ÉNERGIES & MOBILITÉS

Transfert de chaleur et de matière

Projet:

Mesure de coefficients thermophysiques dans des sphères

THIRD EDITION

HEAT CONDUCTION

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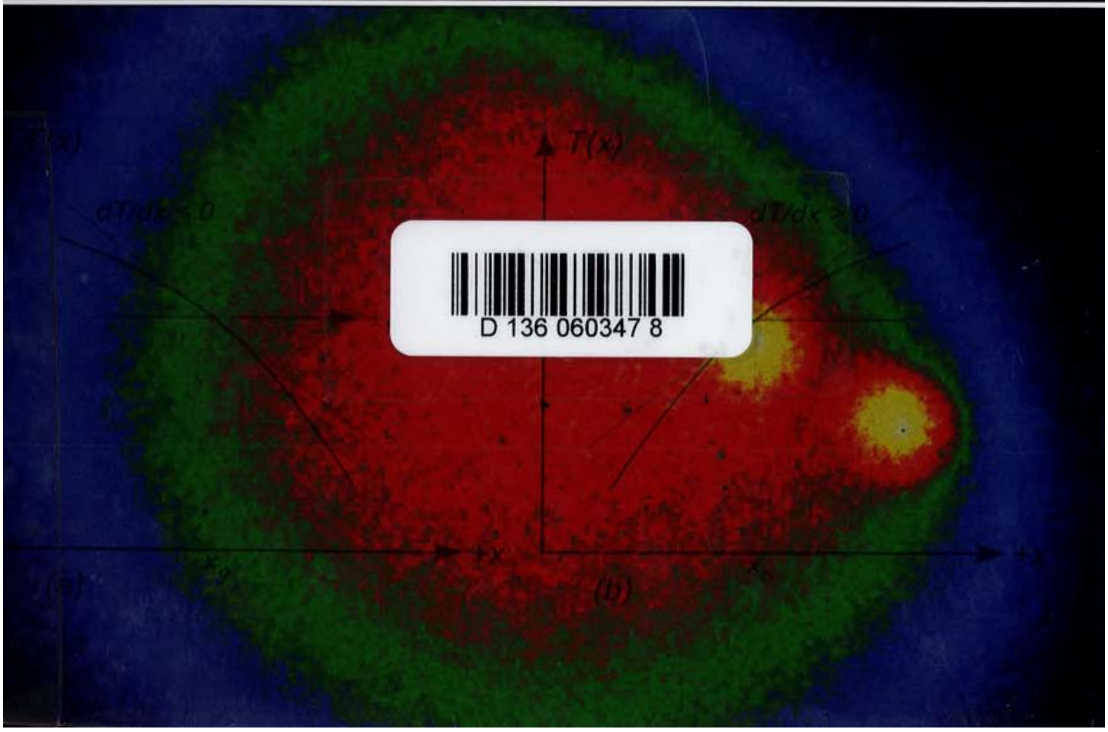

5-3 SOLUTION OF TRANSIENT PROBLEMS

In this section we will solve a range of transient problems in the spherical coordinate system. Our general guidelines established for separation of variables in the rectangular coordinate system equally apply to the spherical coordinate problems. Therefore, transient problems with a homogeneous PDE must always be solved with all homogeneous boundary conditions and with the single nonhomogeneity being the initial condition. If one or more boundary conditions are nonhomogeneous, removal of the nonhomogeneities via a temperature shift may be used, or the principle of superposition must be used to treat the nonhomogeneous boundary conditions separately in a steady-state problem. For a nonhomogeneous PDE (i.e., when heat generation is present), superposition is used to treat the heat generation within a steady-state problem, which is then coupled to a homogeneous transient problem at the initial condition. Prior to solving a range of problems to illustrate the above procedures, a few additional guidelines are offered for the transient heat equation in spherical coordinates.

