence in Grothendieck. A fundamental example of coherent sheaf is the sheaf of germs of holomorphic functions (everything becomes thus connected around the complex variables, see *Subsection* “Techniques de Construction en Géométrie Analytique (1961)” below.) Using previous tools by Serre (sheafification of Riemann-Roch’s statement) and Hirzebruch (topologization of the proof), Grothendieck profits from a quantization (Todd) along the *functorial transit* (Chern) between K -theory and cohomology, from which the classical Riemann-Roch can be derived. The *geometrization* of holomorphic and meromorphic functions in complex variable, the *topologization* present in Riemann surfaces and sheaves, the *arithmetization* of the associated algebraic curves, the *linearization* obtained through group theory, the *categorization* of the structural objects and processes in play, are intertwined in a powerful *unitary vision*.

Even if Grothendieck’s interest in Riemann-Roch goes back to Nancy (the “Themes for Bourbaki’s Seminar 1951–1952” mention a “Kodaira proof of Riemann-Roch” that Grothendieck was to expose; see *Tribu* 25 (1951), p. 6), the deep new abstract conception is developed between 1955, in Kansas, and 1957, in Paris. The correspondence with Serre shows (1) the initial intuition, February 18, 1955 [g1955–1987 5], (2) the final proof, November 1, 1957: “Adjoined a very simple proof of Riemann-Roch, independent of the characteristic” – “I point out to you that ‘morally’ the new method rests on the determination of $K(X)$ and $A(X)$ ” [g1955–1987 57], underscoring thus the emergence of the group $K(X)$, foundation of K -theory, (3) the corrections and expansions, November 12, 1957: “Following your rectifications (...)” – “With the definition of Chern classes for sheaves (...) Riemann-Roch’s theorem can also be stated in the case of non projective varieties (...)” [g1955–1987 59].

Grothendieck’s main contributions in the generalization of Riemann-Roch occur in the elucidation and *natural understanding* of what seemed before a mysterious “*coup de génie*” by Riemann: (i) profound *linearization* of the environment (algebraic geometry, complex variable, arithmetic), thanks to the emergence of the fundamental structures of K -theory (group $K(X)$, ring $A(X)$); (ii) *relativization* of Serre’s and Hirzebruch’s results (sheaves, cohomology, topology, done for a fixed variety) to the case of a morphism $f: X \rightarrow Y$ (*variation of the basis*); (iii) ensuing *naturalization* of the Chern transformation between K (K -theory groups) and H^* (cohomologies), adjusted (*quantized* we would say today) through Todd’s classes. If f_*^K denotes the image of $f: X \rightarrow Y$ under the functor K , and $f_*^H: H^*(X) \rightarrow H^*(Y)$ denotes the image of $f: X \rightarrow Y$ under the cohomological functors, (We follow here the excellent account **Karoubi2**.) we have that the diagram

$$\begin{array}{ccc}
 K(X) & \xrightarrow{f_*^K} & K(Y) \\
 ch_X \downarrow & & \downarrow ch_Y \\
 H^*(X) & \xrightarrow{f_*^H} & H^*(Y)
 \end{array}$$

is *not* commutative, and the Chern morphisms do not work *directly* as a natural transformation between K and H^* . Instead, *indirectly*, through Todd's classes in a generalized Riemann-Roch formula, we do obtain commutativity: $f_*(ch_X(x)T(X)) = ch_Y(f_!(x))T(Y)$ [g1957 54–68]. In the particular case in which Y is reduced to a point, Grothendieck's formula yields a previous formula by Hirzebruch, $f_*^H(ch_X(x)T(X)) = \sum (-1)^k \dim(H^k(X, x))$ (where x is fibered over X), which, in turn, yields Serre's formula, (The left side of equality provides the Euler characteristic of the fibration x , that is, the expression $m - \text{gen}(f) + 1$ if x is the fibration of f over its Riemann surface, while the right side, restricted to $k = 0, 1$, provides the harmonic cohomological difference between the meromorphic and holomorphic functions definable over the surface of f .) a direct expression of the classical Riemann-Roch in cohomological terms. In this way, the *elevation* of a concrete problematic (original Riemann-Roch), through new mediations (Serre's sheafification and cohomologization, Hirzebruch's topologization), leads to a superior abstract environment (sheaves, K -theory) where one can find the *structural archetypical keys* of the transformations which relate number (differential algebraic dimension) and space (topological genus). *Beyond the fundamental concrete types of mathematical knowledge (number, space), some abstract archetypes (sheaves) govern the situation:* a trademark of Grothendieck's *back-and-forth* understanding of mathematical practice, along the *penduli* universal-particular, abstract-concrete, and global-local.

3 Second Period: 1958–1970

Grothendieck's second period is the epoch of conformation of a strong school, which converts the IHES in the *mecca* of mathematical research in the 1960s. Dieudonné, with his proverbial generosity, already Bourbaki's official writer, disposes himself to the orders of the young *genius* (at only 30 years of age), who becomes surrounded with some of the best colleagues and students of his generation (Serre, M. Artin, Deligne, Gabriel, Verdier, Giraud, Illusie, etc.). The gigantic results (EGA) and (SGA), covering close to 7500 pages in a decade (!), set completely new foundations for algebraic geometry, introduce Grothendieck's main generalizations of number (schemes) and space (toposes), and open the way to solve the Weil conjectures.

3.1 The Cohomology Theory of Abstract Algebraic Varieties (1958)

The Edinburgh lecture, *The cohomology theory of abstract algebraic varieties* [g1958], at the International Congress of Mathematicians, constitutes another *major turn* in Grothendieck's vision. (Grothendieck presents one of the plenary lectures at the Congress, already a sign of distinction and recognition (for the moment, mainly for the Riemann-Roch achievement). The Fields medals are delivered in Edinburgh to Roth and Thom; Grothendieck will receive it 8 years later, in Moscow, a meeting that he will not attend in objection to the Soviet violations of human rights (Dieudonné will obtain the medal for him): in fact, his anarchist and fearless mind will always be on the side of the oppressed, regardless of any political dogma.) It is the place where *schemes* emerge, allowing the extension and smoothing of many rigid techniques of algebraic varieties. The movements of schemes *over their basis* turn out to be fundamental, as well as the appearance of *nilpotent* elements, related to smooth differential calculations. In an extremely synthetic *pair of lines* (see below), already in the second page of the conference, Grothendieck offers an amazing view of the *étale* world, a deep weaving of Galois and Riemann, which will be extensively developed along schemes (EGA [g1959–1964]) and toposes (SGA [g1960–1969]). Under the construction of the *étale topos of a scheme*, it will offer the cohomological key to solve Weil's conjectures, 16 years later than the Edinburgh lecture and thousands of pages ahead!

The *heart* of the lecture is directed to one of the central kernels of mathematical thought: the dialectics between the discrete and the continuous. Weil's conjectures (1949) constituted a precise expression of that dialectics, using (after Artin, Hasse, Hecke) a generalized Riemann ζ function along curves over Galois finite fields, and proposing three main properties on the behavior of ζ : rationality, functional equation, and zeroes distribution. After the first emergence of schemes (Serre, Chevalley, Nagata, 1955–1956), where affine spaces were glued, the notion of abstract scheme proposed by Grothendieck glues local rings over the spectrum of *prime* ideals of an *arbitrary* commutative unitary ring (unifying thus the *Riemann trend*, on rings of holomorphic and meromorphic functions, and the *Galois-Dedekind trend*, on rings of algebraic integers). The generalization, a profound enlargement of the notions related to algebraic-topological spaces, turns out to be the needed instrument to study algebraic varieties over a field of characteristic p (discrete environment of the first schemes by Serre-Chevalley-Nagata) through adequate cohomology groups with coefficients in a field of characteristic 0 (continuous environment where one can use Lefschetz techniques to count points). Beyond this, Grothendieck's *genius* appears: the *deep intuition* that *étale* morphisms between schemes form the basis for a (Grothendieck) topology, (An *affine scheme* is the representation sheaf of a (commutative unitary) ring, that is, the sheaf $\pi : E \rightarrow X$ formed on the lower space X by the spectrum of prime ideals P of the ring A , with the Zariski topology (closed sets are algebraic sets), and on the upper space E by local ring fibers $\pi^{-1}(P) = A(A - P)^{-1}$. *Schemes* are sheaves which locally behave as affine Schemes. A morphism between schemes is *étale* if its fibers are of finite type (Galois flavor) and non-ramified

(Riemann flavor). A formal definition of *étale* appears only much later, see [g1960–1969 1.4]) which will allow precisely the construction of the cohomological tools to solve Weil’s conjectures. (The rationality will be proved by Dwork (1960, independently of Grothendieck), the functional equation by Grothendieck (1966), and the zeroes distribution by Deligne (1974). Deligne’s very hard and ingenious proof, worthy of a Fields medal (1978), even if using all of Grothendieck’s machinery, will depart from the main road designed by the master (see *Subsection* “Standard Conjectures on Algebraic Cycles (1968)” below), something that Grothendieck will never accept. It will be the main mathematical reason for the diatribes against his disciple in *Récoltes et semailles* [g1986].)

In the second page of the lecture, Grothendieck writes: “(...) it seems certain now that the Weil cohomology has to be defined by a completely different approach. Such an approach was recently suggested to me by the *connections between sheaf-theoretic cohomology and cohomology of Galois groups on the one hand, and the classification of unramified coverings of a variety on the other* (as explained quite unsystematically in Serre’s tentative Mexico paper) (...)” [g1958 104, Grothendieck’s emphasis]. If the *systematization* of that “revelation” would take many years and many many pages, in the *two underlined lines* one catches the essential kernel of the connections between Galois and Riemann, number and space, algebra and geometry, schemes and toposes. It is a *combined control*, both on the *internal* aspect of the structures given on the fibers of the scheme (“Galois groups”: witnesses of finitude and separability) and on the *external* aspect of superposition (“unramified coverings”) on the fibers. That *double process of smoothening* on sheaves, internal and external, allows to transfer to the cohomology of new abstract constructs, the counting Lefschetz formulas. If the statement of Weil’s conjectures (1949) can be understood as a culmination of *modern* mathematics (tools from abstract algebra and set-theoretic topology to *approach* number theory), its progressive resolution under Grothendieck’s school (1958–1974) can be seen as the beginning of *contemporary* mathematics (tools from homological algebra and categorical topology to *deepen* number theory). A complete methodological and mathematical revolution is thus under way in the higher realms of mathematical practice.

3.2 Techniques de Construction en géométrie analytique (1961)

Grothendieck’s interest in complex variables was present all along his First Period: spaces of holomorphic and entire functions [g1949–1953] [g1953a] [g1953b], Riemann sphere [g1955a], and coherent complex sheaves [g1955–1956] [g1957]. His understanding of the holomorphic/meromorphic realm is taken further in his lectures for the *Séminaire Cartan* (December 19, 1960–May 8, 1961), *Techniques de construction en géométrie analytique* [g1961], where a thorough exploration of the Teichmüller tower is advanced. (Following Henri Cartan, *analytic geometry* referred now to the geometry of analytic (holomorphic) functions. This was very close to Serre’s *GAGA*, Serre1 which was a major influence on Grothendieck.)

