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ÉNERGIES & MOBILITÉS

Transfert de chaleur et de matière

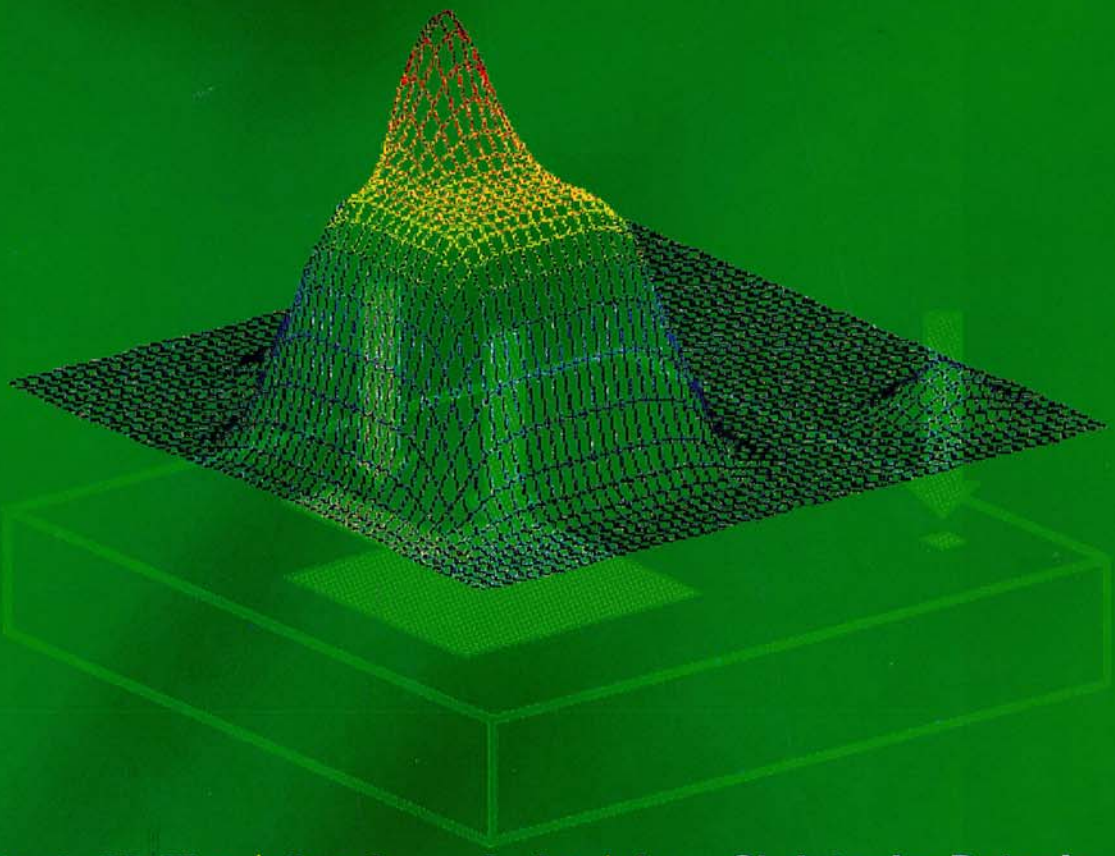
Projet:

Mesure de résistance thermique d'interface par la méthode Flash

WILEY

THERMAL QUADRUPOLES

Solving the Heat Equation through Integral Transforms



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1.3 HOW CAN IT BE WRITTEN?

In order to understand the way the quadrupole method can be implemented, two general heat transfer situations that are based on the notion of thermal resistance or of heat capacities will be briefly presented. They will be used as a base to introduce quadrupoles in a transient one-dimensional case.

1.3.1 Case a: Steady-State Transfer

Let us consider a medium made out of a single homogeneous material and limited by a closed surface of area S as depicted in Figure 1.1. We consider the steady-state heat transfer case where this medium receives heat from the outside through a part S_1 of surface S and releases it through another part S_2. It is considered here that the remaining part of S is insulated from the outside environment. The incoming heat flux through S_1 is called Φ_1 while the outgoing flux through S_2 is called Φ_2. We consider the average temperatures T_1 and T_2 over these two surfaces. Since we are under steady-state conditions, none of the four quantities T_1, T_2, Φ_1 and Φ_2 depends on time: all of them are constant. They are related by the two following equations:

$$T_1 - T_2 = R\Phi_1 \tag{1.1}$$

$$\Phi_1 = \Phi_2 \tag{1.2}$$

where R is the thermal resistance of the flux pipe between S_1 and S_2. This resistance depends on the geometry of the medium and on its thermal conductivity λ.