As introduced in Section 5-1, the last three entries of Table 5-1 concern the transient spherical problems. For the 1-D transient problem, namely, $T = T(r, t)$, the transformation $U = rT$ is used to remove the nonconstant coefficients from the PDE, effectively reducing the spherical heat equation to the form of the Cartesian heat equation, for which we have the necessary tools to solve. The 2-D and 3-D transient problems involve $T = T(r, \mu, t)$ and $T = T(r, \mu, \phi, t)$, respectively. For both instances, we utilize the $V = r^{1/2}T$ transformation, and expect orthogonal functions in all spatial dimensions. For the r dimension, half-integer-order Bessel functions will always arise due to the separation constant introduced to force the Legendre polynomials (or associated Legendre polynomials) in the μ dimension. For the 3-D problem, as with cylindrical systems, the ϕ dimension must always be homogeneous, generating the trigonometric functions $\cos m\phi$ and $\sin m\phi$ for the condition of 2π periodicity for the full sphere ($0 \le \phi \le 2\pi$). The necessity for 2π periodicity will limit the separation constants m to integer values, which will then couple to the order of the associated Legendre polynomials with ϕ dependency. For the spherical coordinate system, the ϕ dimension will always correspond to the full sphere ($0 \le \phi \le 2\pi$), with a hemisphere defined by limiting the domain of the μ dimension, namely, to $0 \le \mu \le 1$. Finally, as described above, ϕ dependency in the absence of μ dependency is not covered from a physical point of view. Similarly, any combination of ϕ or μ dependency in the absence of r dependency is not considered (i.e., physically unrealistic).

Example 5-3 $T = T(r, t)$ for Solid Sphere with Convective Boundary
A solid sphere of radius b is initially at temperature $F(r)$. For $t > 0$, the boundary condition at $r = b$ dissipates heat by convection, with convection coefficient h into a fluid at zero temperature. The mathematical formulation of the problem is given as

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{in} \qquad 0 \le r < b \qquad t > 0 \tag{5-51}$$

$$\text{BC1:} \quad T(r \to 0) \Rightarrow \text{finite} \qquad \text{BC2:} \quad -k\left.\frac{\partial T}{\partial r}\right|_{r=b} = h\, T|_{r=b} \tag{5-52a}$$

$$\text{IC:} \quad T(t = 0) = F(r) \tag{5-52b}$$

Equation (5-51) may be recast in the equivalent form

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rT) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{5-53}$$

We now introduce a new dependent variable $U(r,t)$, using the transform

$$\boxed{U\,(r, t) = rT\,(r, t)} \tag{5-54}$$

With this substitution, the PDE of equations (5-51) and (5-53) becomes

$$\frac{\partial^2 U}{\partial r^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t} \tag{5-55}$$

which is equivalent to the 1-D Cartesian formulation. It is also necessary to transform the boundary conditions. We first consider a change of dependent variable for the spatial derivative of T, namely,

$$\frac{\partial T}{\partial r} = \frac{\partial}{\partial r}\left(\frac{U}{r}\right) = \frac{1}{r}\frac{\partial U}{\partial r} - \frac{1}{r^2}U \tag{5-56}$$

Equation (5-56) has the result of adding an additional term, $1/r^2U$, to each derivative term in a boundary condition; hence both insulated and convective boundary conditions each obtain an additional term. One must also consider the condition of finiteness as $r \to 0$, which now takes the form

$$T(r \to 0) \quad \Rightarrow \quad \lim_{r \to 0} \frac{U(r)}{r} \tag{5-57}$$

which is only finite for the expected trigonometric functions if the numerator is zero in the limit, which then enables the application of L'Hôpital's rule. Thus finiteness at the origin requires $U(r = 0) = 0$. We now summarize the complete transformation of the general $U\,(r, t) = rT\,(r, t)$ formulation in Table 5-2.

In view of the above discussion and Table 5-2, we now reformulate the problem, including both the boundary conditions and initial condition:

$$\frac{\partial^2 U}{\partial r^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t} \quad \text{in} \quad 0 \le r < b \qquad t > 0 \tag{5-58}$$

TABLE 5-2 PDE and Boundary Condition Transformations for $U = rT$

Equation	$T\ (r, t)$	$U\ (r, t)$
PDE	$\dfrac{\partial^2 T}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial T}{\partial r} + \dfrac{g_0}{k} = \dfrac{1}{\alpha}\dfrac{\partial T(r,t)}{\partial t}$	$\dfrac{\partial^2 U}{\partial r^2} + \dfrac{g_0 r}{k} = \dfrac{1}{\alpha}\dfrac{\partial U(r,t)}{\partial t}$
BC1	$T(r = b) = 0$	$U(r = b) = 0$
BC2	$\left.\dfrac{\partial T}{\partial r}\right\|_{r=b} = 0$	$\left.\dfrac{\partial U}{\partial r}\right\|_{r=b} - \dfrac{1}{b}\, U\|_{r=b} = 0$
BC3	$\left.\dfrac{\partial T}{\partial r}\right\|_{r=b} + \dfrac{h}{k}\, T\|_{r=b} = 0$	$\left.\dfrac{\partial U}{\partial r}\right\|_{r=b} + \left(\dfrac{h}{k} - \dfrac{1}{b}\right) U\|_{r=b} = 0$
BC4	$T(r \to 0) \Rightarrow$ finite	$U(r = 0) = 0$