The *container* of the lessons seems to reflect its mathematical *content*. Teichmüller's space T_g is first approached axiomatically, then it is described categorically through categories of analytic spaces, then differential local techniques are applied, and, finally, everything becomes integrated thanks to linearizations, coverings, and quotients. (A good introduction to Teichmüller spaces (1937–1939) can be found in **ImaTan**. The *Teichmüller space* over a given surface provides a parameterization of *all* the complex structures over that surface (it is thus a *holistic, universal, natural* object, of immediate fascination to Grothendieck). If M_g is the *moduli* space of genus g (that is, the set of isomorphism classes of the closed Riemann surfaces of genus g), M_0 is reduced to a point (class of the Riemann sphere), M_1 is reduced to the complex plane (through elliptic curves) and M_g ($g \geq 2$) can be parameterized through $3g - 3$ complex parameters (Riemann 1857). The *Teichmüller space* T_g of genus g is constructed as an adequate covering of M_g , with ramifications associated to the fundamental group of some (= any) closed Riemann surface of genus g . Then T_g acquires globally a *canonical structure* of complex variety, produced from the local fragments of the action of the fundamental group over M_g . The *structure of all* T_g ($g \geq 0$) constitutes what Grothendieck will call later the *Teichmüller Tower* [**g1981**].) The very *text* of the lectures takes on a structural quality – an architectonic *maniera* – with many levels stratified in towers of definitions, lemmas, observations, examples, that reveal the hidden *contrapuntal harmony* of the complex variables environment.

Some of the strategies adopted at the *Séminaire Cartan* highlight the *smoothness* of complex variable spaces: (i) release (“*dégager*”) functorial mechanisms for the use of coherent sheaves, applied in particular to complex variables, (ii) formulate *moduli* space problems in terms of representability of functors, (iii) weave *projectivity* properties with existence theorems, and (iv) bring closer schemes (algebraic geometry) and holomorphic varieties (analytic geometry), in an extended *GAGA* program which profits from the infinitesimal properties of *nilpotent elements in local rings*. On the other hand, the implicit *dialectics* taking place in complex variables act with all their strength: (v) *ascent* (functoriality, globality) and *descent* (projectivity, modularity, locality, nilpotency), (vi) *back-and-forth* between the One (categorical universality) and the Many (concrete categories of schemes and analytic spaces), and (vii) hierarchical stratification of mathematical knowledge along *circulating levels* (differential combinatorics). A highly interesting feature, both mathematical and methodological, of the *Techniques de construction en géométrie analytique* [**g1961**], turns out to be the revelation of *multiple concrete mathematical forms of smoothness* ((i)–(iv)) thanks to a *unity of abstract methodological dialectics* ((v)–(vii)).

3.3 Éléments de Géométrie Algébrique (EGA) (1959–1964)

EGA [**g1959–1964**] can be regarded as one of the major treatises of twentieth century mathematics. Grothendieck's tireless inventivity and Dieudonné's not less inexhaustible writing produce in a few years a huge architectonics which offers,

around schemes, new foundations for algebraic geometry. The manuscripts of EGA are written between 1959 and 1964, and published between 1960 and 1966. (EGA is divided into IV volumes (and 7 parts): *volume I* (228 pp., finished October 17, 1959, published 1960); *volume II* (222 pp., finished July 22, 1960, published 1961); *volume III.1* (167 pp., finished June 28, 1961, published 1961), *volume III.2* (91 pp., finished December 15, 1962, published 1963); and *volume IV.1* (259 pp., finished March 15, 1963, published 1964), *volume IV.2* (231 pp., finished December 15, 1963, published 1965), *volume IV.3* (255 pp., finished November 15, 1964, published 1966). It constitutes an amazing *tour de force* (deepness, discipline, effectiveness) which sums up around 1500 pages and can only be understood under the literal meaning of the *monumental*. We are here in presence of an *ideal of mathematical practice* which has seldom been carried out, in such immensity and concreteness, in the history of mathematics.) The environment of work is the *Institut des Hautes Études Scientifiques* (IHES), created specifically for Dieudonné and Grothendieck. (As Grothendieck recalls, “During the heroic years of the IHES, Dieudonné and me were its only members, and the only ones also to give it credibility and audience in the scientific world, Dieudonné editing the “Publications Mathématiques” (whose first volume appeared already in 1959, the following year of the foundation of the IHES by Léon Motchane), and me directing the “Seminars of Algebraic Geometry” [g1986 1.169].) Around them, a School gradually emerges, with brilliant young doctoral students under Grothendieck (some ten students, among which Deligne, Demazure, Giraud, Illusie, the Raynauds, and Verdier). A *community of mathematical practice* propels then the IHES to the forefront of research.

Some of the strategies present in EGA [g1959–1964] (and further extrapolated to SGA [g1960–1969], see Subsection “Séminaire de Géométrie Algébrique (SGA) (1960–69)” below) include: (i) an extension of the notion of *spectrum*, which allows the development of *smoothness and freeness* along structures and sheaves, and opens the way to a natural “algebraic differentiability” thanks to nilpotent elements; (ii) the construction of a *new path* for the understanding of number theory, which, in a parallel way to the classical reading (numbers $\rightarrow$ points $\rightarrow$ space $\rightarrow$ separation $\rightarrow$ discrete geometry), proposes a new reading (numbers $\rightarrow$ prime ideals $\rightarrow$ spectrum $\rightarrow$ non-separation $\rightarrow$ continuous geometry), and studies a *unification* of the discrete and the continuous; and (iii) the elaboration of a *conceptual filtering sieve*, in order to pass from algebraic varieties of characteristic p (environment of the Weil conjectures) to differential forms in a characteristic 0 environment, thanks to finer topologies on sheaves, schemes, and cohomologies. Then, many of the main *smoother* structures in EGA occur (regular rings, flat modules, smooth algebras, ringed spaces, (pre)schemes, coherent sheaves), and a *free* calculus on them is developed (functors, adjunctions, representation theorems, noetherianity, constructibility).

One of the central objectives of schemes consists in exploring worlds where the *indiscernibles* may become *discerned* (one of the many Galois “Resurrections” in Grothendieck’s output). This happens, for example, when new infinitesimals are introduced to capture differences. Three main techniques enter into play (See **Dieud2** vol. 2, VIII, §§ 27–47.): (a) *relativization*: given a scheme S , study of the

comma category Sch/S of morphisms $X \rightarrow S$, (b) *basification*: study of $X \rightarrow S$ through $S' \rightarrow S$, which produces (via pullbacks) $X \times_S S' \rightarrow S'$, and (c) *representation*: study of $X \rightarrow S$ through the representable functor $h_X: Sch/S \rightarrow Set$. The techniques help to elucidate – beyond our “blind eyes,” as Tarkovsky would later say – some of the transits between the discrete and the continuous, between the finite and the infinite. Given a morphism between schemes $f: X \rightarrow S$, and given $s \in S$, if $\chi(s) = \mathcal{O}_s/\mathfrak{M}_s$ is the residual field over s (quotient of the local ring by its unique maximal ideal) and if one considers the basis change $Spec(\chi(s)) \rightarrow S$, the fiber $f^{-1}(s)$ can be seen as the underlying space of $X_s = X \times_S Spec(\chi(s))$. (See **Dieud2** pp. 198–200.) Thus, X appears parameterized by the X_s , and a *smoothness* strategy consists in determining f from the properties of the X_s . How much does the punctual basis $Spec(\chi(s))$ determine in fact the general basis S ? The *limits* in the category Sch/S can solve the question: the properties of schemes over arbitrary extensions may be reduced to properties over finite type extensions, and the properties of schemes over arbitrary rings can then be understood through properties of finite type algebras over $\mathbb{Z}$. (Dieudonné calls this the “realization of Kronecker’s old dream,” **Dieud2**, p. 200.)

Lefschetz’s formula (valid in characteristic 0) is the fundamental (co)homological tool to count points in varieties (one of the objectives of the Weil conjectures in characteristic p), and thus the *controlled transit* between cohomologies in characteristic p and cohomologies in characteristic 0 becomes central in Grothendieck’s program. (If $f: X \rightarrow X$ is a continuous function over a triangulable compact topological space X , Lefschetz’s formula *counts* the number of fixed points of f thanks to some traces over the homology groups: $\sum_x \in FIX(f) i(f, x) = \sum_{k \geq 0} (-1)^k Tr(f_* \downarrow H_k(X, \mathbb{Q}))$ (where $i(f, x)$ is the index of f in x , intuitively, the number of f windings around x). Through Lefschetz’s formula, one can reckon how the counting of points in a variety (left side of the equation) is related to the understanding of its homology (right side).) The construction of that *conceptual filtering sieve* (approaching characteristic 0 through “limits” of prime characteristic) is obtained thanks to the general *étale* environment, where many cohomologies blossom (*étale*, crystalline, *l*-adic, etc.), more akin to the finer topologies needed to approach the discrete obstructions of number theory. An ample *smoothness/softness theory to capture discrete structures*, or, which is the same, a far-reaching *topologization of algebra*, is at the very heart of the program. We can discover here many of the seeds of Grothendieck’s works in his Third Period (“topological algebra,” see *Section* “Third Period: 1981–1991” below).

It is interesting to compare the two extended decades 1958–1970 and 1981–1991: while the first *ascends* to general forms of smoothness (schemes, toposes) which then descend to the particular (Weil’s conjectures), the second *grows* from particular expressions of smoothness (*moduli* spaces, Teichmüller tower, *dessins d’enfants*) which then produce general frames of unification (n -categories, stacks, derivators, looking for a deep entangling of homology and homotopy). With some perspective, one can thus observe that *all Grothendieck* is looking for a very deep *back-and-forth* between universal-general-archetypical forms and concrete-particular-typical forms, with many *zigzag* movements in the middle of its creative

spectrum: vertical (EGA [g1959–1964], SGA [g1960–1969]), horizontal (*Longue marche* [g1981], *Esquisse* [g1984], see Subsections “*La longue marche à travers la théorie de Galois* (1981),” “*Esquisse d’un programme* (1984)” below), or diagonal (*Stacks* [g1983], *Dérivateurs* [g1991], see Subsections “*Pursuing Stacks* (1983),” “*Les dérivateurs* (1991)” below).