The preceding two equations can be easily given a single matrix equation form:

$$\begin{bmatrix} T_1 \\ \Phi_1 \end{bmatrix} = \begin{bmatrix} 1 & R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} T_2 \\ \Phi_2 \end{bmatrix} \tag{1.3}$$

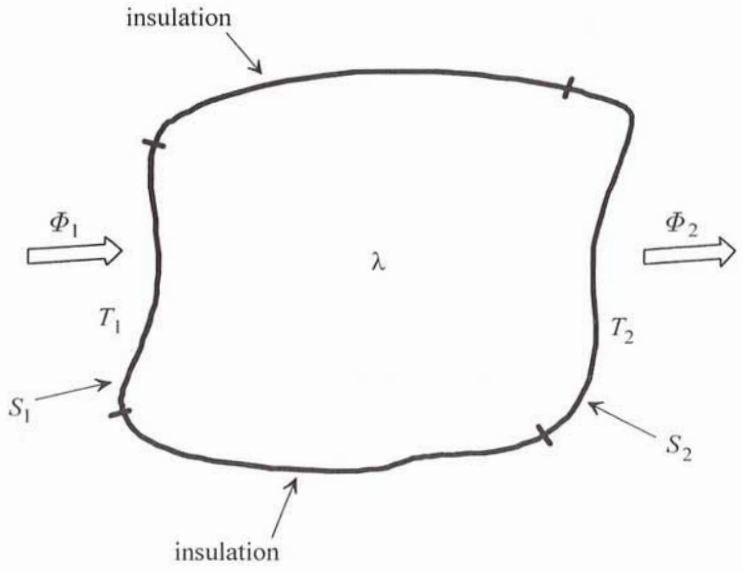

Figure 1.1 The steady-state case – the physical system

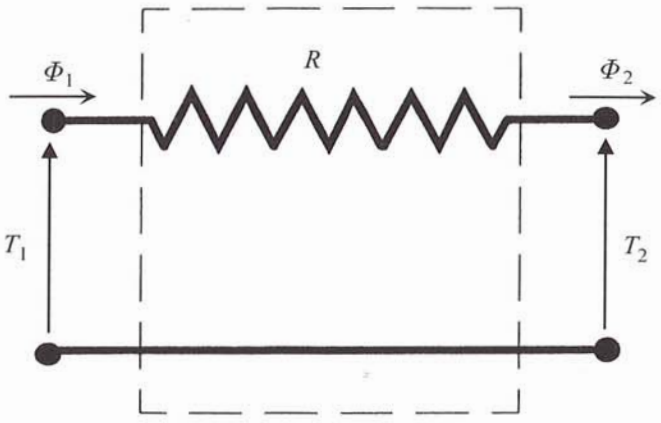

Figure 1.2 The steady-state case – network representation

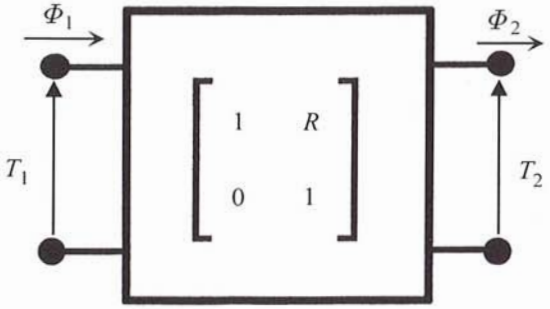

Figure 1.3 The steady-state case – matrix representation

This type of representation now uses the temperature vector $[T \quad \Phi]^t$ as a whole on the two different surfaces (we use the 't' exponent here to denote the transpose of a matrix). In a similar way it is possible to set out an electrical representation of equation (1.3) with the use of a four-terminal network (also called a two-terminal pair network by electrical engineers) presented in Figure 1.2. In this representation one can observe a base line on the lower part of the figure: this simply indicates that the reference temperature is the same on the two sides of the system. There are four terminals in this representation (indicated by bold dots). The two heat fluxes Φ_1 and Φ_2 play the same part as electrical currents; they are represented here by horizontal arrows on the upper horizontal line. The two temperatures T_1 and T_2 are similar to the electrical potentials, which is why they are represented here as vertical arrows with respect to the base line (these vertical arrows stem from the mechanical analogy, where the potential of gravity is an altitude and natural movement of a body occurs in the direction of the slope).

