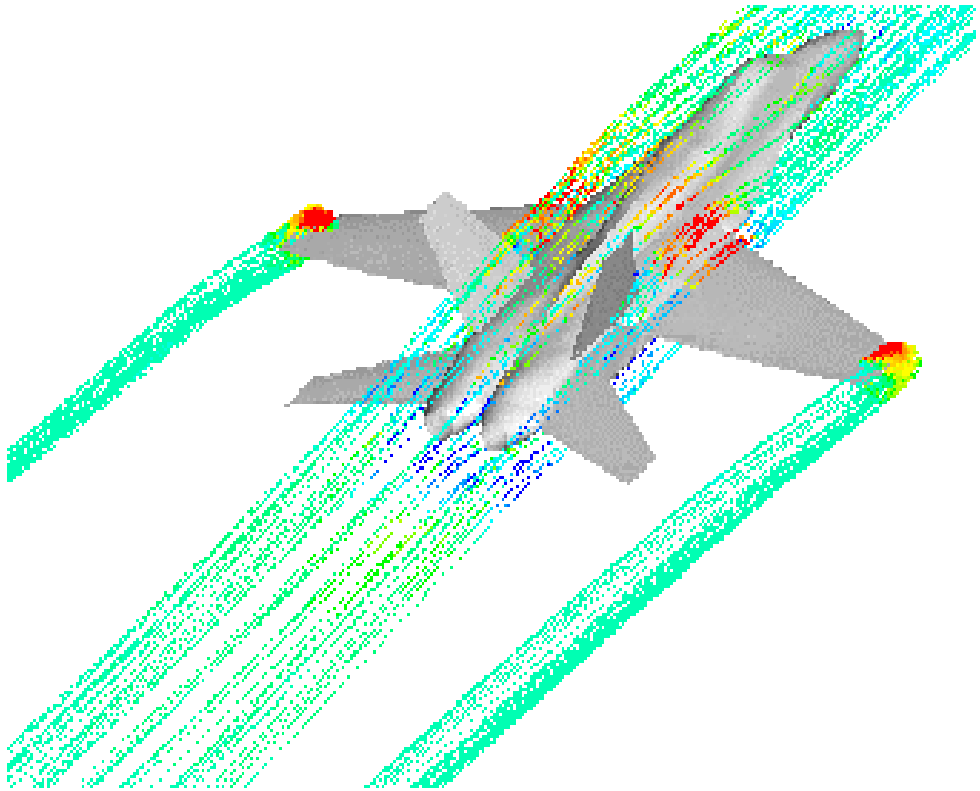
A Turbulence Primer



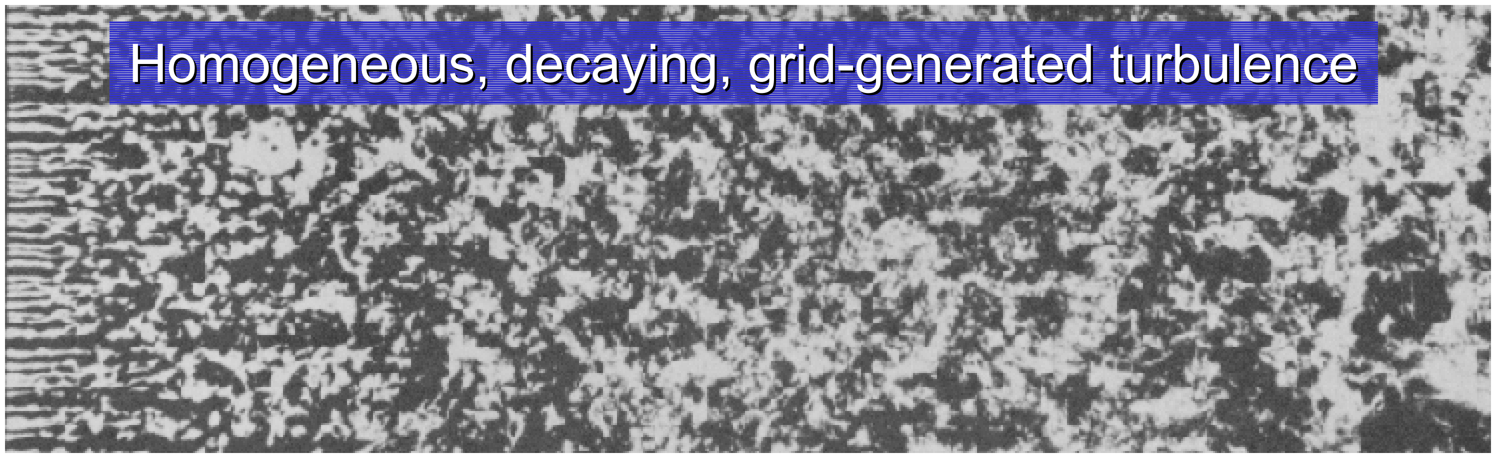
Outline

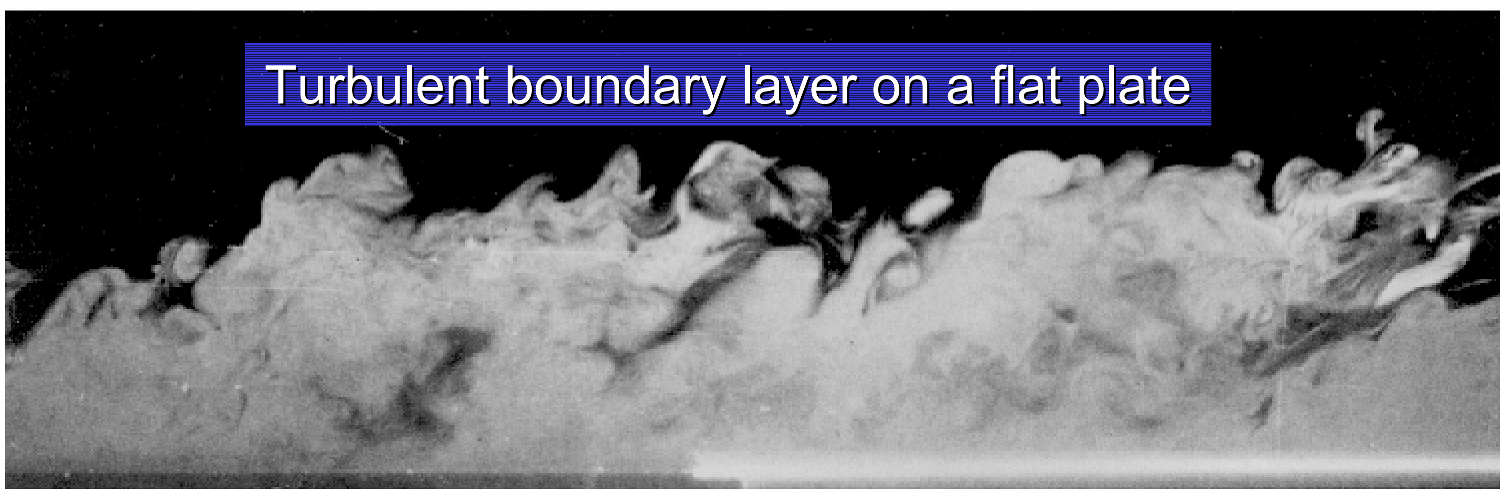
- Background
  - Scales
  - Eliminating the small scales
    - Reynolds Averaging
    - Near Wall Turbulence
    - Filtered Equations
- Examples
  - Comparison with Experiments and DNS
    - Turbulence Models
    - Near Wall Treatments

What is Turbulence?

- Unsteady, irregular (aperiodic) motion in which transported quantities (mass, momentum, scalar species) fluctuate in time and space
- Fluid properties exhibit random variations
  - statistical averaging results in accountable, turbulence related transport mechanisms.
- Contains a wide range of eddy sizes (scales)
  - typical identifiable swirling patterns
  - large eddies ‘carry’ small eddies.

Two Examples of Turbulence





Energy Cascade

- Larger, higher-energy eddies, transfer energy to smaller eddies via vortex stretching.
  - Larger eddies derive energy from mean flow.
  - Large eddy size and velocity on order of mean flow.
- Smallest eddies convert kinetic energy into thermal energy via viscous dissipation.
  - Rate at which energy is dissipated is set by rate at which they receive energy from the larger eddies at start of cascade.

Vortex Stretching

- Existence of eddies implies vorticity (define)
- Vorticity is concentrated along vortex lines or bundles.
- Vortex lines/bundles become distorted from the induced velocities of the larger eddies.
  - As end points of a vortex line randomly move apart
    - vortex line increases in length but decreases in diameter
    - vorticity increases because angular momentum is nearly conserved
  - Most of the vorticity is contained within the smallest eddies.
- Turbulence is a highly 3D phenomenon.

Smallest Scales of Turbulence

- Smallest eddy (Kolmogorov) scales:
  - large eddy energy supply rate ~ small eddy energy dissipation rate $\rightarrow \varepsilon = -dk/dt$.
    - $k \equiv ½(u'^2+v'^2+w'^2)$ is (specific) turbulent kinetic energy $[l^2 / t^2]$.
    - ε is dissipation rate of k $[l^2 / t^3]$.
  - Motion at smallest scales dependent upon dissipation rate, ε, and kinematic viscosity, ν $[l^2 / t]$.
  - From dimensional analysis:

$$\eta = (\nu^3 / \varepsilon)^{1/4}; \quad \tau = (\nu / \varepsilon)^{1/2}; \quad v = (\nu\varepsilon)^{1/4}$$

Small scales vs. Large scales

- Largest eddy scales:
  - Assume $\boldsymbol{l}$ is characteristic of larger eddy size.
  - Dimensional analysis is sufficient to estimate order of large eddy supply rate of k as $k / \tau_{turnover}$.
  - $\tau_{turnover}$ is a time scale associated with the larger eddies
    - the order of $\tau_{turnover}$ can be estimated as $\boldsymbol{l} / k^{1/2}$
- Since $\varepsilon \sim k / \tau_{turnover}$, $\varepsilon \sim k^{3/2} / \boldsymbol{l}$ or $\boldsymbol{l} \sim k^{3/2} / \varepsilon$
- Comparing $\boldsymbol{l}$ with η,

$$\frac{\boldsymbol{l}}{\eta} = \frac{\boldsymbol{l}}{(\nu^3 / \varepsilon)^{1/4}} \approx \frac{\boldsymbol{l}(\boldsymbol{k}^{3/2} / \boldsymbol{l})^{1/4}}{\nu^{3/4}} \approx \mathrm{Re}_T{}^{3/4} \Rightarrow \frac{\boldsymbol{l}}{\eta} >> 1$$

  - where $\mathrm{Re}_T = k^{1/2} * l / \nu$ (turbulence Reynolds number)

Implication of Scales

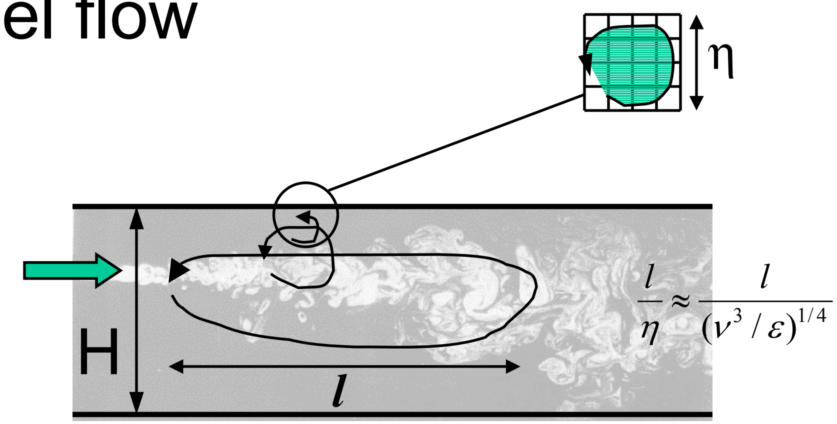
- Consider a mesh fine enough to resolve smallest eddies and large enough to capture mean flow features.
- Example: 2D channel flow
- $N_{cells} \sim (4l\,/\,\eta)^3$

  or

  $N_{cells} \sim (3Re_\tau)^{9/4}$

  where

  $Re_\tau = u_\tau H\,/\,2\nu$



- $Re_H = 30{,}800 \rightarrow Re_\tau = 800 \rightarrow N_{cells} = 4\text{x}10^7$!

Direct Numerical Simulation

- "DNS" is the solution of the time-dependent Navier-Stokes equations without recourse to modeling.

$$\rho\left(\frac{\partial U_i}{\partial t}+U_k\frac{\partial U_i}{\partial x_k}\right)=-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_k}\left(\mu\frac{\partial U_i}{\partial x_j}\right)$$

- Numerical time step size required, $\Delta t \sim \tau$
  - For 2D channel example
    - $Re_H = 30{,}800$
    - Number of time steps ~ 48,000

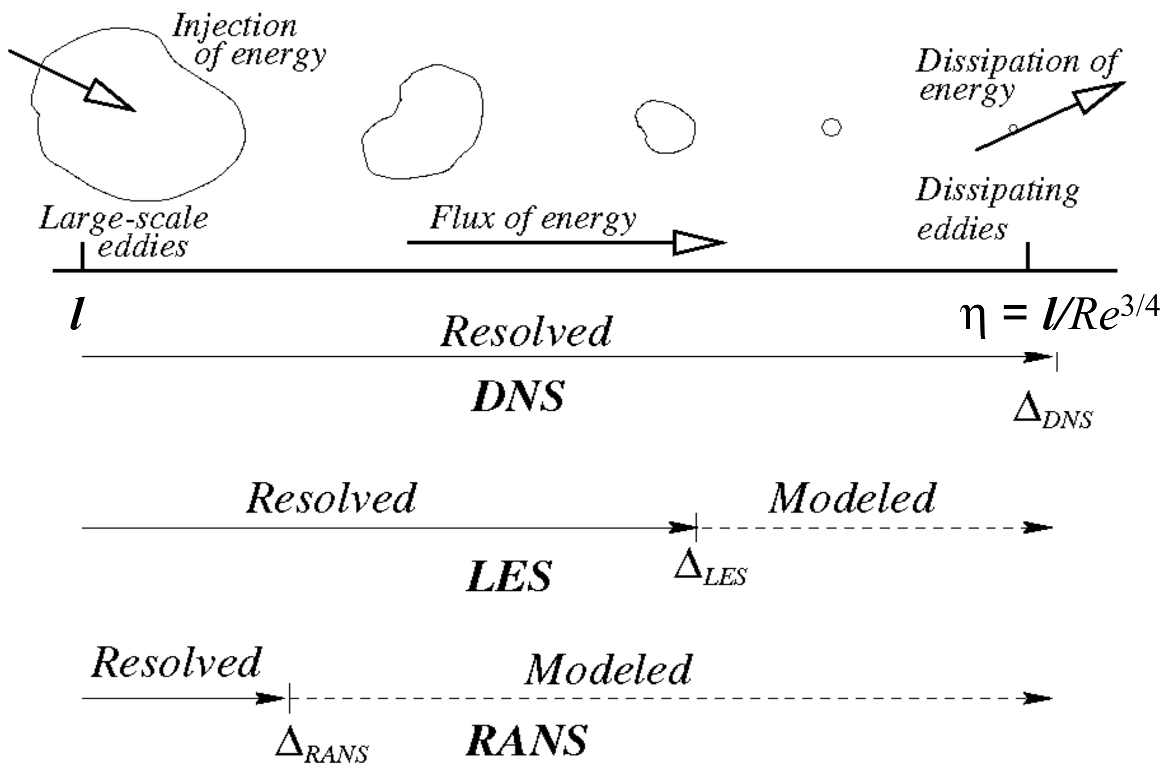
$$\Delta t_{2D\ Channel} \approx \frac{0.003H}{\sqrt{\mathrm{Re}_\tau}\, u_\tau}$$

- DNS is not suitable for practical industrial CFD
  - DNS is feasible only for simple geometries and low turbulent Reynolds numbers
  - DNS is a useful research tool

Removing the Small Scales

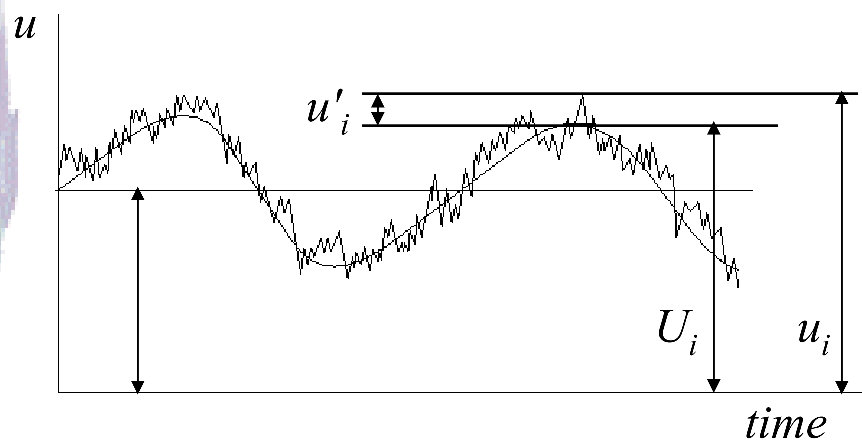
- Two methods can be used to eliminate need to resolve small scales:
  - Reynolds Averaging
    - Transport equations for mean flow quantities are solved.
    - All scales of turbulence are modeled.
    - Transient solution Δt is set by global unsteadiness.
  - Filtering (LES)
    - Transport equations for ‘resolvable scales.’
    - Resolves larger eddies; models smaller ones.
    - Inherently unsteady, Δt set by small eddies.
- Both methods introduce additional terms that must be modeled for closure.

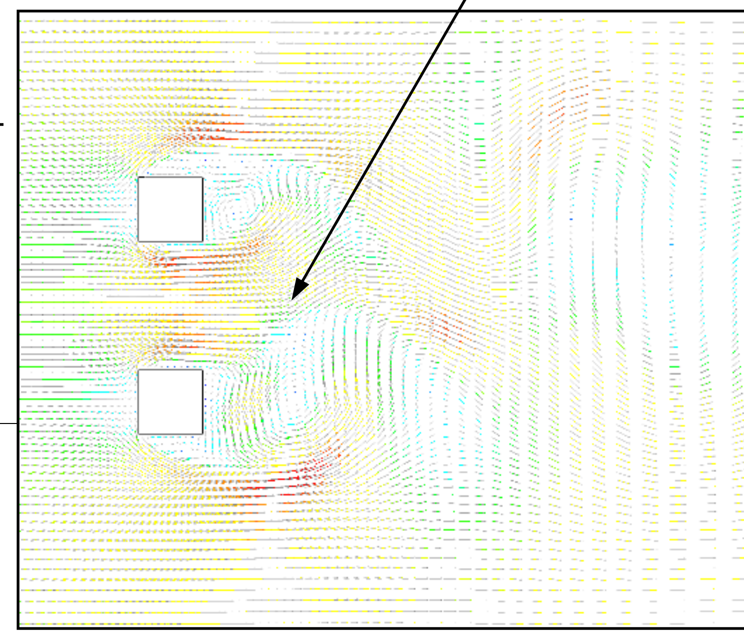
Prediction Methods



RANS Modeling - Velocity Decomposition

- Consider a point in the given flow field:

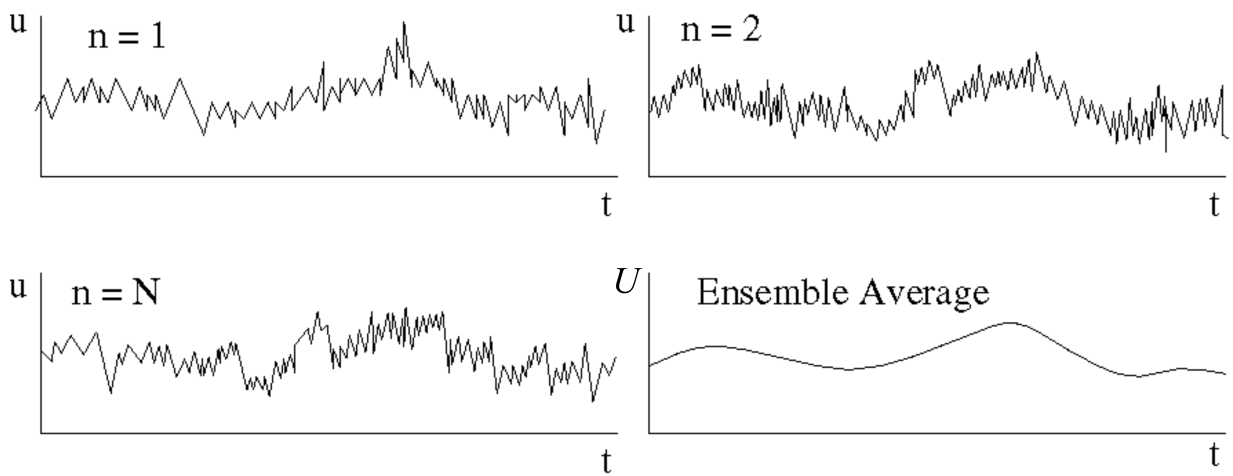




$$u_i(\vec{x},t) = U_i(\vec{x},t) + u_i'(\vec{x},t)$$

RANS Modeling - Ensemble Averaging

- Ensemble (Phase) average; $U_i(\vec{x},t) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} u_i^{(n)}(\vec{x},t)$



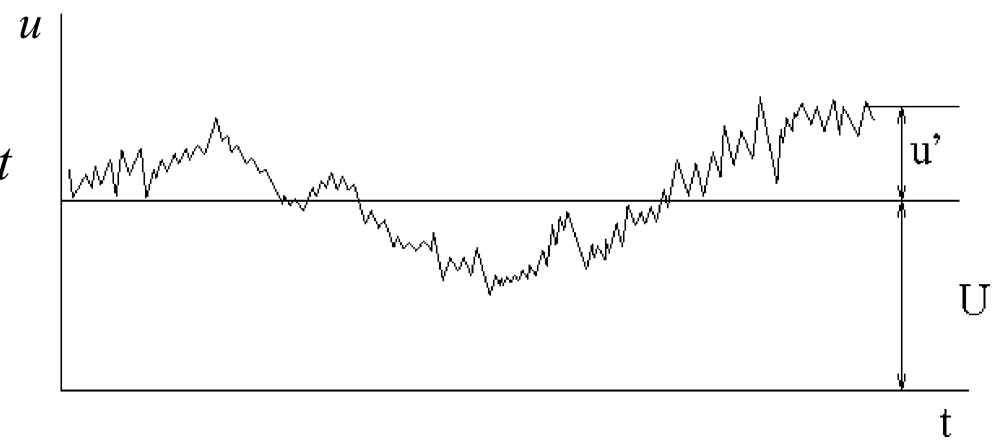
- Applicable to nonstationary flows such as periodic or quasi-periodic flows involving deterministic structures.

RANS Modeling - Time Averaging

- For steady flows, long-time averaging can be used.

$$U_i(\vec{x}) = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} u_i(\vec{x},t)\,dt$$

$$u_i(\vec{x},t) = U_i(\vec{x}) + u_i'(\vec{x},t)$$



  - Often referred to as "Reynolds average" in the literature

Deriving RANS Equations

- Substitute mean and fluctuating velocities in instantaneous Navier-Stokes equations and average:

$$\rho\left(\overline{\frac{\partial(U_i+u_i')}{\partial t}}+\overline{(U_k+u_k')\frac{\partial(U_i+u_i')}{\partial x_k}}\right)=-\overline{\frac{\partial\left(p+p'\right)}{\partial x_i}}+\overline{\frac{\partial}{\partial x_j}\left(\mu\frac{\partial(U_i+u_i')}{\partial x_j}\right)}$$

- Some averaging rules:
  - Given $\phi = \Phi + \phi'$ and $\psi = \Psi + \psi'$

$$\Phi \equiv \overline{\phi};\quad \overline{\phi'} \equiv 0;\quad \overline{\phi\psi} = \Phi\Psi + \overline{\phi'\psi'};\quad \overline{\Phi\,\psi'} = 0;\quad \overline{\phi'\psi'} \neq 0,\ etc.$$

- Mass-weighted (Favre) averaging used for compressible flows.