3.4 Séminaire de Géométrie Algébrique (SGA) (1960–1969)

The *Séminaire de Géométrie Algébrique* (SGA) [g1960–1969] is one of the greatest collaborative efforts in the history of mathematical thought. (SGA is divided into 7 volumes (and 12 parts): *SGA1. Revêtements étales et groupe fondamental* – “Un séminaire dirigé par A. Grothendieck, augmenté de deux exposés de Mme. M. Raynaud” (447 pp., 1960–1961). *SGA2. Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux* – “Alexander Grothendieck (rédigé par un groupe d’auditeurs), augmenté d’un exposé par Mme. Michèle Raynaud” (287 pp., 1962). *SGA3. Schémas en groupes* – “Un séminaire dirigé par M. Demazure et A. Grothendieck, avec la collaboration de M. Artin, J.E. Bertin, P. Gabriel, M. Raynaud, J.P. Serre.” *Tome 1: Propriétés générales des schémas en groupes*; *Tome 2: Groupes de type multiplicatif et structure des schémas en groupes généraux*; *Tome 3: Structure des schémas en groupes réductifs* (564 + 654 + 529 pp., 1962–1964). *SGA4. Théorie des topos et cohomologie étale des schémas* – “Un séminaire dirigé par M. Artin, A. Grothendieck, J.L. Verdier, avec la collaboration de N. Bourbaki, P. Deligne, B. Saint-Donat.” *Tome 1: Théorie des topos (Exposés I à IV)*. *Tome 2: (Exposés V à VIII)*. *Tome 3: (Exposés IX à XIX)* (525 + 418 + 640 pp., 1963–1964). *SGA5. Cohomologie l -adique et fonctions L* – “Dirigé par A. Grothendieck, avec la collaboration de I. Bucur, C. Houzel, L. Illusie, J.P. Jouanolou, J.P. Serre” (484 pp., 1965–1966). *SGA6. Théorie des intersections et théorème de Riemann-Roch* – “Un séminaire dirigé par P. Berthelot, A. Grothendieck, L. Illusie, avec la collaboration de D. Ferrand, J.P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud, J.P. Serre” (700 pp., 1966–1967). *SGA7. Groupes de monodromie en géométrie algébrique. Tome 1* “Dirigé par A. Grothendieck, avec la collaboration de M. Raynaud, D.S. Rim.” *Tome 2* “Par P. Deligne et N. Katz” (523 + 438 pp., 1967–1969). From the point of view of *mathematical practice*, some remarks are in order: the continuous presence of Serre, the ever growing diversity of collaborators, and the funny appearance of Bourbaki (Borgesian autoreference, mention of a collective inside a collective). Just looking at the network of authors, one can sense the driving impulse of Grothendieck and the mystic of the Group. Compared with EGA [g1959–1964], SGA constitutes another *monumental* work which sums up nearly 6000 (!) pages.) In his *Esquisse* [g1972], Grothendieck underlines that, among his topological ideas, “The most fundamental seems to me the extension of general topology, in the spirit of sheaf theory (developed initially by Jean Leray), contained in the topos point of view. I introduced these toposes from 1958 on, for the necessity of defining an l -adic cohomology for algebraic varieties (more generally, for schemes), which would agree with a

cohomological interpretation of the celebrated Weil conjectures. (For a corroboration of that early date, see [g1986 P.24]. On the other hand, in the *Archives Grothendieck* at Montpellier, one doesn't find another temporal register associated with the beginnings of topos theory. For good histories of topos theory, see **Belanger** and **Kromer2**.) In fact, the traditional notion of topological space is not sufficient to treat the case of algebraic varieties over a field different of the complex numbers, since the Zariski topology does not provide reasonable 'discrete' cohomological invariants" [g1972 3–4]. (Again, and always in Grothendieck, the search to *understand the discrete through new continuous tools* will propel many of his most revolutionary points of view. I thank here Alexander Cruz for many conversations around the dialectics (continuous $\leftrightarrow$ discrete) in Grothendieck, and, particularly, around Grothendieck's very original inversion (continuous $\rightarrow$ discrete) expressed along the search for general forms of varieties, and universal topological algebra constructs.)

SGA [g1960–1969] provides a multilayered architectonics where many deep perspectives are brought together: a systematic work on foundations (schemes, sites, toposes), (Given a category C , a *Grothendieck topology* J on C is the synthetic analogue of an analytic (set-theoretic) topology: for each object X of the category, a collection $J(X)$ of morphisms over X is given ("coverings") such that the identity is a covering, coverings of coverings are coverings, and pullbacks of coverings are coverings [g1960–1969 4.1.219]. A *site* is a pair (C, J) , with a category and a Grothendieck topology on it [g1960–1969 4.1.221]. A *sheaf* over a site is the synthetic analogue of an analytic sheaf over a topological space (through some appropriate categorical characterization [g1960–1969 4.1.223]). A *Grothendieck topos* (or just "topos" in this context) is a category equivalent to the category of all sheaves over a site [g1960–1969 4.1.302]. Grothendieck toposes were characterized by exactness properties (Giraud [g1960–1969 4.1.303]). A simplification of those exactness properties (limits, exponentials, subobject classifier) gives rise to Lawvere's *elementary topos* (1970), axiomatized over a first-order language. Any Grothendieck topos is an elementary topos, but not viceversa (e.g., topos of finite sets). Examples of toposes abound [g1960–1969 4.1.311–322]: topos of sets, topos of presheaves, topos of sheaves, topos associated to a topological space, topos associated to a space of operators, topos of actions, classifier topos of a group, initial and final topos, etc. The *étale topos of a scheme* (Grothendieck) is obtained by considering the *étale* morphisms over a given scheme X , which form a Grothendieck topology E ("étale topology"), and taking the topos of sheaves over the site $(Sch/X, E)$. The *étale topos* will turn out to be the central key in solving the Weil conjectures (Grothendieck-Deligne). On the other hand, the *arithmetic topos* may be a key to solve the Riemann hypothesis (**Connes**). In this way, toposes become deeply submerged into some of the most formidable mathematical conjectures of the last two centuries. For a good account of all the logical, *model-theoretic force* of Grothendieck toposes (beyond elementary toposes), see **Caramello**.) a new opening to abstract topological algebra (fundamental algebraic group), a construction of diverse cohomological theories (coherent, *étale*, *l*-adic, De Rham, crystalline... leading to the emergence of motives, see Subsection "Standard Conjectures on

Algebraic Cycles (1968)” below), a wide program of ramifications around the Weil conjectures (Lefschetz local and global theorems, Picard groups, study of homological and homotopical invariants of an algebraic variety [g1972 15]), and, more generally, an understanding of the archetypical connections between number and space. (Beyond Einstein’s *space-time*, Grothendieck’s *space-number* should offer new orientations for the mathematics of the twenty-first century. Time, just a *linear* dimension of number, should be enlarged to *ramified* magnitudes which may explain many of the intricacies of both arithmetical geometry and the cosmos.)

Grothendieck and the IHES school enter then in a new realm which serves as a “synthesis between two worlds, until then contiguous and tightly solidary, but nevertheless separated: the ‘arithmetical’ world, in which reside the so called ‘spaces’ without principle of continuity, and the world of continuous magnitudes, in which reside the ‘spaces’ in the proper sense of the term (...) In the new vision, those worlds separated before, form only one” [g1986 P.30]. Thus, beyond separation, thanks to the abstraction (*freeness, projectivity*) provided by sheaves, a new mathematical *smoothness* governs the interrelations between arithmetic and geometry. Grothendieck’s revolutionary shift is as *simple* as deep: instead of (i) a topological space X , consider (ii) all sheaves over X , and study (iii) the topos T_X of all those sheaves; then *the understanding of the structure T_X turns out to be the true understanding of X* . A double differential and integral process (see Subsection “Relativity and Universality” below) governs the situation: exploring *multiplicity* along all sheaves, and threading *unity* along the exactness properties of the topos.

3.5 Standard Conjectures on Algebraic Cycles (1968)

Around the mid 1960s, (From a semiotical and philosophical perspective, the general method is fascinating: to understand something, consider all possible points of view, and search for a *structural connection* between them. Only then you will be approaching *true knowledge*. A systematic use of the *liberty and multiplicity* of topos theory in our ordinary life would have huge consequences on tolerance, ethics, politics, and social action: an example of how *high mathematical practice* would transform our *everyday practice*.) Grothendieck imagines a “daydreaming” [g1972 17] where a *common form* hopes to unify the diverse cohomologies that have emerged until then. A *motif* (unity) is thought as a conceptual basis (or “heart of the heart” [g1986 P.45]) over which many cohomologies can be represented (multiplicity). Contrary to his previous practices, Grothendieck publishes little on motives, but one of his best renderings is the article *Standard conjectures on algebraic cycles* [g1968]: “We state two conjectures on algebraic cycles, (An *algebraic cycle* over a variety X is a formal linear combination $\sum_i (n_i \cdot V_i)$ (coefficients n_i in $\mathbb{Z}$ or in $\mathbb{Q}$) of subvarieties V_i in X . The cycles, as *decomposition* processes of sophisticated mathematical objects, become central in the emergence of *motives*. In the Riemann-Roch strategy, the *equivalence classes* of cycles are obtained through *rational equivalence* (difference between cycles parameterized through rational functions) [g1957 38].) which arose from an attempt at

understanding the conjectures of Weil on the ζ -functions of algebraic varieties. (...) The first is an existence assertion for algebraic cycles (considerably weaker than the Tate conjectures), and is inspired by and formally analogous to Lefschetz's structure theorem on the cohomology of a smooth projective variety over the complex field. The second is a statement of positivity, generalising Weil's well-known positivity theorem in the theory of abelian varieties. It is formally analogous to the famous Hodge inequalities, and is in fact a consequence of these in characteristic zero" [g1968 193].

Grothendieck delineates the consequences of the two conjectures (for their precise statements, see below), in order to obtain an eventual resolution of Weil's conjectures. The first step consists in recalling the rationality (obtained by Artin and Grothendieck in the SGA [g1960–1969]) that governs the Zeta function Z of a variety X with good *smoothness* properties (smooth, projective, irreducible) of dimension n over a finite field $\mathbb{F}_q$:

$$Z(t) = \frac{L'(t)}{L(t)}, \quad L(t) = \frac{\prod_{i=0}^n L_{2i}(t)}{\prod_{i=1}^n L_{2i-1}(t)}, \quad L_i(t) = \frac{1}{P_i(t)}, \quad P_i(t) = t^{\dim H^i(\bar{X})} Q_i(t^{-1})$$

where Q_i is the characteristic polynomial of the action of the Frobenius endomorphism of X over $H^i(\bar{X})$, H^i denotes the i -th group of l -adic cohomology (l prime different of q), and $\bar{X}$ denotes the extension of X to the algebraic closure of $\mathbb{F}_q$ [g1968 193–194]. The second step consists in signaling that the first conjecture forces the P_i to have entire coefficients (independently of $l \neq q$), and in observing ("thanks to an idea essentially due to Serre" [g1968 194]) that the two conjectures *at the same time* force the eigenvalues of the Frobenius (that is, the reciprocals of the roots of the P_i) to have an absolute value $q^{\frac{i}{2}}$, which proves Weil's conjectures [g1968 194]. (Instead, the path followed by Deligne to prove Weil's conjectures is *completely different*. Grothendieck never really accepted that proof, in his view extremely ingenious but *artificial*, estranged from the *natural path* offered by the standard conjectures (and, in depth, by motives). That *mathematical discontent* is one of the many reasons, in a complex thread of disillusion, that alienated him from Deligne and the mathematical community in general.) Concisely and limpidly, in one page and a half, a vast panorama is drawn, still open today (2020) in many senses.

"The conjecture 1 (of Lefschetz type)" [g1968 195–197], states two equivalent forms ($A(X)$, $B(X)$) of the conjecture. If $C^i(X)$ denotes the set of i -th cohomology classes of algebraic cycles, Grothendieck indicates that one gets (through hyperplane projections: Lefschetz's operator) an homomorphism $(*)$ $H^i(X) \rightarrow H^{2n-i}(X)$ ($i \leq n$) subject to the commutative diagram

$$\begin{array}{ccc} C^j(X) & \longrightarrow & C^{n-j}(X) \\ \downarrow & & \downarrow \\ H^{2j}(X) & \longrightarrow & H^{2n-2j}(X). \end{array}$$

Conjecture $A(X)$ postulates that $(*)$ is always an isomorphism, and that, if $i = 2j$, $(*)$ induces also an isomorphism (through the commutative square) at the level of algebraic cycles $C^j(X) \rightarrow C^{n-j}(X)$ [g1968 196].