An alternative representation of equation (1.3) is shown in Figure 1.3, where the constitutive network is replaced by the 2×2 matrix of equation (1.3).

It is interesting to note that either this matrix or the unique resistance of the network in Figure 1.2 completely characterizes the physical system.

1.3.2 Case b: A Lumped Body in Transient Conditions

The same physical system as above is now considered, but the transient conditions are now studied. The medium is initially at a uniform zero temperature and at time $t = 0$ it begins to

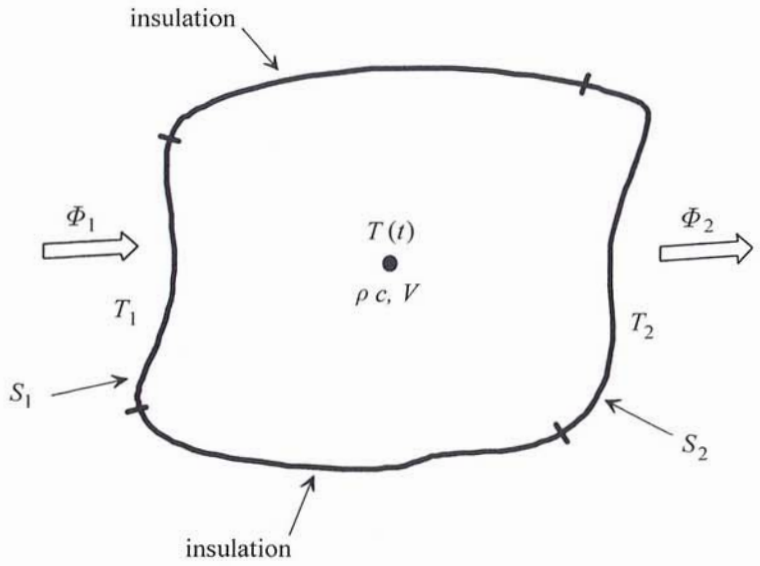

Figure 1.4 Lumped body model – physical system

exchange heat wih the outside environment. It receives a heat flux $\Phi_1(t)$ through its surface S_1 and a flux $\Phi_2(t)$ through S_2, the remaining part of S being insulated. Its conductivity λ is large enough to assume that at any time its temperature $T(t)$ can be taken as uniform, which means that it does not depend on the point considered. A simple heat balance during a time interval $\mathrm{d}t$ around t can be written as

$$\rho c V \frac{\mathrm{d}T}{\mathrm{d}t} = \Phi_1(t) - \Phi_2(t) \quad \text{with } T(0) = 0 \tag{1.4}$$

where ρ is the mass density, c the specific heat and V the volume of the medium (see Figure 1.4).

It is possible to remove the time derivative in equation (1.4) if the Laplace transforms $\theta(p)$ and $\phi_i(p)$ of the temperature and fluxes (for $i = 1$ or 2), respectively, are introduced:

$$\theta(p) = L[T(t)] = \int_0^\infty T(t) \exp(-pt)\, \mathrm{d}t$$

and

$$\phi_i(p) = L[\Phi_i(t)] = \int_0^\infty \Phi_i(t) \exp(-pt)\, \mathrm{d}t \tag{1.5}$$

Equations (1.4) and (1.5) then become

$$C_t\, p\theta(p) = \phi_1(p) - \phi_2(p) \tag{1.6}$$

where $C_t = \rho c V$ is the heat capacity of the medium. The temperatures $T_1(t)$ and $T_2(t)$ of the surfaces S_1 and S_2 are both equal to the temperature $T(t)$ in the medium since this temperature is uniform at any time t. The same is true for their Laplace transforms, i.e.