RANS Equations

- Reynolds Averaged Navier-Stokes equations:

$$\rho\left(\frac{\partial U_i}{\partial t}+U_k\frac{\partial U_i}{\partial x_k}\right)=-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}\left(\mu\frac{\partial U_i}{\partial x_j}\right)+\frac{\partial\left(-\rho\overline{u_i u_j}\right)}{\partial x_j}$$

(prime notation dropped)

- New equations are identical to original except :
  - The transported variables, U, ρ, etc., now represent the mean flow quantities
  - Additional terms appear: $R_{ij}=-\rho\overline{u_i u_j}$
    - R_{ij} are called the Reynolds Stresses
      - Effectively a stress, apparent when rewritten $\rightarrow$ $\dfrac{\partial}{\partial x_j}\left(\mu\dfrac{\partial U_i}{\partial x_j}-\rho\overline{u_i u_j}\right)$
    - **These are the terms to be modeled.**

Turbulence Modeling Approaches

- Boussinesq approach
  - isotropic
  - relies on dimensional analysis
- Reynolds stress transport models
  - no assumption of isotropy
  - contains more “physics”
  - most complex and computationally expensive

The Boussinesq Approach

- Relates the Reynolds stresses to the mean flow by a turbulent (eddy) viscosity, μ_t

$$R_{ij} = -\rho\overline{u_i u_j} = 2\mu_t S_{ij} - \frac{2}{3}\mu_t \frac{\partial U_k}{\partial x_k}\delta_{ij} - \frac{2}{3}\rho k \delta_{ij}; \qquad S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

  - Relation is drawn from analogy with molecular transport of momentum.

$$t_{xy} = -\rho\overline{u''v''} = 2\mu S_{ij}$$

  - Assumptions valid at molecular level, not necessarily valid at macroscopic level
    - μ_t is a scalar (R_{ij} aligned with strain-rate tensor, S_{ij})
    - Taylor series expansion valid if $l_{mfp}|d^2U/dy^2| << |dU/dy|$
    - Average time between collisions $l_{mfp} / v_{th} << |dU/dy|^{-1}$

Modeling μ_t

- Oh well, focus attention on modeling μ_t anyways.
- Basic approach made through dimensional arguments
  - Units of $\nu_t = \mu_t/\rho$ are [m²/s]
  - Typically one needs 2 out of the 3 scales:
    - velocity - length - time
- Models classified in terms of number of transport equations solved, e.g.,
  - zero-equation
  - one-equation
  - two-equation

Zero Equation Model

- Prandtl mixing length model: $\mu_t = \rho l_{mix}^2 \sqrt{2S_{ij}S_{ij}}; \quad S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$
  - Relation is drawn from same analogy with molecular transport of momentum:

$$\mu = \frac{1}{2}\rho \mathrm{v}_{\mathrm{th}} l_{\mathrm{mfp}} \quad \Longleftrightarrow \quad \mu_t = \frac{1}{2}\rho \mathrm{v}_{\mathrm{mix}} l_{\mathrm{mix}}$$

  - The mixing length model:
    - assumes that $\mathrm{v}_{\mathrm{mix}}$ is proportional to l_{mix} & strain rate: $\mathrm{v}_{\mathrm{mix}} \propto l_{\mathrm{mix}}\sqrt{2S_{ij}S_{ij}}$
    - requires that l_{mix} be prescribed
      - l_{mix} must be 'calibrated' for each problem
  - Very crude approach, but economical
    - Not suitable for general purpose CFD though can be useful where a very crude estimate of turbulence is required.

Other Zero Equation Models

- Mixing length observed to behave differently in flows near solid boundaries than in free shear flows.
  - Modifications made to the Prandtl mixing length model to account for near wall flows.
    - Van Driest / Clauser / Klebanoff
      - Reduce mixing length in viscous sublayer (inner boundary layer) with damping factor to effect reduced 'mixing.'
      - Define appropriate mixing length in velocity defect (outer boundary) layer.
      - Account for intermittency dependency.
    - Cebeci-Smith and Baldwin-Lomax
      - Accounts for all of above adjustments in two layer models.
- Mixing length models typ. fail for separating flows.
  - Large eddies persist in the mean flow and cannot be modeled from local properties alone.

One-Equation Models

- Traditionally, one-equation models were based on transport equation for k (turbulent kinetic energy) to calculate velocity scale, $v = k^{1/2}$.
  - Circumvents assumed relationship between v and turbulence length scale (mixing).
  - Use of transport equation allows ‘history effects’ to be accounted for.
- Length scale still specified algebraically based on the mean flow
  - very dependent on problem type
  - approach not suited to general purpose CFD

Turbulence Kinetic Energy Equation

- Exact k equation derived from sum of products of Navier-Stokes equations with fluctuating velocities.
  - (Trace of the Reynolds Stress transport equations)

$$\rho\left(\frac{\partial k}{\partial t}+U_j\frac{\partial k}{\partial x_j}\right)=R_{ij}\frac{\partial U_i}{\partial x_j}-\rho\varepsilon+\frac{\partial}{\partial x_j}\left[\mu\frac{\partial k}{\partial x_j}-\frac{1}{2}\rho\overline{u_i u_i u_j}-\overline{p' u_j}\right]$$

unsteady & convective | production | dissipation | molecular diffusion | turbulent transport | pressure diffusion

  - where $\varepsilon=\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}}$ (incompressible form)

Modeled Equation for k

- The production, dissipation, turbulent transport, and pressure diffusion terms must be modeled.
  - R_{ij} in production term is calculated from Boussinesq formula.
  - Turbulent transport and pressure diffusion:

$$\frac{1}{2}\rho\overline{u_i u_i u_j} + \overline{p' u_j} = -\frac{\mu_t}{\sigma_k}\frac{\partial k}{\partial x_j}$$

Using μ_t/σ_k assumes k can be transported by turbulence as can U.

  - $\varepsilon = C_D k^{3/2}/\boldsymbol{l}$ from dimensional arguments
  - $\mu_t = C_D \rho k^2/\varepsilon$ (recall $\mu_t \propto \rho k^{1/2}\boldsymbol{l}$)
  - C_D, σ_k, and $\boldsymbol{l}$ are model parameters to be specified
    - Necessity to specify $\boldsymbol{l}$ limits usefulness of this model.
- Advanced one equation models are ‘complete’
  - solves for eddy viscosity

Spalart-Allmaras Model Equations

modified turbulent viscosity

$$\rho \frac{D\tilde{\nu}}{Dt} = \rho c_{b1} \tilde{S} \tilde{\nu} + \frac{1}{\sigma_{\tilde{\nu}}} \left[\frac{\partial}{\partial x_j} \left\{ \left(\mu + \rho \tilde{\nu} \right) \frac{\partial \tilde{\nu}}{\partial x_j} \right\} + c_{b2} \rho \left(\frac{\partial \tilde{\nu}}{\partial x_j} \right)^2 \right] - c_{w1} \rho f_w \left(\frac{\tilde{\nu}}{d} \right)^2$$

$$\mu_t \equiv \rho \tilde{\nu} f_{v1}, \quad \mathrm{f}_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3}, \quad \chi \equiv \frac{\tilde{\nu}}{\nu}$$

$$\tilde{S} \equiv S + \frac{\tilde{\nu}}{\kappa^2 d^2} f_{v2}, \quad \mathrm{f}_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}}$$

damping functions

distance from wall

$$f_w = g \left[\frac{1 + c_{w3}^6}{g^6 + c_{w3}^6} \right]^{1/6}, \quad \mathrm{g} = r + c_{w2} \left(r^6 - r \right), \quad r \equiv \frac{\tilde{\nu}}{\tilde{S} \kappa^2 d^2}$$

Wall boundary condition : $\tilde{\nu} = 0$

Spalart-Allmaras Production Term

- Default definition uses rotation rate tensor only:

$$S \equiv \sqrt{2\,\Omega_{\mathrm{ij}}\Omega_{\mathrm{ij}}}; \qquad \Omega_{\mathrm{ij}} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right)$$

- Alternative formulation also uses strain rate tensor:

$$S \equiv \left|\Omega_{\mathrm{ij}}\right| + \mathrm{C}_{\mathrm{prod}}\,\min(0, \left|\mathrm{S}_{\mathrm{ij}}\right| - \left|\Omega_{\mathrm{ij}}\right|)$$

  - reduces turbulent viscosity for vortical flows
  - more correctly accounts for the effects of rotation

Spalart-Allmaras Model

- Spalart-Allmaras model developed for unstructured codes in aerospace industry
  - Increasingly popular for turbomachinery applications
  - "Low-Re" formulation by default
    - can be integrated through log layer and viscous sublayer to wall
    - Fluent's implementation can also use law-of-the-wall
  - Economical and accurate for:
    - wall-bounded flows
    - flows with mild separation and recirculation
  - Weak for:
    - massively separated flows
    - free shear flows
    - simple decaying turbulence

Two-Equation Models

- Two transport equations are solved, giving two independent scales for calculating μ_t
  - Virtually all use the transport equation for the turbulent kinetic energy, k
  - Several transport variables have been proposed, based on dimensional arguments, and used for second equation.
    - Kolmogorov, ω: $\mu_t \propto \rho k / \omega, \ \boldsymbol{l} \propto k^{1/2} / \omega, \ \varepsilon \propto k / \omega$
      - ω is specific dissipation rate
      - defined in terms of large eddy scales that define supply rate of k.
    - Chou, ε: $\mu_t \propto \rho k^2 / \varepsilon, \ \boldsymbol{l} \propto k^{3/2} / \varepsilon$
    - Rotta, $\boldsymbol{l}$: $\mu_t \propto \rho k^{1/2} \boldsymbol{l}, \ \varepsilon \propto k^{3/2} / \boldsymbol{l}$
  - Boussinesq relation still used for Reynolds Stresses.

Standard *k*-ε Model Equations

k-transport equation

$$\rho \frac{Dk}{Dt} = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] + \underbrace{\mu_t S^2}_{\text{production}} - \underbrace{\rho\varepsilon}_{\text{dissipation}}; \qquad S = \sqrt{2S_{ij}S_{ij}}$$

ε-transport equation

$$\rho \frac{D\varepsilon}{Dt} = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right] + \underbrace{\frac{\varepsilon}{k}}_{\text{inverse time scale}}\left(C_{1\varepsilon}\mu_t S^2 - \rho C_{2\varepsilon}\varepsilon\right)$$

coefficients

$\sigma_k, \sigma_\varepsilon, C_{i\varepsilon}, C_{\varepsilon 2}$ ← Empirical constants determined from benchmark experiments of simple flows using air and water.

turbulent viscosity

$$\mu_t = \rho\, C_\mu \frac{k^2}{\varepsilon}$$

Closure Coefficients

- Simple flows render simpler equations from models.
  - Coefficients can be isolated and compared with experiment.
  - e.g.,
    - Uniform flow past grid
      - convection and dissipation terms

$$U\frac{\partial k}{\partial x} = -\varepsilon; \quad U\frac{\partial \varepsilon}{\partial x} = -C_{2\varepsilon}\frac{\varepsilon^2}{k}$$

    - Homogeneous Shear Flow
    - Near-Wall (Log layer) Flow

Standard k-ε Model

- High-Reynolds number model
  - (i.e., must be modified for the near-wall region)
- The term "standard" refers to the choice of coefficients
- Sometimes additional terms are included
  - production due to buoyancy
    - unstable stratification ($\mathbf{g}\cdot\nabla T > 0$) supports k production.
  - dilatation dissipation due to compressibility
    - added dissipation term, prevents overprediction of spreading rate in compressible flows.

$$\rho \frac{Dk}{Dt} = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] + \mu_t S^2 - \rho\varepsilon - g_i \frac{\mu_t}{\rho \Pr_t}\frac{\partial \rho}{\partial x_i} - \rho\varepsilon 2\frac{k}{\gamma RT}$$

Buoyancy production　　Dilatation dissipation

Standard k-ε Model Pros & Cons

- Strengths:
  - robust
  - economical
  - reasonable accuracy for a wide range of flows
- Weaknesses:
  - overly diffusive for many situations
    - flows involving strong streamline curvature, swirl, rotation, separating flows, low-Re flows.
  - cannot predict round jet spreading rate
- Variants of the k-ε model have been developed to address its deficiencies
  - RNG and Realizable

RNG k-ε Model Equations

- Derived using renormalization group theory
  - scale-elimination technique applied to Navier-Stokes equations (sensitizes equations to specific flow regimes)
    - k equation is similar to standard k-ε model
    - Additional strain rate term in ε equation
      - most significant difference between standard and RNG k-ε models
    - Analytical formula for turbulent Prandtl numbers
    - Differential-viscosity relation for low Reynolds numbers
      - Boussinesq model used by default

ε-transport equation

$$\rho\frac{D\varepsilon}{Dt} = \frac{\partial}{\partial x_j}\left[\left(\alpha_\varepsilon \mu_{\text{eff}}\right)\frac{\partial \varepsilon}{\partial x_j}\right] + \frac{\varepsilon}{k}\left(C_{1\varepsilon}\mu_t S^2 - \rho C_{2\varepsilon}^{*}\varepsilon\right) \; where \; \begin{cases} C_{2\varepsilon}^{*} = C_{2\varepsilon} + \dfrac{C_\mu \rho \eta^3 \left(1 - \dfrac{\eta}{\eta_0}\right)}{1+\beta\eta^3} \\ \eta = S\dfrac{k}{\varepsilon} \\ \eta_0, \beta \text{ are coefficients} \end{cases}$$

$$\mu_{\text{eff}} = \mu + \mu_{\text{t}}$$

RNG k-ε Model Pros &Cons

- For large strain rates:
  - where $\eta > \eta_0$, ε is augmented, and therefore k and μ_t are reduced
- Buoyancy and compressibility terms can be included.
- Improved performance over std. k-ε model for
  - rapidly strained flows
  - flows with streamline curvature
- Still suffers from the inherent limitations of an isotropic eddy-viscosity model

Realizable k-ε Model: Motivation

- Ensures positivity of normal stresses

$$\overline{u_\alpha^2} \geq 0$$

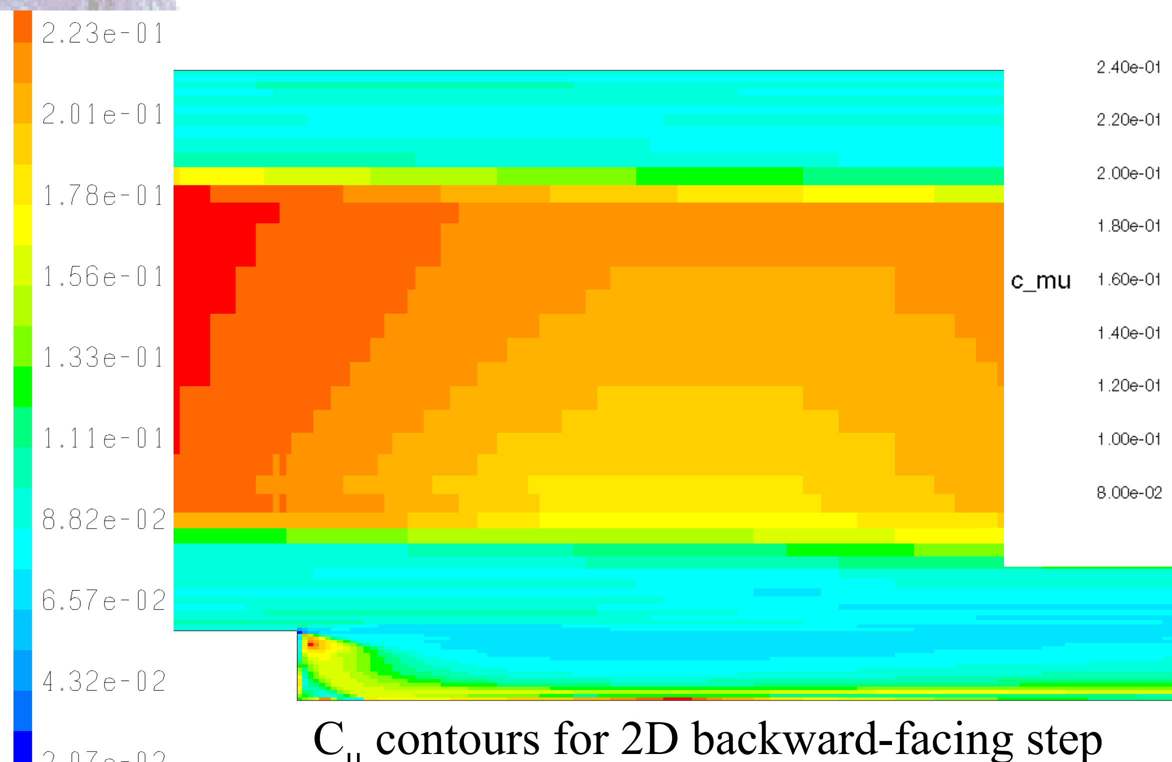
- Ensures Schwarz's inequality of shear stresses

$$\left(\overline{u_\alpha u_\beta}\right)^2 \leq \overline{u_\alpha^2}\,\overline{u_\beta^2}$$

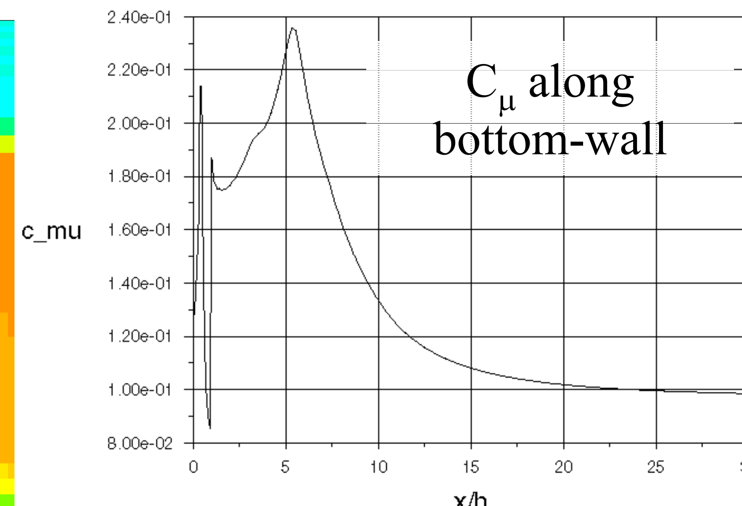
- k equation is same; new formulation for μ_t and ε
- C_μ is variable
- ε equation is based on a transport equation for the mean-square vorticity fluctuation

Realizable k-ε Model: C_μ

- C_μ is not a constant, but varies as a function of mean velocity field and turbulence (0.09 in log-layer $Sk/\varepsilon = 3.3$, 0.05 in shear layer of $Sk/\varepsilon = 6$)



C_μ contours for 2D backward-facing step



Realizable *k*-ε Model: Realizability

How can normal stresses become negative?

- Std. k-ε Boussinesq viscosity relation:

$$-\rho\,\overline{u_i u_j} = \rho\, C_\mu \frac{k^2}{\varepsilon}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3}\rho\, k\, \delta_{\mathrm{ij}}$$

- Normal component:

$$\overline{u^2} = \frac{2}{3}k - 2C_\mu \frac{k^2}{\varepsilon}\frac{\partial U}{\partial x}$$

- Normal stress will be negative if:

$$\frac{k}{\varepsilon}\frac{\partial U}{\partial x} > \frac{1}{3C_\mu} \approx 3.7$$

Realizable k-ε Model Equations

ε-transport equation

$$\rho \frac{D\varepsilon}{Dt} = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right] + C_1 S \rho \varepsilon - C_2 \frac{\rho \varepsilon^2}{k + \sqrt{\nu\varepsilon}}$$

$$C_1 = \max\left[0.43, \frac{\eta}{\eta + 5}\right], \quad \eta = Sk/\varepsilon, \quad C_2 = 1.0$$

turbulent viscosity

$$\mu_t = \rho C_\mu \frac{k^2}{\varepsilon}, \quad C_\mu = \frac{1}{A_0 + A_s \frac{U^* k}{\varepsilon}}$$

where $$A_0 = 4.04, \quad A_s = \sqrt{6}\cos\phi, \quad \phi = \frac{1}{3}\cos^{-1}\left(\sqrt{6}W\right)$$

$$U^* \equiv \sqrt{S_{ij}S_{ij} + \Omega_{ij}\Omega_{ij}}, \; W = \frac{S_{ij}S_{jk}S_{ki}}{\tilde{S}}, \quad \tilde{S} = \sqrt{S_{ij}S_{ij}}$$

Realizable k-ε Model Pros & Cons

- Performance generally exceeds the standard k-ε model
- Buoyancy and compressibilty terms can be included.
- Good for complex flows with large strain rates
  - recirculation, rotation, separation, strong ∇p
- Resolves the round-jet/plane jet anomaly
  - predicts the speading rate for round <u>and</u> plane jets
- Still suffers from the inherent limitations of an isotropic eddy-viscosity model

Why use ε? A k-L Model Case Study

- Other scales can be used
  - Consider using L, the length scale
    - $\mu_t \propto \rho k^{1/2} L, \ \ \varepsilon \propto k^{3/2} / L$
    - L varies linearly near the wall (unlike $\varepsilon \sim 1/y$)
    - L scales simply with distance
    - easy to resolve
  - Easy to use with “2-layer” wall models
  - No production term
- From the chain rule for differentiation:

$$\frac{1}{C_L}\frac{DL}{Dt} = \frac{3}{2}\frac{k^{1/2}}{\varepsilon}\frac{Dk}{Dt} - \frac{k^{3/2}}{\varepsilon^2}\frac{D\varepsilon}{Dt}$$

  - We can use this to transform the k-ε model

k-L Model from the Std. k-ε Model

L-transport equation

$$\rho\frac{DL}{Dt}=\overset{1.5\text{ - }1.44\sim 0}{\cancel{\left(\frac{3}{2}-C_{1\varepsilon}\right)}}\frac{L}{k}\mu_t S^2-C_L\left(\frac{3}{2}-C_{2\varepsilon}\right)\rho k^{1/2}+\frac{\partial}{\partial x_j}\left(\frac{\mu_t}{\sigma_\varepsilon}\frac{\partial L}{\partial x_j}\right)$$

$$-\frac{\mu_t}{\sigma_\varepsilon}\frac{2}{L}\frac{\partial L}{\partial x_j}\frac{\partial L}{\partial x_j}+\frac{3}{2}\left(\frac{1}{\sigma_k}-\frac{1}{\sigma_\varepsilon}\right)\frac{\partial}{\partial x_j}\left(\frac{\mu_t}{\sigma_k}\frac{L}{k}\frac{\partial k}{\partial x_j}\right)$$

$$+\left(\frac{3}{2\sigma_k}-\frac{9}{4\sigma_\varepsilon}\right)\mu_t\frac{L}{k^2}\frac{\partial k}{\partial x_j}\frac{\partial k}{\partial x_j}+\left(\frac{1}{\sigma_\varepsilon}-\frac{1}{\sigma_k}\right)\frac{\mu_t}{k}\frac{\partial k}{\partial x_j}\frac{\partial L}{\partial x_j}$$

- Note additional “cross-derivative terms”
- Can we just neglect them?