$$\text{BC1:} \quad U(r = 0) = 0 \qquad \text{BC2:} \quad \left.\frac{\partial U}{\partial r}\right|_{r=b} + \left(\frac{h}{k} - \frac{1}{b}\right) U|_{r=b} = 0 \tag{5-59a}$$

$$\text{IC:} \quad U(t = 0) = rF(r) \tag{5-59b}$$

For convenience, we will introduce the constant $K = (h/k - 1/b)$ into equation (5-59a). With the successful transformation and with all homogeneous boundary conditions, we are now ready to proceed with separation of the form

$$U(r, t) = R(r)\Gamma(t) \tag{5-60}$$

which after substitution into equation (5-58) yields

$$\frac{1}{R}\frac{d^2 R}{dr^2} = \frac{1}{\alpha\Gamma}\frac{d\Gamma}{dt} = -\lambda^2 \tag{5-61}$$

Solution of separated ODE in the t dimension yields

$$\Gamma(t) = C_1 e^{-\alpha\lambda^2 t} \tag{5-62}$$

while solution of the separated ODE in the r dimension gives

$$R(r) = C_2 \cos\lambda r + C_3 \sin\lambda r \tag{5-63}$$

Boundary condition BC1 yields $C_2 = 0$, while BC2 yields the following transcendental equation:

$$\lambda_n \cot\lambda_n b = -K \quad \rightarrow \quad \lambda_n \quad \text{for} \quad n = 1, 2, 3, \ldots \tag{5-64}$$

The transcendental equation (5-64) may be recast in the form $b\lambda_n \cot \lambda_n b + bK = 0$, which has all real roots if $bK > -1$, and noting that $\lambda_0 = 0$ is an eigenvalue only for the special case of $bK = -1$. When the value of K as defined above is introduced into the inequality $bK > -1$, we find the necessary condition for all real roots given as $bh/k > 0$ (i.e., Bi >0). This result is satisfied with the physical requirements of a positive radius, convection coefficient, and thermal conductivity. We now sum over all possible solutions, yielding the general solution

$$U(r,t) = \sum_{n=1}^{\infty} C_n \sin(\lambda_n r) e^{-\alpha\lambda_n^2 t} \tag{5-65}$$

with $C_n = C_1 C_3$. The initial condition, equation (5-29b), is now applied, yielding

$$U(t=0) = rF(r) = \sum_{n=1}^{\infty} C_n \sin(\lambda_n r) \tag{5-66}$$

We solve for the Fourier coefficients by applying the operator

$$* \int_{r=0}^{b} \sin(\lambda_q r)\, dr$$

which yields the result

$$C_n = \frac{\int_{r=0}^{b} rF(r) \sin(\lambda_n r)\, dr}{\int_{r=0}^{b} \sin^2(\lambda_n r)\, dr} \tag{5-67}$$

The denominator is the norm $N(\lambda_n)$ and may be simplified by case 7 in Table 2-1, with the replacement of H_2 with K. We finally transform the problem back to $T(r,t)$, yielding the result

$$T(r,t) = \frac{U(r,t)}{r} = \sum_{n=1}^{\infty} C_n \frac{\sin(\lambda_n r)}{r} e^{-\alpha\lambda_n^2 t} \tag{5-68}$$

We know that the limit as $r \to 0$ exists and is finite per L'Hôpital's rule. In fact, in this limit, the temperature at the center of the sphere becomes

$$T(r=0,t) = \sum_{n=1}^{\infty} C_n \lambda_n e^{-\alpha\lambda_n^2 t} \tag{5-69}$$

The steady-state solution for the entire sphere is simply $T = 0$, which follows directly from equations (5-68) and (5-69) as $t \to \infty$, and is in agreement with the physics of the problem, namely, equilibration at the fluid temperature of zero.