$$\theta(p) = \theta_1(p) = \theta_2(p) \tag{1.7}$$

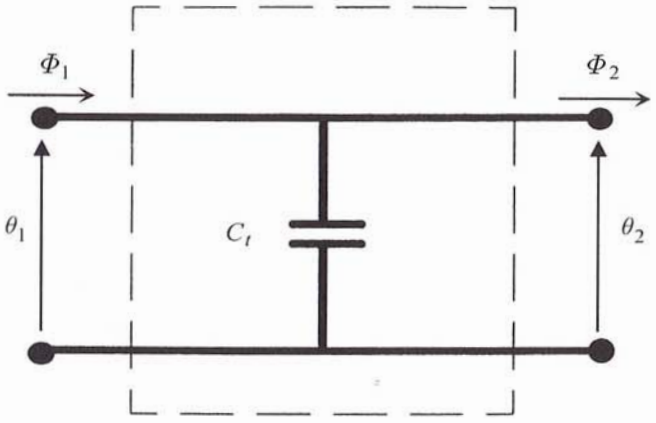

Figure 1.5 Lumped body model – network representation

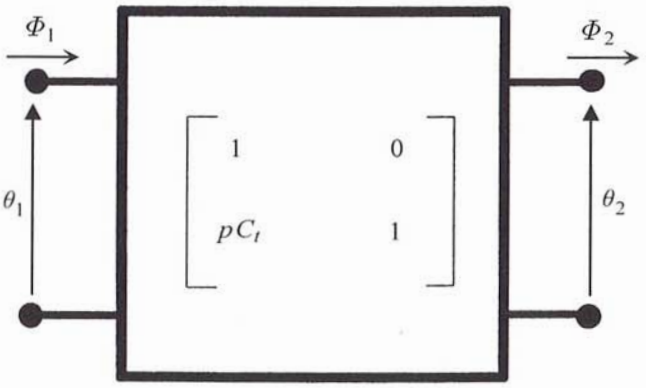

Figure 1.6 Lumped body model – matrix representation

Equations (1.7) and (1.6) give rise to the following matrix equation:

$$\begin{bmatrix} \theta_1 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ C_t p & 1 \end{bmatrix} \begin{bmatrix} \theta_2 \\ \phi_2 \end{bmatrix} \tag{1.8}$$

This equation has exactly the same form as equation (1.3) in the steady state, but the temperature–flux vector $[T \quad \Phi]^t$ at the non-adiabatic boundaries is replaced by its Laplace transform $[\theta \quad \phi]^t$.

We have seen that in steady-state conditions the fluxes are equal, while the temperatures differ. For the lumped body assumption, the temperatures remain equal while the exchanged fluxes differ. By analogy with what is usually done in electricity, there is a four-terminal network representing this last situation (see Figure 1.5). It is also possible to use the matrix representation shown in Figure 1.6.

This matrix, or the unique heat capacity C_t, completely characterizes the thermal system for this model.

1.3.3 Case c: The General Case

The preceding presentation (cases a and b) were based on the decomposition of the boundary of the homogeneous medium into three parts: two active surfaces and an insulated one. The

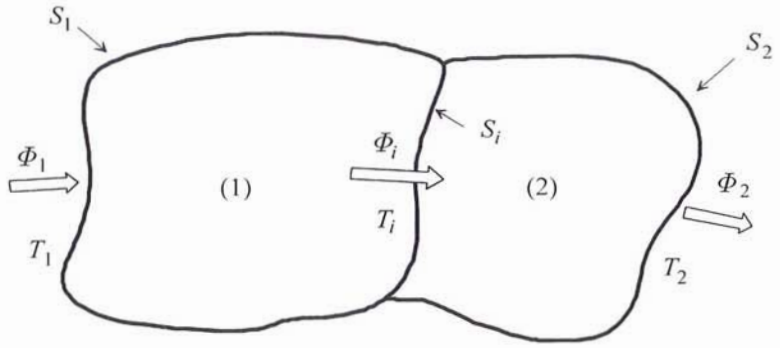

Figure 1.7 Cascade of two media

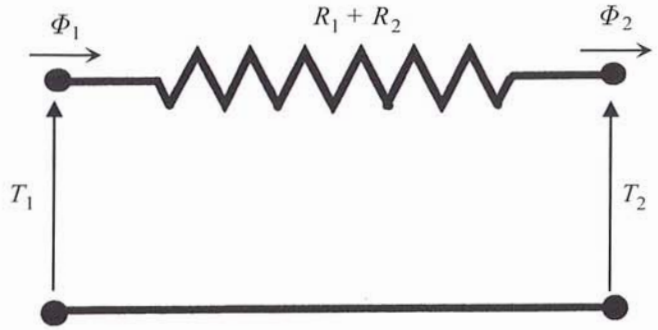

Figure 1.8 Cascade of two media – steady-state case

surface temperature–flux vector representation makes possible the modelling of multi-materials, i.e. systems composed of several different media that can be assembled in cascade for example. The case of a system composed of two different materials (1) and (2) is shown in Figure 1.7. If S_1 and S_2 are the outside surfaces of medium (1) and (2) and S_i their interface, then it is easy to represent the global transfer between S_1 and S_2 by a product of matrices.