Smith's k-L Model

- k-L model of Smith (1994) (high-Re form)

k-transport equation

$$\rho \frac{Dk}{Dt} = \mu_t S^2 - C_L \rho \frac{k^{3/2}}{L} + \frac{\partial}{\partial x_j}\left(\frac{\mu_t}{\sigma k} \frac{\partial k}{\partial x_j} \right)$$

L-transport equation

$$\rho \frac{DL}{Dt} = C_1 \rho k^{1/2} + C_2 \frac{\mu_t}{k} \frac{\partial k}{\partial x_j} \frac{\partial L}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\frac{\mu_t}{\sigma_\varepsilon} \frac{\partial L}{\partial x_j} \right)$$

coefficients

$$C_1, C_2, C_L, C_\mu, \sigma_k, \sigma_L$$

turbulent viscosity

$$\mu_t = \rho {C_\mu}^{1/4} k^{1/2} L$$

k-L Model Pros & Cons

- k-L model is quite successful for compressible wall-bounded shear flows

  But

- Model performs poorly for free shear flows
  - jets (round and plane)
  - far wakes
  - mixing layers
  - radial jets
- Sensitive to inlet and far-field boundary values of L
- The k-ε model does not have this problem!
  - k-ε is robust to inlet and far-field values of ε

k-ω Model of Wilcox

- Wilcox's k-ω model is a popular alternative to k-ε
  - $\omega \sim \varepsilon / k$
  - $\mu_t \propto \rho k / \omega$
- Initially found to be quite sensitive to inlet and far-field boundary values of ω
- Can be used in near-wall region without modification
- Latest version contains several refinements:
  - reduced sensitivity to boundary conditions
  - modifcation for the round-jet/plane-jet anomaly
  - compressibility effects
  - low-Re (transitional) effects

Faults in the Boussinesq Assumption

- Boussinesq: $R_{ij} = 2\mu_t S_{ij}$
  - Is simple linear relationship sufficient?
    - R_{ij} is strongly dependent on flow conditions and history
    - R_{ij} changes at rates not entirely related to mean flow processes
  - R_{ij} is not strictly aligned with S_{ij} for flows with:
    - sudden changes in mean strain rate
    - extra rates of strain (e.g., rapid dilatation, strong streamline curvature)
    - rotating fluids
    - stress-induced secondary flows
- Modifications to two-equation models cannot be generalized for arbitrary flows.

Reynolds Stress Models

- Starting point is the exact transport equations for the transport of Reynolds stresses, R_{ij}.
  - six transport equations in 3d
- Equations are obtained by Reynolds-averaging the product of the exact momentum equations and a fluctuating velocity.
  - $\overline{u'_i NS(u_j) + u'_j NS(u_i)} = 0$
- The resulting equations contain several terms that must be modeled.

Reynolds Stress Transport Equations

Reynolds Stress Transport Eqns.

$$\frac{DR_{ij}}{Dt} = P_{ij} + \Phi_{ij} - \varepsilon_{ij} + \frac{\partial J_{ijk}}{\partial x_k}$$

Generation

$$P_{ij} \equiv \rho\left(\overline{u_i u_k} \frac{\partial U_j}{\partial x_k} + \overline{u_j u_k} \frac{\partial U_i}{\partial x_k} \right)$$

(computed)

Pressure-Strain Redistribution

$$\Phi_{ij} \equiv -\overline{p'\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)}$$

(modeled)

Dissipation

$$\varepsilon_{ij} \equiv 2\mu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}}$$

(related to ε)

Turbulent Diffusion

$$J_{ijk} \equiv \underbrace{\overline{p' u_i}\delta_{jk} + \overline{p' u_j}\delta_{ik}}_{\text{Pressure/velocity fluctuations}} + \underbrace{\rho \overline{u_i u_j u_k}}_{\text{Turbulent transport}}$$

(modeled)

(equations written for incompressible flow w/o body forces)

Dissipation Modeling

- Dissipation rate is predominantly associated with small scale eddy motions.
  - Large scale eddies affected by mean shear.
  - Vortex stretching process breaks eddies down into continually smaller scales
    - The directional bias imprinted on turbulence by mean flow is gradually lost.
    - Small scale eddies assumed to be locally isotropic.

$$\varepsilon_{ij} = \frac{2}{3}\delta_{ij}\varepsilon$$

    - ε is calculated with its own (or related) transport equation.
    - Compressibility and near-wall anisotropy effects can be accounted for.

Turbulent Transport and Pressure Diffusion

- Most closure models combine the pressure diffusion with the triple products and use a simple gradient diffusion hypothesis.

$$\frac{\partial}{\partial x_k}\left(\overline{u_i u_j u_k}\right)+\overline{\frac{p'}{\rho}\left(\delta_{kj}u_i+\delta_{ik}u_j\right)}-\nu\frac{\partial}{\partial x_k}\left(\overline{u_i u_j}\right)=\frac{\partial}{\partial x_k}\left(C_s\frac{k}{\varepsilon}\overline{u_k u_l}\frac{\partial \overline{u_i u_j}}{\partial x_l}\right)$$

Or even a simpler model $=\frac{\partial}{\partial x_k}\left(\frac{C_\mu}{\sigma}\frac{k^2}{\varepsilon}\frac{\partial \overline{u_i u_j}}{\partial x_k}\right)$

- Overall performance of models for these terms is generally inconsistent based on isolated comparisons to measured triple products.
- DNS data indicate that above p' terms are negligible.

Pressure-Strain Modeling

- Pressure-strain term of same order as production
- Pressure-strain term acts to drive turbulence towards an isotropic state by redistributing the Reynolds stresses

$$\Phi_{ij} \equiv -\overline{p'\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}$$

- Decomposed into parts $\Phi_{ij} = \Phi_{ij,1} + \Phi_{ij,2} + \Phi_{ij,w}$

"Slow" Part

$$\frac{1}{\rho}\frac{\partial^2 p'}{\partial x_i \partial x_i} = -u_i \frac{\partial u_j}{\partial x_i} + \overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}}$$

"Rapid Part"

mean gradient

$$\frac{1}{\rho}\frac{\partial^2 p'}{\partial x_i \partial x_i} = -2\frac{\partial u_i}{\partial x_j}\frac{\partial U_j}{\partial x_i}$$

- Model of Launder, Reece & Rodi (1978)

$$\Phi_{ij} = -c_1 b_{ij} + c_2\left[\left(\overline{u_i u_l}\frac{\partial U_j}{\partial x_l} + \overline{u_j u_l}\frac{\partial U_i}{\partial x_l}\right) - \frac{2}{3}\overline{u_l u_m}\frac{\partial U_l}{\partial x_m}\delta_{ij}\right]$$

where $\quad b_{ij} \equiv \frac{\varepsilon}{k}\left(\overline{u_i u_j} - \frac{2}{3}k\,\delta_{ij}\right)$

Pressure-Strain Modeling Options

- Wall-reflection effect
  - contains explicit distance from wall
  - damps the normal stresses perpendicular to wall
  - enhances stresses parallel to wall
- SSG (Speziale, Sarkar and Gatski) Pressure Strain Model
  - Expands the basic LRR model to include non-linear (quadratic) terms
  - Superior performance demonstrated for some basic shear flows
    - plane strain, rotating plane shear, axisymmetric expansion/contraction

Characteristics of RSM

- Effects of curvature, swirl, and rotation are directly accounted for in the transport equations for the Reynolds stresses.
  - When anisotropy of turbulence significantly affects the mean flow, consider RSM.
- More cpu resources (vs. k-ε models) is needed
  - 50-60% more cpu time per iteration and 15-20% additional memory
- Strong coupling between Reynolds stresses and the mean flow
  - number of iterations required for convergence may increase.

Heat Transfer

- The Reynolds averaging process produces an additional term in the energy equation: $\overline{u_i\theta}$
  - Analogous to the Reynolds stresses, this is termed the turbulent heat flux
    - It is possible to model a transport equation for the heat flux, but this is not common practice
    - Instead, a turbulent thermal diffusivity is defined proportional to the turbulent viscosity
      - The constant of proportionality is called the turbulent Prandtl number
      - Generally assumed that Pr_t ~ 0.85-0.9
- Applicable to other scalar transport equations.

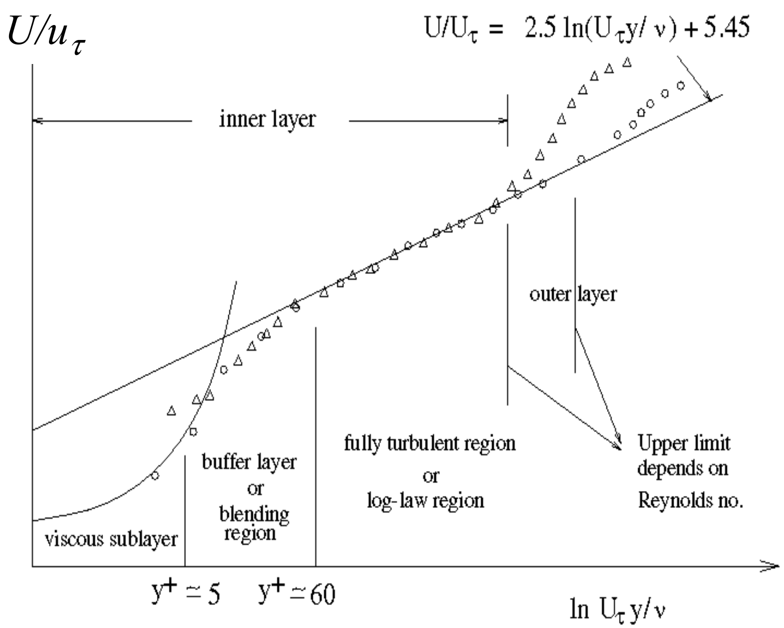
Importance of Near-Wall Turbulence

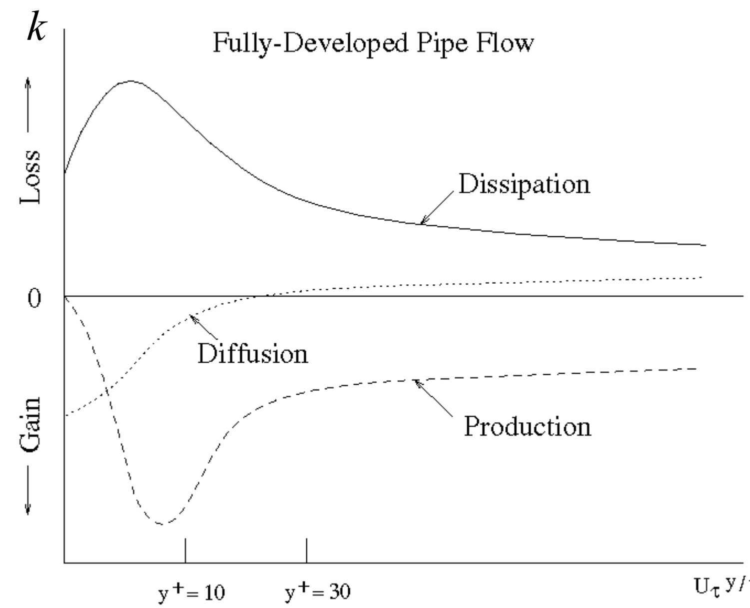
- Walls are main source of vorticity and turbulence.
- Accurate near-wall modeling is important for most engineering applications.
  - Successful prediction of frictional drag for external flows, or pressure drop for internal flows, depends on fidelity of local wall shear predictions.
  - Pressure drag for bluff bodies is dependent upon extent of separation.
  - Thermal performance of heat exchangers, etc., are determined by wall heat transfer whose prediction depends upon near-wall effects.

Near-Wall Modeling Issues

- k-ε and RSM models are valid in the turbulent core region and through the log layer.
  - Some of the modeled terms in these equations are based on isotropic behavior.
    - Isotropic diffusion (μ_t/σ)
    - Isotropic dissipation
    - Pressure-strain redistribution
    - Some model parameters based on experiments of isotropic turbulence.
  - Near-wall flows are anisotropic due to presence of walls.
- Special near-wall treatments are necessary since equations cannot be integrated down to wall.

Flow Behavior in Near-Wall Region

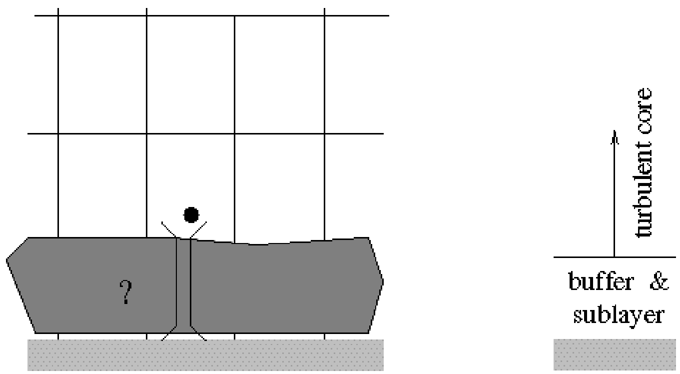




- Velocity profile exhibits layer structure identified from dimensional analysis
  - Inner layer
    - viscous forces rule, $U = f(\rho, \tau_w, \mu, y)$
  - Outer layer
    - dependent upon mean flow
  - Overlap layer
    - log-law applies
- k production and dissipation are nearly equal in overlap layer
  - 'turbulent equilibrium'
- Dissipation >> production in sublayer region

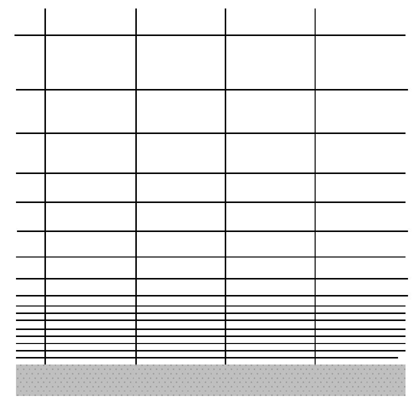
Near-Wall Modeling Approaches

Wall Function Approach



- The viscosity affected region is not resolved.
  - Wall-adjacent cell centroid assumed to be in log layer ($y^+ = 30$).
  - Cell center information bridged by wall function.
  - High Re turbulence models can be used throughout.

Two-Layer Zonal/ Low-Re Approaches



- The near wall region is resolved down to the wall.
  - Wall-adjacent cell assumed to be in sublayer ($y^+ = 1$).
  - Low-Re models are valid throughout.
  - Two-Layer Zonal Model
    - Simpler turbulence model used in viscosity affected region.

Wall Functions

- Wall functions consist of 'wall laws' for mean velocity and temperature and formulas for turbulent quantities.
- 'Universal' Wall Laws
  - Viscous sublayer
    - dimensional analysis (ρ, τ, μ, y)
      - $U = u_\tau f(y^+)$
        - $u_\tau = (\tau_w/\rho)^{1/2}$
        - $y^+ = y u_\tau/\nu$
  - Clauser defect layer
    - $U = U_e - u_\tau g(\eta)$
      - $\eta = y/\Delta$
  - Overlap layer
    - large scale variance ($\nu / u_\tau << \Delta$)
      - $u_\tau f(y^+) = U_e - u_\tau g(\eta)$
      - $u_\tau^2 f'(y^+)/\nu = -u_\tau g'(\eta)/\Delta$ or (x y/u_τ)
      - $y^+ f'(y^+)/\nu = -\eta g'(\eta)$

$$\boxed{u^+ = \frac{1}{\kappa}\ln(y^+) + C;\ \ u^+ = \frac{U}{u_\tau}}$$

- Formulas for k and ε
  - Local turbulent equilibrium
    - $k = u_\tau^2/C_\mu^{1/2}$
    - $\varepsilon = u_\tau^3/\kappa y$
  - Precludes transport of turbulence in log layer
    - k and ε functions of u_τ only

Standard Wall Functions

- Fluent uses Launder-Spalding Wall Functions
  - U = U(ρ, τ, μ, y, k)
    - Introduces additional velocity scale for 'general' application

$$U^* = y^* \quad \text{for} \quad y^* < y_v^*$$
$$U^* = \frac{1}{\kappa}\ln\left(Ey^*\right) \quad \text{for} \quad y^* > y_v^*$$
$$\text{where} \quad U^* \equiv \frac{U_P C_\mu^{1/4} k_P^{1/2}}{\tau_w/\rho} \qquad y^* \equiv \frac{\rho C_\mu^{1/4} k_P^{1/2} y_P}{\mu}$$

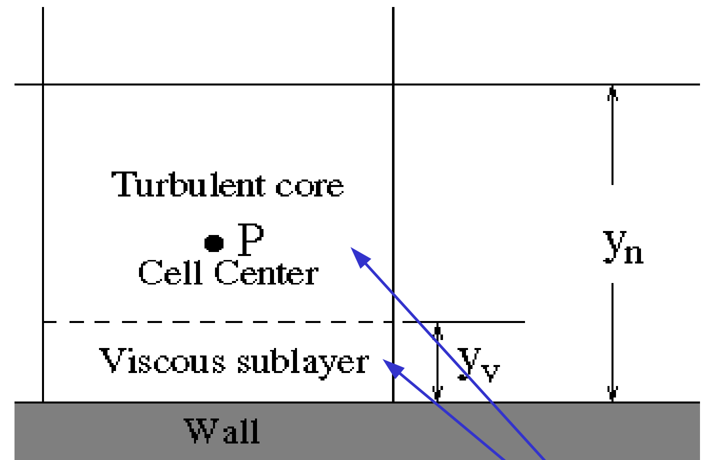
- Estimating k and ε at wall-adjacent cells
  - Generally, k is obtained from solution of k transport equation
    - Cell center is immersed in log layer
    - Local equilibrium (production = dissipation) prevails
    - $\nabla k \cdot \mathbf{n} = 0$ at surface
  - ε also estimated from local equilibrium assumptions
    - $\varepsilon = C_\mu^{3/4} k^{3/2}/\kappa y$
  - Wall functions less reliable when cell intrudes viscous sublayer.

Limitations of Standard Wall Functions

- Wall functions become less reliable when:
  - local equilibrium assumption fails
    - Transpiration through wall
    - strong body forces
    - highly 3D flow
    - rapidly changing fluid properties near wall
  - low-Re flows are pervasive throughout model
  - small gaps are present

Non-equilibrium Wall Functions

- Log-law is sensitized to pressure gradient for better prediction of adverse pressure gradient flows and separation.
- Relaxed local equilibrium assumptions for TKE in wall-neighboring cells.



R_{ij}, k, ε are estimated in each region and used to determine average ε and production of k.

$$\frac{\tilde{U} C_\mu^{1/4} k^{1/2}}{\tau_w / \rho} = \frac{1}{\kappa} \ln\left(E \frac{\rho C_\mu^{1/4} k^{1/2} y}{\mu} \right)$$

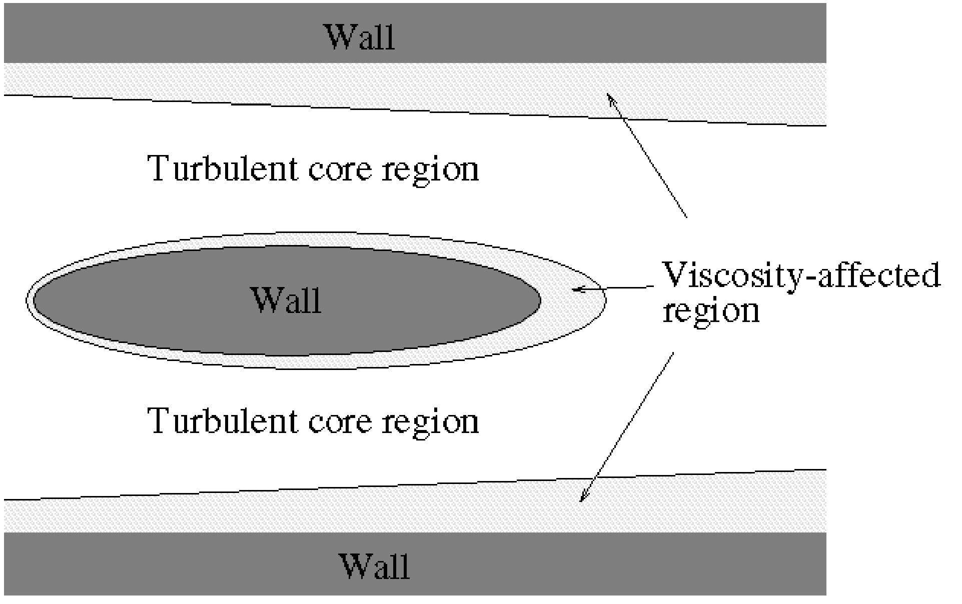
where $$\tilde{U} = U - \frac{1}{2}\frac{dp}{dx}\left[\frac{y_v}{\rho \kappa^* k^{1/2}} \ln\left(\frac{y}{y_v}\right) + \frac{y - y_v}{\rho \kappa^* k^{1/2}} + \frac{y_v^2}{\mu} \right]$$

Two-Layer Zonal Model

- Approach is to divide flow domain into two regions
  - Viscosity affected near-wall region
  - Fully turbulent core region
- Use different turbulence models for each region.
  - One-equation (k) near-wall model for the viscosity affected near-wall region.
  - High-Re k-ε or RSM models for turbulent core region.
- Wall functions, and their limitations, are avoided.

Two-Layer Zones

- The two regions are demarcated on a cell-by-cell basis:
  - $Re_y > 200$
    - turbulent core region
  - $Re_y < 200$
    - viscosity affected region
  - $Re_y = \rho k^{1/2} y/\mu$
  - y is shortest distance to nearest wall
  - zoning is dynamic and solution adaptive



Two-Layer Zones

- In the turbulent core region, the selected high-Re turbulence model is used.
- In the viscosity-affected region, a one-equation model is used.
  - k equation is same as high-Re model.
  - Length scale used in evaluation of μ_t is not from ε.
    - $\mu_t = \rho C_\mu k^{1/2} \boldsymbol{l}_\mu$
    - $\boldsymbol{l}_\mu = c_l y(1-\exp(-Re_y/A_\mu))$
    - $c_l = \kappa C_\mu^{3/4}$
  - Dissipation rate, ε, is calculated algebraically
    - $\varepsilon = k^{3/2}/\boldsymbol{l}_\varepsilon$
    - $\boldsymbol{l}_\varepsilon = c_l y(1-\exp(-Re_y/A_\varepsilon))$

Damping-Function Low-Re Models

- Std. k-ε model modified by damping functions:

$$f_\mu, f_1, f_2$$

ε–transport equation

$$\rho \frac{D\varepsilon}{Dt} = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right] + \frac{\varepsilon}{k}\left(f_1 C_{1\varepsilon}\mu_t S^2 - \rho f_2 C_{2\varepsilon}\varepsilon\right)$$

turbulent viscosity

$$\mu_t = \rho f_\mu C_\mu \frac{k^2}{\varepsilon}$$

- k and ε equations solved on fine mesh (required) right to the wall

Typical Damping Functions

- Damping functions written in terms of Reynolds numbers:

$$\mathrm{Re}_{\mathrm{t}} = \frac{\rho k^2}{\mu\varepsilon} \quad ; \quad \mathrm{Re}_{\mathrm{y}} = \frac{\rho\sqrt{k}\,y}{\mu} \quad ; \quad \mathrm{Re}_{\varepsilon} = \frac{\rho\left(\mu\varepsilon/\rho\right)^{1/4} y}{\mu}$$

- e.g., Abid's model:

$$f_\mu = \tanh(0.008 Re_y)\left(1 + 4 Re_t^{-3/4}\right)$$

$$f_1 = 1$$

$$f_2 = \left[1 - \frac{2}{9}\exp\left(-\frac{Re_t^2}{36}\right)\right]\left[1 - \exp\left(\frac{Re_y}{12}\right)\right]$$

Low-Re k-ε Models

- Several full low-Re k-ε models now available
  - Lam Bremhorst
  - Launder-Sharma
  - Abid
  - Chang et al.
  - Abe-Kondo-Nagano
- Enables modeling of low-Re effects including transitional flows
  - Implementations are problem specific
- Features are not visible in GUI
  - Access from TUI

V2F low-Re k-ε Model

- Durbin (1990) suggests that wall normal fluctuations, $\overline{v^2}$, are responsible for near-wall transport
- $\overline{v^2}$ behaves quite differently to $\overline{u^2}$ and $\overline{w^2}$
  - attenuation of $\overline{v^2}$ is a kinematic effect
  - damping of $\overline{u^2}$ is a dynamic effect
- Model $\mu_t \sim \overline{v^2}T$ instead of $\mu_t \sim kT$
- Requires two additional transport equations:
  - equation for wall-normal fluctuations, $\overline{v^2}$
  - equation for an elliptic relaxation function, f

V2F k-ε Model Equations

ε–transport equation

$$\rho \frac{D\varepsilon}{Dt} = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right] + \frac{1}{T}\left(C_{1\varepsilon}\mu_t\, S^2 - \rho\, C_{2\varepsilon}\varepsilon\right)$$

v^2 -transport equation

$$\rho \frac{D\overline{v^2}}{Dt} = \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial \overline{v^2}}{\partial x_j}\right] + \rho k f - \rho \overline{v^2}\frac{\varepsilon}{k}$$

relaxation equation

$$f - L^2 \frac{\partial^2 f}{\partial x_j \partial x_j} = \frac{C_1}{T}\left(\frac{2}{3} - \frac{\overline{v^2}}{k}\right) + C_2 \frac{\mu_t S^2}{\rho k} + \frac{(N-1)\overline{v^2}}{kT}$$

scales

$$T = \max\left[\frac{k}{\varepsilon}, 6\sqrt{\frac{\nu}{\varepsilon}}\right] \quad ; \quad L^2 = C_L^2 \max\left[\frac{k^3}{\varepsilon^2}, C_\eta \sqrt{\frac{\nu^3}{\varepsilon}}\right]$$

V2F k-ε Model Pros and Cons

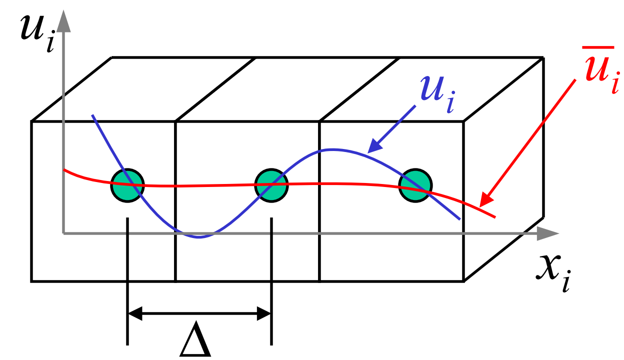
- Very promising results for a wide range of flow and heat transfer test cases
  - at least as good as the best of the damping function approaches in most test cases
- Still an isotropic eddy-viscosity model
  - Can be extended for RSM
- Needs 2 additional equations, so requires more memory and CPU than damping functions

Large Eddy Simulation (LES)

- Recall: Two methods can be used to eliminate the need to resolve small scales.
  - Reynolds Averaging Approach
    - Periodic and quasi-periodic unsteady flows
  - Filtering (LES)
    - Transport equations are filtered such that only larger eddies need be resolved.
      - Difficult to model large eddies since they are
        - anisotropic
        - subject to history effects
        - dependent upon flow configuration, boundary conditions, etc.
    - Smaller eddies are modeled.
      - Typically isotropic and so more amenable to modeling.
    - ‘Random’ unsteadiness can be resolved.