“Conjecture 2 (of Hodge type)” [g1968 197–198], proposes to handle the good behavior of a bilinear form on “primitive cycles.” In detail, if $\xi \in H^2(X)$ is the class of an hyperplane section, if $P^i(X)$ ($i \leq n$) is the “primitive part” of $H^i(X)$ (that is, the kernel of Lefschetz’s morphism induced by ξ^{n-i+1}), and if one defines the set of (classes of) primitive cycles by $C_{Pr}^j = P^{2j} \cap C^j(X)$, one obtains then a bilinear symmetric form $C_{Pr}^j \rightarrow \mathbb{Q} : (x, y) \mapsto (-1)^j K(x \cdot y \cdot \xi^{n-2j})$, where K is the isomorphism $H^{2n}(X) \approx \mathbb{Q}_l$. The conjecture $Hdg(X)$ postulates that the form is positive-definite [g1968 197]. (If V is a real vector space, a bilinear symmetric form $B : V \times V \rightarrow \mathbb{R}$ is *positive-definite* if $B(x, x) \geq 0$ for all x . The positive-definite forms correspond to quadratic forms, basic instruments for arithmetical (Gauss) or metrical (Riemann) mediations, and are associated to central problems in representation and optimization. The emergence of a positive-definite form in algebraic geometry, close to Weil’s conjectures, serves thus as a weaving between arithmetic, algebra, topology, and geometry: a *natural* mediating tool, very attractive to Grothendieck.)

Grothendieck underlines that, in characteristic zero, $Hdg(X)$ is obtained immediately from Hodge theory, and explains the great technical interest of having *simultaneously* the two conjectures in arbitrary characteristic: “ $B(X)$ and $Hdg(X \times X)$ imply, by certain arguments of Weil and Serre, the following: if f is an endomorphism of X such that $f^*(\xi) = q \cdot \xi$ for some $q \in \mathbb{Q}$ (which is necessarily > 0), then the eigenvalues of $f_{H^i(X)}$ are algebraic integers of absolute value $q^{i/2}$. Thus, this implies all of Weil’s conjectures” [g1968 197]. On the other hand, Grothendieck points out that the conjunction $Hdg(X)$ and $B(X)$ forces Hodge’s Λ operator to arise from an algebraic cycle with rational coefficients, *independently of l* [g1968 198]. In this way, plumbing the depths, emerges the vision of some *global universal archetypes* (independent motives), which may then be projected over *local arithmetic strata* (l -cohomologies).

4 Third Period: 1981–1991

Grothendieck’s third period escapes from the stringent exigences of architectural perfection (EGA, SGA), and allows the exploration of conjectural and error-prone wanderings around number and space. *The genius is set entirely free*. Inverting Poincaré’s algebraic topology program, Grothendieck invents *topological algebra*, introducing new topological invariants for algebraic objects and new forms of *smooth* topology (tame topology). The emergence of *anabelian geometry* studies varieties through the action of their fundamental algebraic group. On the other hand, proceeding from low level geometrical objects, Grothendieck offers combinatorial descriptions for Riemann surfaces (“*dessins d’enfants*”) and for their classification, level by level, along actions of the Grothendieck-Teichmüller group. Beyond this, in

a powerful *back-and-forth* between concrete examples and universal theories, he introduces stacks and derivators, in order to study homotopy types and their homological counterparts from a categorical perspective. The 1980s become also the time of his very deep philosophical reflections on *mathematical creativity*, and, particularly, on his own work.

4.1 La Longue Marche à Travers la théorie de Galois (1981)

Grothendieck's retirement from the IHES in 1970 is due to a diversity of factors, not reducible to a single perspective: technical exhaustion (after 20 years of extremely hard work, a point underlined by Cartier and Serre), individual liberation [g1986 1.65], social conscience [g1986 1.159], academic turmoil after May 1968 [g1986 1.156], disappointment with the mathematical community [g1986 1.65], rejection of the military support given to the IHES (the "standard" argument [g1986 L.3, 1.169]). One often hears, with no reason, that Grothendieck abandons the world of mathematics. Nothing could be farther from the truth. He dedicates a great effort to the radical ecologist movement *Survivre et vivre* (1970–1975), but continues steadily to give lessons (*Collège de France* (1970–1972), *Université de Paris-Orsay* (1972–1973), long series of conferences in Romania, Canada, Germany, France, Switzerland, USA), and to foster the mathematical research of his students (*Université de Montpellier* (1973–1984), with even more Ph.D. students there than at the IHES). What seems to be true is that between 1970 and 1980 his own mathematical research is somewhat "muted." Its huge reemergence in the decade 1981–1991 appears then as a kind of "miracle," and one encounters again the inexhaustible Grothendieck at work.

The first expression of Grothendieck's "resurrection" is the unending (and unpublished) manuscript *La longue marche à travers la théorie de Galois* [g1981]. Written between January and June 1981, it covers close to 1600 pages in the principal text and close to 1000 pages in notes and addenda. Some sections of the manuscript have been transcribed (Malgoire, Schneps), but they are difficult to read. They will become integrated later in the synthetic *Esquisse d'un programme* [g1984] (see Subsection "Esquisse d'un programme (1984)" below). Open to the careful observation of concrete examples, *La longue marche* [g1981] combines holistic perspectives (categorical treatments, towers) and case studies (*moduli* spaces, modular curves, *dessins d'enfants*). The study of *algebraic variation* through topological tools and the unveiling of *universal structures* (fundamental algebraic group, Grothendieck-Teichmüller group), behind Galois, Riemann, and Poincaré, become the foundations of Grothendieck's *topological algebra* in the 1980s.

The two central themes of *La longue marche* [g1981] are (i) *anabelian geometry* and (ii) *Grothendieck-Teichmüller theory*. Around (i), the central point is to study a variety (algebra – number theory – Galois), or a scheme (Grothendieck), through its fundamental topological group (topology – geometry – Poincaré), or through its fundamental *algebraic* group (Grothendieck). Given a scheme X (automatically a topological space), one considers the fundamental homotopy group $\pi_1^{\text{top}}(X, x)$

(Poincaré) for some point $x \in X$ (if X is arcwise-connected, one can forget the base point x), and the *profinite completion* of $\pi_1^{\text{top}}(X)$ is by definition the *fundamental algebraic group* $\pi_1^{\text{alg}}(X)$ of X (Grothendieck). This group classifies the *étale*, since its finite continuous orbits are precisely the *étale* coverings of the Scheme. A fundamental example is the projective plane P except three points: in that case, the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}} : \mathbb{Q})$ acts over $\pi_1^{\text{alg}}(P)$ and hides deep arithmetical combinatorial structures (*dessins d'enfants*). The central problem of *topological algebra* becomes how to detect or describe those varieties X completely characterized by $\pi_1^{\text{alg}}(X)$. By definition, those are the *anabelian varieties*, whose geometry (space) and arithmetic (number) are codified in their (non-abelian) fundamental algebraic groups. Grothendieck's main conjecture states that the *moduli* spaces M_g (see note 25 above) constitute the anabelian varieties over $\mathbb{Q}$, and, as a special case, that the hyperbolic curves over algebraic number fields are anabelian. Mochizuki solves in 1996 the particular conjecture (**Mochizuki1**) and launches then his ambitious program of new foundations for *arithmetic geometry* (**Mochizuki2**).

Around (ii), the central object of study becomes the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}} : \mathbb{Q})$ (algebra – number theory – Galois) through its *actions over geometrical objects* (complex variable – geometry – Riemann). In that sense, as its very name shows, the “absolute” Galois group becomes an *archetype* which hopes to be understood through its actions over diverse *types*: curves, *moduli* spaces, fundamental groups (homotopic, algebraic), *dessins d'enfants*, and combinatorial towers. Given a variety $M \in M_{g, \nu}$ (g holes, ν selected points) and given its fundamental homotopy group $\pi_1^{\text{top}}(M)$, Grothendieck considers its profinite completion $\widehat{\pi_1^{\text{top}}(M)}$ and studies the profinite groups of loops, through automorphisms, interior automorphisms, quotients ($C_1(g, \nu)$), discretizations (basis of the group), and characters ($C_2(g, \nu)$). An strategy to understand the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}} : \mathbb{Q})$ consists then in presenting it as a quotient $C_1(g, \nu)/C_2(g, \nu)$ [**g1981** 26–28], *uniformly* over all (g, ν) with $(2g + \nu \geq 3)$. (Schneps indicates that this approximation fails at least when $(g, \nu) = (0, 3)$, due to a later result of Drinfeld. See **Schneps3** p. 62.) Nevertheless, Grothendieck himself observes that this seems to be a “very brutal” strategy [**g1981** 163], and that one should “proceed in a more inductive manner, from the presence of $\text{Gal}(\overline{\mathbb{Q}} : \mathbb{Q})$ [in the Teichmüller tower] (for arithmetico-geometrical reasons), in order to clear up strong properties, and provide finally a purely algebraic characterization” [**g1981** 165]. (The program will be later developed by Drinfeld (1991), through what he will call the *Grothendieck-Teichmüller group* $\widehat{GT}$. A delicate description of $\widehat{GT}$ can be done through subgroups and products of $\widehat{F}_2$ (completion of the free group with two generators), $\widehat{K}_5$ (completion of Artin's braid group), and $\widehat{\mathbb{Z}}$ (completion of the integers). Drinfeld conjectures that $\widehat{GT}$ is also the *automorphism group of the Teichmüller tower*. An injective homomorphism $\text{Gal}(\overline{\mathbb{Q}} : \mathbb{Q}) \hookrightarrow \widehat{GT}$ holds (Drinfeld, Ihara, Belyi). A proof of the open conjecture $\text{Gal}(\overline{\mathbb{Q}} : \mathbb{Q}) \approx \widehat{GT}$ would provide the “purely algebraic characterization” dreamed by Grothendieck. See **Schneps3** pp. 64–66.) In all of this, we can see how Grothendieck *zigzags*, looking for a deep harmonization between vital polarities:

visible/invisible, exterior/interior, discrete/continuous, local/global, layers/completion. A new form of *mathematical practice* emerges along a *new, open, free, tentative, plastic*, style. The conjectural, the inventive strata, the elastic heart, even if imprecise, *command* the theorematical, the solid architectonics, the strict reason. A myriad of questions are then opened for generations to come.

4.2 Pursuing Stacks (1983)

Between 1981 and 1991, Grothendieck will follow concentrated *impulses*, along *continuously flowing* manuscripts, either purely mathematical (*Longue marche* [g1981], *Stacks* [g1983], *Esquisse* [g1984], *Dérivateurs* [g1991]), either methodological, philosophical, and mystical (*Récoltes* [g1986], *Songes* [g1987]). One can sense in the 1980s his strongly burgeoning spirit, never at rest. *Pursuing stacks* [g1983] is the longest Grothendieck manuscript written in English. Proceeding from an *epistolary mood*, that once initiated never stops, it includes three letters to Larry Breen (1975, 58 pp.), a letter to Daniel Quillen (1983, 23 pp.), a table of contents (1983, 10 pp.), and a deep *journal of discovery* (February 27, 1983 – November 4, 1983, 593 pp.) Around higher forms of sheaves and category theory, Grothendieck explores a “*vast synthesis* of homotopical algebra, (co)homological algebra, and topos theory,” in a “*non precise and provocatively fragmentary*” manner (*Archives Grothendieck*, Montpellier, cote 109).