Example 5-4 $T = T(r, t)$ for Solid Sphere with Insulated Boundary
A solid sphere of radius b is initially at temperature $F(r)$. For $t > 0$, the boundary condition at $r = b$ is perfectly insulated. The mathematical formulation of the problem is given as

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{in} \qquad 0 \le r < b \qquad t > 0 \tag{5-70}$$

$$\text{BC1:} \qquad T(r \to 0) \Rightarrow \text{finite} \qquad \text{BC2:} \qquad \left.\frac{\partial T}{\partial r}\right|_{r=b} = 0 \tag{5-71a}$$

$$\text{IC:} \qquad T(t = 0) = F(r) \tag{5-71b}$$

As in the previous problem, we introduce the $U(r, t) = rT(r, t)$ transformation of the PDE, boundary conditions, and initial condition. Using Table 5-2, this yields

$$\frac{\partial^2 U}{\partial r^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t} \qquad \text{in} \qquad 0 \le r < b \qquad t > 0 \tag{5-72}$$

$$\text{BC1:} \qquad U(r = 0) = 0 \qquad \text{BC2:} \qquad \left.\frac{\partial U}{\partial r}\right|_{r=b} - \frac{1}{b}\, U|_{r=b} = 0 \tag{5-73a}$$

$$\text{IC:} \qquad U(t = 0) = rF(r) \tag{5-73b}$$

With the successful transformation, and with all homogeneous boundary conditions, we are now ready to proceed with separation of the form

$$U(r, t) = R(r)\Gamma(t) \tag{5-74}$$

which after substitution into equation (5-72) yields

$$\frac{1}{R}\frac{d^2 R}{dr^2} = \frac{1}{\alpha\Gamma}\frac{d\Gamma}{dt} = -\lambda^2 \tag{5-75}$$

Solution of separated ODE in the t dimension yields the desired solution

$$\Gamma(t) = C_1 e^{-\alpha\lambda^2 t} \tag{5-76}$$

while solution of the separated ODE in the r dimension gives

$$R(r) = C_2 \cos\lambda r + C_3 \sin\lambda r \tag{5-77}$$

Boundary condition BC1 yields $C_2 = 0$, while BC2 yields the following transcendental equation:

$$b\lambda_n \cot\lambda_n b = 1 \quad \to \quad \lambda_n \quad \text{for} \quad n = 0, 1, 2, 3, \ldots \tag{5-78}$$

noting that $\lambda_0 = 0$ is an eigenvalue for this special case of the right-hand side equal to unity (see Appendix II). The general solution is now formed by summing over all possible solutions, yielding

$$U(r,t) = \sum_{n=0}^{\infty} C_n \sin(\lambda_n r) e^{-\alpha\lambda_n^2 t} \tag{5-79}$$

with $C_n = C_1 C_3$. Given the physics of the problem with regard to perfect insulation, as discussed below, it is useful to transform the problem back to $T(r,t)$ prior to considering the initial condition, which gives

$$T(r,t) = \sum_{n=0}^{\infty} C_n \frac{\sin(\lambda_n r)}{r} e^{-\alpha\lambda_n^2 t} \tag{5-80}$$

Examination of equation (5-80) as $t \to \infty$ yields a steady-state temperature equal to zero, which is inconsistent with the physics given that finite energy is initially contained within the sphere, as given by the initial temperature $F(r)$. Unlike with the case of the perfectly insulated cylinder, see Example 4-8, the necessary steady-state temperature does not arise directly from the zero eigenvalue term. Nevertheless, as noted by Carslaw and Jaeger [1, p. 203], for the case of the perfectly insulated sphere, $\lambda_0 = 0$ is an eigenvalue, and an additional term must be added to the solution (see the note at the end of this chapter). With this in mind, we express our general solution in the form

$$T(r,t) = C_0 + \sum_{n=1}^{\infty} C_n \frac{\sin(\lambda_n r)}{r} e^{-\alpha\lambda_n^2 t} \tag{5-81}$$

Ultimately, we expect C_0 to be the steady-state solution, although we continue here using our standard approach to evaluate the constants. We now apply the initial condition as given by equation (5-71b), which yields

$$T(t=0) = F(r) = C_0 + \sum_{n=1}^{\infty} C_n \frac{\sin(\lambda_n r)}{r} \tag{5-82}$$

The orthogonality of the sin function is with respect to a weighting function of unity; hence we first multiply both sides by r, giving

$$rF(r) = C_0 r + \sum_{n=1}^{\infty} C_n \sin(\lambda_n r) \tag{5-83}$$

We now evaluate the Fourier coefficients C_n by applying the operator to both sides

$$* \int_{r=0}^{b} \sin(\lambda_m r)\, dr$$

which yields the following expression:

$$\int_{r=0}^{b} r F(r) \sin(\lambda_m r)\, dr = C_0 \int_{r=0}^{b} r \sin(\lambda_m r)\, dr + \sum_{n=1}^{\infty} C_n \int_{r=0}^{b} \sin(\lambda_n r) \sin(\lambda_m r)\, dr \tag{5-84}$$