With the same notation as above, the steady-state problem can be modelled in the following way:

$$\begin{bmatrix} T_1 \\ \Phi_1 \end{bmatrix} \begin{bmatrix} 1 & R_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & R_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} T_2 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} 1 & R_1 + R_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} T_2 \\ \Phi_2 \end{bmatrix} \tag{1.9}$$

A unique matrix, based on knowledge of the sum of the internal resistances of the two media $R_1 + R_2$, now links the temperature–flux vector, since the interface temperature vector $[T_i \quad \Phi_i]^t$ can be eliminated if the two equations similar to (1.3) are written for each medium (between S_1 and S_i and between S_i and S_2). The corresponding network representation is shown in Figure 1.8.

The transient lumped body counterpart of this case can be derived in the same way:

$$\begin{bmatrix} \theta_1 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ C_{t1}\,p & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ C_{t2}\,p & 1 \end{bmatrix} \begin{bmatrix} \theta_2 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ (C_{t1} + C_{t2})p & 1 \end{bmatrix} \begin{bmatrix} \theta_2 \\ \phi_2 \end{bmatrix} \tag{1.10}$$

It should be noted that the heat capacities of the individual media simply add up when the whole system is considered. Its network representation is shown in Figure 1.9.

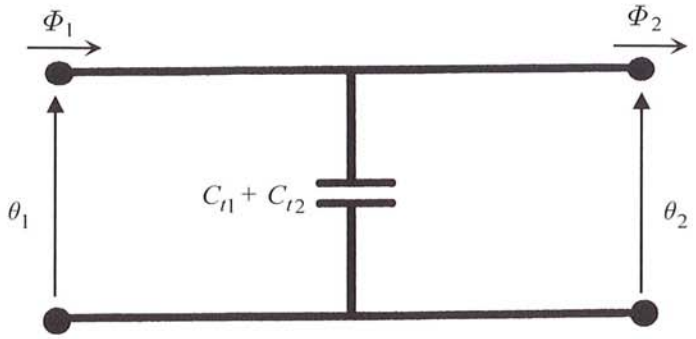

Figure 1.9 Cascade of two media – Lumped body approximation

The preceding two models (1.3) and (1.8) belong to the class of linear heat conduction models. The practical implementation of model (1.3) in the solution of a steady-state problem depends mainly on the way the internal resistance R can be evaluated. It also relies on the notion of average surface temperature, which means that a local temperature cannot be obtained. In that sense it is an external representation of the corresponding physical system, which links the four global variables (surface temperatures and fluxes) thanks to the definition of the thermal resistance of the corresponding flux pipe.

The application of the lumped body model requires the strong assumption of a uniform temperature distribution in the medium. It can be considered as a zero-dimensional model, since no geometrical data enter in the definition of the heat capacity that completely characterizes this system (it is simply the product of its mass by the specific heat c of the material).

The reader might wonder whether it is possible to extend models (1.3) and (1.8) to more general situations. For example, the multidimensional case shown in Figure 1.7 can be considered: transient conditions now apply and the temperature in each medium can no longer be taken as uniform. Since the problem is transient and linear, it is possible to write it in the Laplace domain and a relationship of the following form can be derived:

$$\begin{bmatrix} \theta_1 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} A(p) & B(p) \\ C(p) & D(p) \end{bmatrix} \begin{bmatrix} \theta_2 \\ \phi_2 \end{bmatrix} = \boldsymbol{M}(p) \begin{bmatrix} \theta_2 \\ \phi_2 \end{bmatrix} \tag{1.11}$$

A question may then arise: Can the preceding matrix representations be used to characterize completely the matrix $\boldsymbol{M}$ using knowledge of the thermal resistance and the heat capacity of each medium? For example, is it legitimate to write

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 \\ C_{t1}\,p & 1 \end{bmatrix} \begin{bmatrix} 1 & R_2 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \boldsymbol{M} = \begin{bmatrix} 1 & R_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ C_{t2}\,p & 1 \end{bmatrix} ? \tag{1.12}$$

The answer is no if no further assumption is made about the nature of the two media. The coefficients of the $\boldsymbol{M}$ matrix cannot be derived, in the general case, by a simple multiplication of resistive or capacitative matrices.