Stacks appear in different ways in [g1983] – as coefficients in noncommutative homological algebra, as strata in categorical towers, as unifier bridges between homology and homotopy – but Grothendieck never offers a precise definition of the concept. (A *stack* is considered today as a *generalization of a sheaf* in the context of 2-categories (or bicategories). In an iterated fashion, a n – *stack* is understood as a $(n + 1)$ – *sheaf*. In some cases, *stacks* may be characterized through categorical *fibrations*. In a more precise approach, a *stack* over a site S is a pseudofunctor (preserves identity and composition with respect to isomorphisms) $S^{op} \rightarrow Cat$ (2-category) which satisfies an effective descent for coverings. A *stack of groupoids* takes values in the category of groupoids. A *stack of moduli* takes into account not only equivalence classes in *moduli* spaces but also the *automorphism groups* of those structures. In this way, the development of *stacks* is closely related to Galois’ *ambiguity theory* applied to arithmetic. For a thorough guide of an ever growing panorama, see the many entries in the webpages *nLab* and *Stacks Project*.) This is due in part to the *work in progress maniera*, where the *vanishing lines* of the building become primordial, in detriment of its foundations. (The importance of the *style*, or *maniera*, in *Pursuing stacks* [g1983] is essential: “I sometimes find it difficult to find a proper balance in these notes between the need of working on reasonably firm ground, and working out suggestive terminology and notations for what is coming up, and on the other hand my resolve not to get caught again by the EGA style of work, when it was understood that everything had to be worked out in complete detail and in the greatest generality” [g1983 445].) Nevertheless, Grothendieck’s *vision* may be broken down along some well defined steps: (i) acknowledgment of

the existence of nontrivial automorphism groups (interest of Galois' ambiguity theory), in order to *structure certain structures* (first essential hierarchical iteration); (ii) classification of those structures (e.g., *moduli* spaces) thanks to continuous variations (interest of Riemann's representation theory), in order to *differentiate certain differences* (second essential hierarchical iteration); (iii) consciousness of the ubiquitous role of groupoids (algebraic or topologic) in the preceding themes (structuration, representation, differentiation); and (iv) integration, in higher sheaves, of all previous informations (variation, gluing, ascent/injection, descent/projection). The *coherence* of the program is solidified when one comes to see it as some sort of *methodological sheaf* applied to the very richness of mathematics.

Passages between diverse types of *stacks* become then entirely *natural*, in a *back-and-forth* between (1) *algebraic stacks* (algebraic varieties, schemes, toposes), (2) *differential and topological stacks* (differential varieties, topological spaces, toposes), and (3) n – *stacks* (0 – *stacks* = sheaves, 1 – *stacks* = *stacks*, etc., leading to n -categories and n -groupoids). All these paths offer new horizons: (1) the fine theory of *moduli* spaces, oriented towards arithmetic (Langlands' program), (2) the classification theories in noncommutative algebra, oriented to anabelian geometry (Grothendieck-Teichmüller's program [g1981] [g1984]), (A central example of a *stack* is the (pseudo)functor $S^{op} \rightarrow M_{g,\nu}$ where S is the category of affine schemes over a fixed scheme with the *étale* topology and $M_{g,\nu}$ is the *moduli* space of Riemann surfaces of genus g with ν points. The particular case $(g, \nu) = (1, 1)$ (corresponding to elliptic curves) was first treated by Mumford (1965, introduction of the term “*stack*” in 1969), before Grothendieck considered the general case (g, ν) .) and (3) the homotopical theories of types and categories, oriented towards an alternative foundation of mathematics in higher geometrical orders (Voevodsky's program, HoTT). (*Homotopy Type Theory* (HoTT) looks for a *fourth foundation* of mathematics (after set theory, intuitionistic species, and category theory). A strong geometric intuition (homotopy types) is postulated, *before* considering the combinatorial (sets), the constructive (species), or the relational (categories). An essential part of the program consists in *debilitating equality* (very restrictive for the natural structures of mathematics), and changing it with isomorphisms (“univalence axiom”). For an introduction to HoTT, see VoevH.) We observe here a *continuous thought* expressed through categorical unitary archetypes, *high* enough and *general* enough, to be able to cover the multiplicity of diverse particular types. (A comparison of Hilbert with Grothendieck, easily the two major mathematicians of the twentieth century, may be instructive: while, in Hilbert, each fragment of his production constitutes a well *separated and complete* conglomerate (algebra, number theory, geometry, functional analysis, logic, physics), in Grothendieck all fragments are always intertwined, from the implicit categorical *maniera* of his *Thesis* [g1949–1953] until the overarching “functional analysis” panorama of the *Dérivateurs* [g1991].)

4.3 Esquisse d'un Programme (1984)

Esquisse d'un programme [g1984] constitutes a research project for Grothendieck's admission to the CNRS, after his retirement from Montpellier University (1984). The *Esquisse* (dated January 1984 [g1984 51]) retrieves some of the central ideas of the *Longue marche* [g1981], and extends them to two new paths in *topological algebra*: on one side, *dessins d'enfants*, (A *dessin d'enfants* may be presented as a graph D finite, connected, bicolored, embedded in an oriented compact topological surface X , such that XD is a finite union of disks (see **GirondoGonzalez**, p. 207). Some examples (posterior to Grothendieck) are striking: the (topological) sphere S^2 , with an inscribed *dessin* formed by a black vertex, a white one, and another black, corresponds to the Riemann surface associated to z^2 ; the sphere with a *dessin* formed by vertices black-white-black-white-black corresponds to the Riemann surface associated to $z^2(z - 1)^2$, etc. On the other hand, proto-ideas around the *dessins* can be traced back to the nineteenth century (Hamilton, Klein, etc.) through which Grothendieck combinatorially captures compact Riemann surfaces and studies their behavior under the actions of the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}} : \mathbb{Q})$, and, on the other side, "tame topology" (*topologie modérée*), which hopes to eliminate wild phenomena and artificial counterexamples in classical set-theoretic topology. An "ascent" to higher concepts is provided, in order to "descend" after to lower constructs. In this case, a systematic study of "high" algebraic actions (fundamental algebraic group, anabelian geometry, Grothendieck-Teichmüller group) is explored, in order to understand topological surfaces of "lower" dimension (compact Riemann surfaces, *moduli spaces*).

The *Esquisse d'un programme* [g1984] covers three main subjects: (i) Teichmüller's tower and the absolute Galois group (*Sect. 2* of the typescript), (ii) *dessins d'enfants* (*Sect. 3*), and (iii) tame topology (*Sections 5, 6*). Two secondary topics are considered: regular polyhedra and two-dimensional geometry (*Sections 4, 8*), *stacks* (*Sect. 7*). Finally, some considerations about the personal circumstances of the author are included: balance of research and teaching experience (*Sections 1, 9, 10*). From *higher* perspectives, Grothendieck looks for a general theory of *dévisage* ("unscrewing"), in order to understand the architectonics of stratified mathematical architectures. This leads naturally to the iterative construction of *stacks* and to tame topology, hoping to capture *smooth combinatorial forms* along hierarchical transits. From *lower* perspectives, surfaces, complex algebraic curves over number fields, elliptic curves, modular subgroups, and regular polyhedra, offer many illuminating examples which help to adjust the architectonics. In the *middle ground*, modular multiplicities and their associated groupoids, general varieties over number fields and their associated fundamental groups, Riemann surfaces and their associated *dessins d'enfants*, stratified actions of the absolute Galois group and their invariants, *connect continuous and algebraic variations*, and become ciphered in the fundamental anabelian conjecture about the multiplicities $M_{g, \nu}$ (*stacks*, not varieties). The *Esquisse* [g1984] offers thus a typical Grothendieckian *vision*, conjugating abstraction and concretion, theory and examples, unity and multiplicity,

universality and contingency, continuity and discreteness, globalization and localization. A *yin* drive (see Subsections “Récoltes et semailles (1983–1986),” “La clef des songes (1987–88)” below) towards particular forms of smoothness is complemented with a *yang* drive towards general forms of *dévisage*.

The *arithmetical heart* of the researches collected in the *Esquisse* [g1984] glues together (in a sort of methodological sheaf) the *continuous variations* present in the *moduli* spaces $M_{g, \nu}$ and the *algebraic variations* of $\text{Gal}(\overline{\mathbb{Q}} : \mathbb{Q})$ acting on the Teichmüller tower. In particular, the appearance of the space of elliptic curves $M_{1,1}$ on the first floor of the tower, and its associated modular group $SL(2, \mathbb{Z})$, underlines the arithmetical richness of the environment. On the other side, the *natural* emergence of algebraic curves over number fields along compact Riemann surfaces (Belyi 1979), strikes Grothendieck very deeply. (Belyi’s theorem characterizes the compact Riemann surfaces associated to complex algebraic curves defined over number fields. More precisely, if S is a compact Riemann surface, the following are equivalent: (i) S is defined over $\overline{\mathbb{Q}}$, (ii) there exists a meromorphic function $f: S \rightarrow \mathbb{P}^1$ with at most three ramification points. Belyi’s discovery, “that technically is reduced to such a small thing, exerted in me a very strong impression, and represented a decisive turn in my reflections, a shift of the center of my mathematical interests (...) I do not recall that another mathematical fact had stricken me so strongly (...)” [g1984 12]. Thus, in Grothendieck’s own words, the *decisive turn* towards algebraic geometry (Kansas 1955) and the *decisive turn* towards combinatorial topological algebra (Mormoirion 1984) seem to be two major illuminations. It is interesting to observe that the two circumstances occur far away from the official academic centers of mathematical thought, and after some important psychological crisis. *Mathematical practice*, through immense manuscripts (*Tôhoku* [g1955–1956], *Longue marche* [g1981]), well beyond just mathematical learning, will always be for Grothendieck a way to stabilize his soul.) Beyond this, thanks to Grothendieck’s equivalence between compact Riemann surfaces and oriented topological surfaces with *dessins d’enfants* inscribed on them, the known action of $\text{Gal}(\overline{\mathbb{Q}} : \mathbb{Q})$ over compact Riemann surfaces becomes transferred to a *new combinatorial action* over *dessins d’enfants*, and one can hope to obtain a *smooth combinatorial* description of the absolute Galois group. (This will lead to Drinfeld’s *definition* of the Grothendieck-Teichmüller group $\widehat{GT}$ (see Note 45 above).) Finally, the main goal of anabelian geometry, that is, to control curves over number fields through their fundamental algebraic groups (extending homotopy to the new configurations of topological algebra), achieves the architectonics of the *Esquisse* [g1984]. In this way, once more, Galois, Riemann, and Poincaré, become connected along the depths of Grothendieck’s thought, but now under an arithmetic-algebraic-complex-geometric weaving to which are added an elementary combinatorics (*dessins d’enfants*) and a smooth universal understanding of topological forms (*topologie modérée*).

4.4 Les dérivateurs (1991)

Les dérivateurs [g1991] is Grothendieck's last known manuscript. (Between 1991 and 2014, the year of his death, Grothendieck wrote extensively on many subjects, embracing mathematics, physics, philosophy, mysticism, history, sociology, including a long history of evil in mankind which tracks in particular the monstrosities of concentration camps. Summing up to more than 50,000 pages (20 years is a very long time for a restless mind!), Grothendieck bequeathed his last writings (extremely well organized in careful arranged boxes, as reported by Georges Maltsiniotis) to the *Bibliothèque Nationale de France*. Nevertheless, his heirs have acted differently, and his heritage is being appraised for an eventual sale to a highest bidder (a hideous contradiction with Grothendieck's anarchist and free spirit). When those 50,000 pages will be made available to the public, our understanding of Grothendieck will acquire a new dimension, complementing his extremely complex intellectual portrait through a *Fourth Period* (1991–2014), which should be full of surprises.) The gigantic text (around 2000 pages) was distributed to friends and colleagues, like the other texts of the decade, (In total, the output was immense, summing up circa 8000 pages (!): *mathematical: Longue marche* [g1981]~ 2500 pages; *Pursuing stacks* [g1983]~ 700 pages; *Dérivateurs* [g1991]~ 2000 pages; *philosophical: Récoltes et semailles* [g1986]~ 1400 pages; *La clef des songes* [g1987]~ 1000 pages. The flow of Grothendieck's writings, in solitary confinement, is almost equal in this *Third Period* to the production in the previous *First and Second Periods combined*, even if we count there the help of his multiple IHES collaborators!) and opens new roads for the eclosion of higher category theory. Grothendieck tries to axiomatize *at large*, at a universal global level in the category *Cat* of all categories, some canonical notions in topological algebra. Through a general axiomatics applied to adjunctions, localizations, and actions (oriented by the category *Hot*: *CW*-complexes with homotopy classes as morphisms [g1983]), Grothendieck seeks to recover *both homological algebra* (coefficients, abelian and derived categories) and *homotopical algebra* (limits, *n*-categories, topos constructs), as complementary *slopes* of a unitary whole ("prederivators," sort of localizations of Yoneda's Lemma, independently of basis changes). Beyond isolated efforts (Maltsiniotis, Cisinski, Groth), many of the ideas and techniques of *Les dérivateurs* remain yet to be assimilated and developed by the mathematical community.

