Worksheet 1 - Sets, Proofs, Functions, Equinumerous sets

Computability and Complexity – M1 NLP – 2023–2024

Exercice 1 Composition

Let $f : A \to B$ and $g : B \to C$

1. Prove that if f and g are injective, then $g \circ f$ is injective.
2. Prove that if f and g are surjective, then $g \circ f$ is surjective.
3. Prove that if $g \circ f$ is injective, then f is injective.
4. Prove that if $g \circ f$ is surjective, then g is surjective.
5. Find f and g such that $g \circ f$ is injective but not g
6. Find f and g such that $g \circ f$ is surjective but not f.

Exercice 2 Pairing

The goal is to build another bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. In the table, $f(n, p)$ is written in the cell row n and column p.

	0	1	2	3	4	...
0	0	1	3	6	10	
1	2	4	7	11		
2	5	8	12			
3	9	13				
4	14					
...						

Let D_q be the diagonal going from the cell $(0, q)$ to the cell $(q, 0)$.

- How many elements does D_q contain?
- Find an expression for $f(0, p)$.
- Each cell (n, p) belongs to some D_q. Find an expression for q, in terms of n and p.
- Find an expression for $f(n, p)$.

Exercice 3

Show the two following lemmata in the proof of irrationality of $\sqrt{2}$:

- Any fraction can be cancelled down to a unique simplest form where no more cancellations are possible (for example, 6/36 cancels down to 1/6 and cannot be simplified further).
- For any natural number n, if n^2 is even, then n is even.

Exercice 4 Induction

Show that:

- $\forall n,\ \sum_{i=1}^{n} i^2 = n \times (n+1) \times (2n+1)/6$
- $\forall n,\ 1(1!) + 2(2!) + \ldots + n(n!) = (n+1)! - 1$

Exercice 5 Induction again

Consider Ackermann's function A defined by:

- $A(0, n) = n + 1$
- $A(m + 1, 0) = A(m, 1)$
- $A(m + 1, n + 1) = A(m, A(m + 1, n))$

Show that:

- $A(1, k) = k + 2$
- $A(2, k) = 2k + 3$
- $A(3, k) = 2^{k+3} - 3$

Exercice 6 Reductio ad absurdum

Show that:

- Given a rectangle of 170 square meters, show that its length is greater than 13 meters.
- The square root of a positive irrational number is an irrational number.
- If n is the square of a positive integer, then $2n$ is not the square of an integer.