Filtering

- In finite-volume schemes, the cell size in a mesh can determine the filter width.
  - e.g., in 1-D,



$$\frac{u_i(x+\Delta)-u_i(x-\Delta)}{2\Delta} = \frac{d}{dx}\left(\frac{1}{2\Delta}\int_{x-\Delta}^{x+\Delta} u_i(\xi)\,d\xi\right)$$

    - more or less information is filtered as Δ is varied.

- In general,

$$\overline{u}_i(\vec{x},t) \equiv \frac{1}{V}\iiint_V u_i\left(\vec{\xi},t;\vec{x}\right)d\vec{\xi}, \quad \vec{\xi}\in V$$ where V is the volume of cell

  - where the subgrid scale (SGS) velocity, $u_i' = u_i - \overline{u_i}$

Resolvable-scale filtered velocity

LES - Governing Equations

- The governing equations for LES are obtained by filtering (space-averaging) Navier-Stokes equations:

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0$$

$$\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial (\overline{u}_i \overline{u}_j)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial \overline{u}_i}{\partial x_j} \right) - \frac{\partial \tau_{ij}}{\partial x_j}$$

$$\tau_{ij} \equiv \overline{u_i u_j} - \overline{u}_i \overline{u}_j$$

- The SGS stresses consist of terms that must be modeled: $$\overline{u_i u_j} = \overline{\overline{u}_i \overline{u}_j} + \overline{\overline{u}_i u'_j} + \overline{u'_i \overline{u}_j} + \overline{u'_i u'_j}$$

SGS Stress Modeling

- The subgrid-scale stress is modeled by;

$$\tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij} \cong -2\nu_\Delta \overline{S}_{ij}; \qquad \overline{S}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)$$

- The subgrid-scale eddy viscosity ν_Δ is modeled by:
  - Smagorinsky's subgrid-scale model

$$\nu_\Delta \cong (C_S \Delta)^2 \sqrt{2\overline{S}_{ij}\overline{S}_{ij}}$$

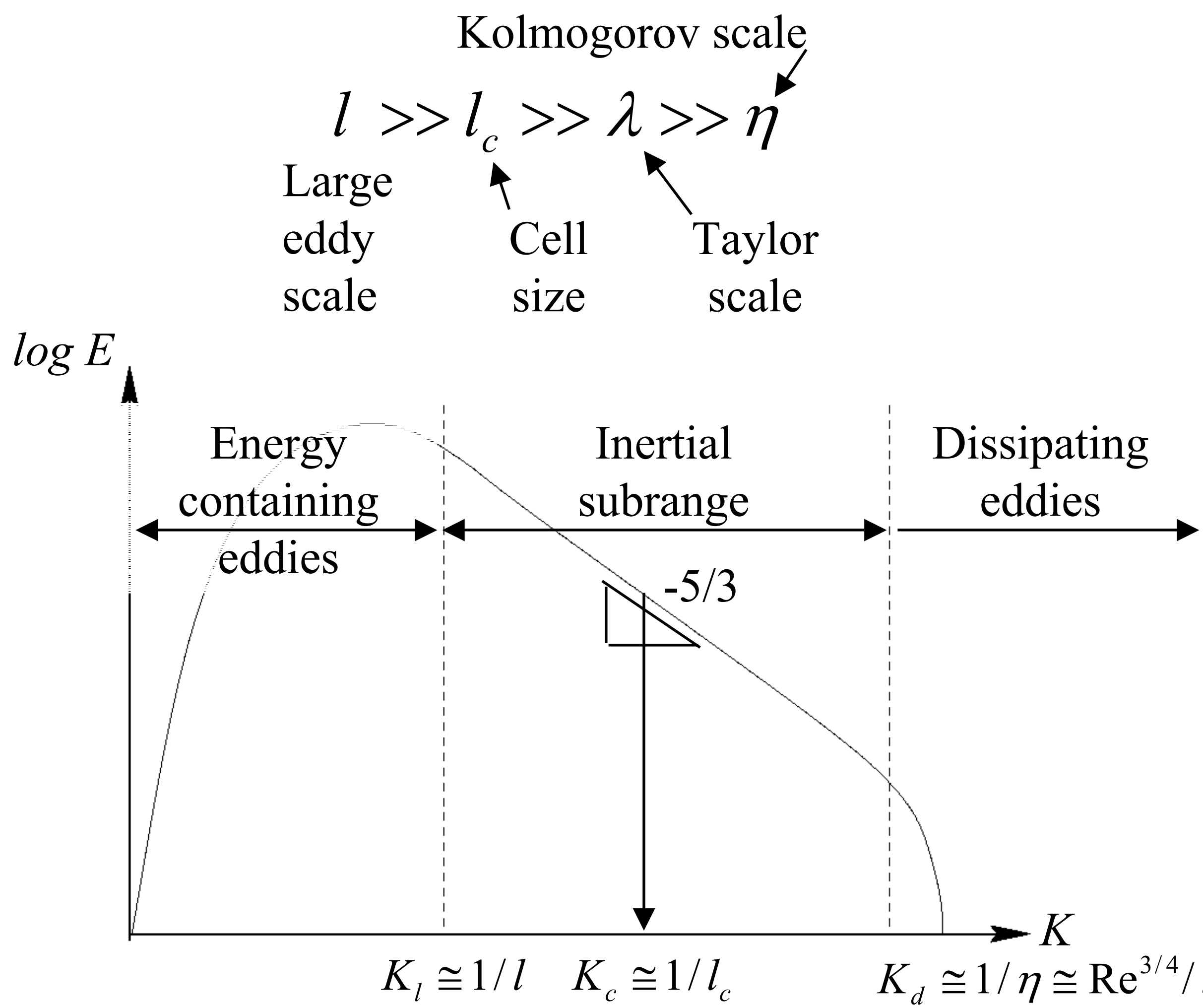
Smagorinsky constant varied from flow to flow

  - RNG-based subgrid-scale model

$$\nu_{eff} \equiv \nu + \nu_\Delta \cong \nu\left[1 + H\left(\frac{{\nu_s}^2 \nu_{eff}}{\nu^3} - C\right)\right]^{1/3}, \quad \nu_S \equiv (C_{RNG}\Delta)^2 \sqrt{2S_{ij}S_{ij}}$$

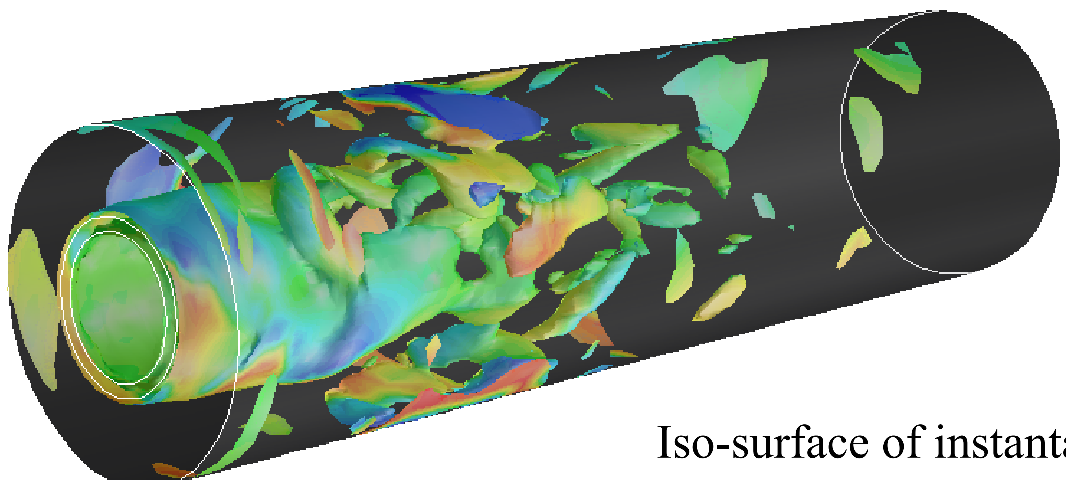
Grid and Time Step Size

- Requires 3-D transient modeling
- Spatial and temporal resolution of scales in "inertial subrange"
  - Not easily defined
  - $\lambda \cong (10\nu k/\varepsilon)^{1/2}$
  - $\tau \cong \lambda/U$



LES Example - Dump Combustor

- A 3-D model of a lean premixed combustor studied by Gould (1987) at Purdue University
- Non-reacting (cold) flow was simulated with a 170K cell hexahedral mesh using second-order temporal and spatial discretization schemes.

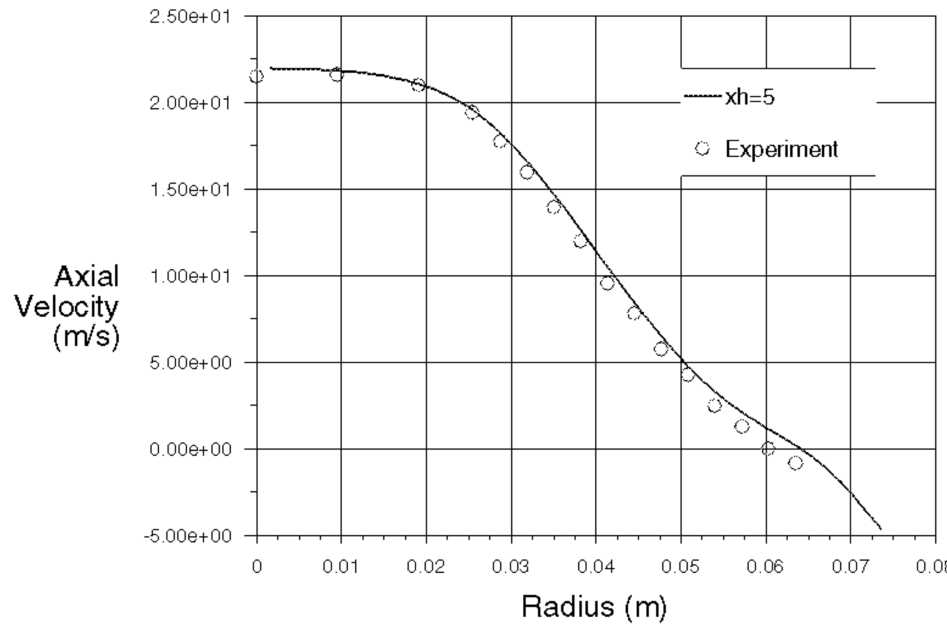


Iso-surface of instantaneous vorticity magnitude colored by velocity angle

LES Examples - Dump Combustor

- Simulation done for $\mathrm{Re}_d = 10^5 \left(\mathrm{Re}_\lambda \approx 150\right)$
- Computed using RNG-based subgrid-scale model

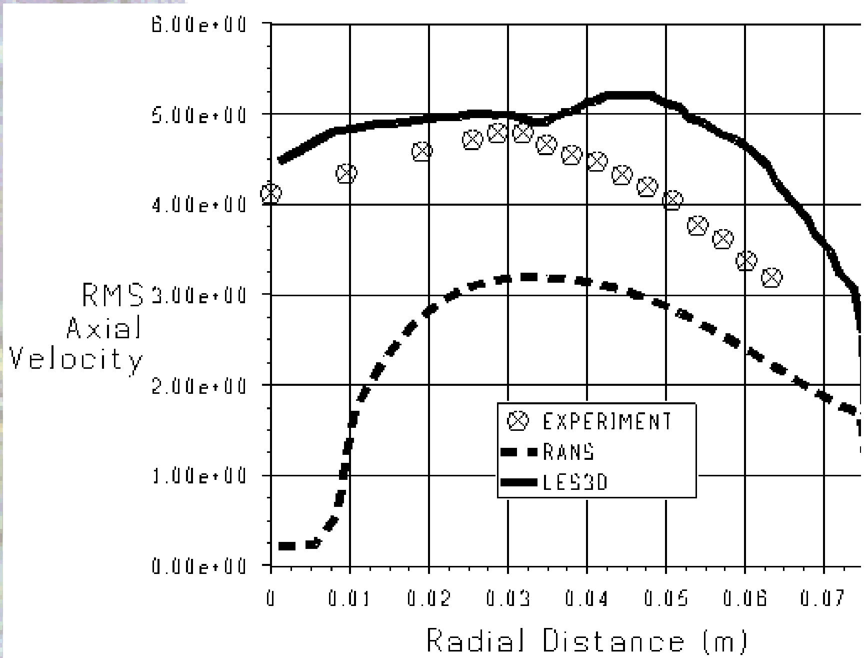
- Mean axial velocity prediction at *x/h* = 5;

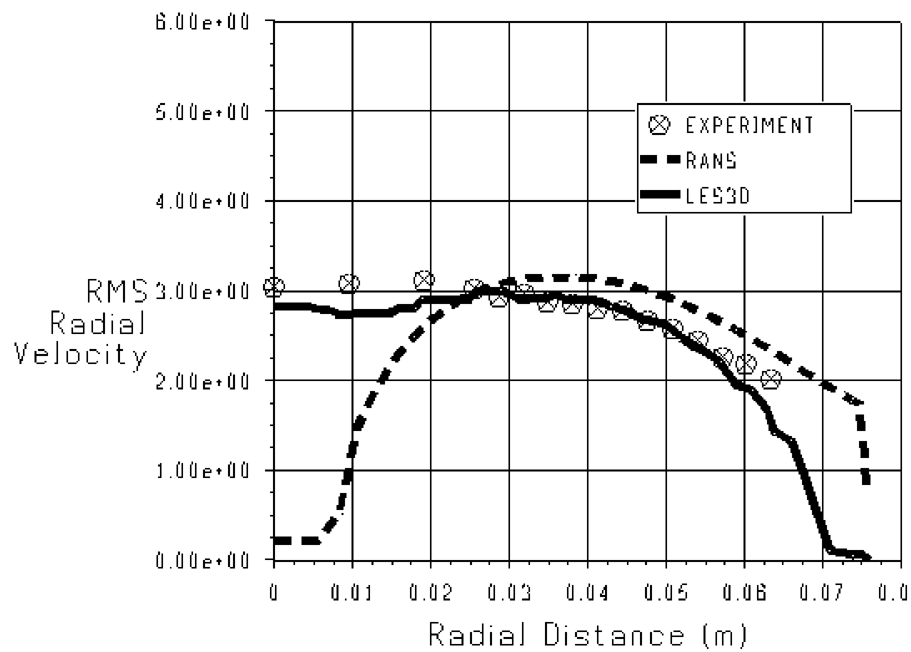


Mean axial velocity at x/h = 5

LES Examples - Dump Combustor

- RMS velocities predictions at *x/h* = 10;





Fluent 6: Turbulence Models

- Customizability enhancement to turbulence modeling via UDFs
  - BCs for turbulence equations
  - source terms (e.g., production and dissipation
  - turbulent viscosity, turbulent Prandtl/Schmidt numbers
- k-ω model (Wilcox, SST)
- "Enhanced" wall function option
  - Aimed at removing y^+ constraint
  - Accounts for the effects of pressure gradient, transpiration, and compressibility

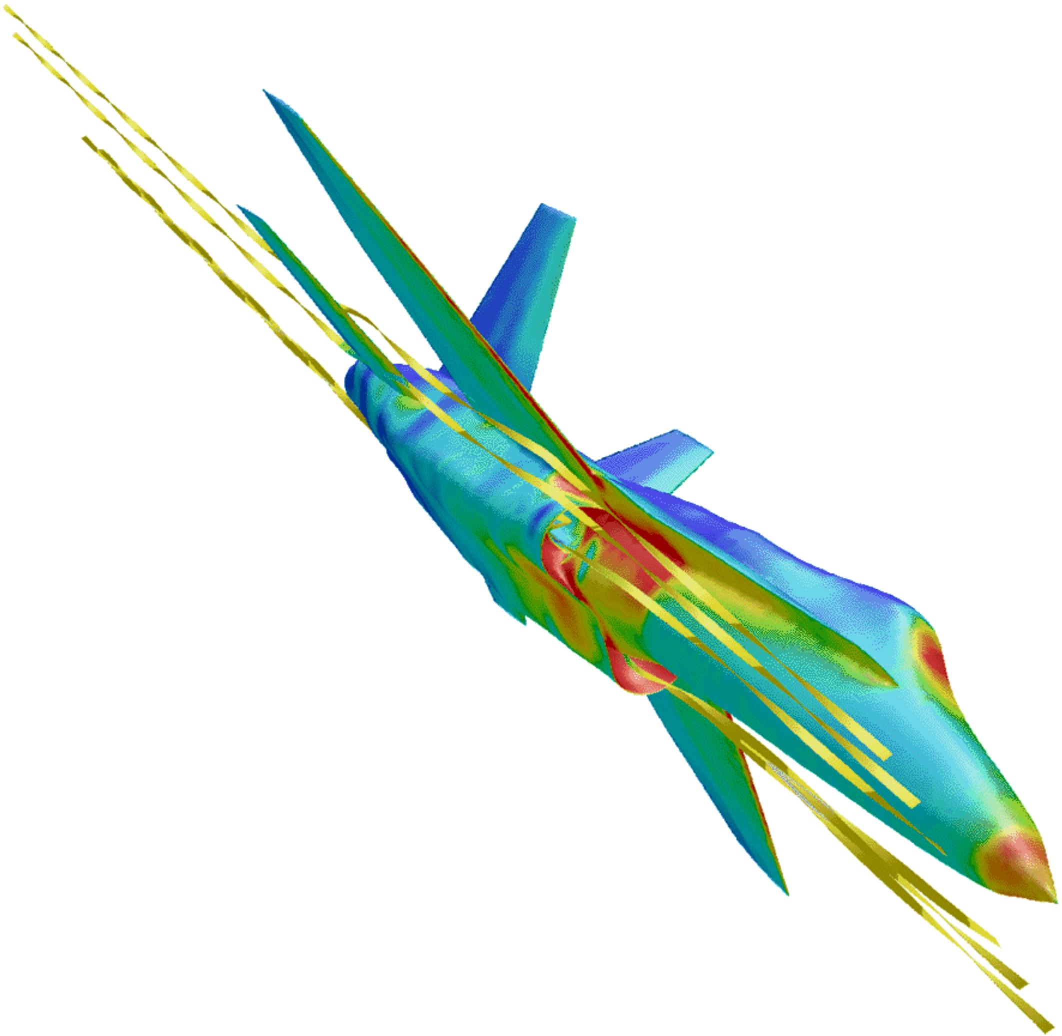
Summary

- CFD codes solve the Reynolds average Navier-Stokes equations
  - Closure problem requires Reynolds stresses to be modeled
  - Boussinesq models:
    - zero equation
      - seldom used for complex problems
    - one-equation
      - works quite well for some classes of flows
    - two-equation (e.g. k-ε)
      - work-horse of industrial CFD
  - Reynolds stress models (7 equations in 3d)
    - needed for complex, anisotropic flows

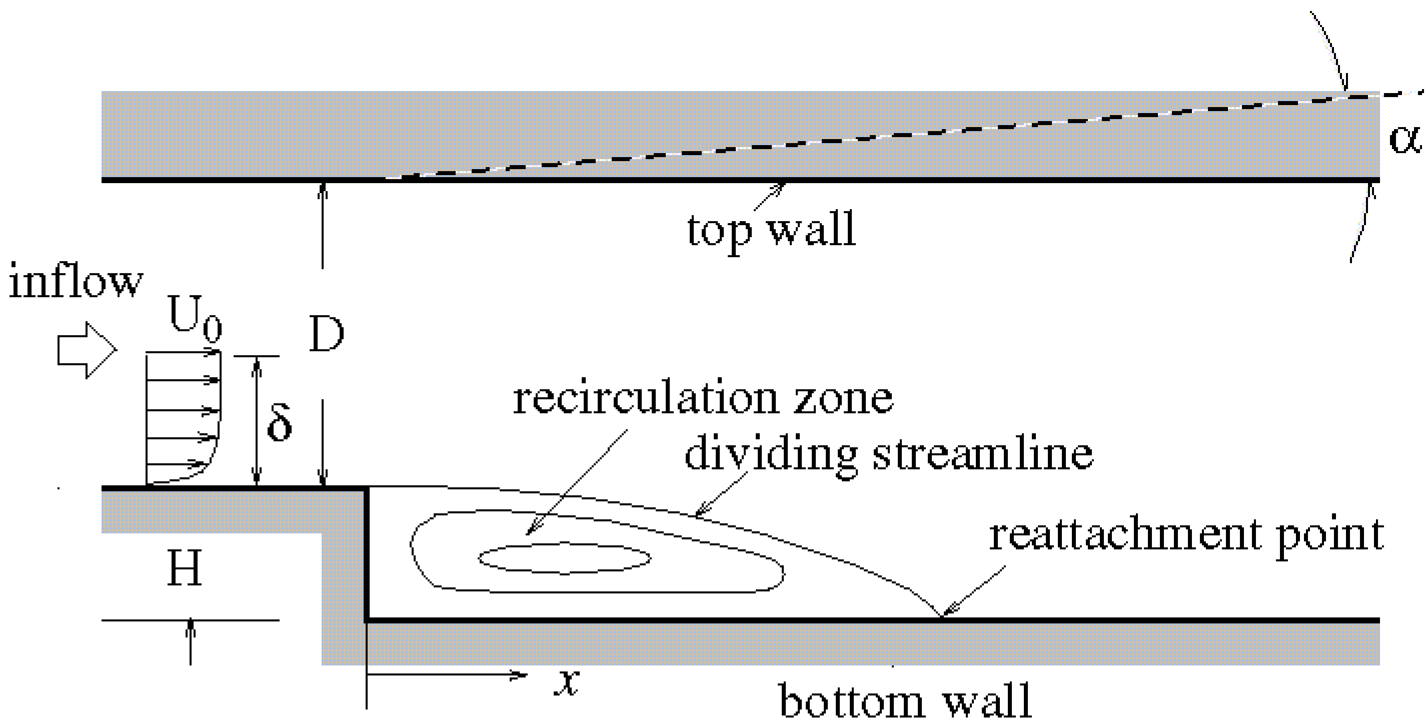
Summary

- Near-wall modeling options
  - wall functions
    - do not resolve viscous sublayer and buffer
    - still can give surprisingly good results
  - two-layer
    - resolves sublayer, so computationally expensive
    - not a true low-Re formulation
  - low-Re models
    - resolves sublayer, so computationally expensive
    - mostly use *ad hoc* damping functions
    - V2F is first low-Re model with potential for being “universal”