However, things simplify a lot in the case of a one-direction heat transfer.

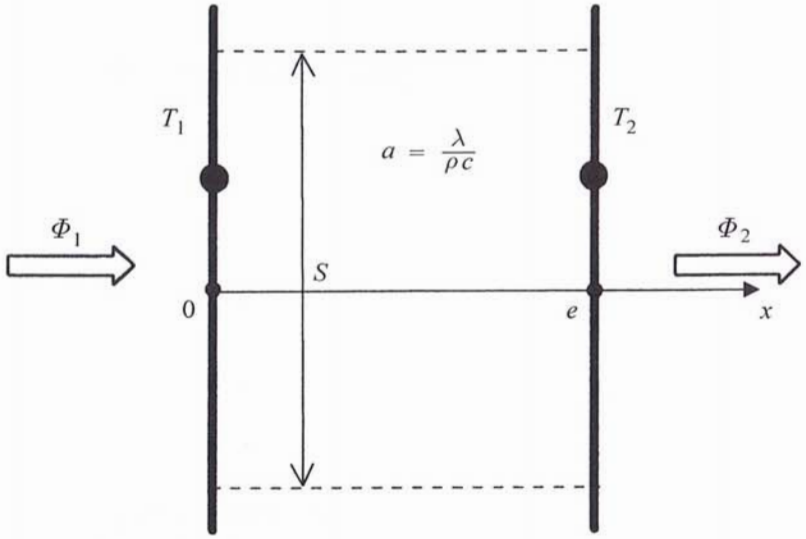

Figure 1.10 One-layer slab in a transient transfer – geometry

1.3.4 Case d: A One-layer Slab in Transient Transfer – the Analytical Quadrupole Matrix

A one-layer slab, made out of a homogeneous solid material, and shown in Figure 1.10, is characterized by its thickness e, its thermal conductivity λ and by its volumetric heat capacity $\rho c, \rho$ being its mass density and c its capacity. We call $a = \lambda/\rho c$ its thermal diffusivity. For a one-direction heat transfer the temperature field inside the slab can be written $T(x,t)$, where x is the coordinate perpendicular to the slab faces and t the time. We assume, furthermore, that the initial temperature in the slab is uniform and equal to T_0. The temperature is the solution of the following heat equation:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{a}\frac{\partial T}{\partial t} \quad \text{with} \quad T = T_0 \text{ for } t = 0 \tag{1.13}$$

Without loss of generality, it is possible to take an initial zero temperature ($T_0 = 0$). T is now the temperature variation with respect to the initial state. Since the problem is linear, a new temperature $T' = T - T_0$ with respect to equation (1.13), with a homogeneous initial condition, can always be defined.

The heat flux Φ at any location x inside the slab is defined for any cross-section S as

$$\Phi = -\lambda S \frac{\partial T}{\partial x} \tag{1.14}$$

The Laplace transforms of both temperature and flux are introduced:

$$\theta(x,p) = \int_0^\infty T(x,t)\exp(-pt)\,\mathrm{d}t \quad \text{and} \quad \phi(x,p) = \int_0^\infty \Phi(x,t)\exp(-pt)\,\mathrm{d}t \tag{1.15}$$

We now consider the Laplace transforms θ_1 and θ_2 of the front ($x = 0$) and rear ($x = e$) face temperatures, respectively, and the transforms ϕ_1 and ϕ_2 of the corresponding heat

fluxes. It can be shown – see Section 3.3.1 – that these four quantities are linked by the following equations:

$$\begin{bmatrix} \theta_1 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \theta_2 \\ \phi_2 \end{bmatrix} = \boldsymbol{M} \begin{bmatrix} \theta_2 \\ \phi_2 \end{bmatrix} \tag{1.16}$$

$$A = D = \cosh(k\,e); \qquad B = \frac{1}{\lambda\, k\, S} \sinh(k\,e) \tag{1.17}$$

$$C = \lambda\, k\, S \sinh(k\,e), \qquad k = \sqrt{p/a} \tag{1.18}$$

The coefficients A, B, C and D depend on the Laplace parameter p, on the thickness e of the slab and on the thermophysical properties of its constitutive material. It is interesting to note that equation (1.16) is valid for *any* boundary condition (in $x = 0$ and $x = e$). It is strictly equivalent to the heat equation and its associated boundary condition (1.13).