Les dérivateurs [g1991] provides many topological techniques and perspectives to capture algebraic notions, through *smooth abstract coverings*: (i) *Hot* covered by *Cat*, (ii) types of homological algebra (coefficients) and types of homotopical algebra (limits) covered by general categorical archetypes, (iii) homotopical formalisms and (co)homological adjunctions covered by an axiomatics of (pre)derivators, (Fundamental examples of (pre)derivators are: (i) presheaves $A \rightarrow \text{Hom}(A^{op}, M)$ (A , M categories), (ii) localizations $A \rightarrow W_A^{-1} \text{Hom}(A^{op}, M)$ (W_A subset of arrows – weak equivalences" – in Yoneda's image of $A \subseteq M^{A^{op}}$), and (iii) a natural functor $\text{HOT}: \text{Cat} \rightarrow \text{Cat}$ associated to a localization of simplices in the category *Hot*. For precise definitions and developments, see **Cisinski**.) (iv) variations (functorialization of

derivation processes) and invariants (intrinsic stability, independence of basis changes) covered by universal procedures in the category of all categories, and (v) resolutions to obstructions to non-canoncity in abelian and derived categories covered by *localizations*. (Grothendieck's problematic begins with observing some *canonicity failures* in the construction of the derived category of an abelian category [g1955–1956]. If A is an abelian category, $C(A)$ is the category of complexes over A , and W is the set of quasi-isomorphisms of $C(A)$ (morphisms $f: X \rightarrow Y$ such that $H_n(X) \approx H_n(Y)$ for all n), the derived category is defined by the (formal) localization $Der(A) = C(A)[W^{-1}]$. Then, many functoriality and adjunction problems emerge, in order to capture the original information of A . For an introduction, see **Maltsiniotis3**.)

In a letter to Thomason (Les Aumettes, 2.4.1991), (Available at www.grothendieckcircle.org, under a folder on derivators.) Grothendieck explains some of the essential points of *Les dérivateurs* [g1991]: universality of abstract axiomatizations in *Cat*, localizations *at large* in *Cat* and comparison with Quillen's model categories, understanding of the additional structure of a derivator with respect to its underlying derived category, correlative study of adjunctions associated to a derivator. The *axiomatic* investigations (See [g1991 I.23–25] (derivator axioms), *Archives Grothendieck*, cote 157–4, pp. 268–350 (localization axioms).) in the ever-evolving document (Again, as happens all along the 1980s, a *diary of invention* is proposed, and one can follow the many ideas, errata, changes, new elaborations, through the evolving manuscript. A detailed study of such *mathematical practice* in a great mathematician, with all its *residues of stratified invention*, is yet to be undertaken. It would provide a magnificent witness of *creativity at large*.) are witnesses of Grothendieck's practice to clear up (“*dégager*”) the universal mechanisms that govern many regions of mathematics. As a consequence, Grothendieck obtains a sort of *general methodology of smoothening singularities* (strong conceptual connections with Hironaka's “resolution of singularities,” 1964), where the *tameness* of the topology, the n -elevation of *stacks*, the unification at large of *derivators*, respond to the desired smoothness. The possibility of *transiting* in higher categories helps to overcome the many *obstructions* in lower regions. Grothendieck's work acquires thus a strong solidarity, going from his very *Thesis* [g1949–1953] until the *Dérivateurs* [g1991]. For developments on this *unity, continuity, and coherence*, see Section “A Synthetic Account of Grothendieck's Work” below.

4.5 Récoltes et Semailles (1983–1986)

Récoltes et semilles [g1986] is perhaps the major analysis ever written on higher mathematical creativity. Beyond Poincaré's *L'invention mathématique* (1908) and Hadamard's *The Psychology of Invention in the Mathematical Field* (1945), (**Poincaré Hadamard**.) Grothendieck's reflections surpass its predecessors both in extent (1500 pages), and in depth (detailed study of the emergence of Grothendieck's main mathematical ideas, a refined approach to one of the two greatest

mathematicians of the twentieth century). (See *Note 50* above.) The subtitle of the work, “Reflections and testimonies on a mathematical past,” indicates its two main watersheds: on one side, reflections on *mathematical practice*, entering in detail inside Grothendieck’s practices, and on another side, testimonies on his own life and his *mathematical school*, expressing a disenchantment with how the “seeds” (*semailles*) of his work had not been adequately “harvested” (*récoltes*) in the period 1970–1983. The *style* of the retrospective is typically grothendieckian: passionate, meticulous, exhaustive, irritating, astonishingly beautiful, both in his autoanalysis and in his accountability of the community. The harsh sincerity of the volume struck his mathematical colleagues, and ended up severing the few remaining ties that Grothendieck had with his previous intellectual environment.

Along this survey, we have had the opportunity to quote many times the manuscript. (The manuscript, not yet published in French, is accessible in many sites online. It has been translated in Japanese (Tsuji, also published), Russian (Fridman and Filkenberg), English (Lisker), and Spanish (Navarro). The best study of *Récoltes et semailles* [g1986] is Herreman.) The text is divided into seven main sections: *Walk Around a Work, or the Boy and the Mother* (65 pp.): January–February 1986; *Letter and Introduction* (56 + xxii pp.): March 1984, May–June 1985, February 1986; *First part. Fatuity and Renovation* (171 pp.): June 1983, February–March 1984; *Second part. The Burial I, or the Clothes of China’s Emperor* (249 pp.): April–June 1984; *Third Part. The Burial II, or the Key of Yin and Yang* (354 pp.): September 1984–January 1985; *Fourth Part. The Burial III, or the Four Operations* (478 pp.): February–June 1985; *The Doors over the Universe. Appendix to the Key of Yin and Yang* (127 pp.): March–April 1986. Acceleration and multiplication point to a *never-ending outpouring* of words, concepts, images, reminiscences. *Récoltes et semailles* [g1986] encompasses the inexhaustible richness of Grothendieck’s persona, underappreciated and ill-understood by the mathematical community, in particular by his great peers, a Serre, a Dieudonné, who never accepted the hermit, the mystic, and the anarchist.

The *complexity* of *Récoltes et semailles* [g1986] captures the complexity of the Grothendieckian world: (1) disaggregation of mathematical thought, in its eternal fight between substance (e.g., spaces, numbers, structures) and form (e.g., cohomologies, motives, derivators); (2) reflections on the inroads to mathematics, and, in particular, on a naive entrance to creativity (freshness, smoothness, childish perception); (3) considerations on mathematical method, and, in particular, on the demands of constant and persevering work (tasks of the architect and the laborer); (4) systematic study of the yin-yang watersheds in creativity, in general, and in mathematical creativity, in particular; (5) journey around the intellectual biography of the author, his central passions, and his tasks in pursuit of alternative life movements (ecology, austerity, oriental philosophy); (6) detailed analysis of the historical and sociological circumstances where that biography is inserted; (7) strong criticism of the degenerescence processes occurring in the Western world in the 1970s and 1980s; (8) stylistic constructions, multi-spatial and multi-temporal, which mirror an inquisitive and open mind, attentive to contemporary trends.

The metaphor of the *rising sea*, “by immersion, absorption, dissolution” [g1986 3.553], acquires a fundamental technical value thanks to Category Theory, when an object (“nut”) is submerged in an adequate category which absorbs it (“sea”). This is the case of the central theorem in the theory, Yoneda’s Lemma, (In a parallel way to Yoneda (c. 1958), the Lemma also appears independently in EGA [g1959–1964 III.1.7].) where a *double iteration* of the rising sea appears: (i) objects are first immersed in their “auras” (representable functors) and (ii) categories of objects are then immersed in presheaf categories. The result provides, for every small category, a *complete* extension through non-representable functors: these can be understood as *phantasmata* which come closer to the invisible. On the other hand, the rising sea occurs also naturally along toposes, when a topological space (or a site) is submerged in the category of *all* sheaves over it. In the topos emerges a powerful “super-structure” (completeness, exponentiation, subobject classifier), invisible from the initial space (or site). In these cases, the *rising sea* reveals a *general* mathematical richness, previously invisible, that can help to solve very difficult *particular* problems (e.g., Weil’s conjectures).

The *Appendix to Récoltes et semailles* [g1986], *The Doors over the Universe* (abbreviated DU), 127 pages written in a month-long euphoria (March 17–April 14, 1986), adopts a sophisticated system of pairings yin/yang to understand human action, in general, and *mathematical practice*, in particular. On the yin side, one finds discovery processes, smoothness, a mother language of intuition. On the yang side, emerge inventive processes, architectonics, the superficial languages of description. Along yang resides reason (“razón” in Spanish), along yin lies the heart (“corazón” in Spanish, *duality* perfectly captured by the prefix “co”). Grothendieck proceeds to construct a *stratified combinatorics* of yin/yang pairs, having in a first level “polygamies” (a yang related to two yins, e.g., sun with moon and earth) and “polyanders” (a yin related to two yangs, e.g., earth with sun and sky) [g1986 DU.7–8]. The first levels become then iterated along higher growing cycles. One of the major interests of (DU) is the use of Grothendieck’s mathematical arsenal (sheaves, schemes, homologies, geometries, combinatorics) to provide an analysis of thought. In this way, *mathematical practice* orients our *common material practice*. In fact, many of the tasks assumed in (DU) refer to gluings, inversions, strata, breaks, thresholds: (i) variations, degrees, intensities, between warm (*yin*) and cold (*yang*) [g1986 DU.16–17]; (ii) inversions, through subgroups, between continent and content, abstraction (*yang*) and concretion (*yin*) [g1986 DU.19–20]; (iii) dialectics multiplicity/unity (“I feel myself as a multiple in search for unity”) [g1986 DU.23]; (iv) diagrams (hexagons, icosahedrons, trees) to capture tonalities yin/yang [g1986 DU.28–32]; (v) dynamics between ideal (*yang*) and real (*yin*) [g1986 DU.36–37]; (vi) zigzag iterations and homologies between unity/mystery (*yin*) and order/simplicity (*yang*) [g1986 DU.37–40]; (vii) tension between discovery of structures (*yin*) and invention of languages (*yang*) [g1986 DU.47–51]; and (viii) “accordion” between exterior (surface, light, *yang*) and interior (deepness, shadow, *yin*) [g1986 DU.51–55], etc. In this way, there appear many implicit inversions and symmetries (Galois connections), and multiplications and unfoldings (Riemann ramifications). As an “Ahab

after the White Whale” [g1986 DU.66], Grothendieck immerses himself in the never-ending complexities of the world.