Turbulent Flow Examples



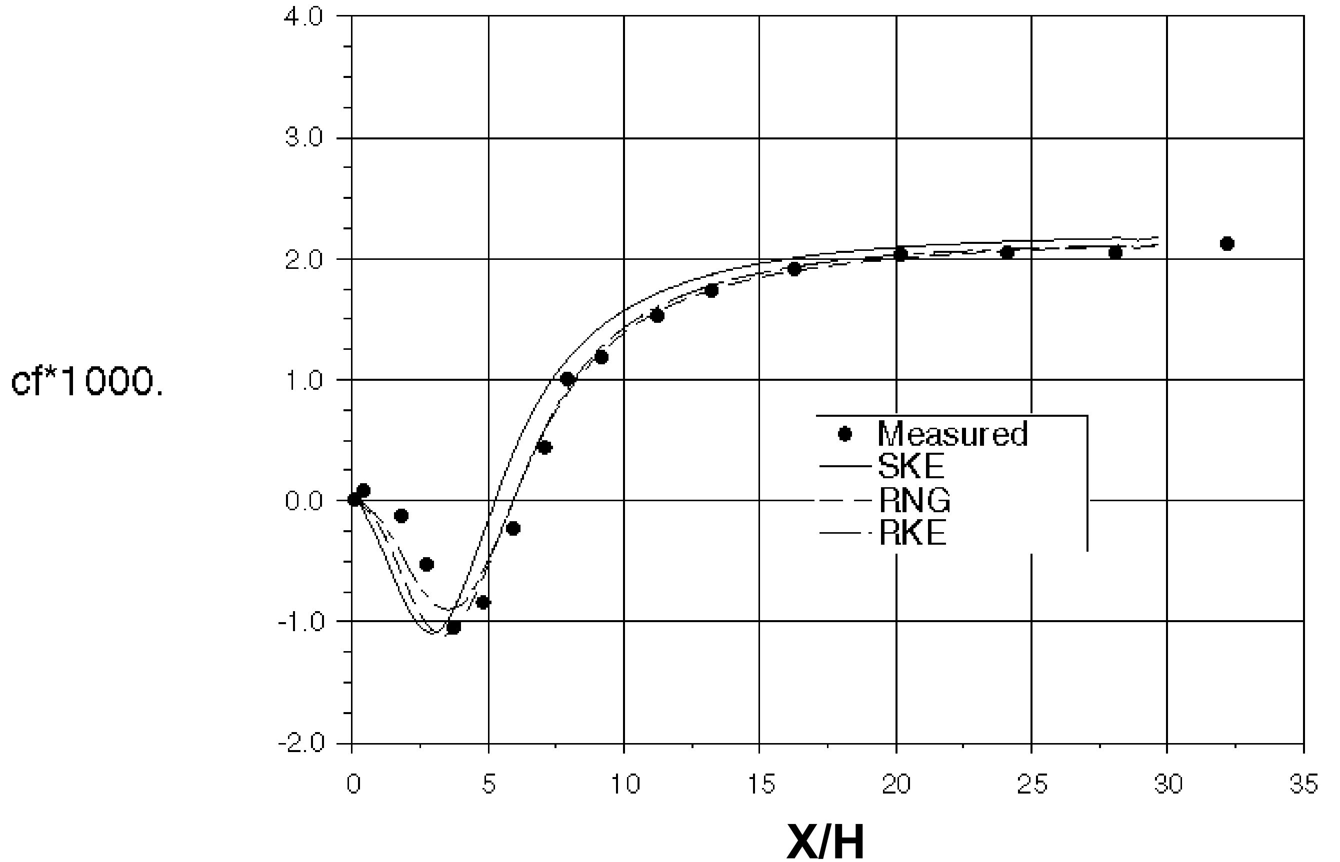
2D Backstep



Driver & Seegmiller (1985)

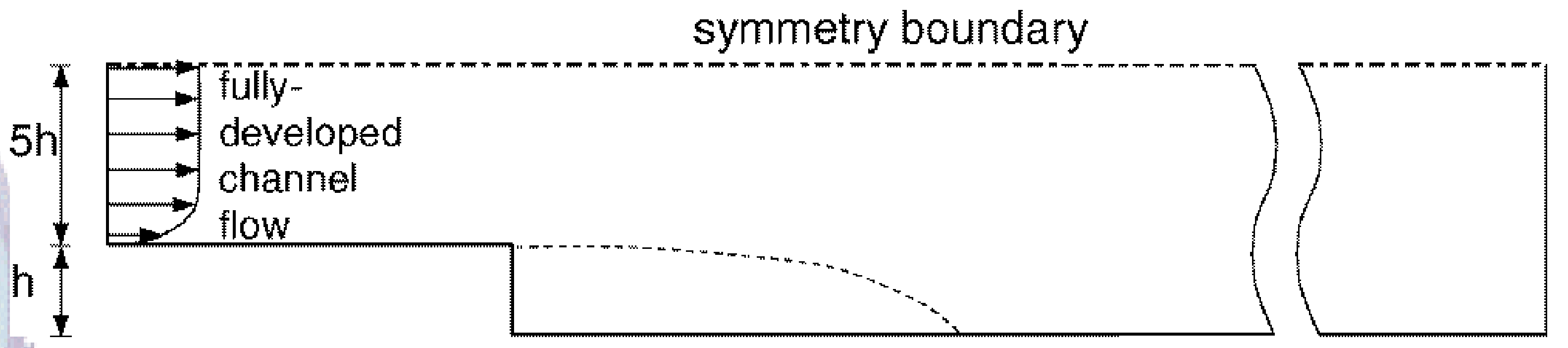
$Re_H = 37{,}400$

2D Backstep: Skin Friction Distribution



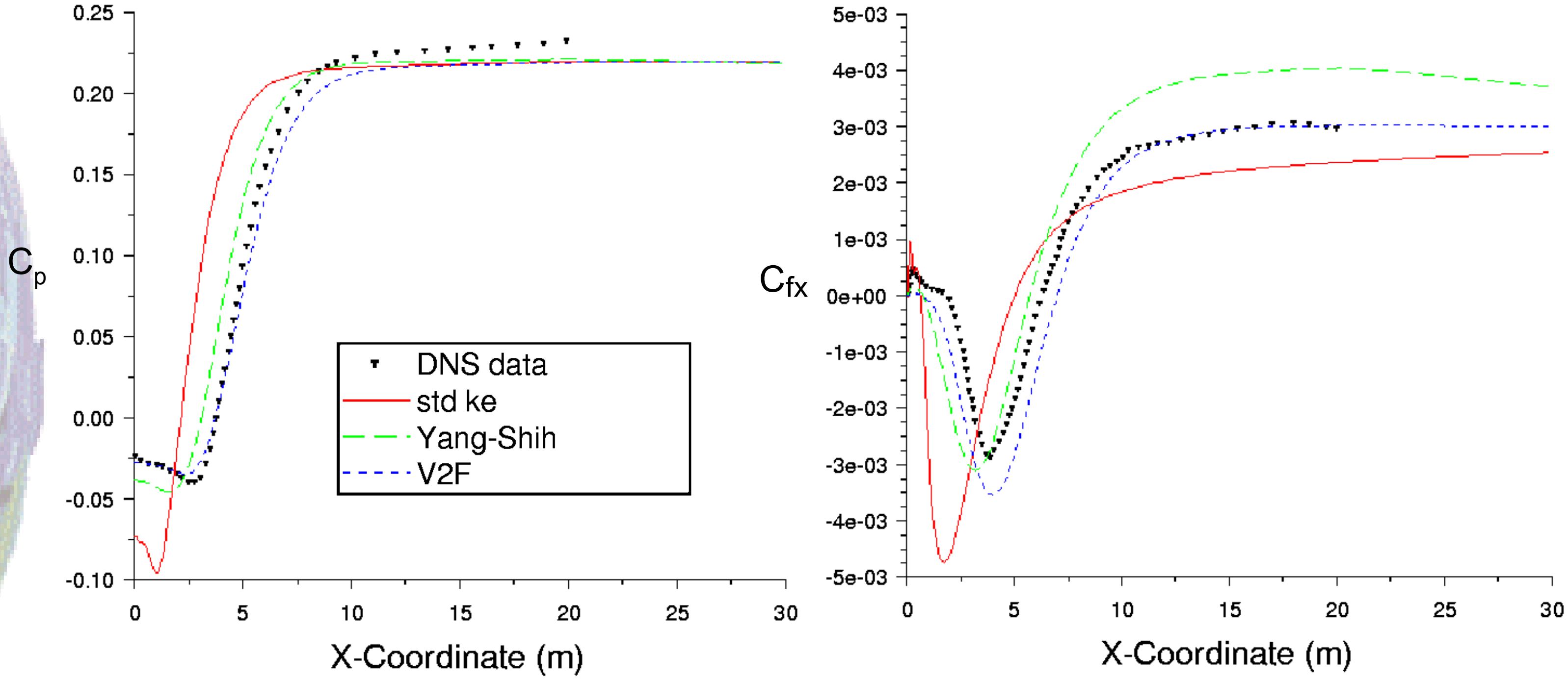
151x91 mesh with 10 nodes less than $y^{+}=10$

Low-Re Backstep



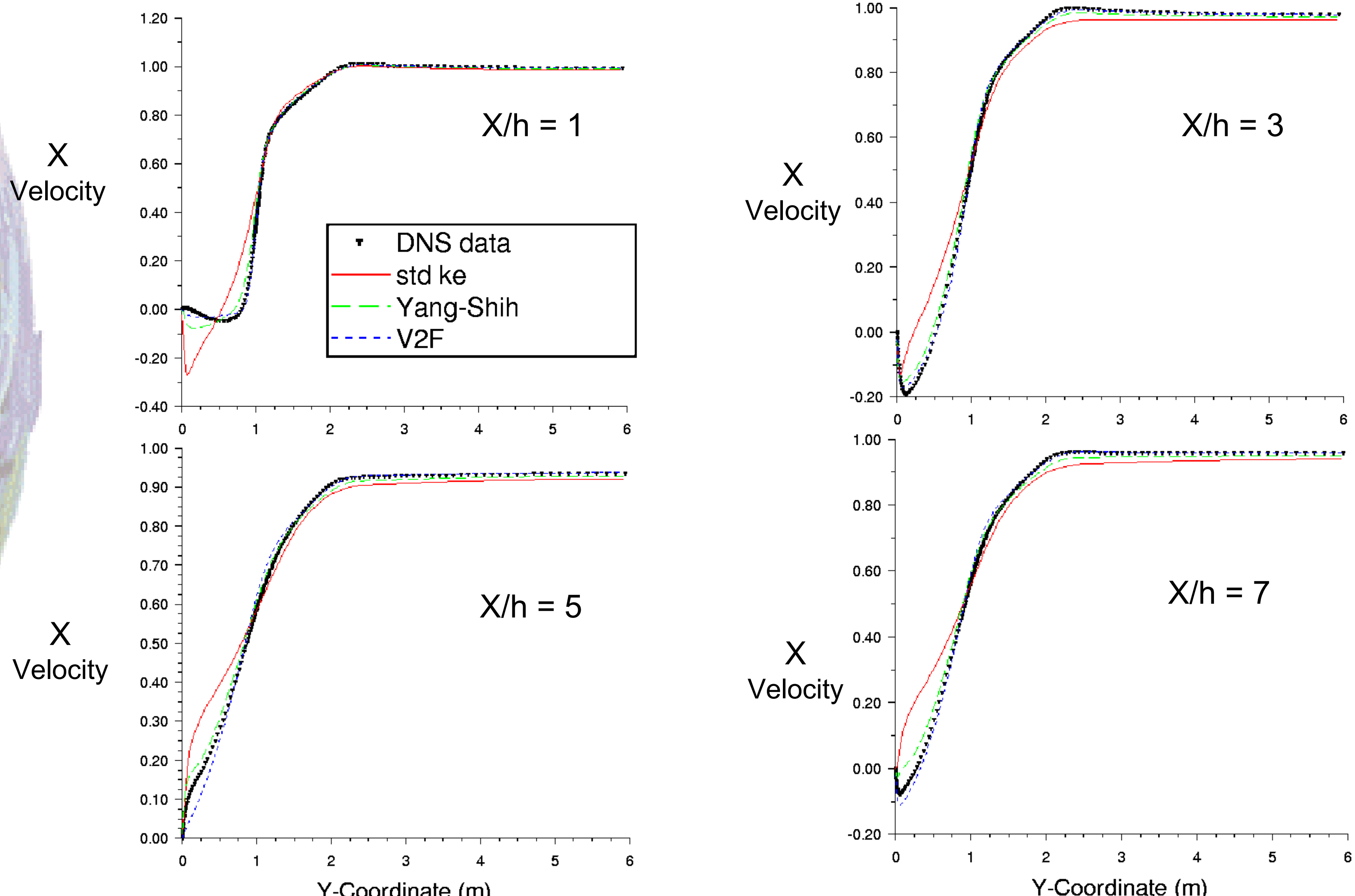
- Re = 5,100
- Comparison with DNS data of Le and Moin (1994)
- Comparison of std. k-ε + 2-layer, Yang-Shih low-Re model and V2F low-Re model

Low-Re Backstep

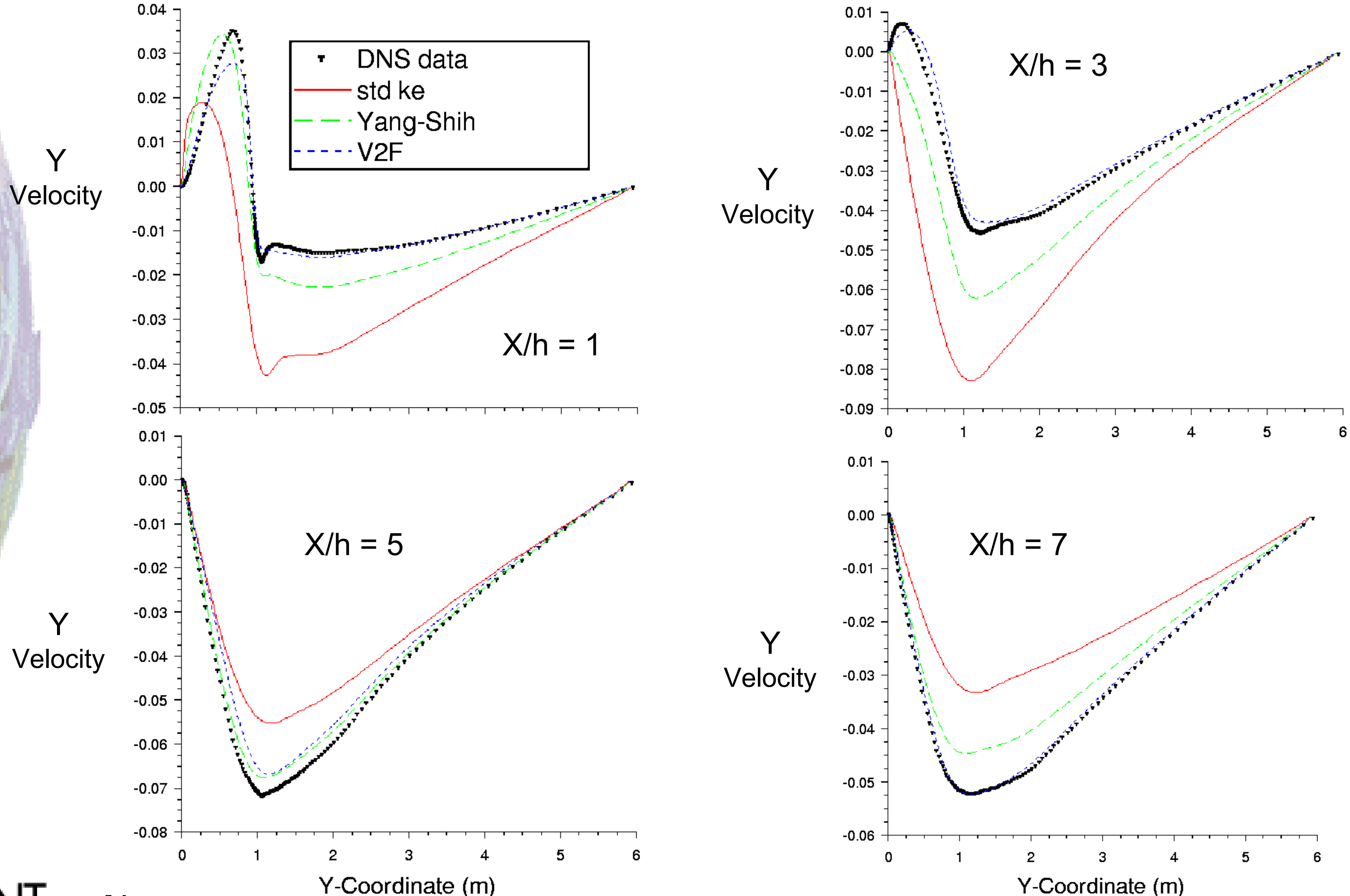


- Pressure coefficient and x-component of skin friction
- 2-layer model less accurate than V2F and Yang-Shih

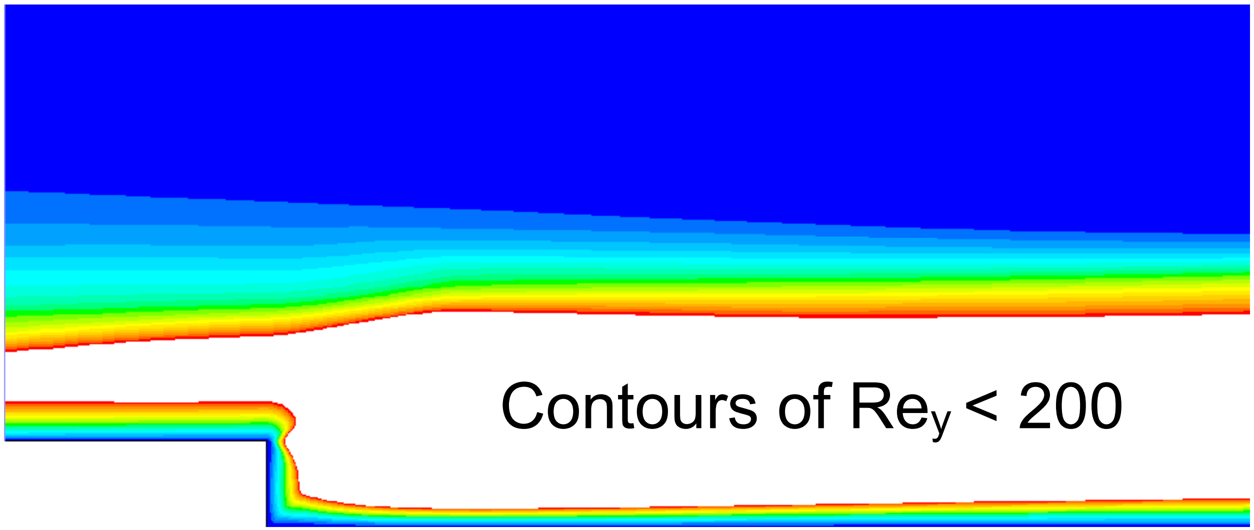
Low-Re Backstep



Low-Re Backstep



Low-Re Backstep

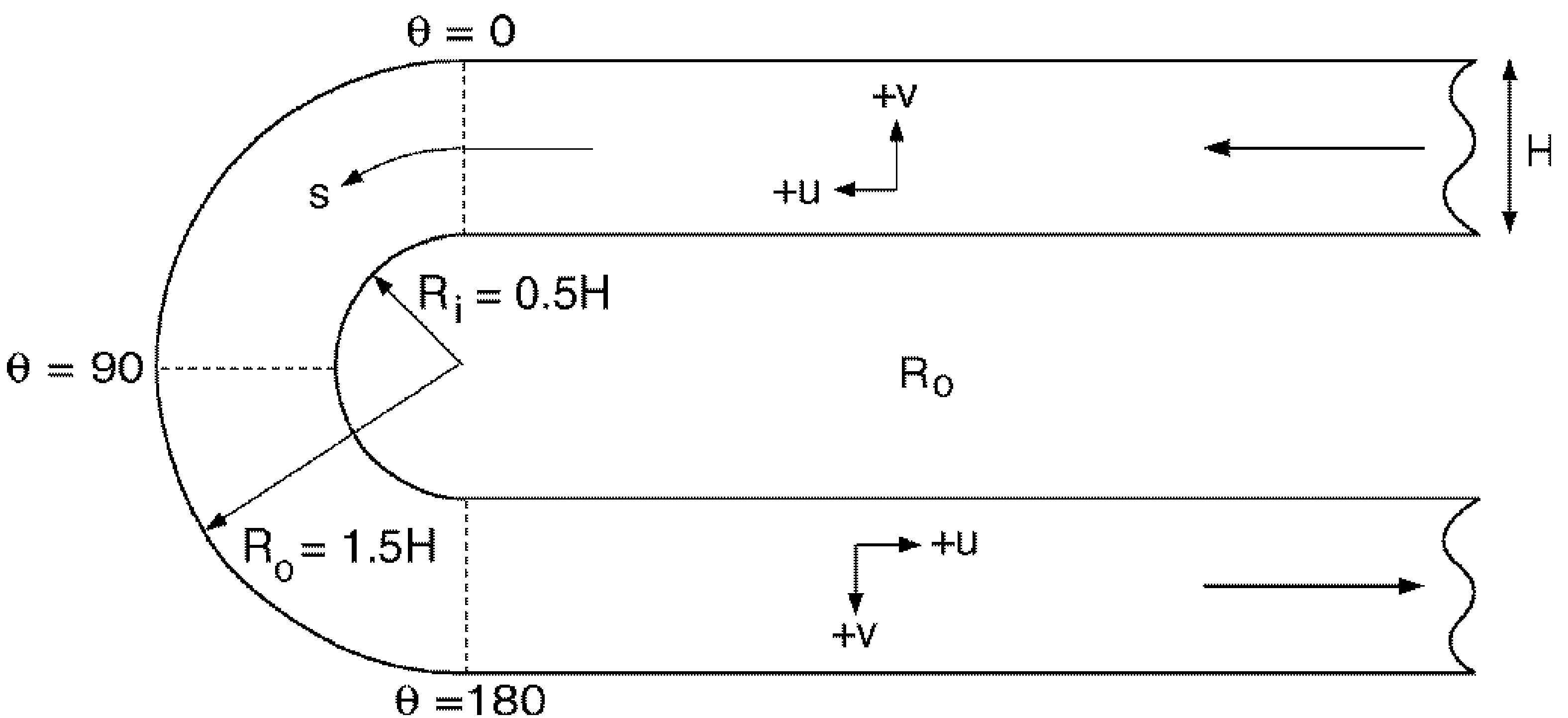


ε and ν_t prescribed algebraically for 2-layer model in region where $Re_y < 200$