We note that the integral $\int_{r=0}^{b} r \sin(\lambda_m r)\, dr$ is identically zero for the eigenvalues as defined by equation (5-78). Using the property of orthogonality for the remaining integral on the right-hand side yields the Fourier coefficients

$$C_n = \frac{\int_{r=0}^{b} r F(r) \sin(\lambda_n r)\, dr}{\int_{r=0}^{b} \sin^2(\lambda_n r)\, dr} \tag{5-85}$$

The norm $N(\lambda_n)$ corresponds to case 7 in Table 2-1 with the replacement of H_2 with $-1/b$; hence the norm becomes

$$\int_{r=0}^{b} \sin^2(\lambda_n r)\, dr = \frac{b\left(\lambda_n^2 + 1/b^2\right) - 1/b}{2\left(\lambda_n^2 + 1/b^2\right)} \tag{5-86}$$

To now evaluate the remaining coefficient C_0, we apply the following operator to both sides of equation (5-83)

$$* \int_{r=0}^{b} r\, dr$$

which yields the following expression:

$$\int_{r=0}^{b} r^2 F(r)\, dr = C_0 \int_{r=0}^{b} r^2\, dr + \sum_{n=1}^{\infty} C_n \int_{r=0}^{b} r \sin(\lambda_n r)\, dr \tag{5-87}$$

As described above, the rightmost integral is zero for all terms in the summation, yielding the expression for C_0 as

$$C_0 = \frac{\int_{r=0}^{b} r^2 F(r)\, dr}{\int_{r=0}^{b} r^2\, dr} = \frac{3}{b^3} \int_{r=0}^{b} r^2 F(r)\, dr \tag{5-88}$$

Equations (5-81), (5-85), and (5-88) complete the problem, where C_0 is now recognized as the steady-state temperature of the sphere. This is readily checked by considering conservation of energy between the initial and final states. For a perfectly insulated sphere, we must conserve total energy, namely,

$$E_{\text{initial}} \equiv E_{\text{final}} \qquad \text{(J)} \tag{5-89}$$

We may evaluate the left-hand term by integrating the initial energy over the full sphere, while the right-hand term follows from $E_{\text{final}} = mcT_{\text{ss}}$, yielding

$$\int_{r=0}^{b} (\rho c) 4\pi r^2 F(r)\, dr = (\rho c)\left(\frac{4}{3}\pi b^3\right) T_{\text{ss}} \tag{5-90}$$

Simplification of equation (5-90) directly yields equation (5-88) with $C_0 \equiv T_{\text{ss}}$; hence our solution of the heat equation, including the steady-state solution, correctly reflects conservation of energy.

Example 5-5 $T = T(r, t)$ for Hollow Sphere

A hollow sphere of inner radius a and outer radius b is initially at temperature $F(r)$ (Fig. 5-3). For $t > 0$, the inner and outer surfaces are maintained at zero temperature.

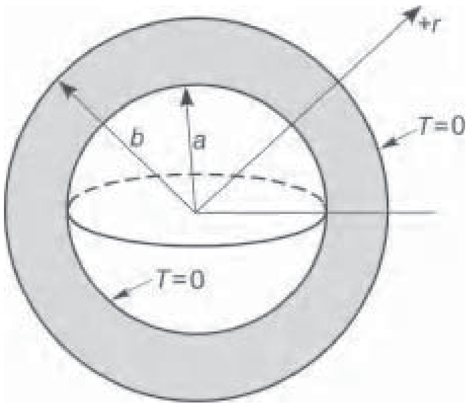

Figure 5-3 Problem description for Example 5-5.

The mathematical formulation of the problem is given as

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{in} \qquad a < r < b \qquad t > 0 \tag{5-91}$$

$$\text{BC1:} \qquad T(r = a) = 0 \qquad \text{BC2:} \qquad T(r = b) = 0 \tag{5-92a}$$

$$\text{IC:} \qquad T(t = 0) = F(r) \tag{5-92b}$$

We introduce the $U(r, t) = rT(r, t)$ transformation of the PDE, boundary conditions, and initial condition. Using Table 5-2, this yields

$$\frac{\partial^2 U}{\partial r^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t} \qquad \text{in} \qquad a < r < b \qquad t > 0 \tag{5-93}$$

$$\text{BC1:} \qquad U(r = a) = 0 \qquad \text{BC2:} \qquad U(r = b) = 0 \tag{5-94a}$$

$$\text{IC:} \qquad U(t = 0) = rF(r) \tag{5-94b}$$