The matrix $\boldsymbol{M}$ with its four coefficients A, B, C and D completely characterizes the system here, in the same way as the thermal resistance R in steady-state heat transfer (case a) or as the heat capacity C_t for the lumped body assumption in transient transfer (case b). This characterization relates to an *external* representation of the system, since the four quantities $\theta_1, \theta_2, \phi_1$ and ϕ_2 are related to the external surfaces of the system and are not internal Laplace temperatures and fluxes. This type of representation is shown in Figure 1.11.

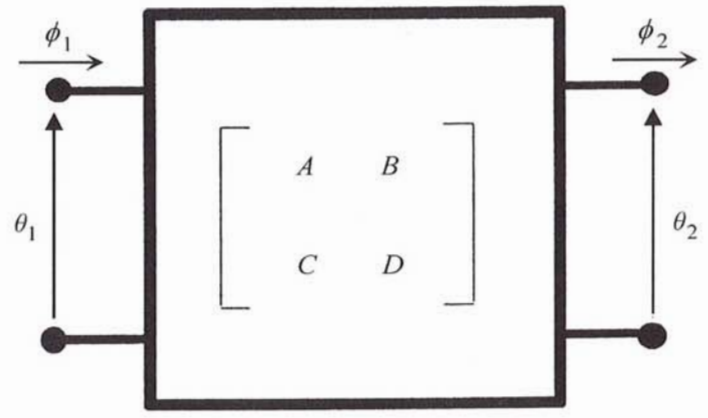

Figure 1.11 One-layer slab in a transient transfer – matrix representation

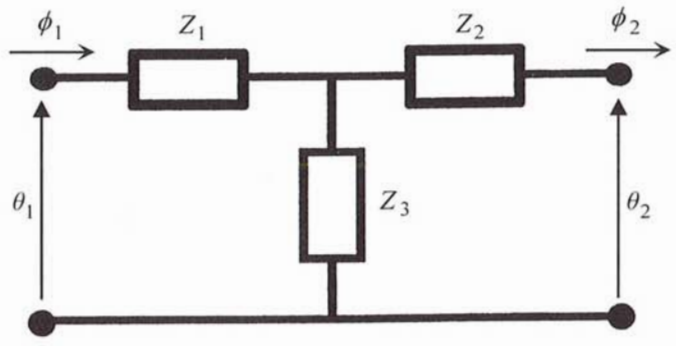

Figure 1.12 One-layer slab in a transient transfer – network representation

A network representation is also available (see Figure 1.12). It uses three thermal impedances Z_1, Z_2 and Z_3 that are related to the quadrupole coefficients A, B, C and D (see Section 3.3.1):

$$Z_1 = \frac{A-1}{C}; \qquad Z_2 = \frac{D-1}{C}; \qquad Z_3 = \frac{1}{C} \tag{1.19}$$

Up to now we have seen that, in a one-dimensional situation, the quadrupole matrix allows us to bridge the gap, in an exact way, between the resistance of the steady-state problem and the heat capacity of the lumped body assumption. The way it can be implemented will be now clarified using concrete examples.

1.4 HOW DOES IT WORK IN CONCRETE EXAMPLES?

1.4.1 Heat Pulse on an Insulated Slab

Equation (1.16) was valid for any type of boundary condition for a slab with parallel faces in a one-dimensional transient heat transfer. In order to calculate a temperature field, it is necessary to set two boundary conditions for this system.