For low Re, much of the flow is in this region

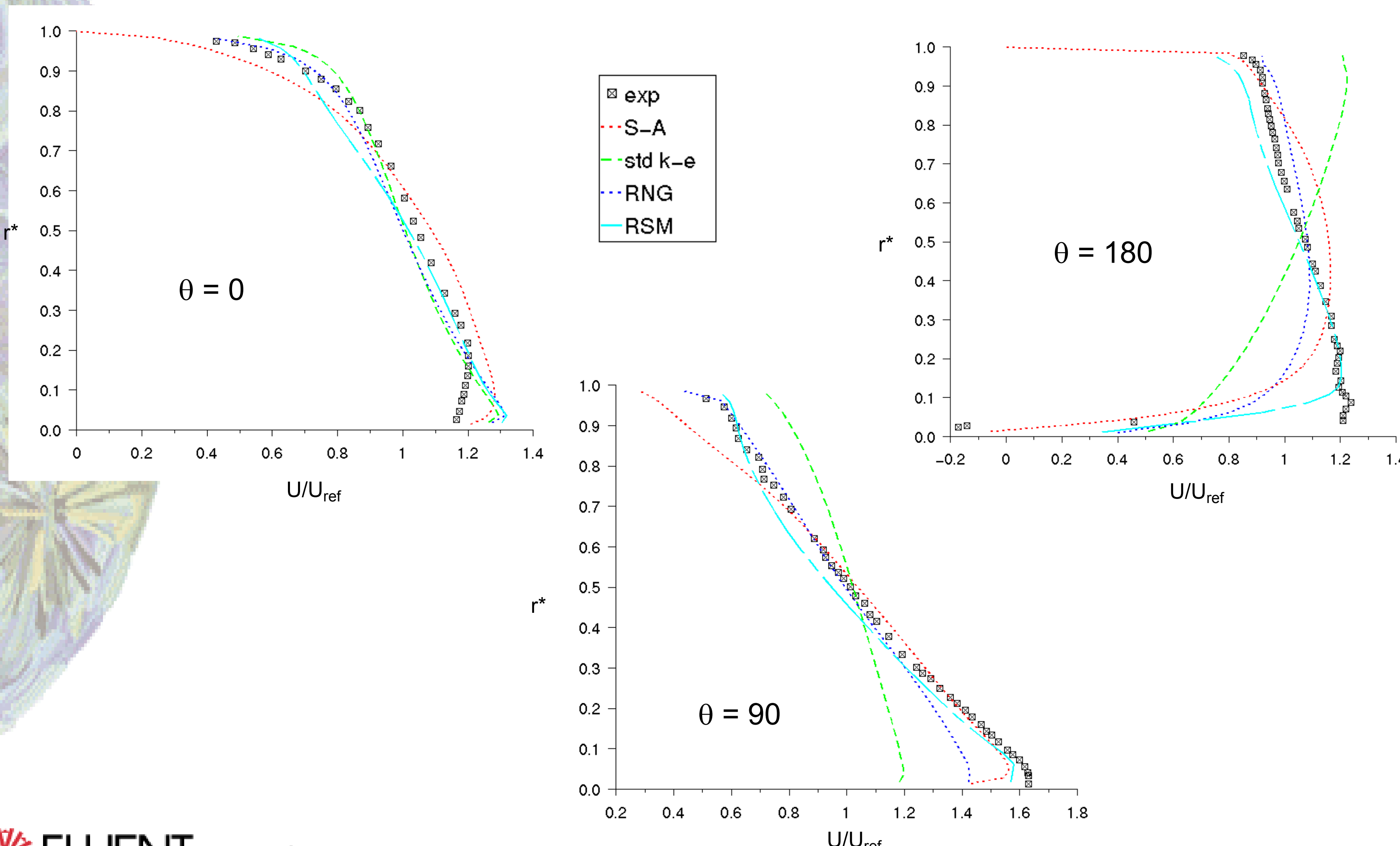
2-layer model is not always a good substitute for a low-Re model

2D U-Bend

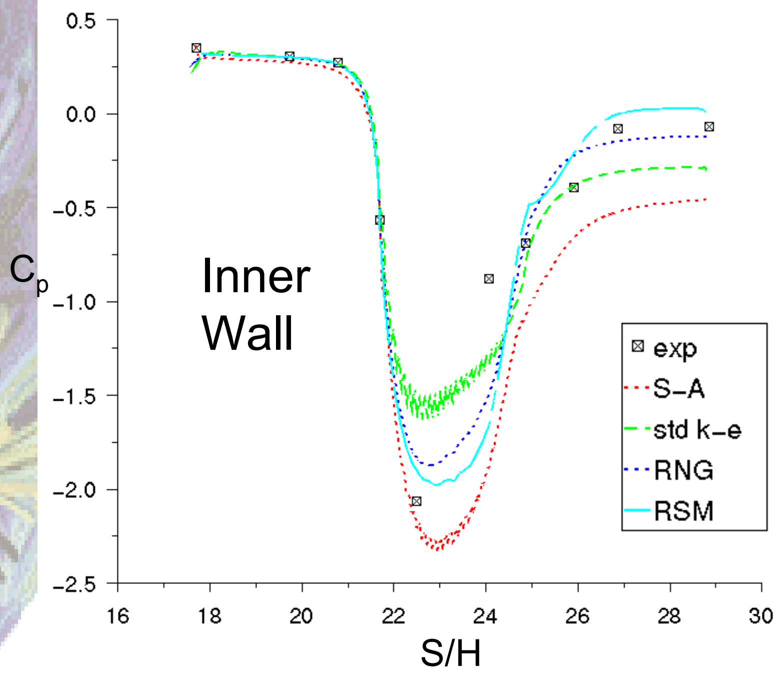


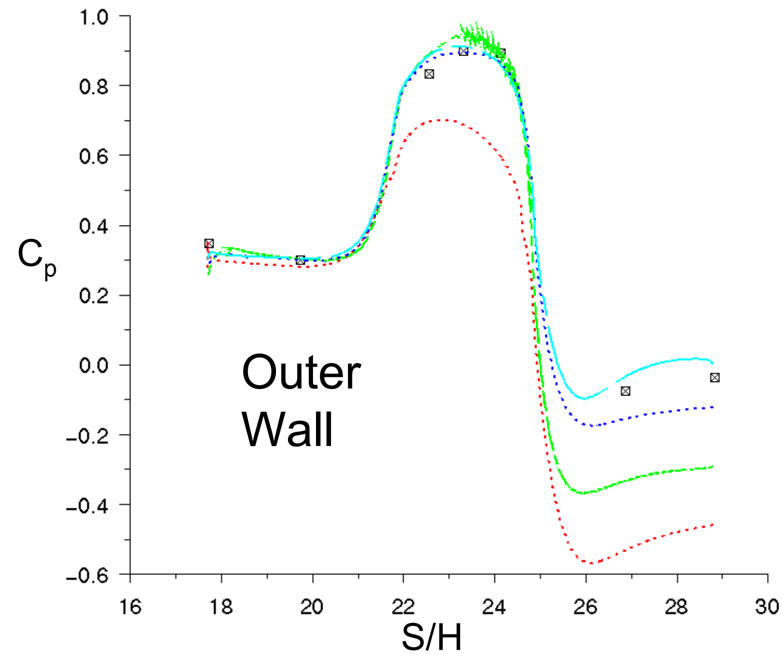
Comparison with exp. data of Monson *et al.* (1990)

Streamwise Velocity Comparisons

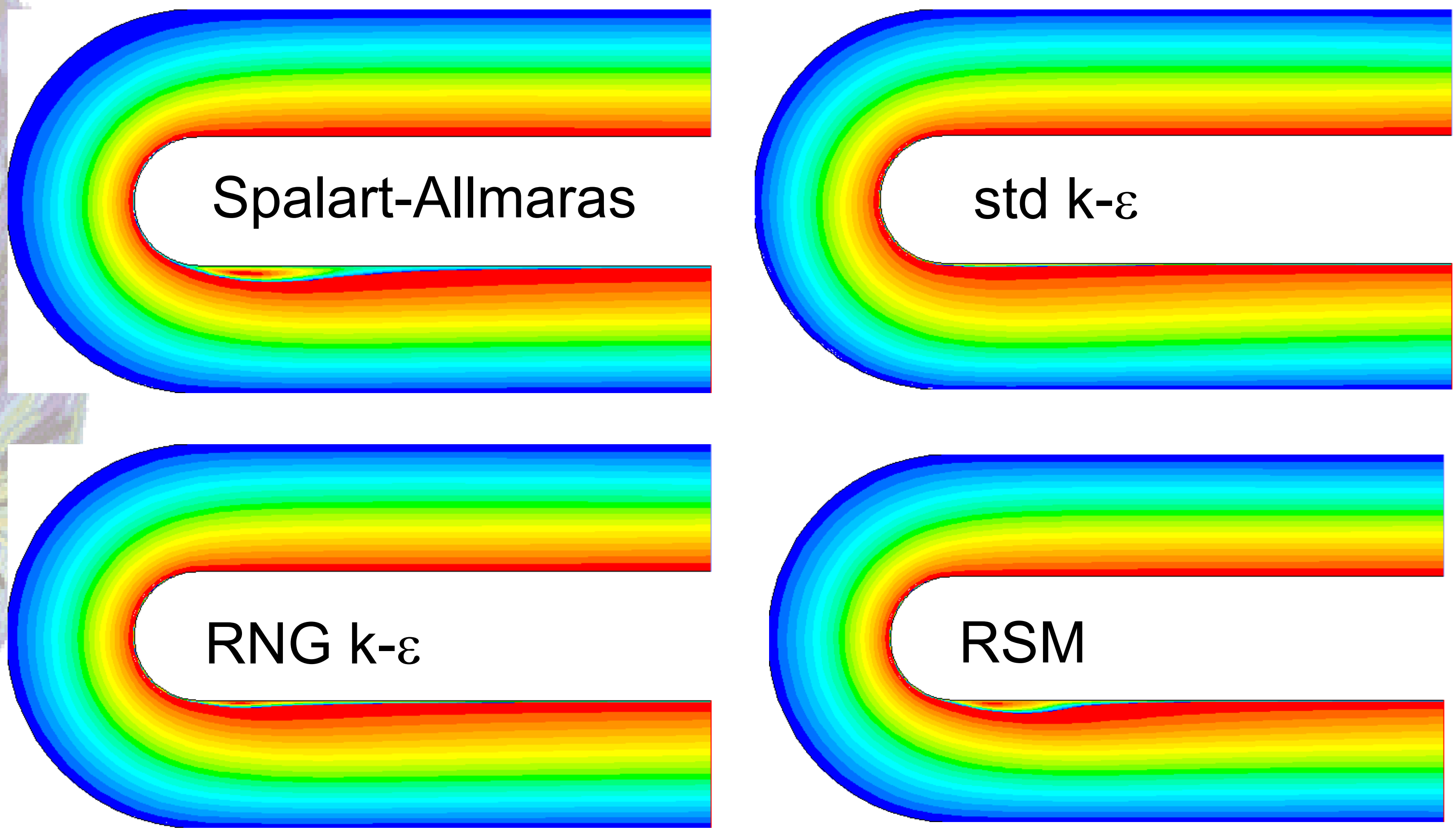


Pressure Coefficients





Stream Function Contours



Lessons from 2-D U-Bend

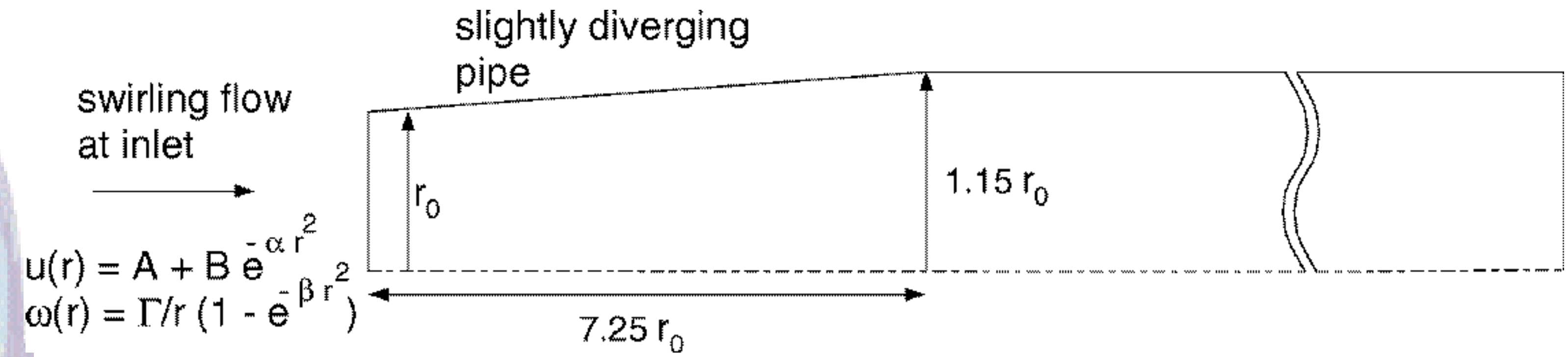
Only the RSM correctly predicts the effects of streamline curvature

Std. k-e does not predict any separation

RNG k-e predicts slight separation

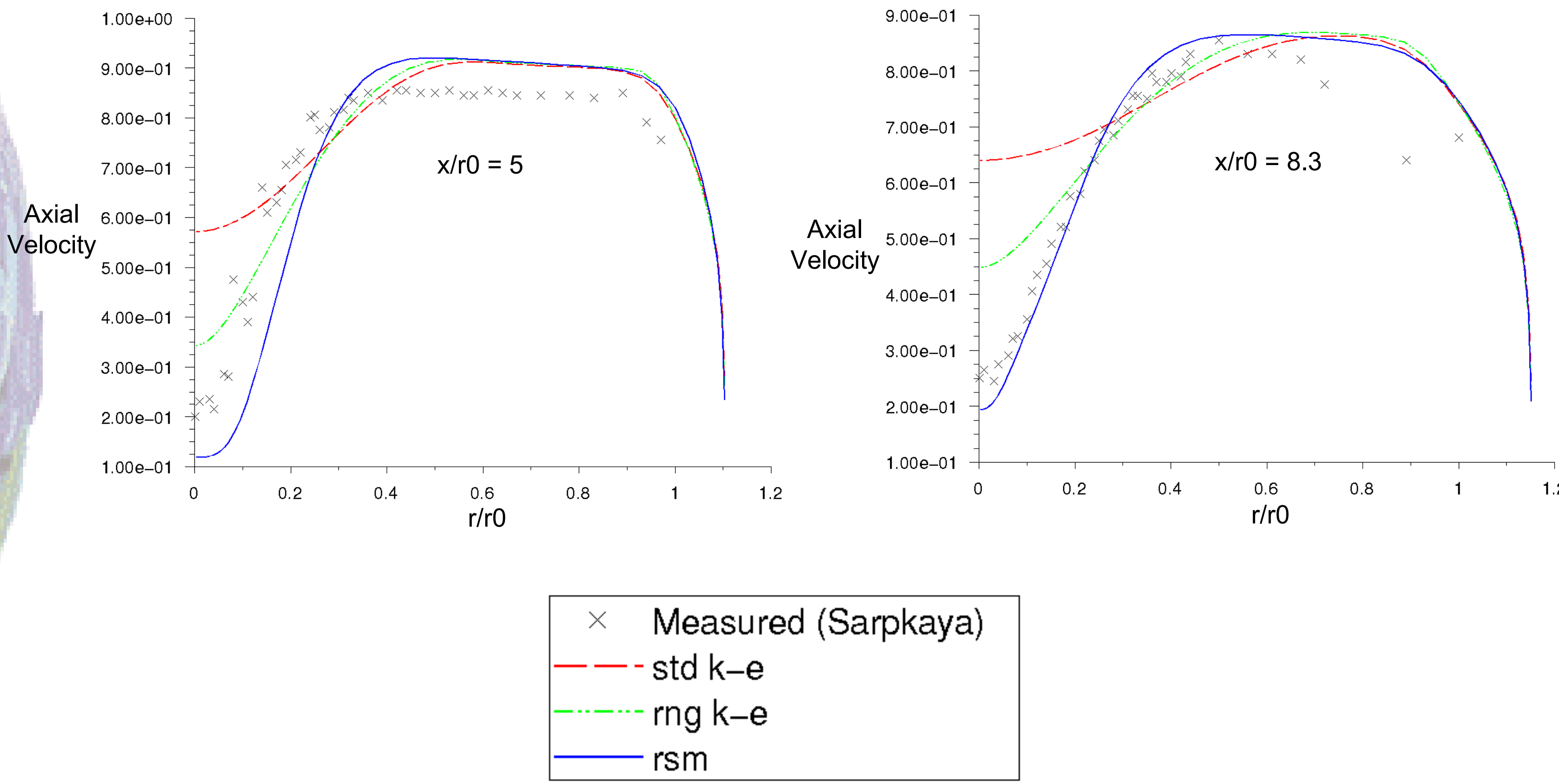
Both RSM and Spalart-Allmaras predict significant separation

Turbulent Vortex Breakdown

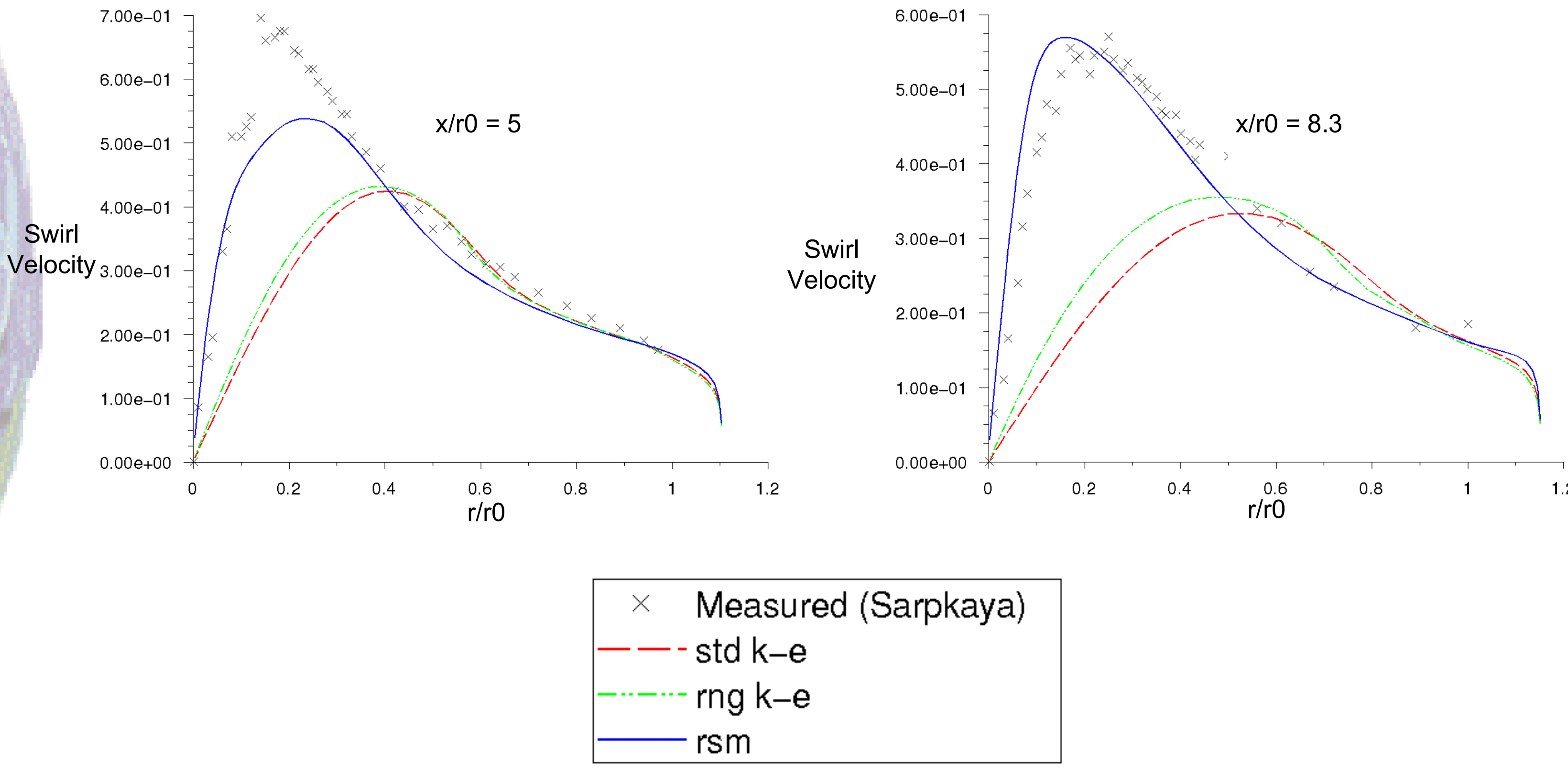


- Comparison with experimental data of Sarpkaya (1999)
- 2D axisymmetric calculation
- Simulation courtesy of R. Spall, Utah State University

Comparisons of Axial Velocity Profiles



Comparisons of Swirl Velocity Profiles



Lessons from Turbulent Vortex Breakdown

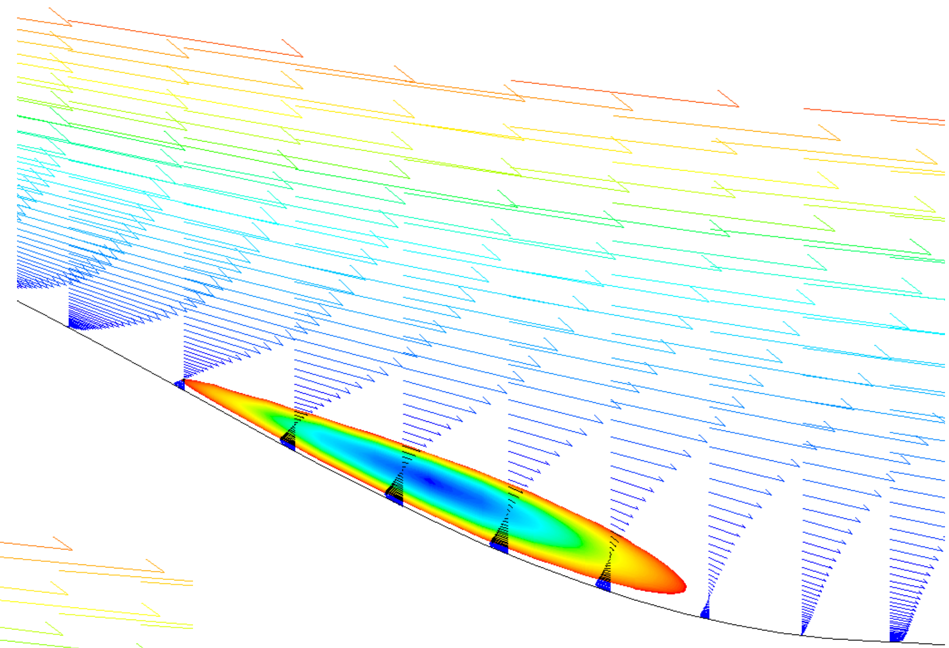
- k-ε model cannot predict vortex breakdown
  - in high strain rates, turbulent kinetic energy increases and increases turbulent viscosity
  - RNG k-ε model is better (additional strain-rate term, and an ad hoc swirl correction, reduce the turbulent viscosity) but not acceptable
- RSM results show significant improvement for this and many other swirling flow cases

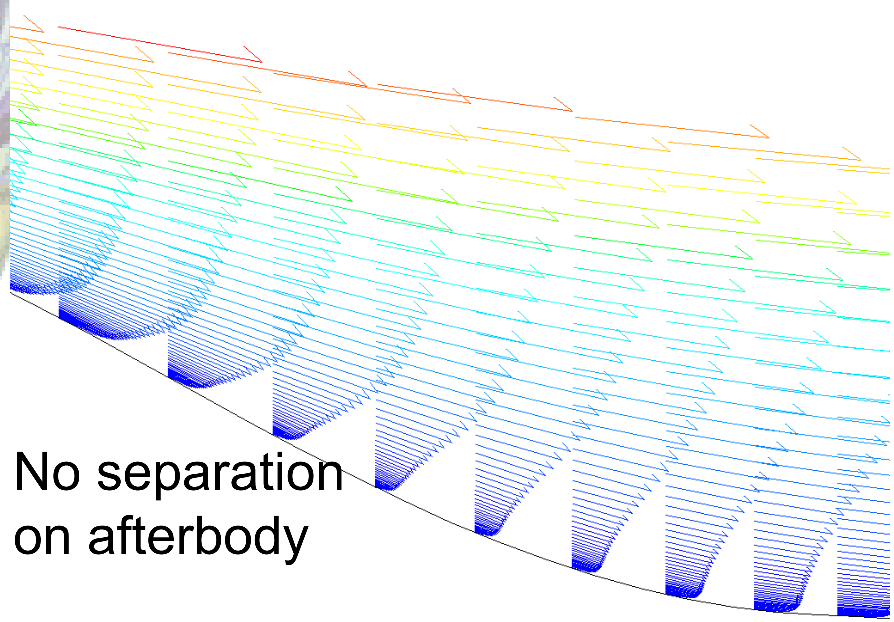
Axisymmetric Afterbody

- Compared to experiment of Huang *et al.*
- Spalart-Allmaras, std k-ε, RNG k-ε, realizable k-ε and RSM
- Run on both coarse and fine quad meshes
  - wall functions on coarse meshes
  - 2-layer formulation on fine mesh with k-ε and RSM
  - low-Re formulations investigated