4.6 La Clef Des Songes (1987–1988)

La clef des songes, ou dialogue avec le Bon Dieu [g1987] studies the spiritual wanderings of the author, in an ever meticulous, extremely detailed manner, the Grothendieckian Way. After a precise register of his dreams in previous years (more than a “thousand” [g1987 6], Grothendieck’s numbers always amaze), the author proceeds to a *dévissage* of the oniric realm, along extremely diverse strata and thresholds. The emergence of multiple types of spirituality (breath, faith, heart), intellectuality (eyes, language, reason), and sensibility (hands, passion, eros) offers a fascinating immersion in Grothendieck’s mind. This leads him to the “revelation” of an archetype – a Dreamer, a Hand, an Eye, forms of the *Bon Dieu* – which is *projected* into his own dreams, writings, and visions. *Iterating* this self-reflexive mood, there emerges, naturally, a key (*clef*) to understand both Grothendieck’s Self and the World. (As in *The Doors over the Universe* (see above), it is enthralling to see how Grothendieck uses, beyond mathematics, the very central methodologies of his mathematical thought: *dévissage*, multiplication, projection, iteration, etc.) The central text of *La clef des songes* [g1987] (315 pages) is complemented with many notes (691 pages), which contain a long final development (circa 500 pages), *Les mutants*, where Grothendieck proposes a sort of genealogy of his spiritual Masters, and their influence on his own libertarian thought.

The manuscript is divided into six parts and many notes, summarized in what follows. *I. All the dreams are creation of the Dreamer* [g1987 1–46]: the dream and the knowledge of the self, discovery of the Dreamer, childhood, dreams proceed from the Dreamer, instants of truth, the key of the big dream, the voice of reason and the other voice, acts of knowledge and acts of faith, the spark and the flame, work and conception, the rhythm of creativity, cycles of Eros, emotion and thought. *II. God is the Dreamer* [g1987 47–79]: God is the Dreamer, lost knowledge, the incredible Good News, brothers in starvation, meeting the Dreamer, prohibited questions, reunion with God, only one Dreamer exists, the Creator and the Web, God is not defined nor proven, a new multiplication Table. *III. Voyage to Memphis (1): wandering* [g1987 80–136]: my parents, God’s splendor, Rudi, a fall of wonders, God by sound reason, lost encounters, the calling, the turn, faith and mission, death, God speaks in low voice. *IV. Aspects of a mission (1): a freedom song* [g1987 137–182]: unthinkable convergence, testimony as a call for the discovery of one’s self, Eros or power, Eye or sense, vision as testimony, soul and labor, man as a creator, creativity and repression, free creativity and interior work. *V. Aspects of a mission (2): spiritual knowledge* [g1987 183–268]: inclusive knowledge, beauty and contemplation, pain and shadows, the soul of things and the man without soul, the mentality of the flock, the use of Times, creativity and interior voice, creativity and listening, the tree of good and evil, archaic spirituality, law, ignorance, truth and creativity, resistance and suffering. *VI. Voyage to Memphis (2): seeds for a mission*

[g1987 269–315]: acting, separation, beginning without end, hope, Tempest, the new man, surface and deepness, the call of silence, the messenger, crossing of the desert and revelation, seeds and harvest, tasks and gestation. *Notes* (notes numbered 1–57) [g1987 N1–N176]: Eros, the One and the Infinite, wisdom of the body and action of God, the role of dreams, archetypes and God’s manifestations, dream and free will, mystic experience and self-knowledge, miracles and reason, truth and knowledge, mathematics and imponderables, belief and faith, the boy and the mystic, my friend the “*Bon Dieu*,” mission and creativity, Hell and Providence, eros and spirit, the great innovators, the creative boy, “unthinkable May 68,” creativity and maturity, error and ignorance, doubts and security, work and desire. *The mutants* (notes numbered 58–145) [g1987 N177–N691]: register and study of spiritual guides and freedom thinkers, Fujii Guruji, Gandhi, Walt Whitman, the mutants (1: spirit, justice, freedom), the place of Riemann, A. S. Neill, Edward Carpenter, Félix Carrasquer, the mutants (2: power, diversity, sex), Solvic (sic = Slovik), Steiner, Krishnamurti, the mutants (3: self-knowledge, reconciliation, crisis of civilization, hope), Darwin, Freud.

Through this list, one can see how Grothendieck is mainly concerned with *creativity*, and how the different layers of creativity *expand* from his own experiences and dreams, towards a superior archetype (God), passing through many intermediate Masters (*mutants*: the visionary leaders of a “Great Mutation” needed for a redemption of mankind, a chronological list of Grothendieck’s 18 *mutants* appears in [g1987 N480–N482]). The *multiplicative richness* of Grothendieck’s ideas covers history, phenomenology, metaphysics, psychoanalysis, sociology, religion, and mystic thought. As if it were a *universal sheaf* in action, Grothendieck looks for a common blowing, a unitary life, a universal harmonics (“global sections”), beyond the most particular concretions, blades of grass, daily dreams, sensible perceptions (“local sections”). *Content and form* are combined in a uniquely *smooth* state of grace in *La clef des songes* [g1987]: on one hand (content), the search for a common stem in the Tree of Life, found with passion in Whitman and Darwin, and reflected along Grothendieck’s main mathematical unitary archetypes (nuclear spaces [g1949–1953], projective norms [g1953c], K -theory group [g1957], schemes [g1959–1964], toposes [g1960–1969], motives [g1967], anabelian varieties [g1981], derivators [g1991]); on the other hand (form), the development of a *free language*, disentangled from bonds and prejudices (types), and plastically open to “listen to things” (archetypes) which transcend language itself.

5 A Synthetic Account of Grothendieck’s Work

Grothendieck’s mathematical work shows a strong equilibrium between (1) high conceptualizations and (2) concrete examples. In the first trend, the use of Category Theory is ubiquitous; we will call *categorical reason* this tendency to the global and the general. In the second trend, the uses of complex variables, algebraic structures and topological forms pervade the work; we will call *mathematical heart* this support on the local and the particular. Grothendieck’s work can then be understood

as an *adjunction* between reason and heart, in Spanish, between *razón* and *corazón*. Since *corazón* = *co-razón*, we can sense a full *duality* (“co”) between intelligibility and sensibility, something clearly stated in *Récoltes et semailles* and *La clef des songes*. The duality extends to a deep understanding of the polarities *étalé* and *éale* (all Grothendieck may be ciphered in an accent!), *relativity and universality*, *types and archetypes*.

5.1 Mathematical Heart and Categorical Reason

The *back-and-forth* between the particular and the universal, the concrete and the abstract, the typical and the archetypical, the Many and the One, the local and the global, the examples and the theories – paradigm of a *pendular mathematical practice* – pervades Grothendieck’s work. We will call *mathematical heart* the environment of his local technical strengths (functional analysis, topology, complex variables, sheaves, co/homology, algebraic geometry, etc.), and we will call *categorical reason* the environment of his global conceptual visions (schemes, topos, motives, *stacks*, *n*-categories, derivators, etc.) Following a pendular movement between “mathematical heart” and “categorical reason,” a full dialectics (“co”) is established between *co-razón* (= *corazón* = heart) and *razón* (= reason).

In Grothendieck’s *First Period 1949–1957*, the mathematical heart appears in *topology* (A: topological vector spaces, B: sheaves, C: non-separation problems), *functional analysis* (D: Hilbert and Banach spaces, E: nuclear spaces, F: approximation problems), *complex variables* (G: Riemann-Roch, H: spaces of holomorphic functions, I: Riemann sphere), *abstract algebra* (J: tensor products, K: stability properties, L: co/homology). The twelve forces (A–L) support Grothendieck’s originary intuitions and secure the *concrete richness* of his mathematics. A *calibration* of forces along the main productions of the decade ([g1949–1953], [g1953c], [g1955–1956], [g1957]) shows that themes (C) and (K) are ubiquitous, and that themes (B) and (J) reveal some of the favorite *universal* constructions of the young mathematician.

Along Grothendieck’s *Second Period 1958–1970*, three new forces occur in the *mediation* between mathematical heart and categorical reason (X: schemes, Y: topos, Z: topos *éale* of a scheme). Emerging from the Edinburgh lecture [g1958], they propagate through the main IHES writings ([g1961], [g1959–1964], [g1960–1969]), converging finally in the standard conjectures [g1968]. A calibration of themes (A–L) and (X–Z) inside those texts shows how Grothendieck in this “communitary” period passes to work at high levels of abstraction (“categorical reason,” X–Z), to the detriment of the initial concrete seeds (“mathematical heart,” A–L) that fostered first his imagination.

Finally, Grothendieck’s *Third Period 1981–1991* shows a strong return to classical themes at the heart of mathematics (α : *moduli* spaces of Riemann surfaces, β : absolute Galois group, δ : fundamental groups / groupoids, γ : homotopy types). A fascinating back-and-forth is constructed between the very concrete in arithmetic (anabelian varieties [g1981]) and the very abstract in homology and homotopy

(derivators [g1991]), going through unexpected combinatorial mediations (*stacks* and *n*-categories [g1983], *dessins d'enfants* and tame topology [g1984]). In these years, a calibration of the themes (A-L), (X-Z), and (α - γ) shows how “corazón” and “razón” become tightly entangled, producing an extremely powerful integrative *co-reason* (= “co-razón”).

The *continuity* of Grothendieck’s thought becomes evident when one ponders his reflections on “gluing” *space-number*. In a *First Period*, *varieties of space* (topological vector spaces, spaces of holomorphic functions, Riemann surfaces) *resonate* with *varieties of number* (tensor products, rings, co/homologies), before becoming fully integrated at *two unification levels*, first as singular sheaves, then as plural sheaf categories. In this second level (*Second Period*), emerge (1) sheaf categories with *abelian fibers* (fibers are abelian groups, the case of *abelian categories*), (2) sheaf categories with *local fibers* (fibers are local rings, the case of *scheme categories*), (3) sheaf categories with *set-theoretic fibers* (fibers are sets, the case of *toposes*), and (4) sheaf categories with *arithmetic fibers* (fibers are finite and separable residuated fields, the case of *étale toposes*). In this last environment, the fine techniques of entangling space and number provide the adequate “rising sea” (“categorical reason”) to solve Weil’s conjectures (“mathematical heart”). On the other hand, those algebraic topology or algebraic geometry forces become *inverted* in the *Third Period*, where the new constructs of topological algebra (fundamental algebraic group, anabelian varieties, Grothendieck-Teichmüller group, etc.) complement Grothendieck’s deep explorations of the extremely complex and stratified *space-number* realm.

5.2 Étale and étale

An *extreme synthesis* allows to condense a fair amount of Grothendieck’s work in the *behavior of an accent!* In fact, the wide difference between the terms *étalé* and *étale* (very often confused, even between Grothendieck scholars) captures two of the main watersheds of Grothendieck’s thought. On one side, (1) *étalé* summons the “*espace étalé*” of Cartan’s school [g1955–1956], that is, a *sheaf* as an unfolded space. On the other side, (2) *étale* calls for the “*topologie étale*” of Grothendieck’s school, that is, the smooth, folded category-theoretic topology used in the resolution of Weil’s conjectures [g1960–1969]. The opposition is extreme: (1) the *étalé* alludes to multiplicity, ramification, unfolding of a given sheaf, while (2) *étale* alludes to unity, smoothness, folding in a category of schemes. The *pendular movement between the Many and the One* is double: in (1), over the particular (one sheaf), one studies the variation of topological sections, while in (2), over the general (multiple schemes in a category), one observes the stability of arithmetical fibers – (1) reveals the transit from the One to the Many (*étalé*) while (2) detects, inversely, the passage from the Many to the One (*étale*).