It is now assumed that the front face of the slab ($x = 0$) is excited by a Dirac heat pulse of total energy density Q (in $\mathrm{J\,m^{-2}}$) while its rear face ($x = e$) remains insulated. These boundary conditions are typical of the heat pulse method for the measurement of the thermal diffusivity a. In the time domain these conditions are

$$\Phi = QS\delta(t) \quad \text{at } x = 0 \tag{1.20}$$

$$\Phi = 0 \quad \text{at } x = e \tag{1.21}$$

where $\delta(t)$ is the Dirac distribution, which can physically be considered as the limit, as $\varepsilon \to 0$, of the rectangular function of width ε:

$$d_\varepsilon(t) = 0 \quad \text{for } t \leq 0 \text{ and for } t > \varepsilon \tag{1.22}$$

$$d_\varepsilon(t) = 1/\varepsilon \quad \text{for } 0 < t \leq \varepsilon \tag{1.23}$$

These boundary conditions can be translated into the Laplace domain (see Table 9.1 in Chapter 9 for the usual Laplace transforms):

$$\phi = QS \quad \text{at } x = 0 \tag{1.24}$$

$$\phi = 0 \quad \text{at } x = e \tag{1.25}$$

It should be noted that time t (in seconds) has been replaced by the Laplace parameter p (in hertz), but the space variable x remains unmodified.

The system (1.16) can be written as

$$\theta_1 = A\theta_2 + B\phi_2; \qquad \phi_1 = C\theta_2 + D\phi_2 \tag{1.26}$$

where the Laplace boundary conditions (1.24) and (1.25), $\phi_1 = QS$ and $\phi_2 = 0$ have to be taken into account. Expression (1.26) therefore constitutes a linear system of two equations whose unknowns are the Laplace temperatures of the two faces, θ_1 and θ_2. Its solution is

$$\theta_2 = \frac{QS}{C} = \frac{Q}{\lambda\sqrt{p/a}\sinh(e\sqrt{p/a})} \tag{1.27}$$

for the rear face Laplace temperature, and

$$\theta_1 = QS\,\frac{A}{C} = \frac{Q}{\lambda\sqrt{p/a}\tanh(e\sqrt{p/a})} \tag{1.28}$$

for the front face Laplace temperature.

Internal Laplace temperatures can also be found by the same technique. The preceding notations can be slightly modified at this point: the Laplace temperature and flux at any point $x(0 \leq x \leq e)$ are now denoted by $\theta(x)$ and $\phi(x)$, respectively ($\theta(0) = \theta_1$, $\phi(0) = \phi_1$, $\theta(e) = \theta_2$, $\phi(e) = \phi_2$). If a layer of the slab defined by the coordinates x_1 and x_2 $(0 \leq x_1 < x_2 \leq e)$ is considered, then we can write the following basic quadrupole equation:

$$\begin{bmatrix} \theta(x_1) \\ \phi(x_1) \end{bmatrix} = \begin{bmatrix} A(w) & B(w) \\ C(w) & D(w) \end{bmatrix} \begin{bmatrix} \theta(x_2) \\ \phi(x_2) \end{bmatrix} = \boldsymbol{M} \begin{bmatrix} \theta_2 \\ \phi_2 \end{bmatrix} \tag{1.29}$$

where the four coefficients A, B, C and D are defined by equations (1.17) and (1.18), simply by replacing the thickness of the slab e by the thickness of the layer w (with $w = x_2 - x_1$).

Writing this equation for $x_1 = e/2$ and $x_2 = e$, it is possible to calculate the midslab Laplace temperature and flux:

$$\theta(e/2) = A(e/2)\theta(e) = \frac{Q}{2\lambda\sqrt{p/a}\sinh\left(\frac{e}{2}\sqrt{p/a}\right)} \tag{1.30}$$

$$\phi(e/2) = C(e/2)\theta(e) = \frac{QS}{2}\,\frac{1}{\cosh\left(\frac{e}{2}\sqrt{p/a}\right)} \tag{1.31}$$

The preceding three temperatures and the midslab flux can be calculated, in the time domain, by numerical inversion of the Laplace transforms (1.27), (1.28), (1.30) and (1.31).

Numerical inversions of these transforms are presented in detail in Chapter 9. We have decided to use Stehfest's algorithm here (see Section 9.3.2). If $F(p)$ is the known Laplace transform of the function $f(t)$, then an approximate value of this function at time t can be calculated:

$$f(t) \cong \frac{\ln(2)}{t} \sum_{i=1}^{n} \nu_i F(i\ln(2)/t) \tag{1.32}$$

The coefficients ν_i are given in Chapter 9 for different values of the total number of terms n.