Axisymmetric Afterbody

Spalart-Allmaras model (fine mesh)



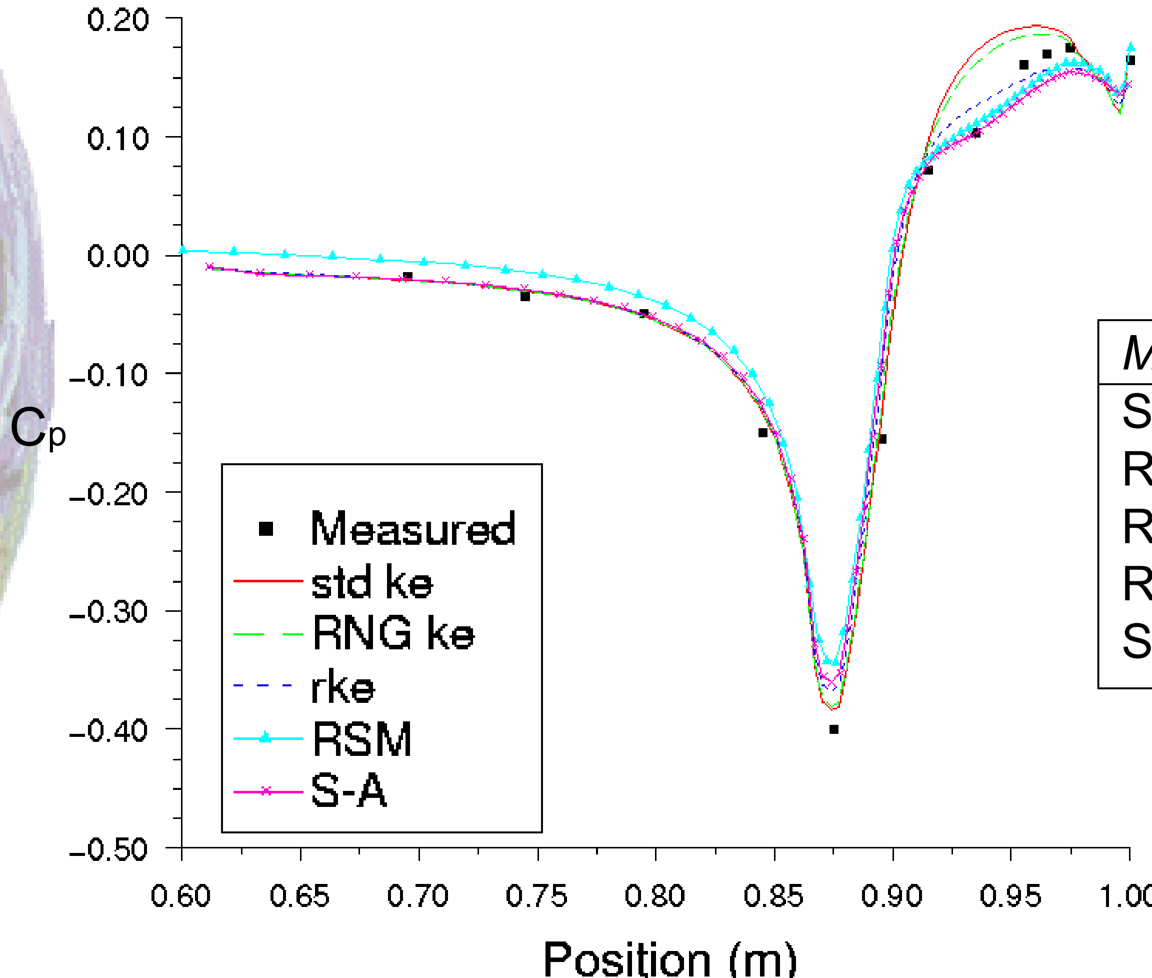


No separation on afterbody

Std. k-ε model + 2-layer (fine mesh)

Axisymmetric Afterbody

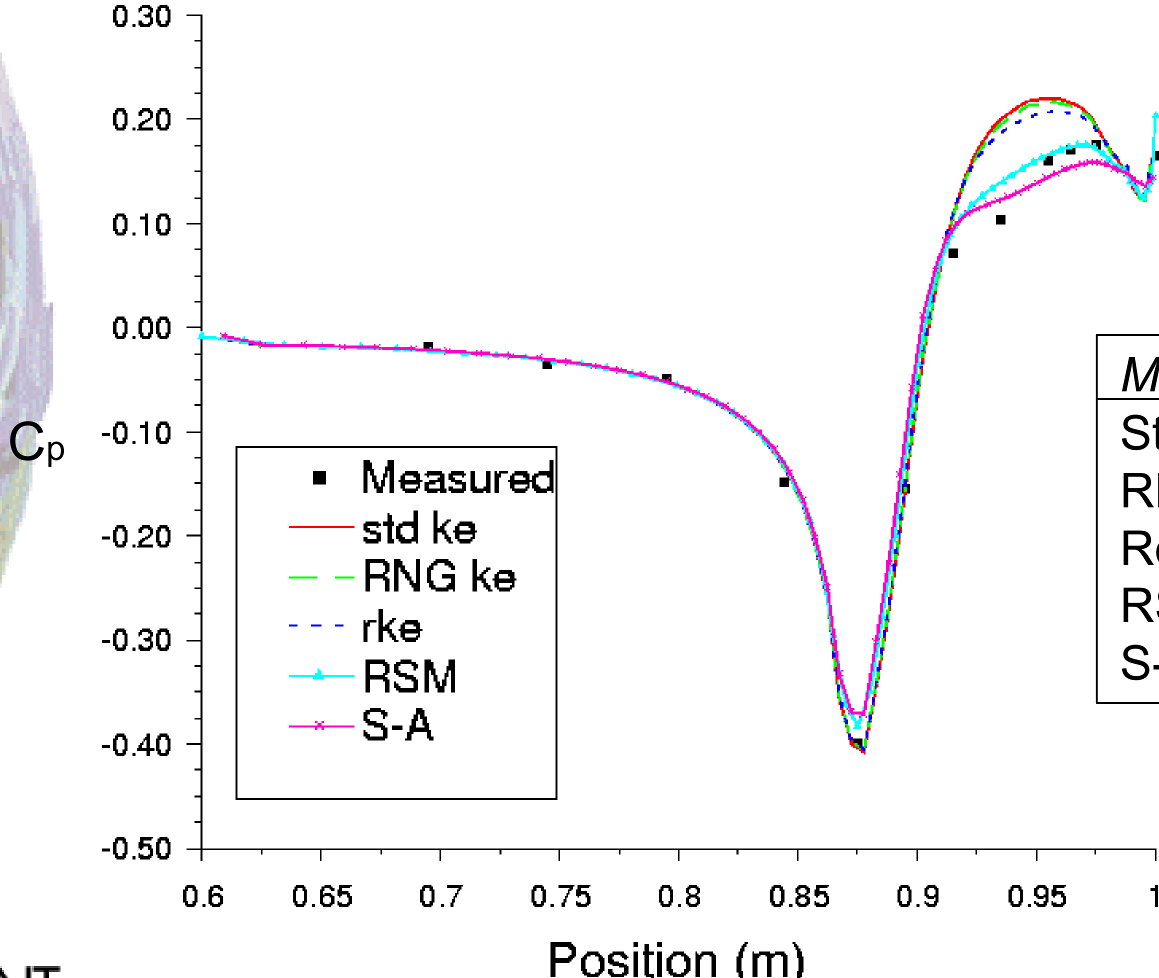
- Pressure coeff. on coarse mesh ($y^+ \sim 40$) using wall functions



Model	*Separates?*
Std k-ε	n
RNG k-ε	n
Real. k-ε	y
RSM	y
S-A	y

Axisymmetric Afterbody

- Pressure coeff. on fine mesh ($y^+ \sim 0.5$) using two-layer model



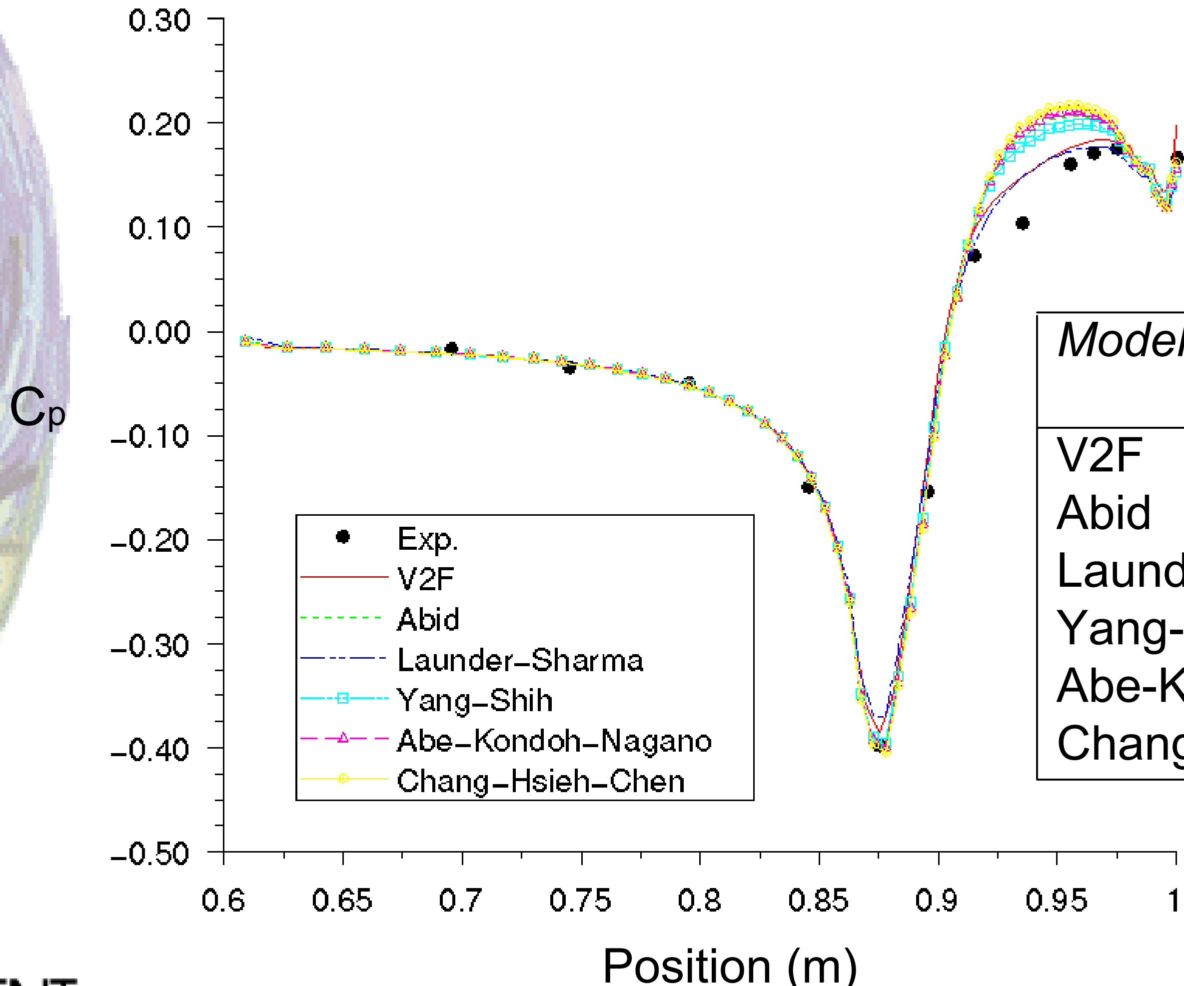
Model	*Separates?*
Std k-ε	n
RNG k-ε	n
Real. k-ε	y
RSM	y
S-A	y

Axisymmetric Afterbody

- Spalart-Allmaras gives consistent results on both meshes
- Separation not predicted by std. k-ε on either mesh
- RSM separates on both meshes
  - C_p on body somewhat overpredicted on coarse mesh
  - "Wall reflection" term, or quadratic pressure-strain term, necessary to obtain coarse mesh separation
- Subtle separation illustrates effect of near-wall treatment
  - Realizable k-ε has smaller separation bubble on fine mesh
- Difficult to get grid-independent solutions using wall functions --- would a low-Re formulation work?

Axisymmetric Afterbody

- Pressure coeff. on fine mesh using Low-Re k-ε models

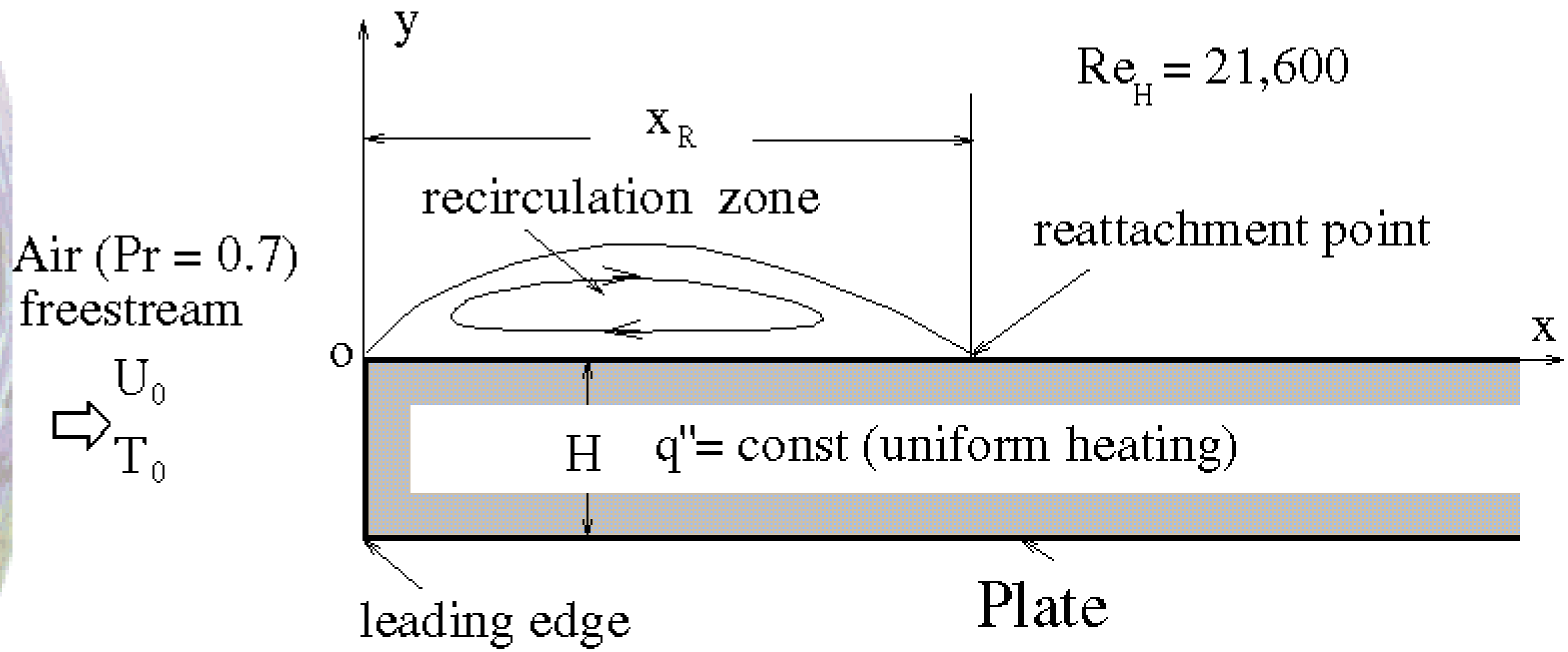


Model	*Sepa-rates?*
V2F	y
Abid	n
Launder-Sharma	n
Yang-Shih	n
Abe-Kondo-Nagano	n
Chang-Hsieh-Chen	n

Axisymmetric Afterbody

- Low-Re models using damping functions do not predict the separation
- Durbin's V2F (4-equation) model predicts separation

Turbulent Heat Transfer Over a Blunt Plate

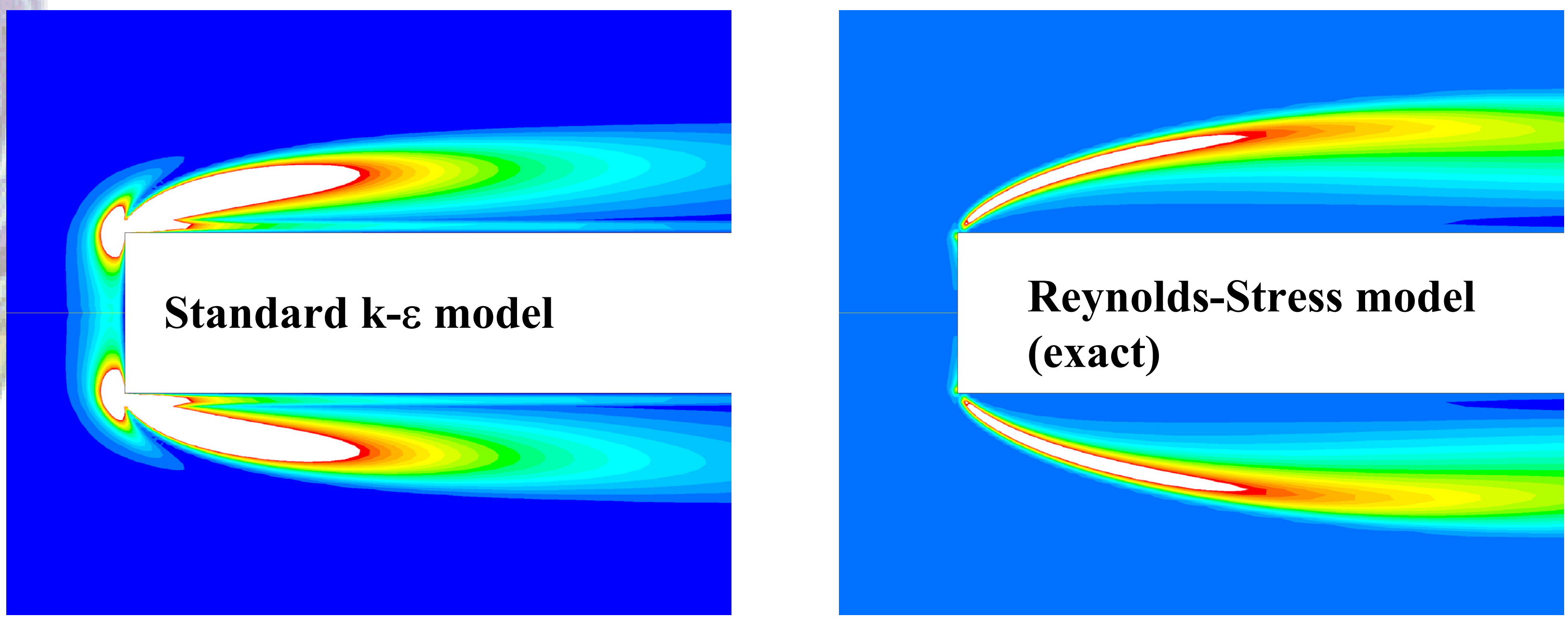


Ota & Kan

151x75 quad mesh

Blunt Plate

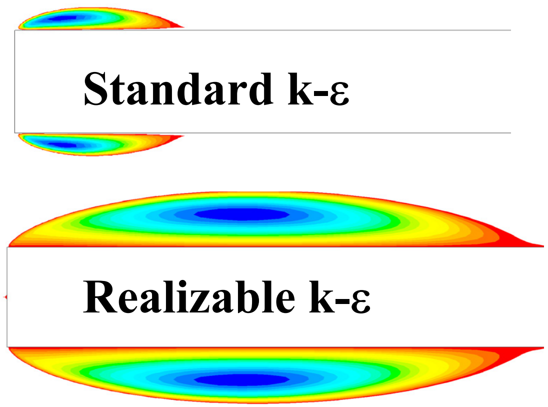
- The standard k-ε model gives spuriously large turbulent kinetic energy on the front face, underpredicting the size of the recirculation.

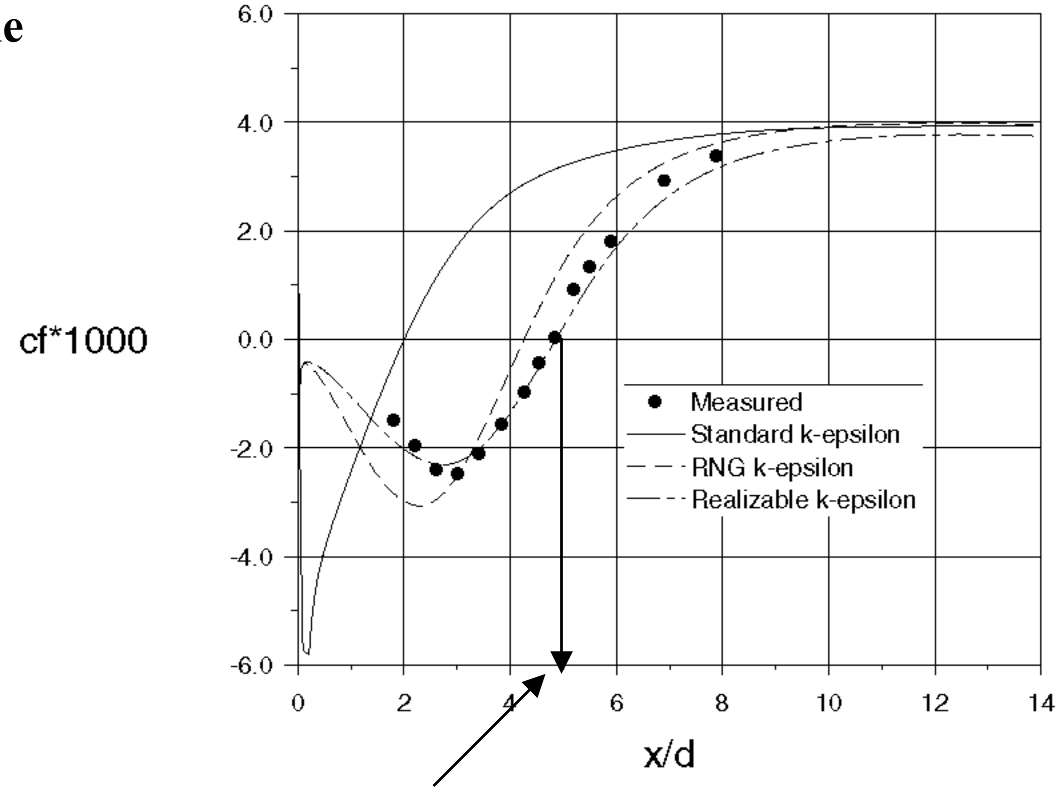


Contours of TKE production

Blunt Plate

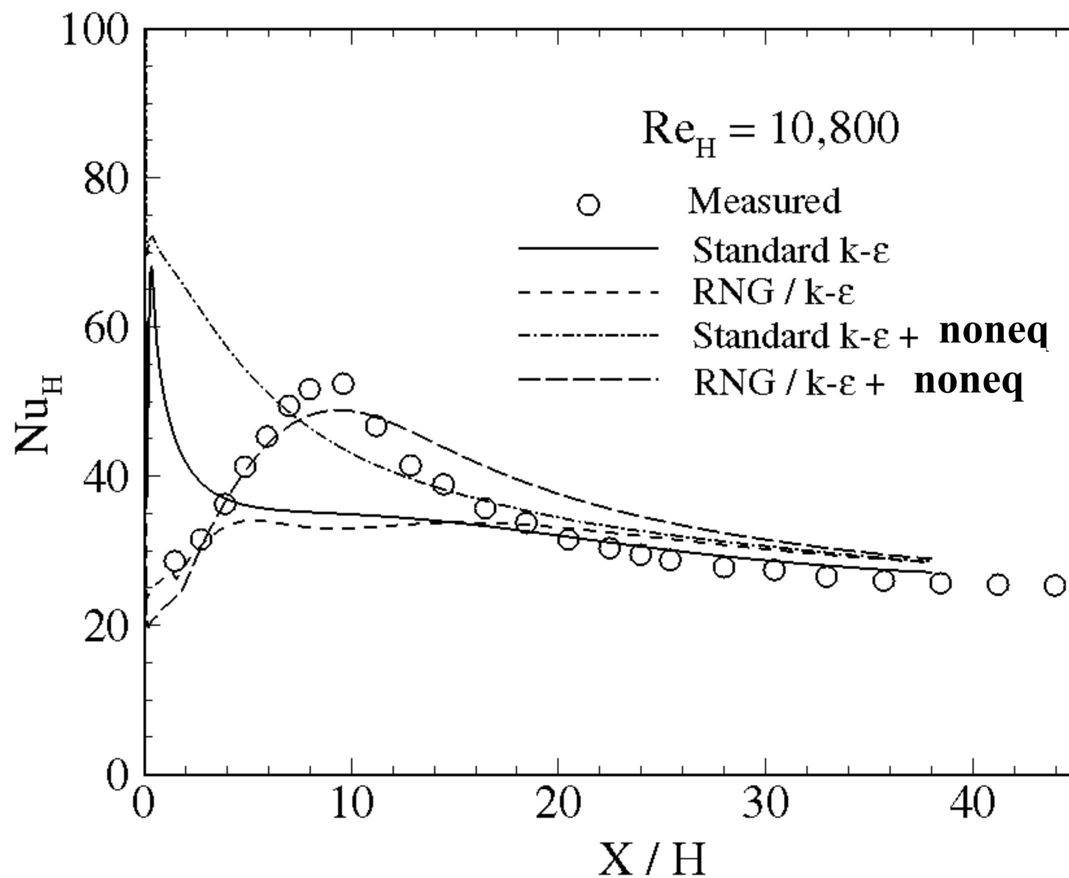
Predicted separation bubble





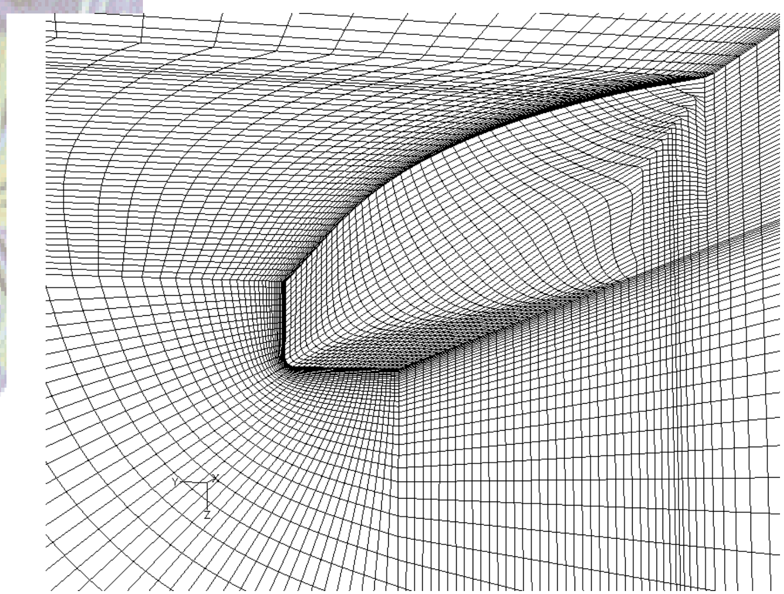
Experimentally observed reattachment point is at x/d = 4.7