The *étalé* strategy (1) governs Grothendieck’s world, traversed in all directions by sheaves. The main problem, understanding the *mixture space-number*, is explored through universal sheaf constructions (*K*-theory, topos), which explain the *types* of

space and number as projections of sheaf-like *archetypes*. On the other hand, the *étale* vision (2) enters in harmonic conjugacy with the *flat or rising sea* (Victor Hugo's "*mer étale*"), (Grothendieck acknowledges that "in my work I am as yin, as sea and movement, as one can be" [g1986 3.595]. The "*étale* sea" appears in Victor Hugo's *Les travailleurs de la mer* (1865–1866): "*Le niveau uniforme du varech sur toutes les roches marquait la ligne de flottaison de la marée pleine et de la mer étale,*" **Hugo1** p. 678.) where general categorical settings (abelian categories, topos, derivators) literally flood our understanding. The architectonic tensions between (1) *étalé* and (2) *étale* help to situate Grothendieck's major inventions. In the universal, abstract environment of the (1) *étalé*, live the main *strata and hierarchizations*, the staggered flats of the building: tensor norms in Banach spaces [g1953c], iterated sheaves [g1959–1964], Teichmüller's tower [g1961] [g1981], homotopy types and *n*-categories [g1983], etc. On the other side, in the universal, abstract environment of the (2) *étale*, live the main *coverings*, the high perspectives from the building's top: nuclear spaces [g1949–1953], abelian categories [g1955–1956], *K*-theory [g1957], toposes [g1960–1969], motives [g1967] [g1968], tame topology [g1984], derivators [g1991], etc.

Two strong varieties of *mathematical practice* are then combined in Grothendieck's work: (1) unfolding, unscrewing ("*dévisser*"), and towering (close to a "mathematical heart"), complemented with (2) folding, screwing, and covering (closer to "categorical reason"). The *back-and-forth* methodology between (1) and (2) allows to explore a multitude of *apparently* wild and unlike situations, and to find *natural universal invariants* beyond the variable multitudes of mathematics and the World. A new (1) *differential* and (2) *integral calculus* is thus proposed at a very high conceptualization level.

5.3 Relativity and Universality

The detailed *practice* of "relative mathematics" is a permanent feature of Grothendieck's "doing". (We use the term "doing" ("hacer" in Spanish) in the sense of Javier de Lorenzo, who, since his *Introducción al estilo matemático* (1971), insisted in the importance of constructing a mathematical philosophy attentive to the advances of the discipline, and carefully oriented towards *mathematical practice*. See **Lorenzo1 Lorenzo2** De Lorenzo anticipated thus by 40 years (!) the present-day movement (fashion?) of the "*philosophy of mathematical practice*." For a complete vision of De Lorenzo's work until the year 2000, see **Zal10**.) In fact, the *localization* and *variation* of the objects occur early in comma categories [g1955–1956], in local rings along coherent sheaves and in local projections of the *K*-theory group over (co)homology [g1957], or in local schemes over a given basis [g1959–1964]. These methods become then extended to any form of construction which may be deemed "Grothendieckian." *Relativity* constitutes one of the sine qua non forces of his thought. But on the other hand, its natural counterpart – to look for invariants beyond variation, universals beyond relativization, globals beyond localization – is what really defines his enterprise. His *systematic finding of archetypes beyond types*

(see Subsection “Types and Archetypes” below) distinguishes him from many previous great mathematicians.

The finding is not accidental and is due to the *systematic use of sheaves and categories* in Grothendieck’s thought. Both an *epoch* (when sheaves and categories were emerging) and an exceptional *character* (vision, courage, solitude, discipline) were required to provide a deep new notion, a true revolution both in mathematics and philosophy: the ingenuous glimpse and the strenuous practice of the concept of a *relative universal*.

The notion immediately seems a contradiction in terms (how can something universal be relative and vice versa?), but it acquires a sense as simple as profound when one conjugates the dialectics *abstract-concrete* in Category Theory. The essential obstruction to overcome is to be able to imagine the idea of *the Universal without the Absolute*, something that can be obtained when universality is linked to a *practice of dynamical abstraction*. On one hand, *concrete categories* (classes of structured sets, e.g., groups, rings, ordered sets, topological spaces, differential varieties, etc.) provide very diverse distinctions, and capture the *relativity of mathematical practice*. But on the other hand, *abstract categories* (classes of morphisms with associative composition and identities) offer integral paths, and capture the *universality* of mathematical thought, thanks to simple and general definitions that can be provided through the extremely powerful quantifier “exists unique” ($\exists!$). (For example, an “initial object” 0 (defined by the property $\forall X \exists ! 0 \rightarrow X$) is a relative universal in abstract categories, which incarnates in many ways in concrete categories: in the empty set $0/$ in the category of sets, in the trivial group with one element $\{e\}$ in the category of groups, in the ring of integers $\mathbb{Z}$ in the category of (commutative unitary) rings, in a minimum m in the category associated to a partially ordered set, etc. On the other hand, 0 *does not* incarnate in the category of infinite sets (due to the infinitude, and thus non-unicity, of morphisms from ω , the only plausible candidate for an initial object). The *unifying structure* (A) of the abstract universal is thus *distributed* along the *variegated multiplicity* (B) of its very diverse concrete relativizations. We are here again in presence of a high conceptual *integral (A) and differential (B) calculus*. In a certain sense, our “blind eyes” (Tarkovsky) become then disposed to see: *beyond types emerge archetypes, beyond appearances we can grasp a “universal breath.”*)

The *relative universals* are the *clef de voûte* of Grothendieck’s cathedral. Thanks to them, a *cohesion* of multiple elements is obtained, and the cathedral’s arches are well adjusted, without collapsing. The *topos* notion [g1960–1969] constitutes perhaps the richer relative universal of Grothendieck’s opus. Its construction mixes many levels along the adjunctions between the pairs relativization/universalization and localization/globalization: (i) in an arbitrary category C with pullbacks, a Grothendieck topology J uses localization and variation techniques, giving way to a site (C, J) ; (ii) a sheaf on a site combines localizations over the base and a search for global sections in the unfolded (*étalé*) space; (iii) the collection of all sheaves on the site (Grothendieck topos) universalizes in a new unitary structure all possible types of relativizations. The result offers a generalization of space (topological space $\rightarrow$ site $\rightarrow$ topos), whose *relative universal* (topos) is extended well beyond the initial

source (topological space). In the *ascending* injection to the universal, and, further, in the *descending* projection to the relative, many technical tools are invented (e.g., *étale* cohomology) which help to understand the very concrete (Weil's conjectures) thanks to the very abstract (topos of a scheme). In particular, in this example, the *conjunction of the étalé and étale visions* can be seen as a strong partial weaving between the universal (topological sheaf – *étalé* world) and the relative (arithmetical scheme – *étale* world).

5.4 Types and Archetypes

The *Tôhoku* [g1955–1956] introduced sheaves and categories into the landscape of “real” mathematics. (Along with **Corfield** we understand by “real” mathematics, the corpus of the hard-working mathematicians along their *mathematical practice*: combinatorics, number theory, abstract algebra, algebraic geometry, topology, complex variables, functional analysis, differential geometry, etc., well beyond just sets and logic (the reduced, impoverished, realm of analytical philosophy). Sheaves were invented in 1942 by Leray, and categories were invented in 1945 by Mac Lane and Eilenberg. Nevertheless, their actual use in “real” mathematics, as signaled by the very Mac Lane (see **Kromer2** p. 158), is due to Grothendieck.) Grothendieck's contributions to Category Theory in this paper are impressive on their own (introduction of subobjects, adjoint functors, equivalences, representable functors, additive and abelian categories, generators, infinitary axioms), but the central points of the *Tôhoku* deal with two kinds of *archetypes related to mathematical practice*: (1) to prove that in an abelian category with a generator and a suitable infinitary axiom, any object (“type”) can be embedded into an injective object (“archetype”) and (2) to prove that the cohomology of a space, with coefficients in a sheaf (“types”), can be reconstructed as a projection of a formalism of derived functors (“archetypes”). In that way, “projective” (co)homology, one of the main mathematical investigations in the period 1895–1950, can be regarded as a seed of an even more fundamental theme: a theory of abstract derivators in suitable categories [g1991].

Grothendieck's *Tôhoku* strategy is very interesting in terms of the duality *abstraction-concretion*. First, categories abstract universal properties holding in the diversity of mathematical regions, through a series of *axioms*: associativity and identities (all categories), existence of an abelian group structure in the *Hom*-sets (additive categories, e.g., category of holomorphic fibered spaces over a Riemann surface), existence of kernels, cokernels, and adequate factorizations through them (abelian categories, e.g., category of sheaves with fibers abelian groups), etc. In this approach, the *axiomatic abstract definitions* have their expected *concrete models*. Second, in the abstract setting, that is in general categories which satisfy the axioms, not yet “reified” in concrete models, “archetypes” occur in a *natural way* (definitions and properties related to the quantifier “exists only” $\exists!$), and only afterwards they are projected into existential objects along concrete categories. Thus, abstract categories are “doubly archetyped” (*inside* categories, free objects appear; *outside* categories,

free objects incarnate in apparently very different disguises), while concrete categories become just partial modelizations of general categorical transits. Third, in the abstract settings, many general theorems occur thanks to a *free machinery* which has escaped from the particularity of the concrete models.

Grothendieck's *La longue marche à travers la théorie de Galois* [g1981] sets a new research program, after his works of the First (1949–1957) and Second (1958–1970) Periods. His attention is now turned to “low-level” complexity objects, that is, surfaces in general and Riemann surfaces in particular, modular curves, complex towers, and combinatorial approximations to the absolute Galois group. As we have seen (Section “La longue marche à travers la théorie de Galois (1981)” above), one of his main conjectures states that the anabelian varieties over $\mathbb{Q}$ are essentially the isomorphism classes of Riemann surfaces. In this way, an *archetypal* property (anabelianity) is expected to capture some of the main *types* of modern mathematical thought, the Riemann surfaces. On the other hand, *Pursuing stacks* [g1983] sets the stage for a profound connection between homotopy and homology, and a *hierarchy* of concepts, objects and techniques is explored through the many levels of homological and homotopical groups. In that sense, an n -stack can be understood as an n -truncated homotopy type, which in turn can be imagined as a $(n + 1)$ -sheaf. The *iteration of types* gives rise to ∞ -categories, where all information on n -morphisms (morphisms, functors, natural transformations, etc.) is piled-up. ∞ -categories act then as *very large archetypes* which cover huge varieties of models. Finally, *Les dérivateurs* [g1991] proposes several *axiomatic* treatments of the constructions first imagined in *Pursuing stacks*. Axioms are imposed on classes W of morphisms in Cat , to try to model some properties of weak equivalences in the category Hot of CW -complexes with homotopies between them. The program consists in trying to *elevate types in Hot to archetypes in Cat*. Axioms *free* the lower level constructions (say, in Hot), in order to get *smoothness* properties at higher levels (say, in Cat). In an *abstract, free, archetypal* environment, derivators capture some of the main *type constructions* of homotopy (coming from toposes and homotopy groups) and homology (coming from abelian categories and homology groups).

The *archetype/type dialectics* between abstract categorical constructions and structured, concrete models, considerably enriches the panorama of *mathematical practice*. The abstract and the concrete, the universal and the particular, categorical reason and mathematical heart, become all necessary to *nurture mathematical imagination*. Mathematics has to explore all possible worlds, beyond mere actuality, but, on the other hand, many of her greatest achievements occur when a general architecture illuminates an actual conjecture. The *back-and-forth between types and archetypes*, elevated to a method in Grothendieck's work, helps us to understand the irreducible complexity of mathematical thought.

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