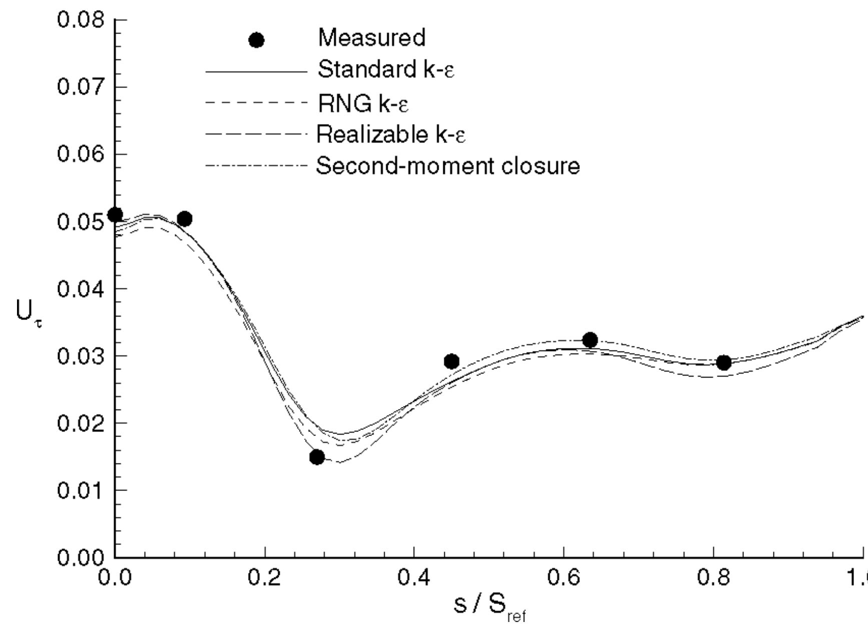
Heat Transfer Over a Blunt Plate



HSVA Ship Model

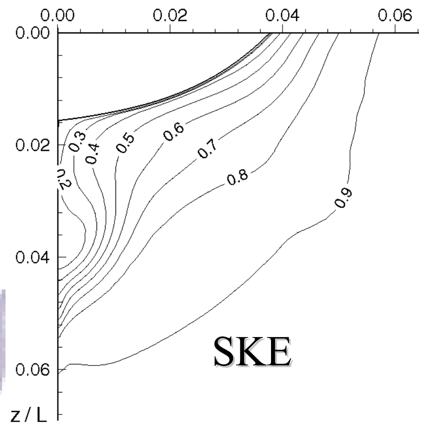
- Measured in an wind tunnel with a double-body model
  - ReL = 5 x 106
  - 70x41x56 hex mesh was used.
- Computed using different turbulence models with the standard wall functions
- Girthwise friction-velocity at x/L = 0.945



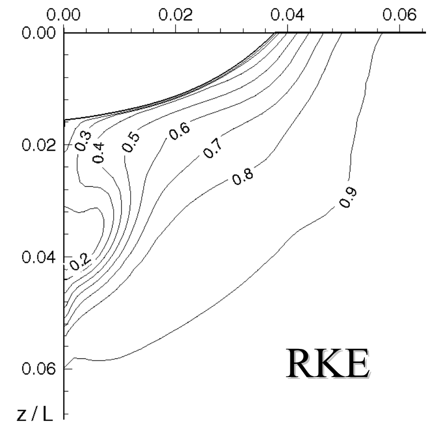


HSVA Ship Model

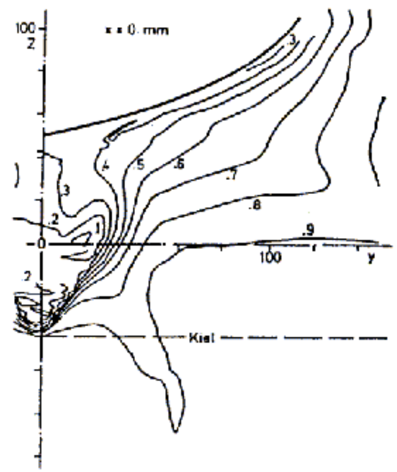
U-contours at *x/L* = 0.974



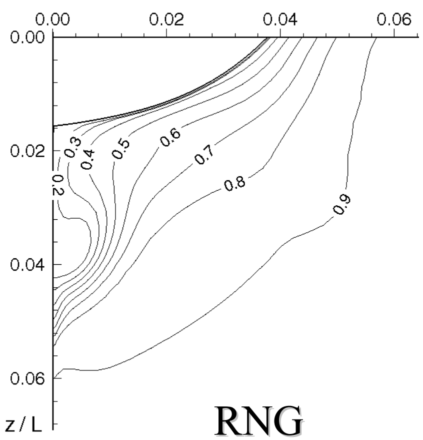
SKE



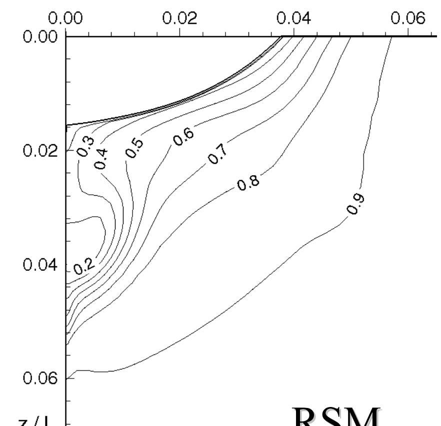
RKE



Measured



RNG

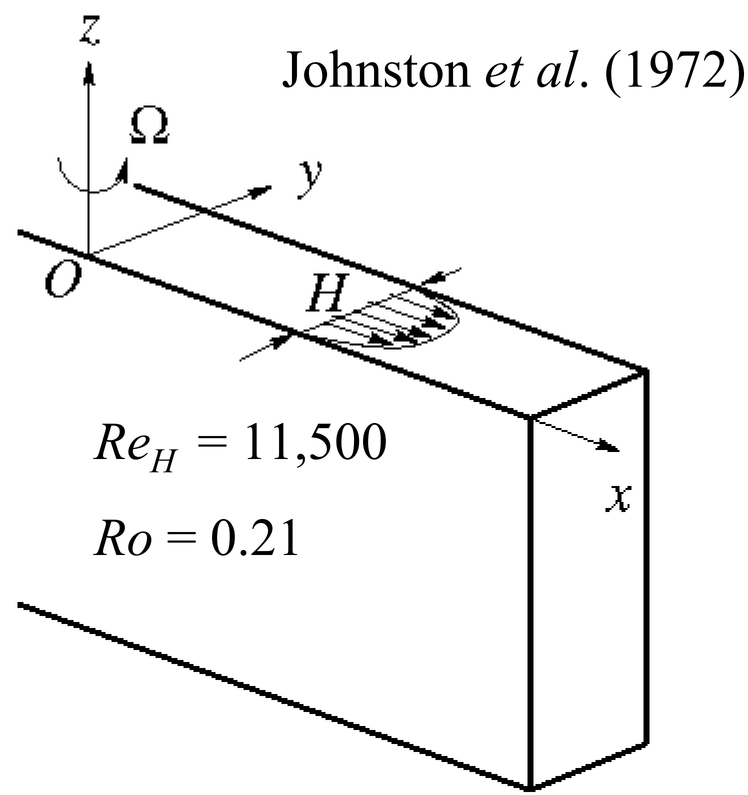


RSM

Flow in a Rotating Channel

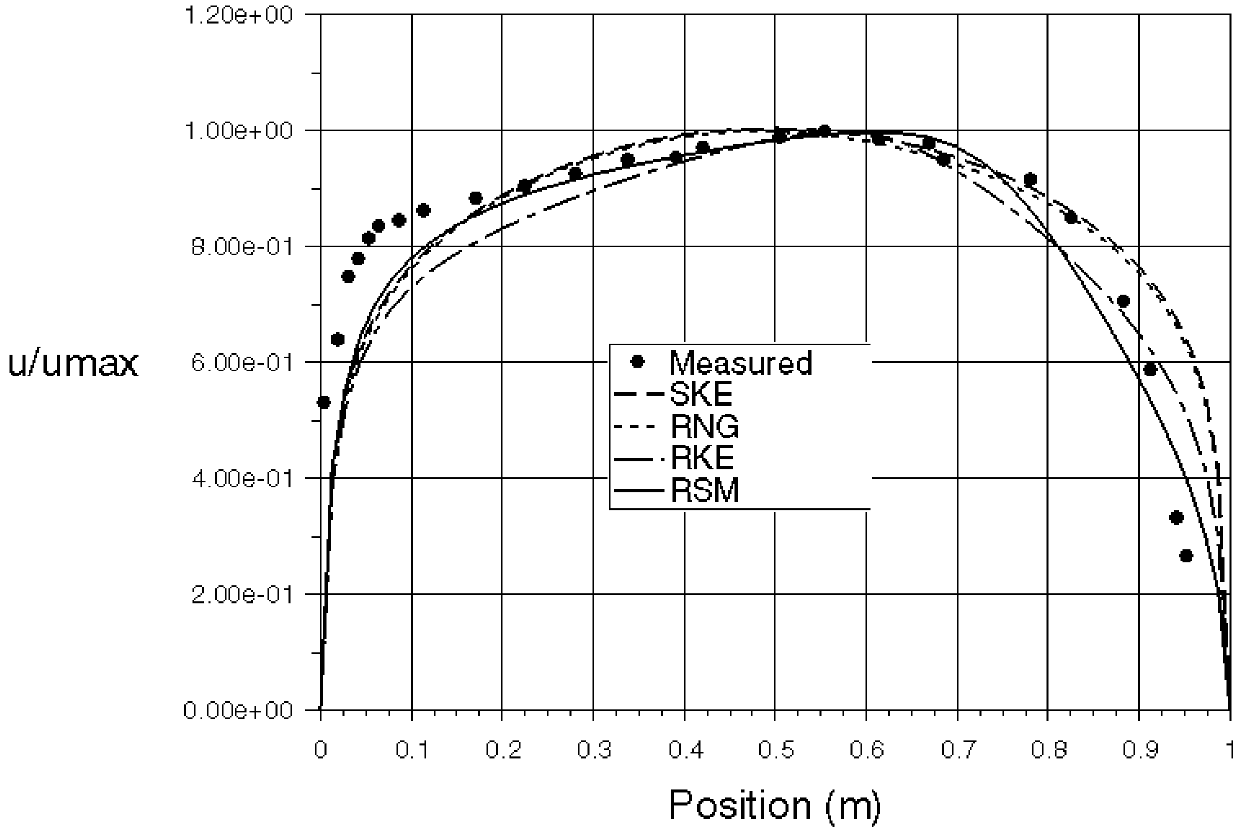
- Represents flows through rotating internal passages (e.g. turbomachinery applications)
- Rotation affects mean axial momentum equation through turbulent stresses.
- Rotation makes mean axial velocity asymmetrical.
- Computations are carried out using SKE, RNG, RKE and RSM models are with the standard wall functions.

Flow configuration;



Flow in a Rotating Channel

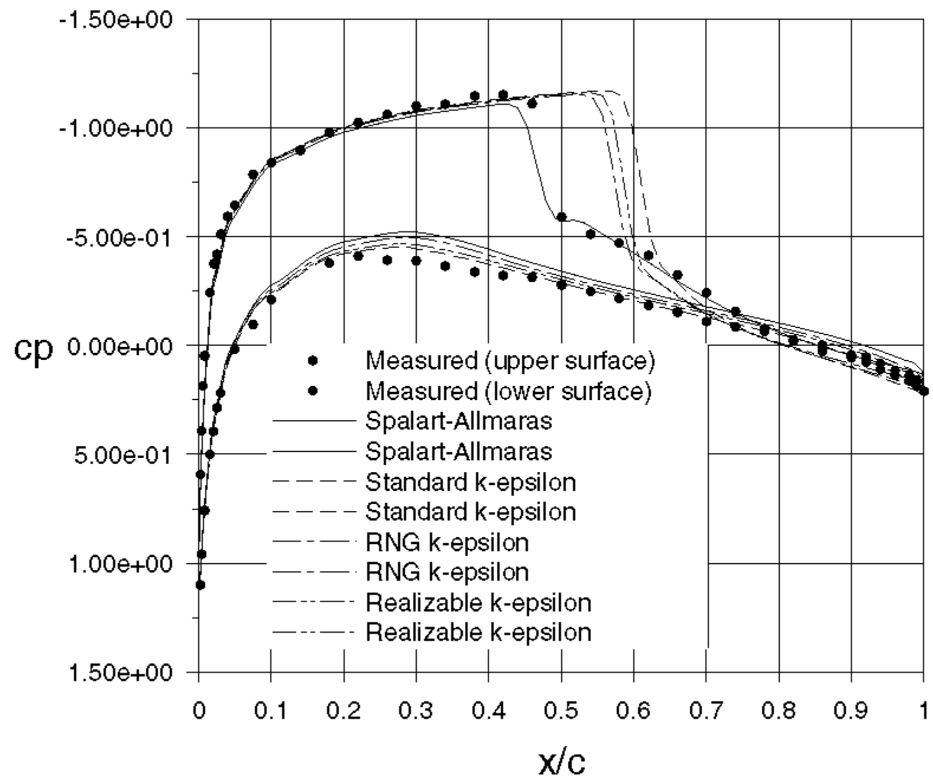
Predicted axial velocity profiles (Re_H = 11.500, Ro = 0.21)



NACA 0012 Airfoil

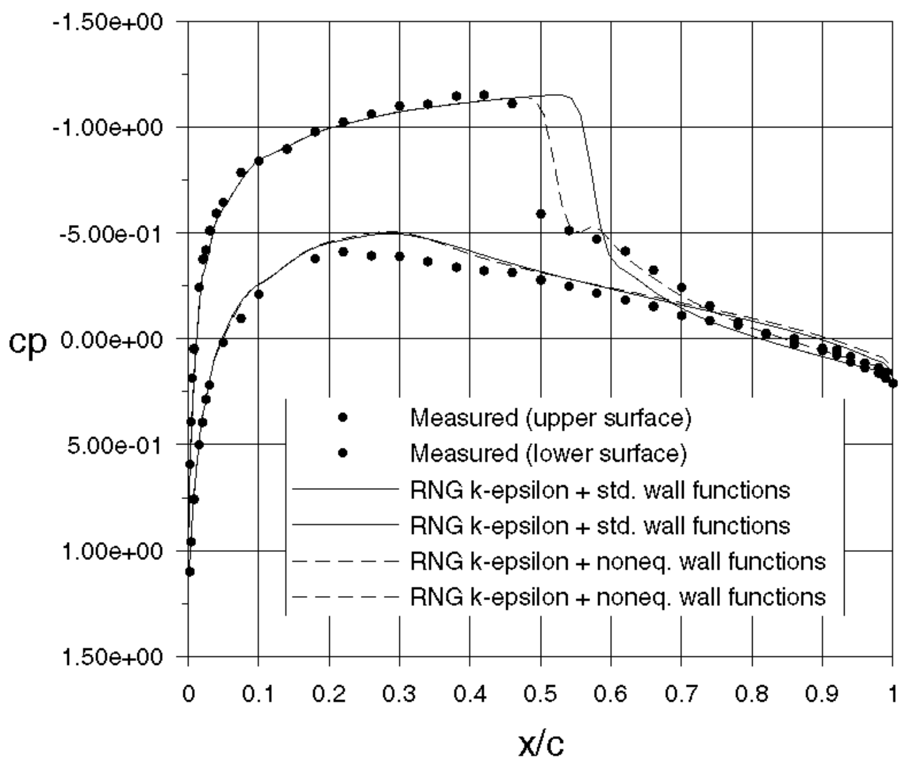
- Transonic flow (Ma = 0.8, $Re_c = 9 \times 10^6$) with boundary-layer separation
  - a = 2.26 degrees
  - Computed using different one-and two-equation turbulence models (S-A, SKE, RNG, RKE) with standard wall functions

C_p predictions by standard wall functions

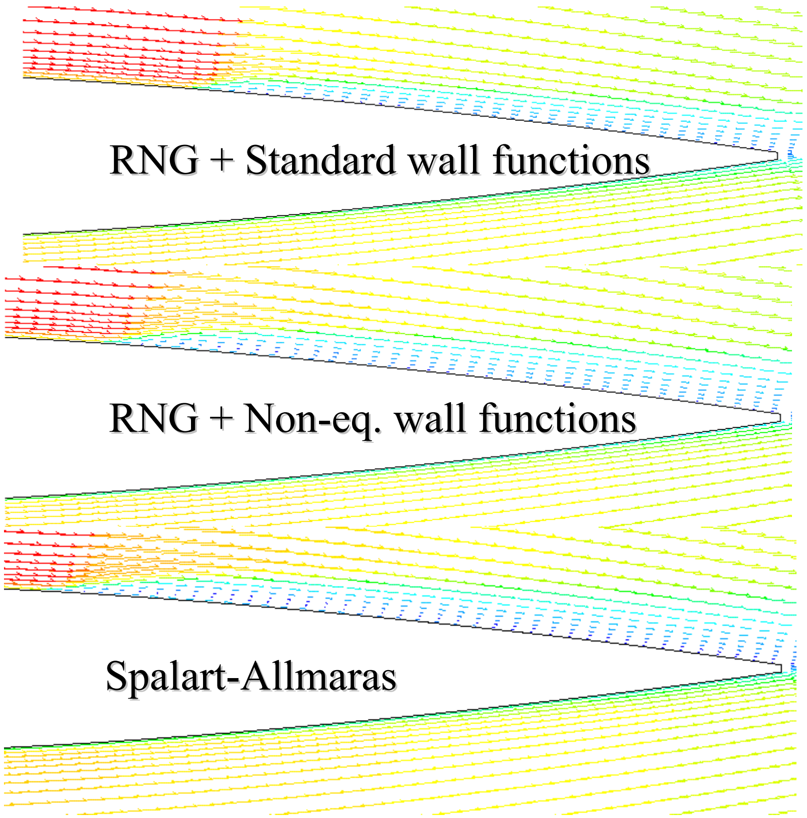


NACA 0012 Airfoil

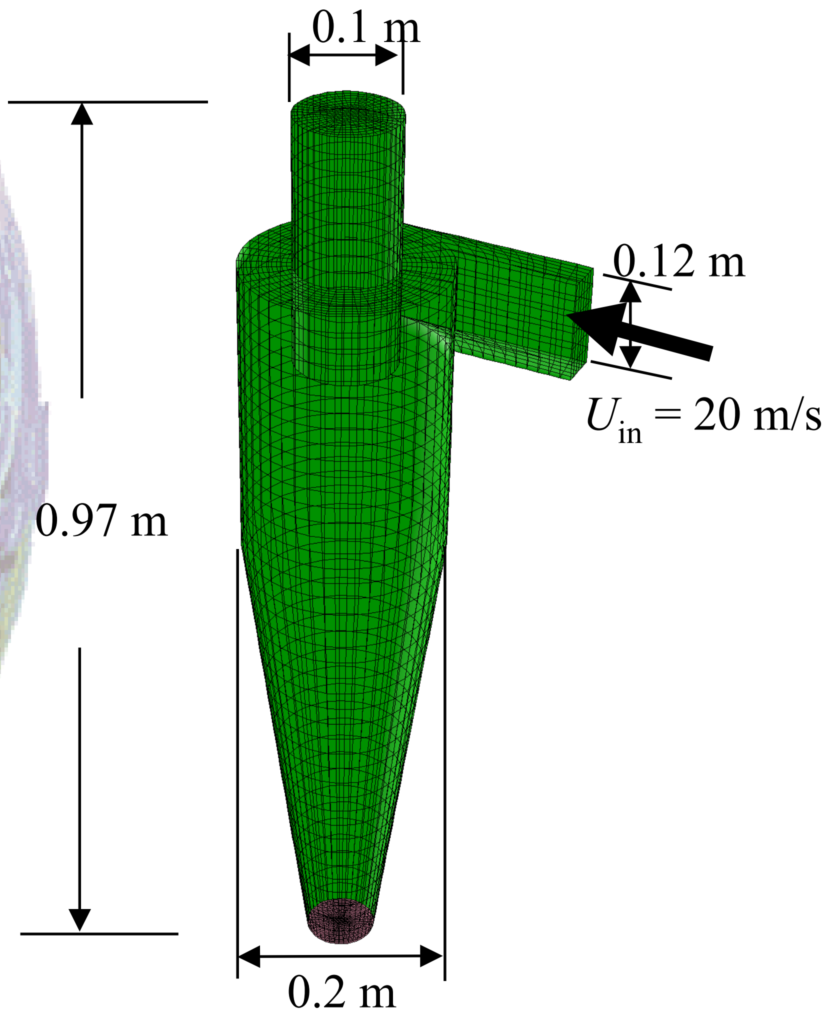
- C_p predictions by RNG model with non-equilibrium wall functions



- Predictions of B.L. separation



Flow in a Cyclone



- 40,000 cell hexahedral mesh
- High-order upwind scheme was used.
- Computed using SKE, RNG, RKE and RSM models with the standard wall functions
- Represents highly swirling flows (W_{max} = 1.8 U_{in})

Flow in a Cyclone

- Tangential velocity profile predictions at 0.41 m below the vortex finder